Design and Analysis
of Algorithms I

Master Method

Proof (Part II)

The Story So Far/Case 1

= 1 for all j

- Total work: $\leq cn^d \times \underbrace{\sum_{j=0}^{\log_b n} \boxed{\left(\frac{a}{b^d}\right)^j}}_{= (\log_b n + 1)}$ $\quad (*)$

=1

$$If \quad a = b^d, \; then$$

$$(*) = cn^d(\log_b n + 1)$$

$$= O(n^d \log n)$$

[end Case 1]

Basic Sums Fact

For $r \neq 1$, we have

$$1 + r + r^2 + r^3 + \ldots + r^k = \frac{r^{k+1} - 1}{r - 1}$$

Proof : by induction (you check)

Upshot:

Independent of k

1. If r<1 is constant, RHS is <= $\frac{1}{1-r}$ = a constant

   I.e., 1st term of sum dominates

2. If r>1 is constant, RHS is <= $r^k \cdot (1 + \frac{1}{r-1})$

   I.e., last term of sum dominates

Case 2

- Total work: $\leq cn^d \times \sum_{j=0}^{\log_b n} (\frac{a}{b^d})^j$ (*)

:= r

<= a constant (independent of n)
[by basic sums fact]

$If \quad a < b^d \quad [RSP < RWS]$

$$= O(n^d)$$

[total work dominated by top level]

Case 3

- Total work: $\leq cn^d \times \sum_{j=0}^{\log_b n} (\frac{a}{b^d})^j$ (*)

$:= r > 1$

<= constant * largest term

$If \quad a > b^d \quad [RSP > RWS]$

$(*) = O(n^d \cdot (\frac{a}{b^d})^{\log_b n})$

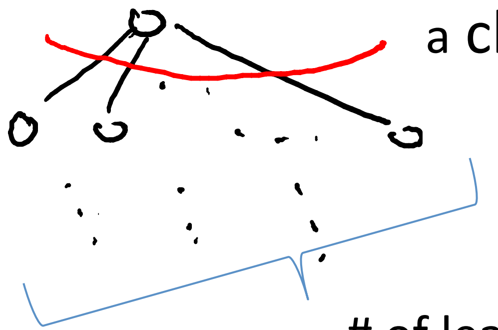
$Note \quad : \quad b^{-d \log_b n} = (b^{log_b n})^{-d} = n^{-d}$

$So \quad : \quad (*) = O(a^{\log_b n})$

Level 0

Level 1

Level $\log_b n$



a children

# of leaves = $a^{\log_b n}$

Which of the following quantities is equal to $a^{\log_b n}$?

- ○ The number of levels of the recursion tree.
- ○ The number of nodes of the recursion tree.
- ○ The number of edges of the recursion tree.
- ○ The number of leaves of the recursion tree.

Case 3 continued

- Total work: $\leq cn^d \times \sum_{j=0}^{\log_b n} (\frac{a}{b^d})^j$ (*)

$$So: \quad (*) = O(a^{\log_b n}) = O(\#\ leaves)$$

$$Note: \quad a^{\log_b n} = n^{\log_b a}$$

More intuitive

Simpler to apply

$$[Since \quad (\log_b n)(\log_b a) = (\log_b a)(\log_b n)]$$

[End Case 3]

The Master Method

If $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

then

$$T(n) = \begin{cases} O\left(n^d \log n\right) & \text{if } a = b^d \quad \text{(Case 1)} \\ O(n^d) & \text{if } a < b^d \quad \text{(Case 2)} \\ O(n^{\log_b a}) & \text{if } a > b^d \quad \text{(Case 3)} \end{cases}$$

Tim Roughgarden