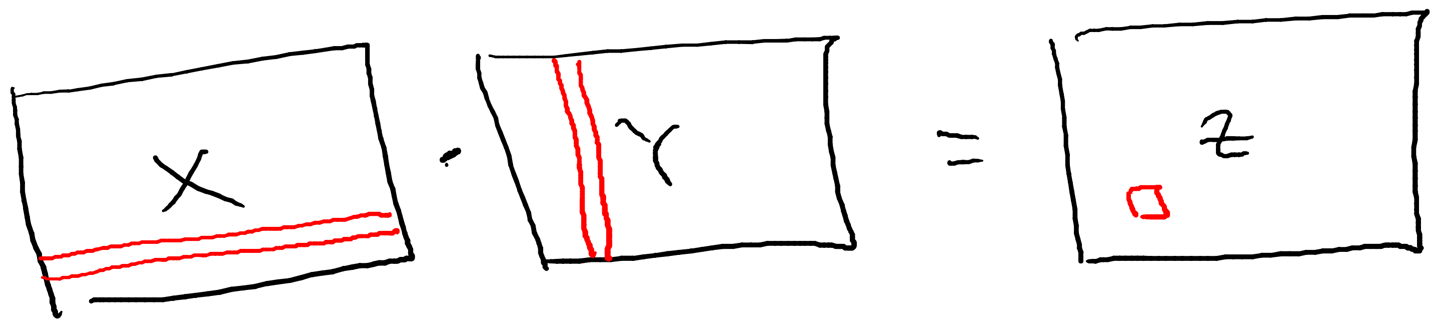
Design and Analysis of Algorithms I

Divide and Conquer

Matrix Multiplication

Matrix Multiplication



(all $n \times n$ matrices)

where $z_{ij} = \left(\begin{matrix} i^{th} \text{ row} \\ \text{of } X \end{matrix}\right) \cdot \left(\begin{matrix} j^{th} \text{ column} \\ \text{of } Y \end{matrix}\right)$

$$= \sum_{k=1}^{n} x_{ik} \cdot y_{kj}$$

Note: input size $= \Theta(n^2)$

Example (n=2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}$$

$\hookrightarrow O(n)$

What is the asymptotic running time of the straightforward iterative algorithm for matrix multiplication?

- $\theta(n \log n)$
- $\theta(n^2)$
- $\theta(n^3)$
- $\theta(n^4)$

The Divide and Conquer Paradigm

1) DIVIDE into smaller subproblems.

2) CONQUER subproblems recursively.

3) COMBINE solutions of subproblems into one for the original problem.

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Applying Divide and Conquer

Idea: write $X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ and $Y = \left(\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right)$

[where A through H are all $\frac{n}{2} \times \frac{n}{2}$ matrices]

Then: (you check)

$$X \cdot Y = \left(\begin{array}{c|c} AE + BG & AF + BH \\ \hline CE + DG & CF + DH \end{array}\right)$$

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Recursive Algorithm #1

$$X \cdot Y = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix}$$

<u>Step1</u>: recursively compute the 8 necessary products.

<u>Step2</u>: do the necessary additions ($\Theta(n^2)$ time)

<u>Fact</u>: run time is $\Theta(n^3)$. [follows from master method]

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Strassen's Algorithm (1969)

Step 1: recursively compute only 7 (cleverly chosen) products

Step 2: do the necessary (clever) additions + subtractions (still $\Theta(n^2)$ time)

Fact: better than cubic time! [see Master Method lecture]

$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $\quad$ $Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$

The Details

The Seven Products: $P_1 = A(F-H)$, $P_2 = (A+B)H$, $P_3 = (C+D)E$, $P_4 = D(G-E)$, $P_5 = (A+D)(E+H)$, $P_6 = (B-D)(G+H)$, $P_7 = (A-C)(E+F)$

Claim: $X \cdot Y = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix} = \begin{pmatrix} P_5+P_4-P_2+P_6 & P_1+P_2 \\ P_3+P_4 & P_1+P_5-P_3-P_7 \end{pmatrix}$

Proof: $AE + \cancel{AH} + \cancel{DE} + \cancel{DH} + \cancel{DG} - \cancel{DE} - \cancel{AH} - \cancel{BH} + BG + \cancel{BH} - \cancel{DG} - \cancel{DH} = AE + BG$ QED!

Question: where did this come from? (remains open!)

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