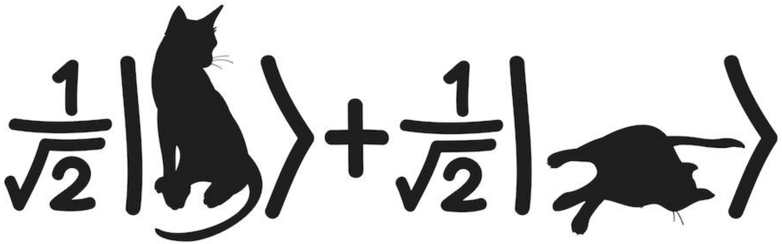
Quantum Mechanics & Quantum Computation



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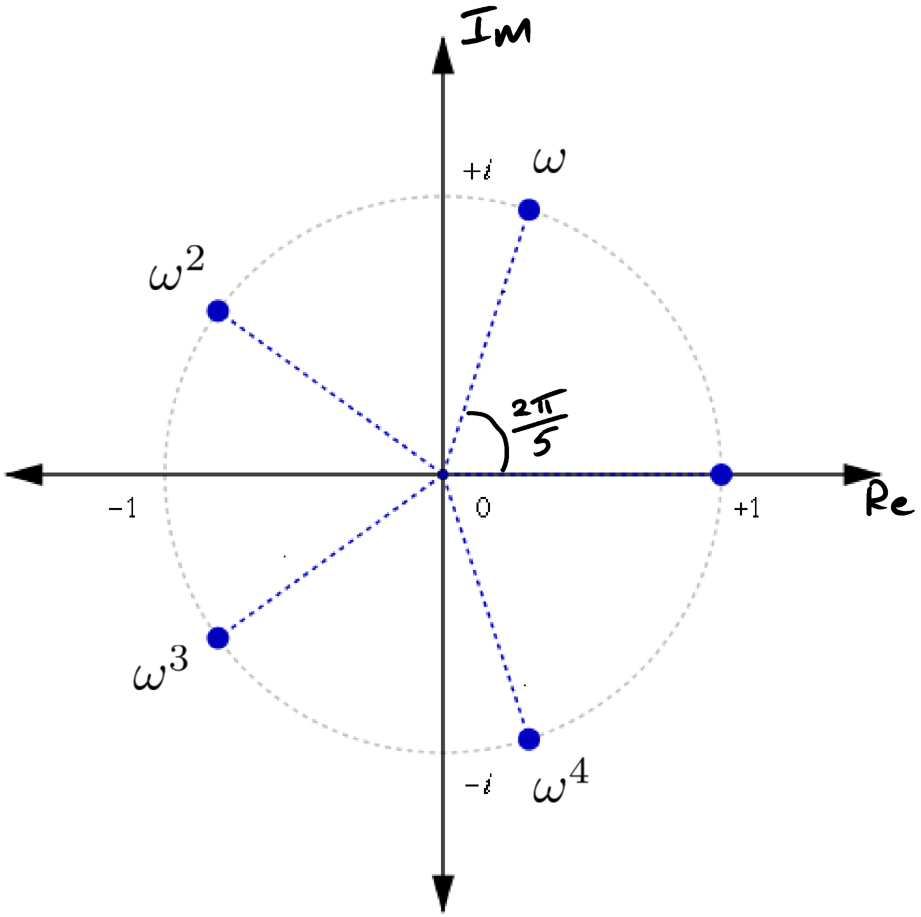
Lecture 13: Quantum Fourier Transform

Definition

Fourier transform



$$\omega^N = 1 \quad \omega \in \mathbb{C}$$

$\omega = e^{\frac{2\pi i}{N}}$

ω^d is ~~also~~ an N-th root:

$i = \sqrt{-1}$

$$(\omega^d)^N = \omega^{dN} = 1$$

Fourier transform

$$\frac{1}{N}\left(1 + \omega^{j} + \omega^{2j} + \cdots + \omega^{(N-1)j}\right) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{o.w.} \end{cases}$$

F_N is unitary.

$$F_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)^2} \end{pmatrix}$$

$$\omega = e^{2\pi i/N}$$

$$F_N = \begin{array}{cc} & k \\ j & \begin{pmatrix} & \vdots & \\ \cdots & \omega^{jk} & \\ & & \end{pmatrix} \end{array} \qquad (F_N)_{jk} = \omega^{jk}$$

Example: $\quad N=4 \quad \omega = i = \sqrt{-1}$

$$F_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} = \frac{1}{2}|0\rangle + \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle - \frac{i}{2}|3\rangle$$

$$\alpha_0|0\rangle + \underline{\alpha_1|1\rangle} + \alpha_2|2\rangle + \alpha_3|3\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$F_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 \cdots & \omega^{N-1} \\ \vdots & & & \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N-1} \end{pmatrix}$$

$O(N^2)$ steps

Fast Fourier Transform

FFT : $O(N \log N)$

$n = \log N$

$$\underbrace{\bullet \;\bullet \;\bullet \;\bullet \;\bullet}_{n}$$

$$\sum_j \alpha_j |j\rangle \longrightarrow \sum_k \beta_k |k\rangle$$

exponential

QFT $O(n^2)$ steps $= O(\log^2 N)$

Measure : see $\underline{\underline{k}}$ with probability $|\beta_k|^2$