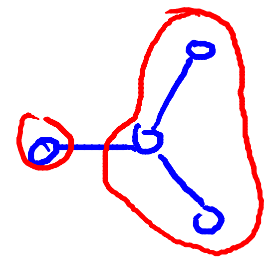
Design and Analysis of Algorithms I

Contraction Algorithm

Counting Mininum Cuts

The Number of Minimum Cuts



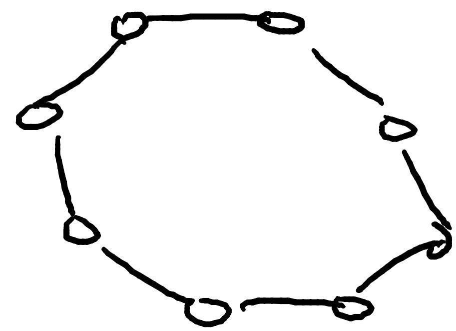
<u>Note</u>: a graph can have multiple min cuts.
[e.g., a tree with n vertices has
(n-1) minimum cuts]

<u>Question</u>: what's the largest number of min cuts that a graph with n vertices can have?

<u>ANSWER</u>: $\binom{n}{2} = \frac{n(n-1)}{2}$

The Lower Bound

Consider the n-cycle.



(n=8)

Note: each pair of the n edges defines a distinct minimum cut (with two crossing edges).

$\Rightarrow$ has $\geq \binom{n}{2}$ min cuts

The Upper Bound

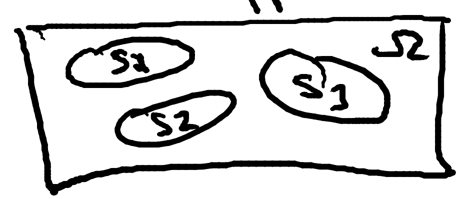
Let $(A_1,B_1), (A_2,B_2), \ldots, (A_t,B_t)$ be the min cuts of a graph with n vertices.

By the Contraction Algorithm analysis (without repeated trials):

$$\Pr[\underbrace{\text{output} = (A_i,B_i)}_{S_i}] \geq \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}$$ for all $i = 1,2,\ldots,t$.

<u>Note</u>: S_i's are <u>disjoint</u> events (i.e., only one can happen)

$\Rightarrow$ their probabilities sum to at most 1



<u>Thus</u>: $\frac{t}{\binom{n}{2}} \leq 1 \Rightarrow t \leq \binom{n}{2}$.

QED!