Chapter 1

Introduction

Nature at the atomic level behaves very differently from the World we observe around us. As such, it often flies in the face of our physical intuition. To comprehend this behavior requires a new kind of physics called **quantum mechanics**, which exhibits a number of unusual features:

- Complete knowledge of a system's state is forbidden - A measurement of the state of the system reveals only a small amount of information about the state. Moreover, the very act of making the measurement disturbs the state.

- Quantum entities do not have trajectories We can only say for a photon or electron that it started at A and was measured at B at a later time. We cannot speak about the trajectory or path of the photon from A to B.

- It is inherently probabilistic - Measurement outcomes are inherently probabilistic. If identical states are measured the outcome may be different each time.

- Wave-particle duality Unlike classical systems which can usually be described in terms of particles or waves, quantum systems exhibit some properties of each but behave in their own unique way.

These features may seem bizarre, and some of you might have trouble accepting the claim that Nature behaves in accordance with such a theory. This is a natural response - to quote the great physicist Niels Bohr, "Anyone who is not shocked by quantum theory has not understood it."

In today's lecture, we will highlight these aspects of quantum mechanics in the context of the iconic double slit experiment.

By using the experiment as a means for comparing and contrasting the behavior of classical waves and particles against the behavior of quantum entities, we hope to motivate the existence of these strange properties, and help develop an intuition for how they work.

1.1 The Double Slit Experiment

A great deal of insight into the quantum theory can be gleaned by addressing the question, *is light transmitted by particles or waves?* Until quite recently, the evidence strongly favored wave-like propagation. Diffraction of light, a wave interference phenomenon, was observed as long ago as 1655 by Grimaldi. In fact, a rather successful theory of wave-like light propagation, due to Huygens, was developed in 1678. Perhaps the most striking confirmation of the wave nature of light was the double-slit interference experiment performed by Young in 1802. However, a dilemma began in the late 19th century when theoreticians such as Wien calculated how might light should be emitted by hot objects (*i.e.,* blackbody radiation). Their wave-based calculation differed dramatically from what was observed experimentally. At about the same time, the 1890's, it was noticed that the behavior of electrons kicked out of metals by light, the photoelectric effect, was strikingly inconsistent with any existing wave theory. In the first decade of the 20th century, blackbody radiation and the photoelectric effect were explained by treating light not as a wave phenomenon, but as particles containing discrete packets of energy, which we now call *photons.*

To illustrate this seeming paradox, let us recall Young's double-slit experiment, which consists of a source of light, an intermediate screen with two very thin identical slits, and a viewing screen; see Figure 1.1. If only one slit is open then intensity of light on the viewing screen is maximum on the straight line path and falls off in either direction. However, if both slits are open, then the intensity oscillates according to the familiar interference pattern predicted by wave theory. These facts can be very convincingly explained, both qualitatively and quantitatively, by positing that light travels in waves.

Suppose, however, that you were to place photodetectors at the viewing screen, and turn down the intensity of the light source until the photodetectors only occasionally record the arrival of a photon, then you would make a very surprising discovery. To begin with, you would notice that as you turn down the intensity of the source, the magnitude of each click remains constant, but the time between successive clicks increases. You could infer that light is emitted from the source as discrete particles (photons) — the intensity of

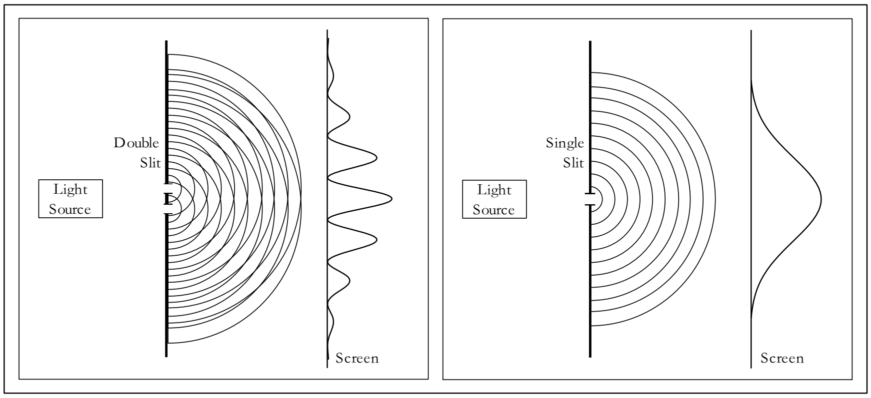

Figure 1.1: Double- and single-slit diffraction. Notice that in the double-slit experiment the two paths interfere with one another. This experiment gives evidence that light propagates as a wave.

light is proportional to the rate at which photons are emitted by the source. And since you turned the intensity of the light source down sufficiently, it only emits a photon once every few seconds. You might now ask the question, once a photon is emitted from the light source, where will it hit the viewing screen. The answer is no longer deterministic, but probabilistic. You can only speak about the probability that a photodetector placed at point x detects the photon. So what is the probability that the photon is detected at point x in the setup of the double slit experiment with the light intensity turned way down? If only a single slit is open, then plotting this probability of detection as a function of x gives the same curve as the intensity as a function of x in the classical Young experiment. So far this should agree with your intuition, since the photon should randomly scatter as it goes through the slit. What happens when both slits are open? Our intuition would strongly suggest that the probability we detect the photon at x should simply be the sum of the probability of detecting it at x if only slit 1 were open and the probability if only slit 2 were open. In other words the outcome should no longer be consistent with the interference pattern. If you were to actually carry out the experiment, you would make the very surprising discovery that the probability of detection does still follow the interference pattern. Reconciling this outcome with the particle nature of light appears impossible, and this is the basic dilemma we face.

Before proceeding further, let us try to better understand in what sense the outcome of the experiment is inconsistent with the particle nature of light. Clearly, for the photon to be detected at x, either it went through slit 1 and ended up at x or it went through slit 2 and ended up at x. And the probability of seeing the photon at x should then be the sum of the probabilities of the two cases. The nature of the contradiction can be seen even more clearly at "dark" points x, where the probability of detection is 0 when both slits are open, even though it is non-zero if either slit is open. This truly defies reason! After all, if the photon has non-zero probability of going through slit 1 and ending up at x, how can the existence of an additional trajectory for getting to x possibly decrease the probability that it arrives at x?

Quantum mechanics provides a way to reconcile both the wave and particle nature of light. Let us sketch how it might address the situation described above. Quantum mechanics introduces the notion of the complex amplitude $\psi_1(x) \in \mathbb{C}$ with which the photon goes through slit 1 and hits point x on the viewing screen. The probability that the photon is actually detected at x is the square of the magnitude of this complex number: $P_1(x) = |\psi_1(x)|^2$. Similarly, let $\psi_2(x)$ be the amplitude if only slit 2 is open. $P_2(x) = |\psi_2(x)|^2$.

Now when both slits are open, the amplitude with which the photon hits point x on the screen is just the sum of the amplitudes over the two ways of getting there: $\psi_{12}(x) = \psi_1(x) + \psi_2(x)$. As before the probability that the photon is detected at x is the squared magnitude of this amplitude: $P_{12}(x) = |\psi_1(x) + \psi_2(x)|^2$. The two complex numbers $\psi_1(x)$ and $\psi_2(x)$ can cancel each other out to produce destructive interference, or reinforce each other to produce constructive interference or anything in between.

Some of you might find this "explanation" quite dissatisfying. You might say it is not an explanation at all. Well, if you wish to understand how Nature behaves you have to reconcile yourselves to this type of explanation — this wierd way of thinking has been successful at describing (and understanding) a vast range of physical phenomena. But you might persist and (quite reasonably) ask "but how does a particle that went through the first slit know that the other slit is open"? In quantum mechanics, this question is not well-posed. Particles do not have trajectories, but rather take all paths simultaneously (in superposition). As we shall see, this is one of the key features of quantum mechanics that gives rise to its paradoxical properties as well as provides the basis for the power of quantum computation. To quote Feynman, 1985, "The more you see how strangely Nature behaves, the harder it is to make a model that explains how even the simplest phenomena actually work. So theoretical physics has given up on that."

1.2 Basic Quantum Mechanics

Feynman also said, "I think I can safely say that nobody understands quantum mechanics." Paradoxically, quantum mechanics is a very simple theory, whose fundamental principles can be stated very concisely and are enshrined in the three basic postulates of quantum mechanics - indeed we will go through these postulates over the course of the next two chapters. The challenge lies in understanding and applying these principles, which is the goal of the rest of the book (and will continue through more advanced courses and research if you choose to pursue the subject further):

- The superpostion principle: this axiom tells us what are the allowable (possible) states of a given quantum system. An addendum to this axiom tells us given two subsystems, what the allowable states of the composte system are.
- The measurement principle: this axiom governs how much information about the state we can access.
- Unitary evolution: this axoim governs how the state of the quantum system evolves in time.

In keeping with the philosophy of the book, we will introduce the basic axioms gradually, starting with simple finite systems, and simplified basis state measurements, and building our way up to the more general formulations. This should allow the reader a chance to develop some intuition about these topics.

1.3 The Superposition Principle

Consider a system with k distinguishable (classical) states. For example, the electron in a hydrogen atom is only allowed to be in one of a discrete set of energy levels, starting with the ground state, the first excited state, the second excited state, and so on. If we assume a suitable upper bound on the total energy, then the electron is restricted to being in one of k different energy levels — the ground state or one of $k-1$ excited states. As a classical system, we might use the state of this system to store a number between 0 and $k-1$. The superposition principle says that if a quantum system can be in one of two states then it can also be placed in a linear superposition of these states with complex coefficients.

Let us introduce some notation. We denote the ground state of our k-state system by $|0\rangle$, and the succesive excited states by $|1\rangle, \ldots, |k-1\rangle$. These are the k possible classical states of the electron. The superposition principle tells us that, in general, the quantum state of the electron is $\alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_{k-1} |k-1\rangle$, where $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ are complex numbers normalized so that $\sum_j |\alpha_j|^2 = 1$. α_j is called the *amplitude of the state* $|j\rangle$. For instance, if $k = 3$, the state of the electron could be

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{2} |1\rangle + \frac{1}{2} |2\rangle$$

or

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{2} |1\rangle + \frac{i}{2} |2\rangle$$

or

$$|\psi\rangle = \frac{1+i}{3} |0\rangle - \frac{1-i}{3} |1\rangle + \frac{1+2i}{3} |2\rangle .$$

The superposition principle is one of the most mysterious aspects about quantum physics — it flies in the face of our intuitions about the physical world. One way to think about a superposition is that the electron does not make up its mind about whether it is in the ground state or each of the $k-1$ excited states, and the amplitude α_0 is a measure of its inclination towards the ground state. Of course we cannot think of α_0 as the probability that an electron is in the ground state — remember that α_0 can be negative or imaginary. The measurement priniciple, which we will see shortly, will make this interpretation of α_0 more precise.

1.4 The Geometry of Hilbert Space

We saw above that the quantum state of the k-state system is described by a sequence of k complex numbers $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{C}$, normalized so that $\sum_j |\alpha_j|^2 = 1$. So it is natural to write the state of the system as a k dimensional vector:

$$|\psi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{pmatrix}$$

The normalization on the complex amplitudes means that the state of the system is a unit vector in a k dimensional complex vector space — called a Hilbert space.

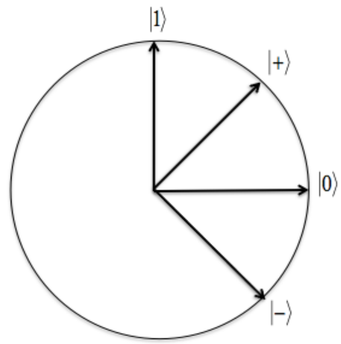

Figure 1.2: Representation of qubit states as vectors in a Hilbert space.

But hold on! Earlier we wrote the quantum state in a very different (and simpler) way as: $\alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_{k-1} |k-1\rangle$. Actually this notation, called Dirac's ket notation, is just another way of writing a vector. Thus

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |k-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

So we have an underlying geometry to the possible states of a quantum system: the k distinguishable (classical) states $|0\rangle, \ldots, |k-1\rangle$ are represented by mutually orthogonal unit vectors in a k-dimensional complex vector space. i.e. they form an orthonormal basis for that space (called the standard basis). Moreover, given any two states, $\alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_{k-1} |k-1\rangle$, and $\beta |0\rangle + \beta |1\rangle + \cdots + \beta |k-1\rangle$, we can compute the inner product of these two vectors, which is $\sum_{j=0}^{k-1} \alpha_j^* \beta_j$. The absolute value of the inner product is the cosine of the angle between these two vectors in Hilbert space. You should verify that the inner product of any two basis vectors in the standard basis is 0, showing that they are orthogonal.

The advantage of the ket notation is that the it labels the basis vectors explicitly. This is very convenient because the notation expresses both that the state of the quantum system is a vector, while at the same time explicitly writing out the physical quantity of interest (energy level, position, spin, polarization, etc).

1.5 Bra-ket Notation

In this section we detail the notation that we will use to describe a quantum state, $|\psi\rangle$. This notation is due to Dirac and, while it takes some time to get used to, is incredibly convenient.

Inner Products

We saw earlier that all of our quantum states live inside a Hilbert space. A Hilbert space is a special kind of vector space that, in addition to all the usual rules with vector spaces, is also endowed with an inner product. And an inner product is a way of taking two states (vectors in the Hilbert space) and getting a number out. For instance, define

$$|\psi\rangle = \sum_k a_k |k\rangle ,$$

where the kets $|k\rangle$ form a basis, so are orthogonal. If we instead write this state as a column vector,

$$|\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}$$

Then the inner product of $|\psi\rangle$ with itself is

$$\langle \psi, \psi \rangle = \begin{pmatrix} a_0^* & a_1^* & \cdots & a_{N_1}^* \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix} = \sum_{k=0}^{N-1} a_k^* a_k = \sum_{k=0}^{N-1} |a_k|^2$$

The complex conjugation step is important so that when we take the inner product of a vector with itself we get a real number which we can associate with a length. Dirac noticed that there could be an easier way to write this by defining an object, called a "bra," that is the conjugate-transpose of a ket,

$$\langle \psi | = |\psi\rangle^\dagger = \sum_k a_k^* \langle k| .$$

This object acts on a ket to give a number, as long as we remember the rule,

$$\langle j| \, |k\rangle \equiv \langle j|k\rangle = \delta_{jk}$$

Now we can write the inner product of $|\psi\rangle$ with itself as

$$\begin{aligned}\langle\psi|\psi\rangle &= \left(\sum_j a_j^* \langle j|\right)\left(\sum_k a_k |k\rangle\right)\\ &= \sum_{j,k} a_j^* a_k \langle j|k\rangle\\ &= \sum_{j,k} a_j^* a_k \delta_{jk}\\ &= \sum_k |a_k|^2\end{aligned}$$

Now we can use the same tools to write the inner product of any two states, $|\psi\rangle$ and $|\phi\rangle$, where

$$|\phi\rangle = \sum_k b_k |k\rangle .$$

Their inner product is,

$$\langle\psi|\phi\rangle = \sum_{j,k} a_j^* b_k \langle j|k\rangle = \sum_k a_k^* b_k$$

Notice that there is no reason for the inner product of two states to be real (unless they are the same state), and that

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* \in \mathbb{C}$$

In this way, a bra vector may be considered as a "functional." We feed it a ket, and it spits out a complex number.

The Dual Space

We mentioned above that a bra vector is a *functional* on the Hilbert space. In fact, the set of all bra vectors forms what is known as the *dual space.* This space is the set of *all* linear functionals that can act on the Hilbert space.

1.6 The Measurement Principle

This linear superposition $|\psi\rangle = \sum_{j=0}^{k-1} \alpha_j |j\rangle$ is part of the private world of the electron. Access to the information describing this state is severely limited —

in particular, we cannot actually measure the complex amplitudes α_j. This is not just a practical limitation; it is enshrined in the measurement postulate of quantum physics.

A measurement on this k state system yields one of at most k possible outcomes: i.e. an integer between 0 and $k-1$. Measuring $|\psi\rangle$ in the standard basis yields j with probability $|\alpha_j|^2$.

One important aspect of the measurement process is that it alters the state of the quantum system: the effect of the measurement is that the new state is exactly the outcome of the measurement. I.e., if the outcome of the measurement is j, then following the measurement, the qubit is in state $|j\rangle$. This implies that you cannot collect any additional information about the amplitudes α_j by repeating the measurement.

Intuitively, a measurement provides the only way of reaching into the Hilbert space to probe the quantum state vector. In general this is done by selecting an orthonormal basis $|e_0\rangle, \ldots, |e_{k-1}\rangle$. The outcome of the measurement is $|e_j\rangle$ with probability equal to the square of the length of the projection of the state vector ψ on $|e_j\rangle$. A consequence of performing the measurement is that the new state vector is $|e_j\rangle$. Thus measurement may be regarded as a probabilistic rule for projecting the state vector onto one of the vectors of the orthonormal measurement basis.

Some of you might be puzzled about how a measurement is carried out physically? We will get to that soon when we give more explicit examples of quantum systems.

1.7 Qubits

Qubits (pronounced "cue-bit") or quantum bits are basic building blocks that encompass all fundamental quantum phenomena. They provide a mathematically simple framework in which to introduce the basic concepts of quantum physics. Qubits are 2-state quantum systems. For example, if we set $k = 2$, the electron in the Hydrogen atom can be in the ground state or the first excited state, or any superposition of the two. We shall see more examples of qubits soon.

The state of a qubit can be written as a unit (column) vector $\binom{\alpha}{\beta} \in \mathbb{C}^2$. In Dirac notation, this may be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \text{with} \quad \alpha, \beta \in \mathbb{C} \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = 1.$$

This linear superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is part of the private world of the electron. For us to know the electron's state, we must make a measurement. Making a measurement gives us a single classical bit of information — 0 or 1. The simplest measurement is in the standard basis, and measuring $|\psi\rangle$ in this $\{|0\rangle, |1\rangle\}$ basis yields 0 with probability $|\alpha|^2$, and 1 with probability $|\beta|^2$.

One important aspect of the measurement process is that it alters the state of the qubit: the effect of the measurement is that the new state is exactly the outcome of the measurement. *I.e.*, if the outcome of the measurement of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ yields 0, then following the measurement, the qubit is in state $|0\rangle$. This implies that you cannot collect any additional information about α, β by repeating the measurement.

More generally, we may choose any orthogonal basis $\{|v\rangle, |w\rangle\}$ and measure the qubit in that basis. To do this, we rewrite our state in that basis: $|\psi\rangle = \alpha'|v\rangle + \beta'|w\rangle$. The outcome is v with probability $|\alpha'|^2$, and $|w\rangle$ with probability $|\beta'|^2$. If the outcome of the measurement on $|\psi\rangle$ yields $|v\rangle$, then as before, the the qubit is then in state $|v\rangle$.

Examples of Qubits

Atomic Orbitals

The electrons within an atom exist in quantized energy levels. Qualitatively these electronic orbits (or "orbitals" as we like to call them) can be thought of as resonating standing waves, in close analogy to the vibrating waves one observes on a tightly held piece of string. Two such individual levels can be isolated to configure the basis states for a qubit.

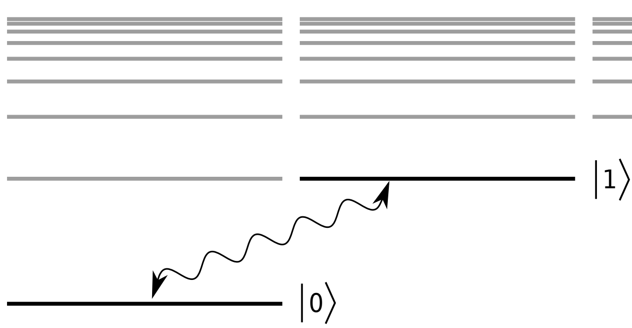

Figure 1.3: Energy level diagram of an atom. Ground state and first excited state correspond to qubit levels, $|0\rangle$ and $|1\rangle$, respectively.

Photon Polarization

Classically, a photon may be described as a traveling electromagnetic wave. This description can be fleshed out using Maxwell's equations, but for our purposes we will focus simply on the fact that an electromagnetic wave has a *polarization* which describes the orientation of the electric field oscillations (see Fig. 1.4). So, for a given direction of photon motion, the photon's polarization axis might lie anywhere in a 2-d plane perpendicular to that motion. It is thus natural to pick an orthonormal 2-d basis (such as $\vec{x}$ and $\vec{y}$, or "vertical" and "horizontal") to describe the polarization state (i.e. polarization direction) of a photon. In a quantum mechanical description, this 2-d nature of the photon polarization is represented by a qubit, where the amplitude of the overall polarization state in each basis vector is just the projection of the polarization in that direction.

The polarization of a photon can be measured by using a polaroid film or a calcite crystal. A suitably oriented polaroid sheet transmits x-polarized photons and absorbs y-polarized photons. Thus a photon that is in a superposition $|\phi\rangle = \alpha\,|\mathsf{x}\rangle + \beta\,|\mathsf{y}\rangle$ is transmitted with probability $|\alpha|^2$. If the photon now encounters another polariod sheet with the same orientation, then it is transmitted with probability 1. On the other hand, if the second polaroid sheet has its axes crossed at right angles to the first one, then if the photon is transmitted by the first polaroid, then it is definitely absorbed by the second sheet. This pair of polarized sheets at right angles thus blocks all the light. A somewhat counter-intuitive result is now obtained by interposing a third polariod sheet at a 45 degree angle between the first two. Now a photon that is transmitted by the first sheet makes it through the next two with probability 1/4.

To see this first observe that any photon transmitted through the first filter is in the state, $|0\rangle$. The probability this photon is transmitted through the second filter is 1/2 since it is exactly the probability that a qubit in the state $|0\rangle$ ends up in the state $|+\rangle$ when measured in the $|+\rangle, |-\rangle$ basis. We can repeat this reasoning for the third filter, except now we have a qubit in state $|+\rangle$ being measured in the $|0\rangle, |1\rangle$-basis — the chance that the outcome is $|0\rangle$ is once again 1/2.

Spins

Like photon polarization, the spin of a (spin-1/2) particle is a two-state system, and can be described by a qubit. Very roughly speaking, the spin is a quantum description of the magnetic moment of an electron which behaves like a spin-

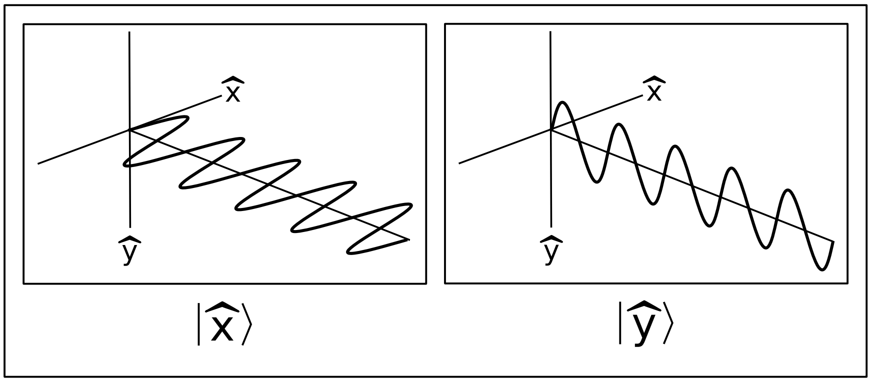

Figure 1.4: Using the polarization state of light as the qubit. Horizontal polarization corresponds to qubit state, $|\hat{x}\rangle$, while vertical polarization corresponds to qubit state, $|\hat{y}\rangle$.

ning charge. The two allowed states can roughly be thought of as clockwise rotations ("spin-up") and counter clockwise rotations ("spin-down"). We will say much more about the spin of an elementary particle later in the course.

Measurement Example I: Phase Estimation

Now that we have discussed qubits in some detail, we can are prepared to look more closely at the measurement principle. Consider the quantum state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle .$$

If we were to measure this qubit in the standard basis, the outcome would be 0 with probability 1/2 and 1 with probability 1/2. This measurement tells us only about the norms of the state amplitudes. Is there any measurement that yields information about the phase, θ?

To see if we can gather any phase information, let us consider a measurement in a basis other than the standard basis, namely

$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad \text{and} \qquad |-\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

What does $|\phi\rangle$ look like in this new basis? This can be expressed by first writing,

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \qquad \text{and} \qquad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Now we are equipped to rewrite $|\psi\rangle$ in the $\{|+\rangle, |-\rangle\}$-basis,

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle) \\ &= \frac{1}{2}(|+\rangle + |-\rangle) + \frac{e^{i\theta}}{2}(|+\rangle - |-\rangle) \\ &= \frac{1+e^{i\theta}}{2}|+\rangle + \frac{1-e^{i\theta}}{2}|-\rangle \quad .\end{aligned}$$

Recalling the Euler relation, $e^{i\theta} = \cos\theta + i\sin\theta$, we see that the probability of measuring $|+\rangle$ is $\frac{1}{4}((1+\cos\theta)^2 + \sin^2\theta) = \cos^2(\theta/2)$. A similar calculation reveals that the probability of measuring $|-\rangle$ is $\sin^2(\theta/2)$. Measuring in the $(|+\rangle, |-\rangle)$-basis therefore reveals some information about the phase θ.

Later we shall show how to analyze the measurement of a qubit in a general basis.

Measurement example II: General Qubit Bases

What is the result of measuring a general qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, in a general orthonormal basis $|v\rangle, |v^\perp\rangle$, where $|v\rangle = a|0\rangle + b|1\rangle$ and $|v^\perp\rangle = b^*|0\rangle - a^*|1\rangle$? You should also check that $|v\rangle$ and $|v^\perp\rangle$ are orthogonal by showing that $\langle v^\perp|v\rangle = 0$.

To answer this question, let us make use of our recently acquired bra-ket notation. We first show that the states $|v\rangle$ and $|v^\perp\rangle$ are orthogonal, that is, that their inner product is zero:

$$\begin{aligned}
\left\langle v^{\perp} | v \right\rangle &= (b^* |0\rangle - a^* |1\rangle)^{\dagger} (a |0\rangle + b |1\rangle) \\
&= (b \langle 0| - a \langle 1|)^{\dagger} (a |0\rangle + b |1\rangle) \\
&= ba \langle 0|0\rangle - a^2 \langle 1|0\rangle + b^2 \langle 0|1\rangle - ab \langle 1|1\rangle \\
&= ba - 0 + 0 - ab \\
&= 0
\end{aligned}$$

Here we have used the fact that $\langle i|j\rangle = \delta_{ij}$.

Now, the probability of measuring the state $|\psi\rangle$ and getting $|v\rangle$ as a result is,

$$\begin{aligned}
P_{\psi}(v) &= |\langle v|\psi\rangle|^2 \\
&= |(a^* \langle 0| + b^* \langle 1|) (\alpha |0\rangle + \beta |1\rangle)|^2 \\
&= |a^* \alpha + b^* \beta|^2
\end{aligned}$$

Similarly,

$$\begin{aligned}
P_{\psi}(v^{\perp}) &= \left| \left\langle v^{\perp} | \psi \right\rangle \right|^2 \\
&= |(b \langle 0| - a \langle 1|) (\alpha |0\rangle + \beta |1\rangle)|^2 \\
&= |b\alpha - a\beta|^2
\end{aligned}$$

Unitary Operators

The third postulate of quantum physics states that the evolution of a quantum system is necessarily unitary. Geometrically, a unitary transformation is a rigid body rotation of the Hilbert space, thus resulting in a transformation of the state vector that doesn't change its length.

Let us consider what this means for the evolution of a qubit. A unitary transformation on the Hilbert space $\mathbb{C}^2$ is specified by mapping the basis states $|0\rangle$ and $|1\rangle$ to orthonormal states $|v_0\rangle = a |0\rangle + b |1\rangle$ and $|v_1\rangle = c |0\rangle + d |1\rangle$. It is specified by the linear transformation on $\mathbb{C}^2$:

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

If we denote by $U^\dagger$ the conjugate transpose of this matrix:

$$U^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

then it is easily verified that $UU^\dagger = U^\dagger U = I$. Indeed, we can turn this around and say that a linear transformation U is unitary if and only if it satisfies this condition, that

$$UU^\dagger = U^\dagger U = I.$$

Let us now consider some examples of unitary transformations on single qubits or equivalently single qubit quantum gates:

- Hadamard Gate. Can be viewed as a reflection around $\pi/8$ in the real plane. In the complex plane it is actually a π-rotation about the $\pi/8$ axis.

  $$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

  The Hadamard Gate is one of the most important gates. Note that $H^\dagger = H$ – since H is real and symmetric – and $H^2 = I$.

- Rotation Gate. This rotates the plane by θ.

  $$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- NOT Gate. This flips a bit from 0 to 1 and vice versa.

  $$NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Phase Flip.

  $$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  The phase flip is a NOT gate acting in the $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ basis. Indeed, $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$.

How do we physically effect such a (unitary) transformation on a quantum system? To explain this we must first introduce the notion of the Hamiltonian acting on a system; you will have to wait for three to four lectures before we get to those concepts.

1.8 Problems

Problem 1

Show that

$$HZH = X$$

Problem 2

Verify that

$$U^\dagger U = UU^\dagger = I$$

for the general unitary operator,

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$