Quantum Mechanics & Quantum Computation

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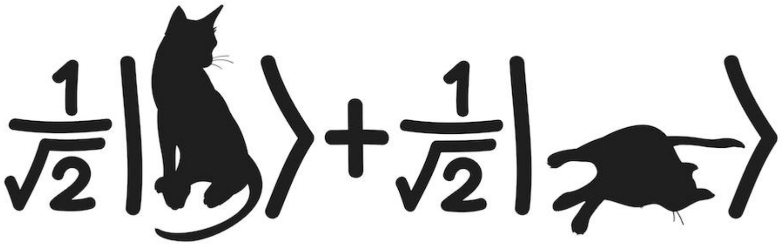


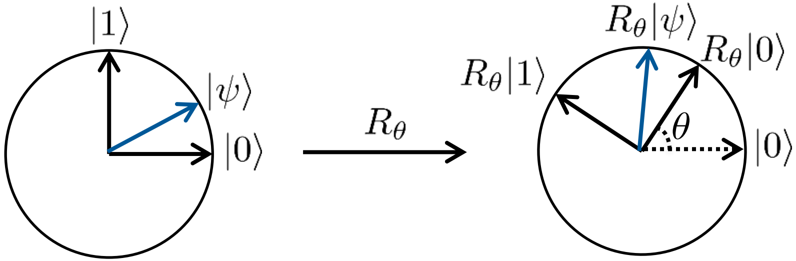
Lecture 5: Quantum Gates

Unitary Transforms

Rotation Matrix

- Rotation of the space is a linear transformation. Represent by a matrix:

- Example



$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$R_\theta |\psi\rangle$

Unitary transformations

$$a, b, c, d \in \mathbb{C}$$

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$|\psi\rangle \longrightarrow U|\psi\rangle$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \longrightarrow \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle$$

$$UU^\dagger = U^\dagger U = I$$

$$|1\rangle \longrightarrow c|0\rangle + d|1\rangle.$$

$$U^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

$$|a|^2 + |b|^2 = 1 = |c|^2 + |d|^2$$

$$a^*c + b^*d = 0$$

$$U^\dagger U = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & \boxed{0} \\ \boxed{0} & 1 \end{pmatrix}$$

Linearity

$$U|0\rangle = |\phi_0\rangle = a|0\rangle + b|1\rangle$$

$$U|1\rangle = |\phi_1\rangle = c|0\rangle + d|1\rangle$$

$$\begin{aligned} U(\alpha|0\rangle + \beta|1\rangle) &= \alpha|\phi_0\rangle + \beta|\phi_1\rangle \\ &= \alpha(a|0\rangle + b|1\rangle) + \beta(c|0\rangle + d|1\rangle) \\ &= (\alpha a + \beta c)|0\rangle + (\alpha b + \beta d)|1\rangle. \end{aligned}$$

Axioms of quantum mechanics

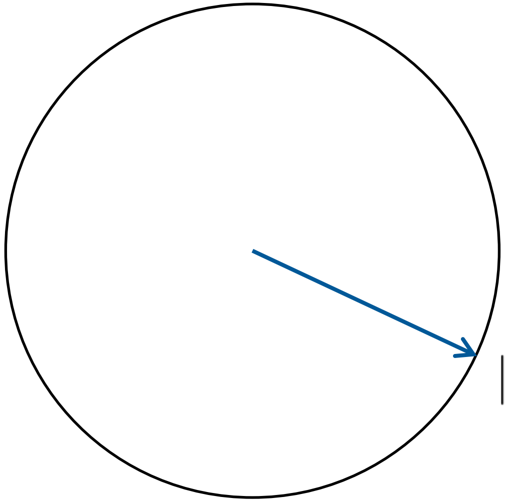
Axiom 1. Superposition

Axiom 2. Measurement

Axiom 3. Unitary evolution

Axiom 1. Superposition

- Allowable states of k-level system: unit vector in a k-dimensional complex vector space (called a Hilbert space).



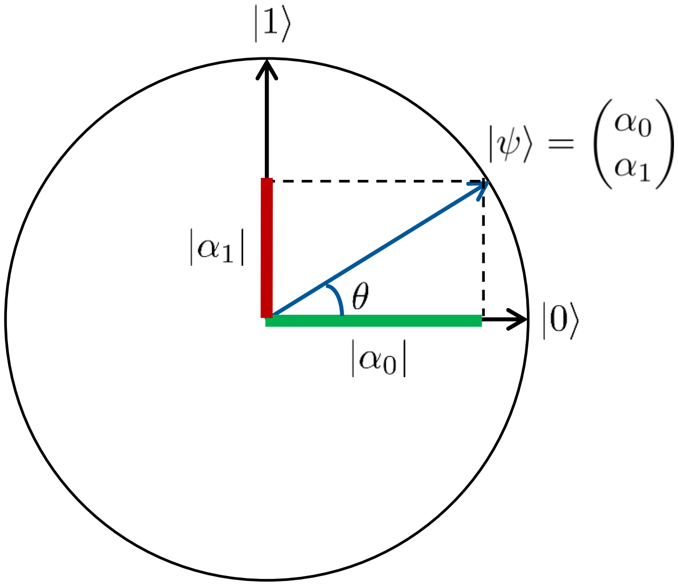
$$|\psi\rangle = \alpha_0|0\rangle + \cdots + \alpha_{k-1}|k-1\rangle = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} \in \mathbb{C}^k$$

Axiom 2. Measurement

- A measurement is specified by choosing an orthonormal basis.

- The probability of each outcome is the square of the length of the projection onto the corresponding basis vector.

- The state collapses to the observed basis vector.



If $|u_0\rangle, |u_1\rangle, \ldots, |u_{k-1}\rangle$ was the chosen basis and $|\psi\rangle = \alpha_0|u_0\rangle + \alpha_1|u_1\rangle + \cdots + \alpha_{k-1}|u_{k-1}\rangle$

u_i with probability $|\alpha_i|^2 = |(|\psi\rangle, |u_i\rangle)|^2$

New state is $|u_i\rangle$

Axiom 3. Unitary evolution

- Quantum systems evolve by (rigid-body) rotation of the Hilbert space.