Quantum Mechanics & Quantum Computation

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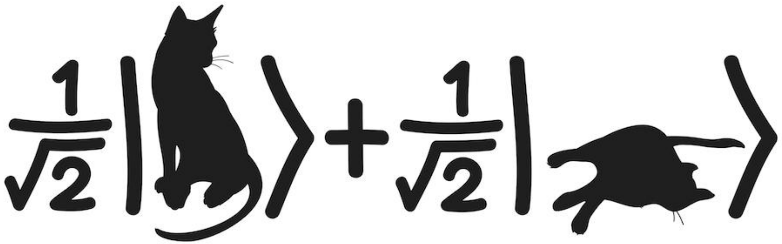


Lecture 7: Observables and Schrödinger's equation

Schrödinger's equation (part 1)

Axiom of unitary evolution

- **Unitary evolution axiom**: a quantum system evolves by a unitary rotation of the Hilbert space.

$$UU^\dagger = U^\dagger U = I$$

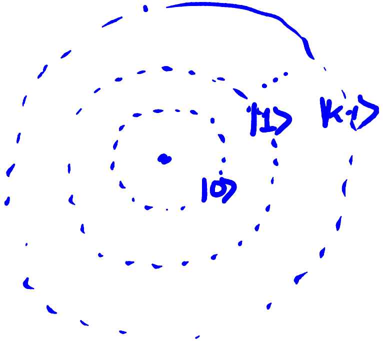
- But… by *which* unitary rotation?

This is described by **Schrödinger's equation,**
"the quantum equation of motion"

Schrödinger's equation

- Energy observable H, called the Hamiltonian of the system.
  - Its eigenvectors $|\phi_i\rangle$'s are the states with definite energy.
  - The eigenvalues λ_i's are the energy of the corresponding state.

- Example $H = \begin{pmatrix} -\frac{1}{2} & \frac{5}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{pmatrix}$
  - $|+\rangle$ with energy = 2
  - $|-\rangle$ with energy = −3



$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_{k-1}|k-1\rangle$

$$H = \begin{pmatrix} E_0 & & & 0 \\ & E_1 & & \\ & & \ddots & \\ 0 & & & E_{k-1} \end{pmatrix}$$

Schrödinger's equation

- Energy observable H, called the Hamiltonian of the system.
  - Its eigenvectors $|\phi_i\rangle$'s are the states with definite energy.
  - The eigenvalues λ_i's are the energy of the corresponding state.

- Schrödinger's equation:

$|\psi(t)\rangle$ = state of system at time t
Given $|\psi(0)\rangle$ & H

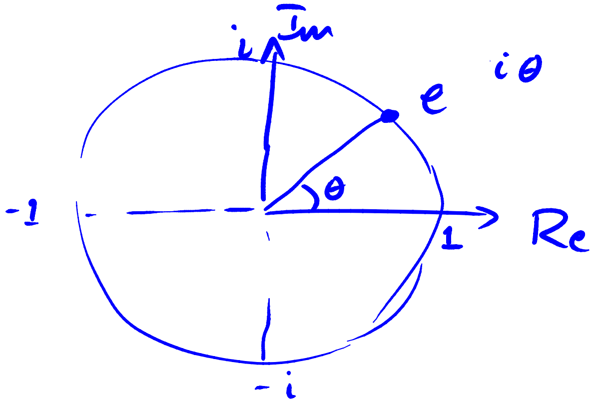
$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

Solving Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

$|\psi(0)\rangle = |\phi_j\rangle$ where $|\phi_j\rangle$ is some eigenvector of H with a corresponding eigenvalue $|\lambda_j\rangle$

Then $|\psi(t)\rangle = e^{-\frac{i\lambda_j t}{\hbar}}|\phi_j\rangle$



$$H|\phi_j\rangle = \lambda_j|\phi_j\rangle$$

$$\Rightarrow \quad |\psi(t)\rangle = a(t)|\phi_j\rangle$$

$$i\hbar\frac{d\, a(t)\cancel{|\phi_j\rangle}}{dt} = H(a(t)|\phi_j\rangle)$$

$$= a(t)\,\lambda_j\,\cancel{|\phi_j\rangle}$$

$$\frac{d\,a(t)}{a(t)} = -i\frac{\lambda_j}{\hbar}\,dt$$

$$a(t) = e^{-\frac{i\lambda_j t}{\hbar}}$$

Solving Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

In general: $|\psi(0)\rangle = \sum_j \alpha_j|\phi_j\rangle$

$$|\psi(t)\rangle = \sum_j \alpha_j e^{-\frac{i\lambda_j t}{\hbar}}|\phi_j\rangle$$

$UU^\dagger = U^\dagger U = I$

In the eigenbasis, we can write

$$|\psi(t)\rangle = \underbrace{\begin{pmatrix} e^{-\frac{i\lambda_1 t}{\hbar}} & & 0 \\ & \ddots & \\ 0 & & e^{-\frac{i\lambda_k t}{\hbar}} \end{pmatrix}}_{} |\psi(0)\rangle$$

$= U(t)$

$B = e^A$

B has same evectors

evalue of A is λ_j

evalue of B = e^{λ_j}

unitary $U(t) = e^{-\frac{iHt}{\hbar}}$ (shorthand notation)

Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

In general: $|\psi(0)\rangle = \sum \alpha_j|\phi_j\rangle$

$$|\psi(t)\rangle = \sum \alpha_j e^{-\frac{i\lambda_j t}{\hbar}}|\phi_j\rangle$$

Example $|\psi(0)\rangle = |0\rangle$

$H = X$

What is $|\psi(t)\rangle$?

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{\frac{-it}{\hbar}}|+\rangle + \frac{1}{\sqrt{2}}e^{+it/\hbar}|-\rangle$$

We know that X's eigenvectors are :

$|+\rangle$ with eigenvalue 1

$|-\rangle$ with eigenvalue -1