Design and Analysis of Algorithms I

Divide and Conquer

Closest Pair II

Correctness Claim

$\min\{d(p_1,q_1), d(p_2,q_2)\}$

Claim: Let $p \in Q$, $q \in R$ be a split pair with $d(p,q) < \delta$.

Then: (A) p and q are members of S_y.

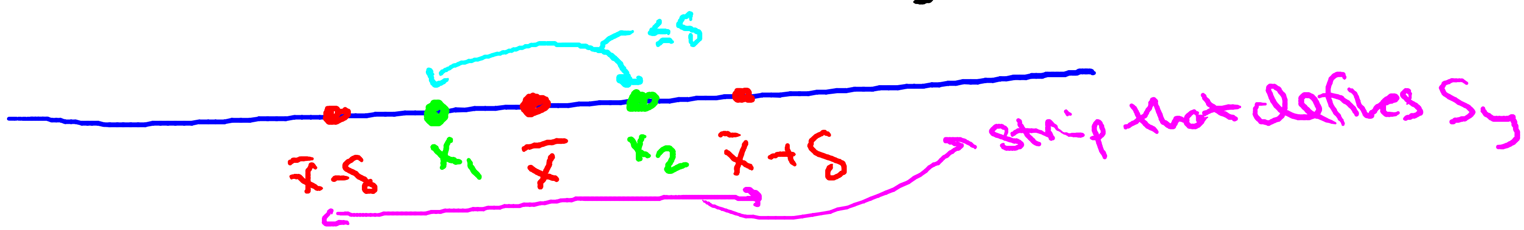
(B) p and q are at most 7 positions apart in S_y.

Proof of Correctness Claim (A)

Let $p=(x_1,y_1)\in Q$, $q=(x_2,y_2)\in R$, $d(p,q)<\delta$.

Note: Since $d(p,q)<\delta$, $|x_1-x_2|<\delta$ and $|y_1-y_2|<\delta$.

Proof of (A) [p and q are members of S_y — i.e., $x_1,x_2\in[\bar{x}-\delta,\bar{x}+\delta]$]



Note: $p\in Q \Rightarrow x_1\le\bar{x}$ and $q\in R \Rightarrow x_2\ge\bar{x}$.

$\Rightarrow x_1,x_2\in[\bar{x}-\delta,\bar{x}+\delta]$

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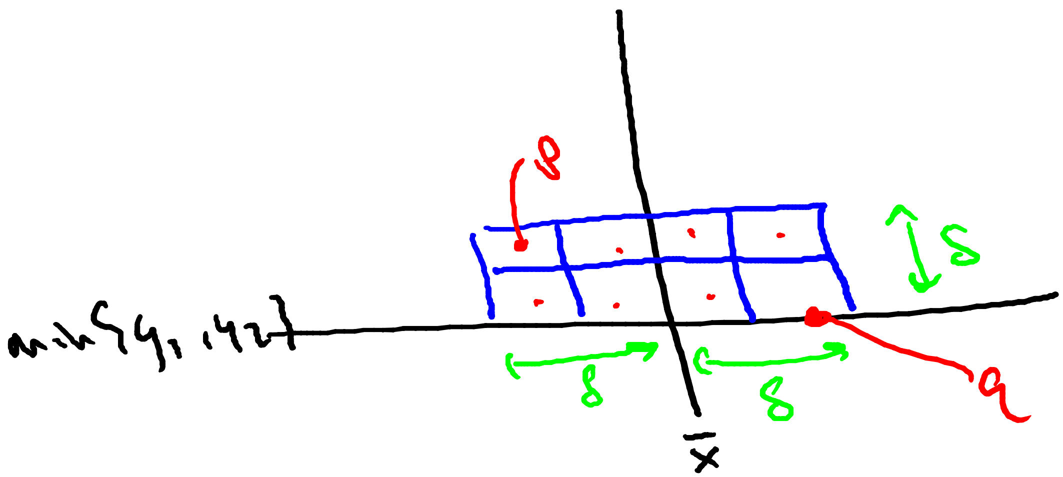
Proof of Correctness Claim (B)

(B): $p=(x_1,y_1)$ and $q=(x_2,y_2)$ are at most 7 positions apart in S_y.

Key picture: draw $\frac{\delta}{2} \times \frac{\delta}{2}$ boxes with center $\bar{x}$ and bottom $\min\{y_1,y_2\}$.



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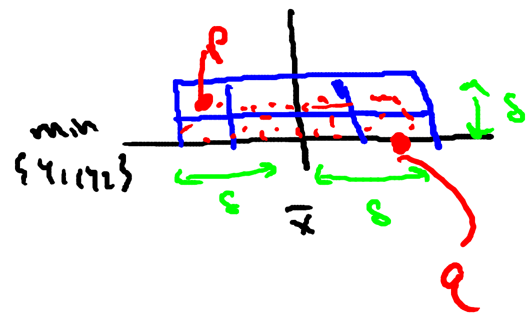
Proof of Correctness Claim (B)

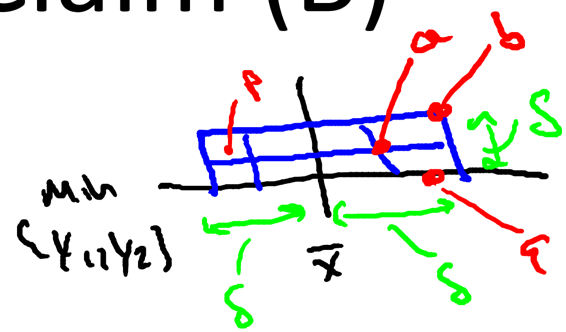


Lemma: all points of S_y with y-coordinate between those of p and q, inclusive, lie in one of these 8 boxes.

Proof: First, recall y-coordinates of p, q differ by $< \delta$.
Second, by definition of S_y, all have x-coordinates between $\bar{x}-\delta$ and $\bar{x}+\delta$.

qed.

Proof of Correctness Claim (B)



Lemma 2: At most one point of P in each box.

Proof: by contradiction.

Suppose a, b lie in the same box. Then:

(i) a, b are either both in Q or both in R

(ii) $d(a,b) \leq \frac{\delta}{2} \cdot \sqrt{2} < \delta$

But (i) and (ii) contradict the definition of δ (as smallest distance between pairs of points in Q or in R).

q.e.d.

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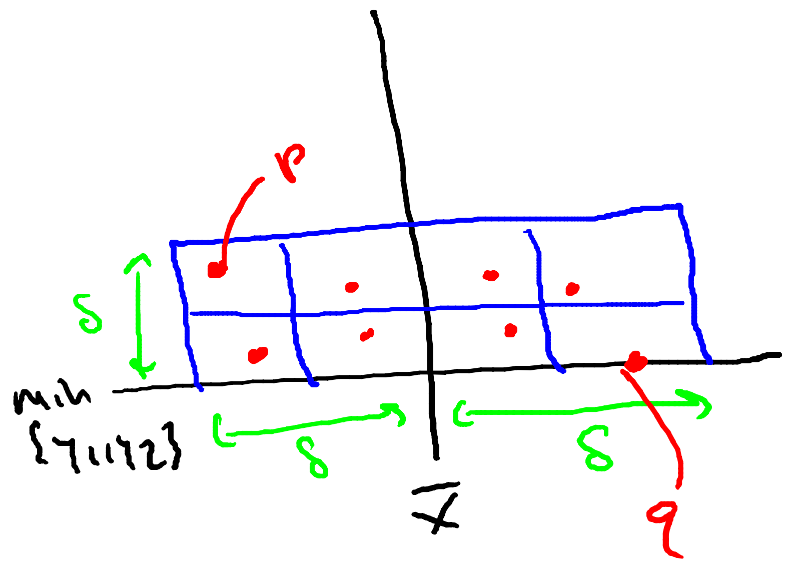
Final Wrap-Up

Lemmas 1 and 2 $\Rightarrow$ at most 8 points in this picture (including p and q).

$\Rightarrow$ positions of p, q in S_y differ by at most 7

QED!



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