Quantum Mechanics & Quantum Computation

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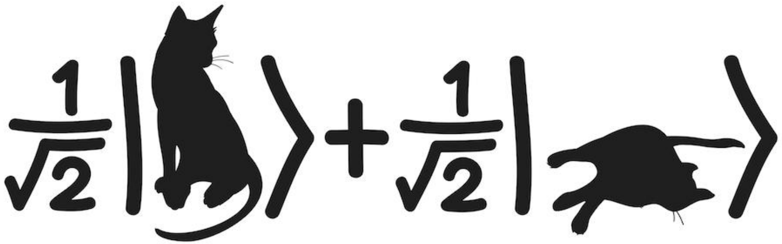


Lecture 7: Bra-ket notation, Eigenvectors, Tensor products

Hermitian Matrices

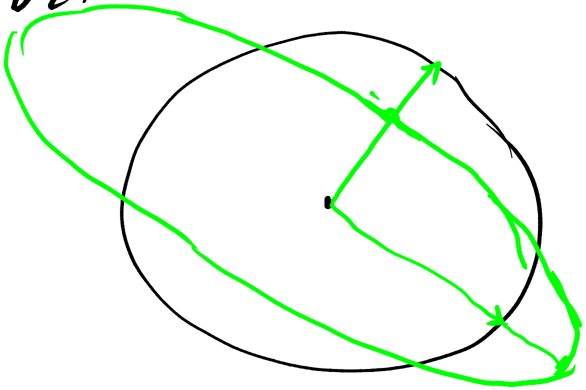
A Hermitian iff $A = A^\dagger$

$$\begin{bmatrix} 1 & 1+i \\ 1-i & -2 \end{bmatrix}$$

A Hermitian & Real $\Rightarrow$ symmetric

$|\phi\rangle$ evector of A if $A|\phi\rangle = \lambda|\phi\rangle$ $\quad \lambda \in \mathbb{C}$

Spectral Theorem: A Hermitian $\Rightarrow$ A has an orthonormal set of evectors with real evalues.
$|\phi_0\rangle, \ldots, |\phi_{K-1}\rangle$ $\quad \lambda_0, \ldots, \lambda_{K-1}$



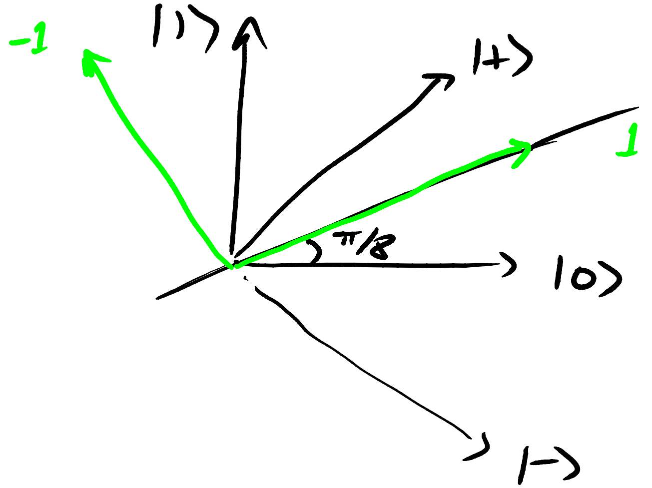
A maps sphere into an ellipsoid.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \underset{1}{|+\rangle}, \ \underset{-1}{|-\rangle}$$

$$X|+\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} = |+\rangle \qquad \lambda_+ = 1$$

$$X|-\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = -|-\rangle \qquad \lambda_- = -1.$$

$$H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$



$$H|\phi\rangle = \lambda|\phi\rangle$$

$$\underbrace{(H - \lambda I)}_{}|\phi\rangle = 0$$

$$\det(H - \lambda I) = 0$$

$$\det\begin{pmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{pmatrix} = 0 = \left(\frac{1}{\sqrt{2}} - \lambda\right)\left(-\frac{1}{\sqrt{2}} - \lambda\right) - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$= -\frac{1}{2} + \lambda^2 - \frac{1}{2} = 0$$

$$\lambda^2 = 1 \qquad \lambda = \pm 1.$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} a_0 + \frac{1}{\sqrt{2}} a_1 = a_0$$

$$\left(1 - \frac{1}{\sqrt{2}}\right) a_0 = \frac{1}{\sqrt{2}} a_1$$

$$(\sqrt{2} - 1) a_0 = a_1$$

$$\frac{a_1}{a_0} = \sqrt{2} - 1 .$$

A has orthonormal evectors $|\phi_0\rangle \cdots |\phi_{k-1}\rangle$

real evalues $\lambda_0 \cdots \lambda_{k-1}$.

Write A

$\rightarrow$ In $|\phi_0\rangle \cdots |\phi_{k-1}\rangle$ basis $= \begin{pmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_{k-1} \end{pmatrix} = \Lambda$

$$A = \underline{U \Lambda U^{+}} \qquad U = \begin{pmatrix} | & | & & | \\ |\phi_0\rangle & |\phi_1\rangle & \cdots & |\phi_{k-1}\rangle \\ | & | & & | \end{pmatrix}$$

$$A = \lambda_0 |\phi_0\rangle\langle\phi_0| + \lambda_1 |\phi_1\rangle\langle\phi_1| + \cdots + \lambda_{k-1} |\phi_{k-1}\rangle\langle\phi_{k-1}|$$

$$= \sum_{j=0}^{k-1} \lambda_j P_j \qquad P_j = |\phi_j\rangle\langle\phi_j|$$