



Design and Analysis
of Algorithms I

QuickSort

Analysis II: The Key Insight

Average Running Time of QuickSort

QuickSort Theorem: for every input array of length n ,
the average running time of QuickSort (with random pivots)
is $O(n \log n)$.

Note: holds for every input. {no assumptions on the data}

- recall our guiding principles!
- "average" is over random choices made by the algorithm
(i.e., pivot choices)

The Story So Far

$C(\sigma)$ = # of comparisons Quicksort makes with pivots σ

$X_{ij}(\sigma)$ = # of times z_i, z_j get compared

z_{i^*, j^*} smallest entries in array

Recall: $E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Pr[X_{ij}=1] = \Pr[z_i, z_j \text{ get compared}]$

Key Claim: $\forall i < j, \Pr[z_i, z_j \text{ get compared}] = \frac{2}{(j-i+1)}$

Proof of Key Claim

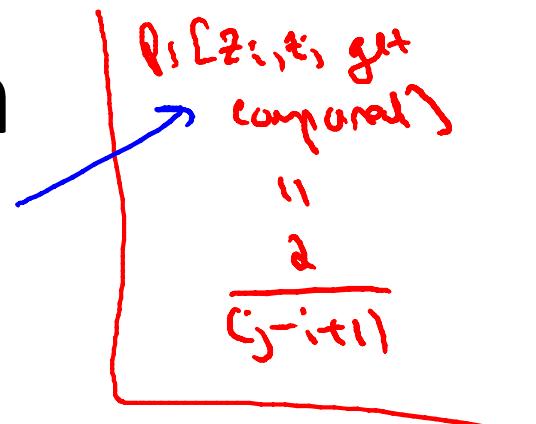
Fix z_i, z_j with $i < j$.

Consider the set $z_i, z_{i+1}, \dots, z_{j-1}, z_j$.

Inductively: as long as none of these are chosen as a pivot, all are passed to the same recursive call.

Consider the first among $z_i, z_{i+1}, \dots, z_{j-1}, z_j$ that gets chosen as a pivot.

- ① if z_i or z_j gets chosen first, then z_i and z_j get compared key insight
- ② if one of z_{i+1}, \dots, z_{j-1} gets chosen first, then $z_i \notin z_j$ are never compared [split into different recursive calls]



Proof of Key Claim (con'd)

- ① z_i or z_j chosen first \Rightarrow they get compared \leftarrow
- ② One of z_{i+1}, \dots, z_{j-1} chosen first \Rightarrow z_i, z_j never compared

Note: Since pivots always chosen uniformly at random, each of $z_i, z_{i+1}, \dots, z_{j-1}, z_j$ is equally likely to be the first.

$$\Rightarrow \Pr\{z_i, z_j \text{ get compared}\} = \frac{2}{(y-i+1)}$$

choices that lead
to Case ①²
total # of choices

QED!

$$\text{So: } E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{2}{(y-i+1)}$$

(E)

[still need to show
this is $O(n \log n)$]