Algorithms: Design and Analysis, Part II

Minimum Spanning Trees

Proof of the Cut Property

The Cut Property

Assumption: distinct edge costs.

$\boxed{\text{CUT PROPERTY:}}$ Consider an edge e of G. Suppose there is a cut (A,B) such that e is the cheapest edge of G that crosses it. Then e belongs to the MST of G.

Proof Plan

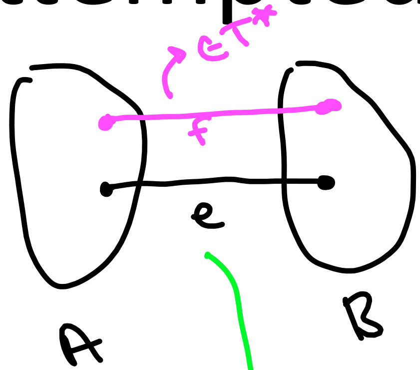
Will argue by contradiction, using an exchange argument. [compare to scheduling application]

Suppose there is an edge e that is the cheapest one crossing a cut (A,B), yet e is not in the MST T^*.

Idea: exchange e with another edge in T^* to make it even cheaper (contradiction).

Question: which edge to exchange e with?

Attempted Exchange



cheapest edge of G crossing (A,B); also, not in T^* (so $c_e < c_f$)

Note: since T^* is connected, must contain an edge f ($\neq e$) crossing (A,B).

Idea: exchange e & f to get a spanning tree cheaper than T^* (contradiction).

Exchanging Edges

Question: Let T^* be a spanning tree of G, $e \notin T^*$, $f \in T^*$. Is $T^* \cup \{e\} - \{f\}$ a spanning tree of G?

(T^* = pink edges)

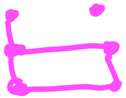
(A) Yes, always

(B) No, never

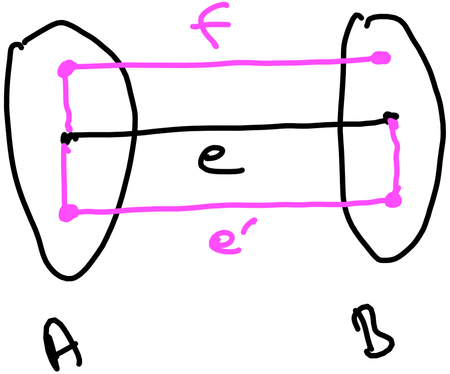
(C) If e is the cheapest edge crossing some cut, then yes.

(D) Maybe, maybe not (depending on the choice of e & f)

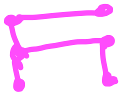
exchanging e and f yields



(not a spanning tree)



exchanging e and e' yields



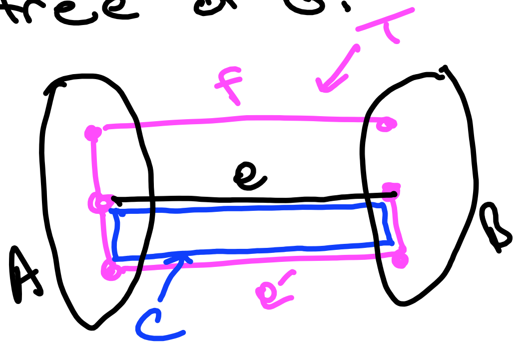
(a spanning tree)

Tim Roughgarden

Smart Exchanges

Hope: Can always find suitable edge e' so that exchange yields bona fide spanning tree of G.

How?: Let C = cycle created by adding e to T^*



By the Double-Crossing Lemma: Some other edge e' of C [with $e' \neq e$ and $e' \in T^*$] crosses (A,B).

You Check: $T = T^* \cup \{e\} - \{e'\}$ is also a spanning tree.

Since $c_e < c_{e'}$, T cheaper than purported MST T^*, contradiction.

QED!

Tim Roughgarden