Design and Analysis of Algorithms I

Linear-Time Selection

Deterministic Selection (Algorithm)

The Problem

Input: array A with n distinct numbers and a number $i \in \{1, 2, \ldots, n\}$.

for simplicity

Output: i^{th} order statistic

(i.e., i^{th} smallest element of input array)

10	8	2	4

3^{rd} order statistic

Example: median.

($i = \frac{n+1}{2}$ for n odd,

$i = n/2$ for n even)

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Randomized Selection

RSelect (array A, length n, order statistic i)

(0) if n=1 return A[1]

(1) choose pivot p from A uniformly at random

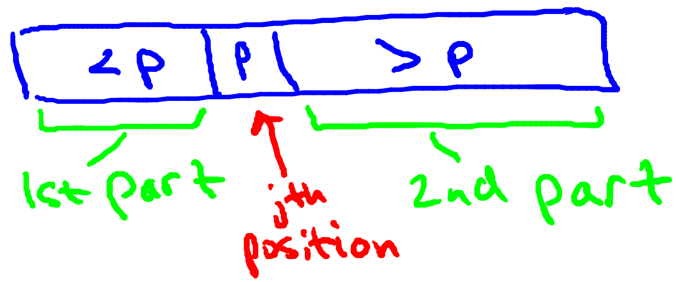
(2) partition A around p

let j = new index of p

(3) if j=i return p

(4) if j>i return RSelect (1st part of A, j-1, i)

(5) [if j<i] return RSelect (2nd part of A, n-j, i-j)



Guaranteeing a Good Pivot

Recall: "best" pivot = the median! (seems circular!)

Goal: find pivot guaranteed to be pretty good.

Key idea: use "median of medians"!

A Deterministic ChoosePivot

ChoosePivot(A,n)

- logically break A into $n/5$ groups of size 5 each
- sort each group (e.g., using MergeSort)
- copy $n/5$ medians (i.e., middle element of each sorted group) into new array C
- recursively compute median of C (!)
- return this as pivot

The DSelect Algorithm

DSelect(array A, length n, order statistic i)

1. Break A into groups of 5, sort each group
2. C = the n/5 "middle elements"
3. p = DSelect(C,n/5,n/10) [recursively computes median of C]
4. Partition A around p
5. If j = i return p
6. If j < i return DSelect(1st part of A, j-1, i)
7. [else if j > i] return DSelect(2nd part of A, n-j, i-j)

ChoosePivot

Same as before

How many recursive calls does DSelect make?

- 0
- 1
- 2
- 3

Running Time of DSelect

DSelect Theorem: for every input array of length n, DSelect runs in O(n) time.

Warning: not as good as RSelect in practice

1) worse constants 2) not-in-place

History: from 1973.

Blum - Floyd - Pratt - Rivest - Tarjan

'95 '78 '02 '06

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Deterministic Selection (Analysis I)

The DSelect Algorithm

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7. [else if j > i] return DSelect(2nd part of A, n-j, i-j)

Choose Pivot

Same as in RSelect

What is the asymptotic running time of step 1 of the DSelect algorithm?

- $\theta(1)$
- $\theta(\log n)$
- $\theta(n)$
- $\theta(n \log n)$

Note: sorting an array with 5 elements takes ≤ 120 operations.

[Why 120? take $m=5$ in our $6m(\log_2 m+1)$ bound for MergeSort]

$\underbrace{6m(\log_2 m+1)}_{} \to 6\cdot 5\cdot(\underbrace{\log_2 5}_{\leq 3}+1) \leq 120$

So: $\leq \underbrace{n/5}_{\#\text{groups}} \cdot \underbrace{120}_{\text{ops per group}} = 24n = O(n)$ for all groups

The DSelect Algorithm

DSelect(array A, length n, order statistic i)

1. Break A into groups of 5, sort each group → $\Theta(n)$
2. C = the n/5 “middle elements” → $\Theta(n)$
3. p = DSelect(C,n/5,n/10) [recursively computes median of C] → $T(\frac{n}{5})$
4. Partition A around p → $\Theta(n)$
5. If j = i return p
6. If j < i return DSelect(1st part of A, j-1, i)
7. [else if j > i] return DSelect(2nd part of A, n-j, i-j)

$T(?)$

The Key Lemma

Key Lemma: 2nd recursive call (in line 6 or 7) guaranteed to be on an array of size $\leq \frac{7}{10}n$ (roughly).

Upshot: Can replace "?" by "$\frac{7}{10}n$".

Rough Proof: Let $k = \frac{n}{5}$ = # of groups.

Let x_i = i^{th} smallest of the k "middle elements".

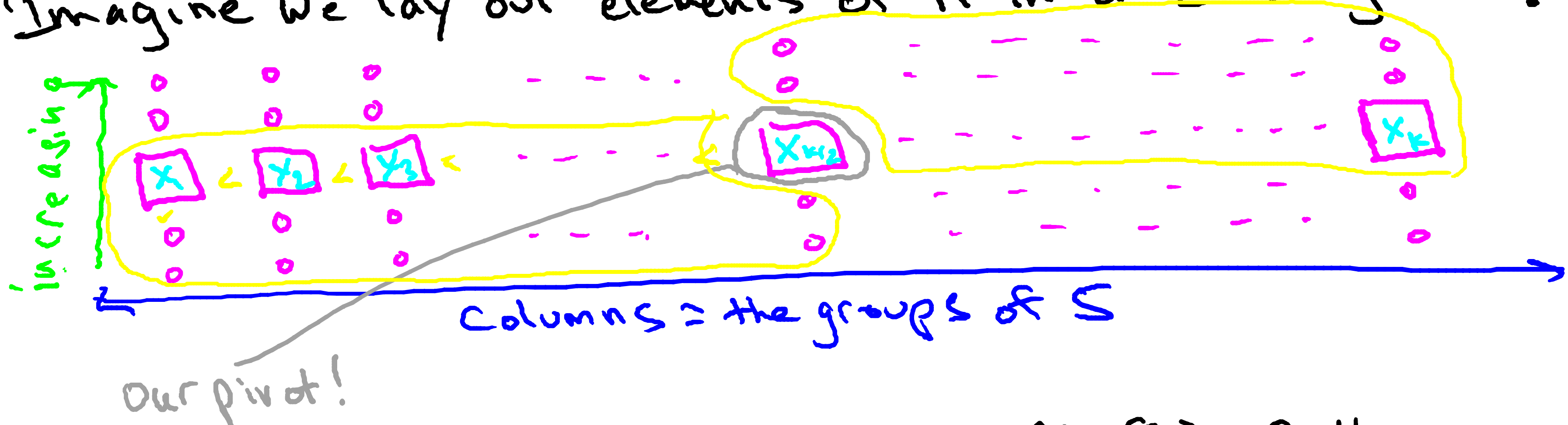
[so pivot = $x_{k/2}$]

Goal: $\geq 30\%$ of input array smaller than $x_{k/2}$, $\geq 30\%$ is bigger.

Rough Proof of Key Lemma

Thought experiment:

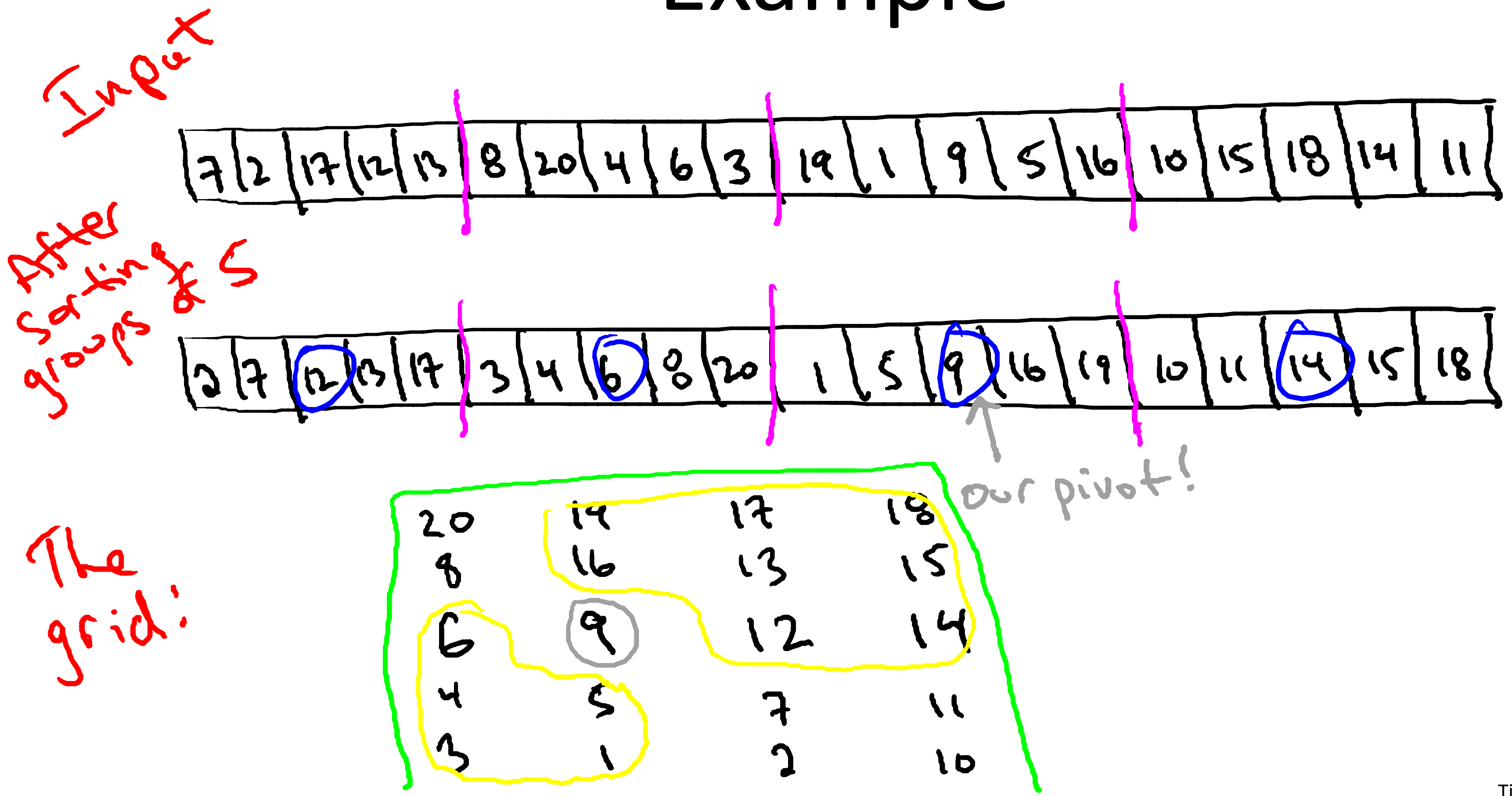
Imagine we lay out elements of A in a 2-D grid:



Key point: $x_{k/2}$ bigger than 3 out 5 (60%) of the elements in ≈ 50% of the groups

⇒ bigger than 30% of A (Similarly, smaller than 30% of A)

Example



Input
7 2 17 12 13 8 20 4 6 3 19 1 9 5 16 10 15 18 14 11
After sorting groups of 5
2 7 12 13 17 3 4 6 8 20 1 5 9 16 19 10 11 14 15 18
our pivot!
The grid:
20 14 17 18
8 16 13 15
6 9 12 14
4 5 7 11
3 1 2 10

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Deterministic Selection (Analysis II)

Rough Recurrence (Revisited)

Let $T(n)$ = maximum running time of DSelect on an input array of length n.

There is a constant $c \geq 1$ such that:

① $T(1) = 1$

② $T(n) \leq \underbrace{cn}_{\text{sorting the groups, partition}} + \underbrace{T\left(\frac{n}{5}\right)}_{\text{recursive call in line 3}} + \underbrace{T(?)}_{\text{recursive call in line 6 or 7}}$

$? \leq \frac{7}{10}n$ by key Lemma

Rough Recurrence (Revisited)

$T(1) = 1$, $T(n) \leq \underline{c}n + T(n/5) + T(\frac{7}{10}n)$

↳ constant $c \geq 1$

Note: different-sized subproblems ⇒ can't use Master method!

Strategy: "hope and check".

Hope: there is some constant a [independent of n] such that $T(n) \leq an$ $\forall n \geq 1$.

[if true, then $T(n) = O(n)$ and algorithm is linear-time]

Analysis of Rough Recurrence

$T(1) = 1$

$T(n) \le c\, n + T(n/5) + T(7/10\, n)$ constant ≥ 1

Claim: Let $a = 10c$.

Then $T(n) \le an$ for all $n \ge 1$. $\Rightarrow$ DSelect runs in $O(n)$ time.

Proof: by induction on n.

Base case: $T(1) = 1 \le a \cdot 1$ (since $a \ge 1$)

Inductive Step: $[n > 1]$ **Inductive Hypothesis**: $T(k) \le ak \;\; \forall k < n$

We have $T(n) \underset{\text{given}}{\le} cn + T(n/5) + T(\frac{7}{10}n)$

$\underset{\text{ind hyp}}{\le} cn + a(n/5) + a(7/10\, n)$

$= n(c + \frac{9}{10}a) \overset{\text{choice of } a}{=} an$

QED!