Design and Analysis of Algorithms I

Graph Primitives

Dijkstra's Algorithm: Why It Works

Dijkstra's Algorithm

this array only to help explanation!

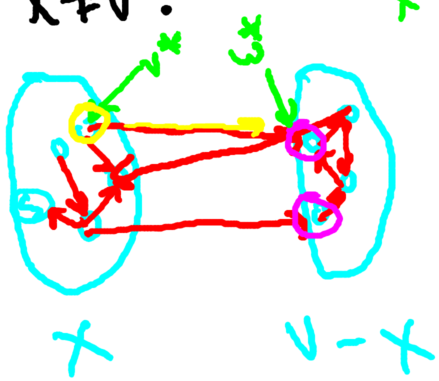
Initialize:

- $X = \{s\}$ [vertices processed so far]
- $A[s] = 0$ [computed shortest path distances]
- $B[s] =$ empty path [computed shortest paths]

Main Loop

- while $X \neq V$:

- need to grow X by one node



Main Loop cont'd

- among all edges $(v,w) \in E$ with $v \in X$, $w \notin X$, pick the one that minimizes

$$A[v] + \ell_{vw}$$

(Dijkstra's greedy criterion)

already computed in earlier iteration

[call it (v^*, w^*)]

- add w^* to X
- set $A[w^*] := A[v^*] + \ell_{v^*w^*}$
- set $B[w^*] := B[v^*] \cup (v^*, w^*)$

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Correctness Claim

Theorem [Dijkstra] For every directed graph with nonnegative edge lengths, Dijkstra's algorithm correctly computes all shortest-path distances.

[i.e., $\underbrace{A[v]}_{\text{what algorithm computes}} = \underbrace{L(v)}_{\text{true shortest path distance from } s \text{ to } v} \quad \forall v \in V$]

Proof: by induction on the number of iterations.

Base case: $A[s] = L(s) = 0$ (correct ✓)

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Proof

Inductive step:

Inductive Hypothesis: all previous iterations correct.

i.e., for all $v \in X$, $A[v] = L(v)$ and $B[v]$ is a true shortest s-v path in G.

In current iteration:

$\in X$ $\notin X$

we pick an edge (v^*, w^*) and we add w^* to X.

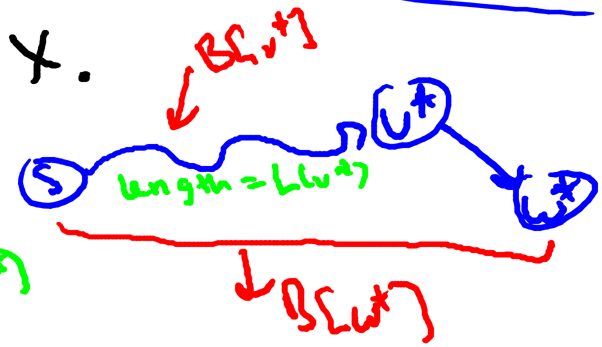
we set $B[w^*] = B[v^*] \cup (v^*, w^*)$

↳ has length by $L(v^*)$

has length $L(v^*) + \ell_{v^*w^*}$

$L(v^*)$ by I.H.

Also: $A[w^*] = A[v^*] + \ell_{v^*w^*} = L(v^*) + \ell_{v^*w^*}$



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Proof (con'd)

Upshot: in current iteration, we set:

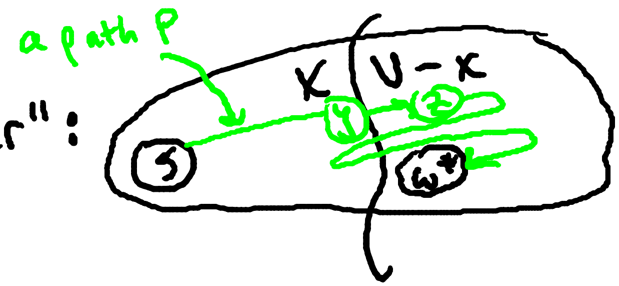
(1) $A[w^*] = L(v^*) + \ell_{v^*w^*}$

(2) $B[w^*]$ = an $s \to w^*$ path with length

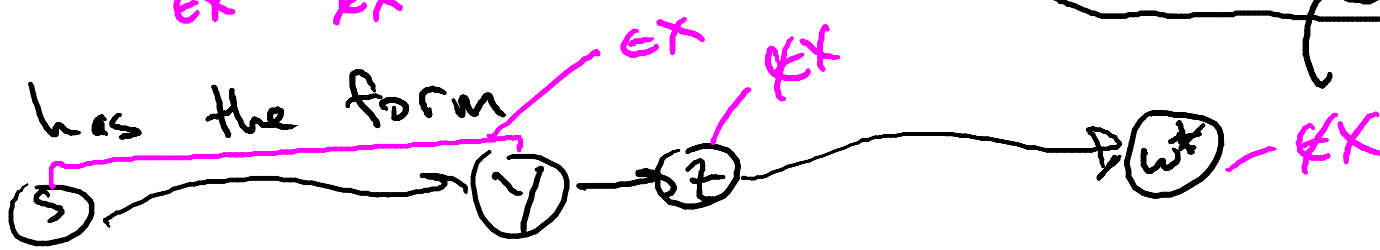
To finish proof: need to show that every $s - w^*$ path has length $\geq L(v^*) + \ell_{v^*w^*}$ (if so, our path is the shortest!)

So: let P = any $s \to w^*$ path. Must "cross the frontier":

$\in X$ $\notin X$



and so has the form



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Proof (con'd)

So: every $s \to w^*$ path P has to have the form

$\in X$ ≥ length of shortest s-y path $= L(y) = A[y]$ by inductive hypothesis (since $y \in X$)

length $= \ell_{yz}$

$\notin X$

length ≥ 0 (since no negative edges!)

($y \in X$, $z \notin X$)

Total length of path P: at least $A[y] + c_{yz}$

$\Rightarrow$ by Dijkstra's greedy criterion, $A[v^*] + \ell_{v^*w^*} \leq A[y] + \ell_{yz} \leq$ length of P.

length of our path!

QED!