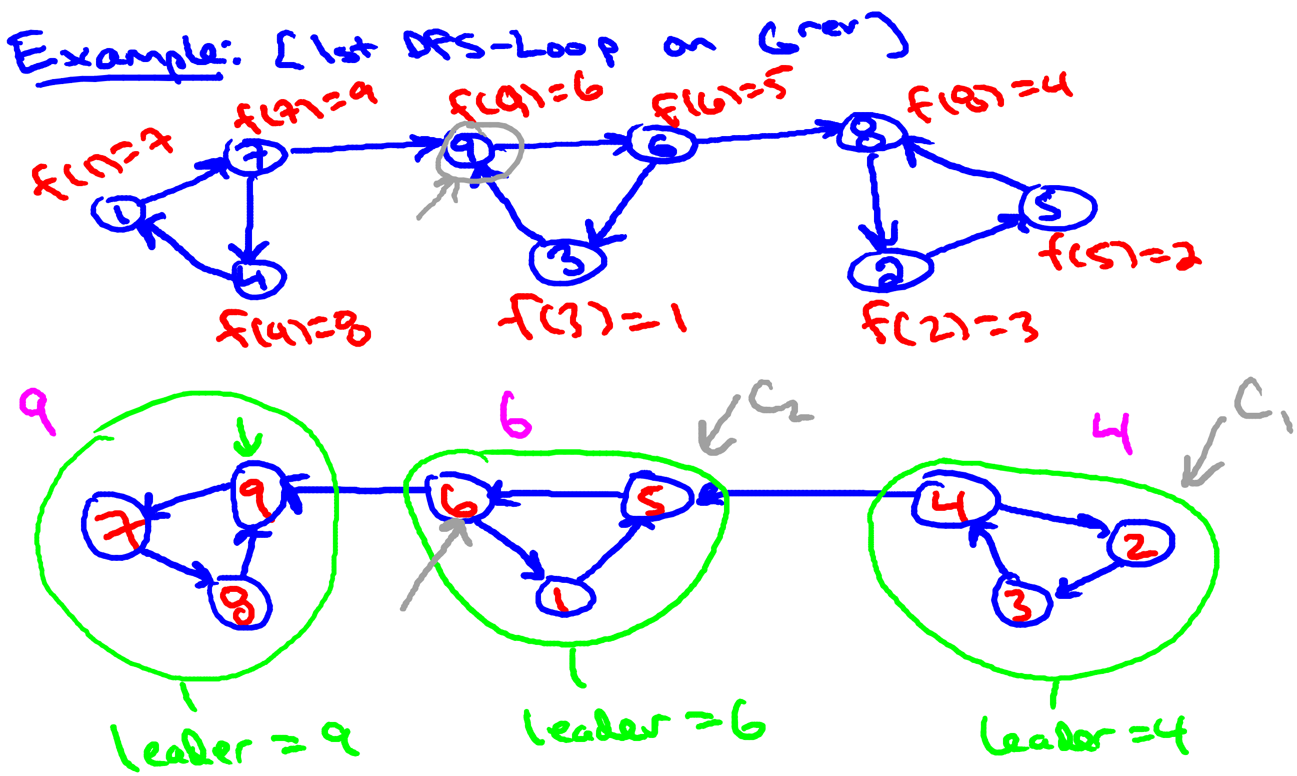
Design and Analysis of Algorithms I

Graph Primitives

Correctness of Kosaraju's Algorithm

Example Recap

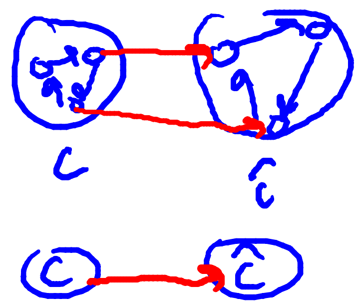


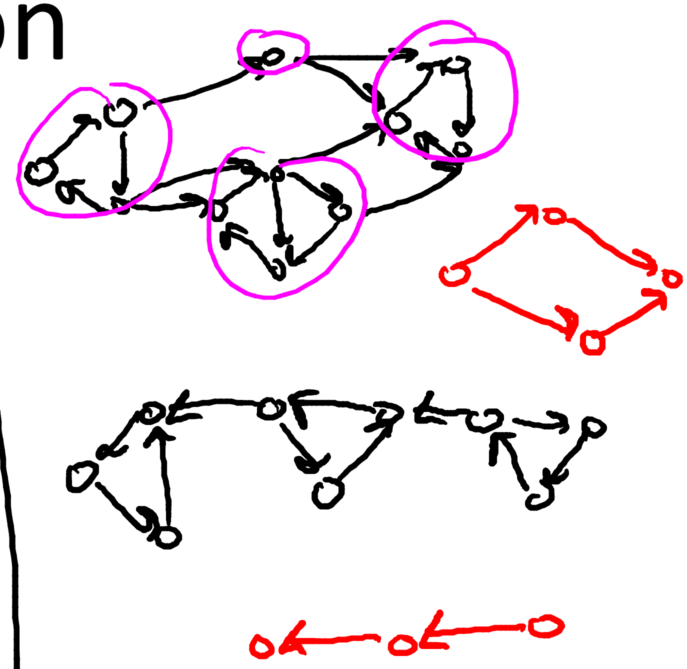
Observation

Claim: the SCCs of a directed graph G induce an acyclic "meta-graph":

- meta-nodes = the SCCs $C_1, \ldots, C_k$ of G
- $\exists$ arc $C \to \hat{C}$ $\iff$ $\exists$ arc $(i,j) \in G$ with $i \in C$, $j \in \hat{C}$

Why acyclic?: a cycle of SCCs would collapse into one.



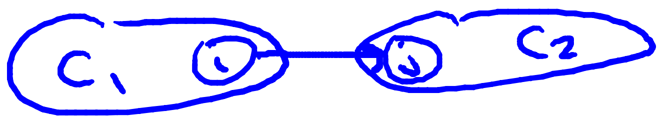


What how are the SCC of the original graph G and its reversal G^{rev} related?

- In general, they are unrelated.
- Every SCC of G is contained in an SCC of G^{rev}, but the converse need not hold.
- Every SCC of G^{rev} is contained in an SCC of G, but the converse need not hold.
- They are exactly the same.

Key Lemma

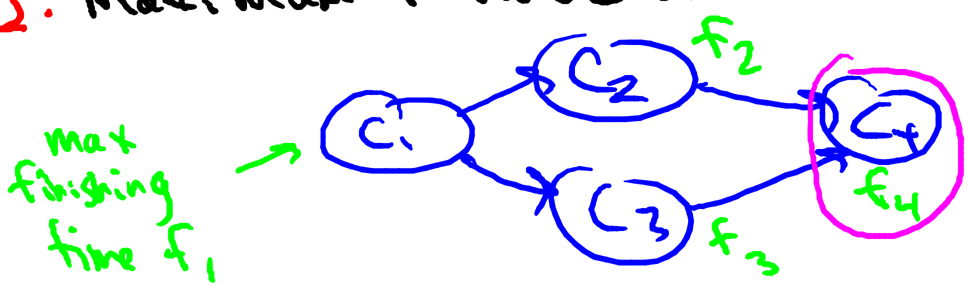
Lemma: Consider two "adjacent" SCCs in G:



Let $f(v)$ = finishing times of DFS-Loop in G^{rev}.

Then: $\boxed{\max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)}$

Corollary: Maximum f-value of G must lie in a "sink SCC".

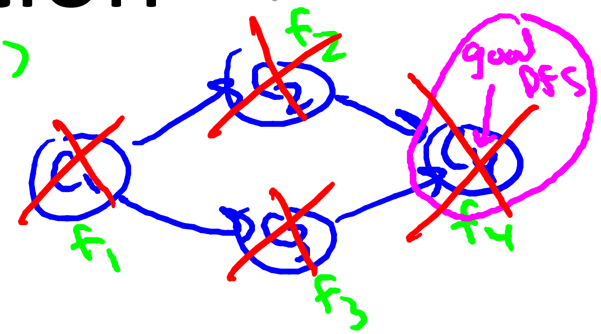


$f_1 < f_2, f_3 < f_4$

Tim Roughgarden

Correctness Intuition

(see notes for formal proof)



By Corollary: 2nd pass of DFS-Loop begins somewhere in a sink SCC C^*.

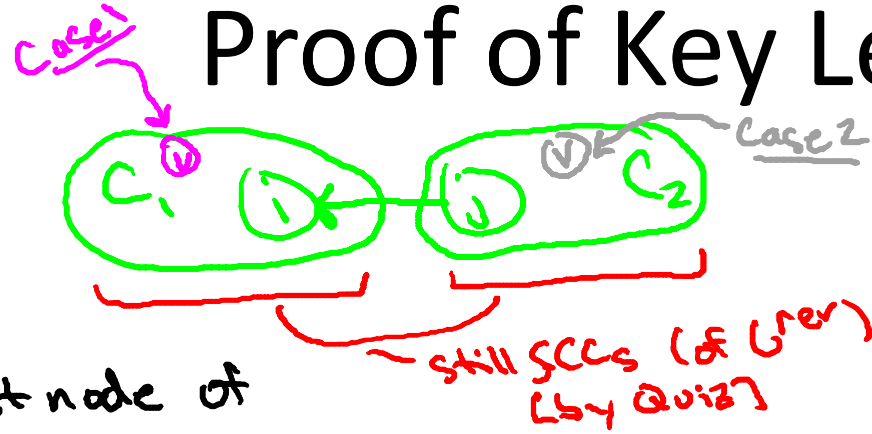
=> first call to DFS discovers C^* and nothing else!

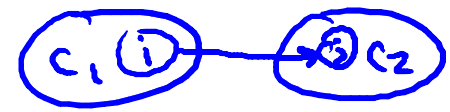
=> rest of DFS-Loop like recursing on G with C^* deleted [starts in a sink node of $G - C^*$]

=> successive calls to DFS(G,i) "peel off" the SCCs one by one [in reverse topological order of the "meta-graph" of SCCs]

Proof of Key Lemma

in G^{rev}:





$\max_{v \in C_1} f(v) < \max_{v \in C_2} f(v)$

Let v = 1st node of $C_1 \cup C_2$ reached by 1st pass of DFS-Loop (on G^{rev}).

Case1 [$v \in C_1$]: all of C_1 explored before C_2 ever reached.

Reason: no paths from C_1 to C_2 (since meta-graph is acyclic).

$\Rightarrow$ all f-values in C_1 less than all f-values in C_2

Case2 [$v \in C_2$]: DFS(G^{rev}, v) won't finish until all of $C_1 \cup C_2$ completely explored $\Rightarrow f(v) > f(w)$ for all $w \in C_1$ QED!