Design and Analysis of Algorithms I

QuickSort

Analysis III: Final Calculations

Average Running Time of QuickSort

Quicksort Theorem: for every input array of length n, the average running time of QuickSort (with random pivots) is $O(n \log n)$.

Note: holds for every input. [no assumptions on the data]

- recall our guiding principles!
- "average" is over random choices made by the algorithm (i.e., pivot choices)

The Story So Far

$$E[C] = 2 \underbrace{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{(j-i+1)}}_{\Theta(n^2) \text{ terms}} \quad (*)$$

how big can this be?

$\leq n$ choices for i

Note: for each fixed i, the inner sum is

$$\sum_{j=i+1}^{n} \frac{1}{(j-i+1)} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

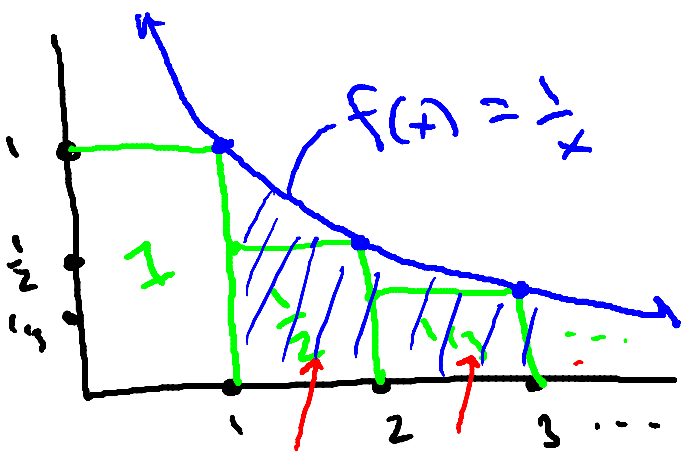
So: $E[C] \leq 2 \cdot n \cdot \sum_{k=2}^{n} \frac{1}{k}$

Claim: this is $\leq \ln n$.

Completing the Proof

$E[C] \le 2n \sum_{k=2}^{n} 1/k$ $\le \ln n$ | **Claim:** $\sum_{k=2}^{n} 1/k \le \ln n$

Proof of claim:



So: $\sum_{k=2}^{n} 1/k \le \int_1^n \frac{1}{x}\,dx$

$= \ln x \Big|_1^n$

$= \ln n - \cancel{\ln 1}$ $\rightarrow 0$

$= \ln n.$

qed (claim)

So:
$E[C] \le 2n \ln n$

QED!