Design and Analysis of Algorithms I

QuickSort

Analysis I: A Decomposition Principle

Necessary Background

Assumption: you know and remember (finite) sample spaces, random variables, expectation, linearity of expectation. For review:

- Probability Review I (video)
- Lehman-Leighton notes (free PDF)
- Wikibook on Discrete Probability

Average Running Time of QuickSort

Quicksort Theorem: for every input array of length n, the average running time of QuickSort (with random pivots) is $O(n \log n)$.

Note: holds for every input. [no assumptions on the data]

- recall our guiding principles!
- "average" is over random choices made by the algorithm (i.e., pivot choices)

Preliminaries

Fix input array A of length n.

<u>Sample space Ω</u> = all possible outcomes of random choices in QuickSort (i.e., pivot sequences).

<u>Key random variable</u>: for $\sigma \in \Omega$,

$C(\sigma)$ = # of comparisons between two input elements made by QuickSort (given random choices σ).

<u>Lemma</u>: running time of QuickSort dominated by comparisons.

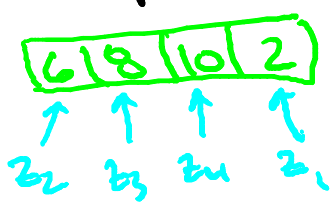
↳ $\exists$ constant c s.t. $\forall \sigma \in \Omega$ $RT(\sigma) \le c \cdot C(\sigma)$. (see notes)

Remaining goal: $E[C] = O(n \log n)$.

Building Blocks

Note: can't apply Master Method [random, unbalanced subproblems]

[A = fixed input array]



Notation: z_i = i^{th} smallest element of A.

For $\sigma \in \Omega$, indices $i<j$, let

$X_{ij}(\sigma)$ = # of times z_i, z_j get compared in QuickSort with pivot sequence σ.

Fix two elements of the input array. How many times can these two elements get compared with each other during the execution of QuickSort?

- 1
- 0 or 1
- 0, 1, or 2
- Any integer between 0 and $n - 1$

Reason: two elements compared only when one is the pivot, which is excluded from future recursive calls.

Thus: each X_{ij} is an "indicator" (i.e., 0-1) random variable.

A Decomposition Approach

So: $C(\sigma)$ = # of comparisons between input elements.

$X_{ij}(\sigma)$ = # of comparisons between z_i and z_j

Thus: $\forall \sigma$, $C(\sigma) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}(\sigma)$

complicated

simple!

By linearity of expectation: $E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$

Since $E[X_{ij}] = 0 \cdot \Pr[X_{ij}=0] + 1 \cdot \Pr[X_{ij}=1] = \Pr[X_{ij}=1]$

Thus: $E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[z_i, z_j \text{ get compared}]$ (*)

next video

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A General Decomposition Principle

1. Identify random variable Y that you really care about.
2. Express Y as sum of indicator random variables:

$$Y = \sum_{\ell=1}^{m} X_\ell$$

3. Apply linearity of expectation:

$$E[Y] = \sum_{\ell=1}^{m} Pr[X_\ell = 1]$$

"just" need to understand those!

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