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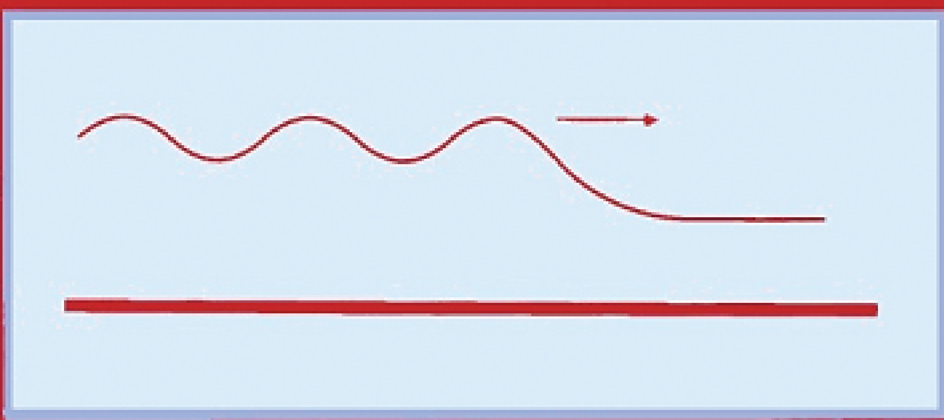
A Modern Introduction to the Mathematical Theory of Water Waves



R.S. JOHNSON

The theory of water waves has been a source of intriguing mathematical problems for at least 150 years. Virtually every classical mathematical technique appears somewhere within its confines. The aim of this book is to introduce mathematical ideas and techniques that are directly relevant to water-wave theory (although a formal development is not followed), enabling the main principles of modern applied mathematics to be seen in a context that both has practical overtones and is mathematically exciting.

Beginning with the introduction of the appropriate equations of fluid mechanics, the opening chapters go on to consider some classical problems in linear and nonlinear water-wave theory. This sets the scene for a study of more modern aspects, including problems that give rise to soliton-type equations. The book closes with an introduction to the effects of viscosity.

All the mathematical developments are presented in the most straightforward manner, with worked examples and simple cases carefully explained. Exercises, further reading, and historical notes on some of the important characters round off the book and help to make this an ideal text for either advanced undergraduate or beginning graduate courses on water waves.

A Modern Introduction to the Mathematical Theory of Water Waves

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To my parents Dorothy and Eric,
to my sons Iain and Neil,
and last but first to my wife Ros.

Contents

Preface

The theory of water waves has been a source of intriguing – and often difficult – mathematical problems for at least 150 years. Virtually every classical mathematical technique appears somewhere within its confines; in addition, linear problems provide a useful exemplar for simple descriptions of wave propagation, with nonlinearity adding an important level of complexity. It is, perhaps, the most readily accessible branch of applied mathematics, which is the first step beyond classical particle mechanics. It embodies the equations of fluid mechanics, the concepts of wave propagation, and the critically important rôle of boundary conditions. Furthermore, the results of a calculation provide a description that can be tested whenever an expanse of water is to hand: a river or pond, the ocean, or simply the household bath or sink. Indeed, the driving force for many workers who study water waves is to obtain information that will help to tame this most beautiful, and sometimes destructive, aspect of nature. (Perhaps 'to tame' is far too bold an ambition: at least to try to make best use of our knowledge in the design of man-made structures.) Here, though, we shall – without apology – restrict our discussion to the many and varied aspects of water-wave theory that are essentially mathematical. Such studies provide an excellent vehicle for the introduction of the modern approach to applied mathematics: complete governing equations; nondimensionalisation and scaling; rational approximation; solution; interpretation. This will be the type of systematic approach that is adopted throughout this text.

The comments that we have offered above describe the essential character of the study of water waves, particularly as it appeared during its first 120 years. However, the last 25 or 30 years have seen an altogether amazing explosion in the complexity of mathematical theories for water waves. The development of *soliton theory*, which itself started life in the context of water waves, has completely transformed many aspects of the mathematical description of nonlinear wave propagation. If it was needed, soliton theory has certainly brought the theory of water waves into the era of modern applied mathematics. This book, it is hoped, presents the material in a way that emphasises the mathematical aspects of classical water-wave theory, and also provides a description of the intrinsic relation between soliton theory and water waves.

This book is based on material which has been taught to either final-year honours mathematicians or to MSc students at the University of Newcastle upon Tyne, at various times over the last 20 years or so. The topics in classical water-wave theory (mainly in Chapter 2) are a considerable extension of those taught, in four or five different lecture courses, by the author during his time at Newcastle. The material on soliton theory is based on an introductory course given to MSc students in Applied Mathematics (and which also provided one of the bases for the book *Solitons: an introduction*, written jointly with Professor Philip Drazin). In all these courses, the aim has been to introduce mathematical ideas and techniques directly, rather than to present a formal and rigorous development. This approach, which is very much in the British tradition, enables the main principles of modern applied mathematics to be seen in a context that both has practical overtones and is mathematically exciting. It is intended that this text will provide an introduction to the theory of water waves (and associated mathematical techniques) to final-year undergraduate students in mathematics, physics, or engineering, as well as to postgraduate students in similar areas. Some of the more elementary material could be taught in the second year of some undergraduate programmes. However, it must be emphasised that there is no attempt to provide such an extensive treatment that the borders of current research are reached, although the book may allow the student to go some way in this direction. It should also be clear that *ad hoc* attempts to describe complicated phenomena are not part of our remit, important though some of these studies are. Furthermore, mainly in the interests of space, a section on numerical methods, which certainly play a rôle in the broader aspects of water-wave theory, is not included.

Chapter 1 introduces the appropriate equations of fluid mechanics, together with the relevant boundary conditions that are needed to describe water waves. In addition, the ideas of nondimensionalisation, scaling and asymptotic expansions are briefly explored, as are simple concepts in wave propagation. A student with a background in elementary fluid mechanics, and some knowledge of simple mathematical methods, could ignore this chapter and move directly to Chapter 2. (The only essential background that the student requires is in advanced calculus (for example, some familiarity with vector calculus), and in the methods of applied mathematics (for example, methods of solution of some classical ordinary and partial differential equations).) Chapter 2 looks, first, at some of the classical problems in linear water-wave theory. These include the speed of gravity and capillary waves, the effects of variable depth and the ship-wave pattern; the application of ray theory to problems where the background flow slowly varies is also developed. The second part of this chapter is devoted to nonlinear problems, but still those that are generally regarded as classical. In this area we include the Stokes wave, nonlinear long waves via the method of characteristics (and Riemann invariants), the hydraulic jump and bore, waves on a sloping beach and the solitary wave. Many additional examples and applications can be explored through the exercises at the end of the chapter.

Chapters 3 and 4 are devoted to the more modern aspects: problems that give rise to soliton-type equations. These are, first, the equations that belong to the Korteweg–de Vries family; some relevant results from soliton theory are quoted, and these are used to help in the interpretation of the various equations and solutions that arise. The applications are extended to include the effects of shear and variable depth. Then the Nonlinear Schrödinger family of equations is discussed in a similar fashion, although the rôles of an underlying shear flow or variable depth are treated less fully for this family, mainly because the calculations are very much more involved. For both families, some two-dimensional configurations of waves are also discussed.

The final chapter provides a brief introduction to the rôle and effects of viscosity, as they are relevant in a few water-wave phenomena. This is intended to add a broader view to water-wave theory; all the previous discussions here are solely for an inviscid fluid (but the flow is sometimes allowed to be rotational).

All the mathematical developments are presented in the most straightforward manner, with worked examples and simple cases carefully explained. Many other aspects, relevant calculations and additional examples are provided in the numerous exercises at the end of each chapter. Also at the end of each chapter is a section of further reading which indicates where more information can be found about some of the topics; these references include both research papers and other texts. Sections are numbered following the decimal system, and equations are numbered according to the chapter in which they appear: for example, equation (1.2) is equation 2 in Chapter 1. The exercises are numbered in a similar fashion (for example, Q2.3), as are the answers and hints at the end of the book (for example, A2.3). Also provided at the end of the book is a fairly extensive bibliography and author index, and also a collection of brief historical notes on some of the important characters who have worked on the theory of water waves. The quotations at the beginning of each chapter, and at the start of some sections, are taken from the poetical works of Alfred, Lord Tennyson.

I wish to put on record my very grateful thanks for the typing of the manuscript to Mrs Heather Bliss, Mrs Helen Bell and Miss Jackie Tait, who all played a part, but most particularly to Mrs Susan Cassidy, who carried by far the major burden. This she did with great efficiency, speed, dedication and, throughout, with the greatest good humour when faced with (a) my handwritten manuscript and (b) my changes of mind. The originals of the figures were produced on my PC, using a combination of *Mathematica* and *KeyDraw*, and printed on my Hewlett-Packard DeskJet printer (so I carry full responsibility for their clarity and accuracy). Finally, I wish to thank Cambridge University Press, and particularly Professor David Crighton, for their encouragement to write this text (and their patience when I got behind the planned schedule).

RSJ
Newcastle upon Tyne
December 1996

1

Mathematical preliminaries

For nothing is that errs from law
In Memoriam A.H.H. LXXIII

Science moves, but slowly slowly, creeping on from point to point
Locksley Hall

Before we commence our presentation of the theory of water waves, we require a firm and precise base from which to start. This must be, at the very least, a statement of the relevant governing equations and boundary conditions. However, it is more satisfactory, we believe, to provide some background to these equations, albeit within the confines of an introductory and relatively brief chapter. The intention is therefore to present a derivation of the equations for inviscid fluid mechanics (*Euler's equation* and the equation of *mass conservation*) and a few of their properties. (The corresponding equations for a viscous fluid – primarily the *Navier–Stokes equation* – appear in Appendix A.) Coupled to these general equations is the set of boundary (and initial) conditions which select the water-wave problem from all other possible solutions of the equations. Of particular importance, as we shall see, are the conditions that define and describe the surface of the fluid; these include the *kinematic condition* and the rôles of *pressure* and *surface tension*. Some rather general consequences of coupling the equations and boundary conditions will also be mentioned.

Once we have available the complete prescription of the water-wave problem, based on a particular model (such as for inviscid flow), we may 'normalise' in any manner that is appropriate. It turns out to be very convenient – and is indeed typical of the applied mathematical approach – to introduce a suitable set of *nondimensional variables*. Further, a useful next step (which is particularly significant for our work in Chapters 3 and 4) is to *scale* the variables with respect to the small parameters thrown up by the nondimensionalisation. All this will enable us to characterise, in a rather precise way, the various types of approximation that we shall employ. In the process, we shall give a summary of the equations that represent different approximations of the full water-wave problem.

Throughout, we take the opportunity to present all the relevant equations in both rectangular Cartesian and cylindrical coordinates.

In the final stage of this preliminary discussion we provide a brief overview of some of the ideas that will permeate many of the problems that we shall encounter. This involves a simple introduction to the mathematics of wave propagation, where we describe the important phenomena associated with the *nonlinearity*, *dispersion* and *dissipation* of the wave. Further, much of our work in the newer aspects of water-wave theory will be with small-amplitude waves and with the slow evolution of wave properties; these may occur separately or together. In order to extract useful and relevant solutions in these cases, we shall require the application of asymptotic methods. Here we present an introduction to the use of *asymptotic expansions*, which will include both near-field and far-field asymptotics and the method of multiple scales.

These mathematical preliminaries may cover material already familiar to some readers, in whole or in part. Those with a background in fluid mechanics could ignore Section 1.1, whereas, for example, those who have received a basic course in wave propagation and elementary asymptotics could ignore Section 1.4. In Chapter 2, and thereafter, we start by giving a summary of the equations and boundary conditions that are relevant to each topic under discussion; this, at its simplest level, is all that is necessary to begin those studies.

1.1 The governing equations of fluid mechanics

In these derivations we shall use a vector notation and the methods of the vector calculus. (The tensor calculus is used in the brief derivation of the Navier–Stokes equation given in Appendix A, although the resulting equation is also written there in terms of vectors.) Here we shall derive the equations of mass conservation and motion (Newton's Second Law) in the absence of thermal changes (which are altogether irrelevant in the propagation of water waves). Any energy equation is therefore a consequence of only the motion (through Newton's Second Law) without any contributions from the thermodynamics of the fluid.

The notation that we shall adopt is the conventional one: at any point in the fluid, the velocity of the fluid is $\mathbf{u}(\mathbf{x}, t)$ where $\mathbf{x}$ is the position vector and t is a time coordinate. The density (mass/unit volume) of the fluid is $\rho(\mathbf{x}, t)$ (but for water-wave applications, as we shall mention later, we take $\rho =$ constant); the pressure at any point in the fluid is $P(\mathbf{x}, t)$. If the choice of coordinates is the familiar right-handed rectangular Cartesian system, then we write

$$\mathbf{x} \equiv (x, y, z) \quad \text{and} \quad \mathbf{u} \equiv (u, v, w).$$

We shall assume that $\mathbf{u}$, ρ, and P are continuous functions (in $\mathbf{x}$ and t) – usually called the *continuum hypothesis* – and that they are also suitably differentiable functions.

1.1.1 The equation of mass conservation

Imagine a volume V, which is bounded by the surface S, within (and totally occupied by) the fluid. We treat V as fixed relative to some chosen inertial frame, so that the fluid in motion may cross the imaginary surface S. Given that the density of the fluid is $\rho(\mathbf{x}, t)$, then the rate of change of mass in V is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho \, \mathrm{d}v\right)$$

where $\int_{\mathrm{V}} \mathrm{d}v$ represents the triple integral over V. Now, let $\mathbf{n}$ be the outward unit normal on S (see Figure 1.1) so that the outward velocity component of the fluid across S is $\mathbf{u} \cdot \mathbf{n}$. Thus the net rate at which mass flows *out* of V is

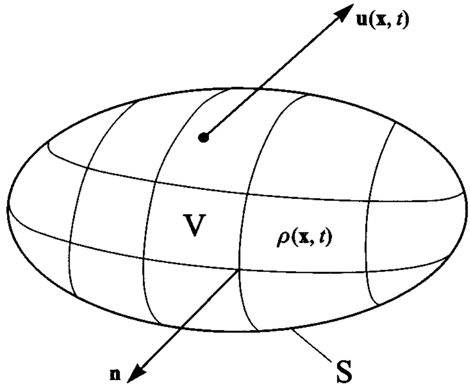


Figure 1.1. The volume V bounded by the surface S; $\rho(x, t)$ is the density of the fluid, $\mathbf{u}(\mathbf{x}, t)$ is the velocity at a point in the fluid and $\mathbf{n}$ is the outward normal on S.

$$\int_S \rho \mathbf{u} \cdot \mathbf{n} \mathrm{d}s,$$

where this is the double integral over S.

Under the fundamental assumption that matter (mass) is neither created nor destroyed anywhere in the fluid, the rate of change of mass in V is brought about only by the rate of mass flowing *into* V across S, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_V \rho \, \mathrm{d}v\right) = -\int_S \rho \mathbf{u} \cdot \mathbf{n} \mathrm{d}s.$$

This equation is rewritten by the application of *Gauss' theorem* (the *divergence theorem*) to the integral on the right, to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_V \rho \, \mathrm{d}v\right) + \int_V \nabla \cdot (\rho \mathbf{u}) \mathrm{d}v = 0$$

where ∇ is the familiar *del* operator (used here in the *divergence* of $\rho \mathbf{u}$). Further, since V is fixed in our coordinate system, the only dependence on t is through $\rho(\mathbf{x}, t)$, so we may write

$$\int_V \left\{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right\} \mathrm{d}v = 0. \tag{1.1}$$

(We shall write more about *differentiation under the integral sign* later; see also Q1.30, Q1.31.) Now equation (1.1) is clearly applicable to any V totally occupied by the fluid, so the limits (represented symbolically by V) of the triple integral are therefore arbitrary; the integral is then always zero (for a continuous integrand, which we assume here) only if

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{1.2}$$

This equation, (1.2), is one form of the *equation of mass conservation* (sometimes called the *continuity* equation, referring to the continuity of matter). (The argument that takes us from (1.1) to (1.2) can be rehearsed in the simple example

$$\int_a^b f(x) \mathrm{d}x = 0 \quad \text{for arbitrary } a, b \Rightarrow f(x) = 0;$$

this is left as an exercise.)

It is usual to expand (1.2) as

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\rho = 0,$$

and then introduce

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{1.3}$$

the *material* (or *convective*) *derivative*; see Q1.5 and Section 1.2.1. Equation (1.2) therefore becomes

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla \cdot \mathbf{u} = 0, \tag{1.4}$$

from which we see that for an incompressible flow defined by

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \tag{1.5}$$

we have

$$\nabla \cdot \mathbf{u} = 0. \tag{1.6}$$

(A function ($\mathbf{u}$) which satisfies equation (1.6), so that the divergence of $\mathbf{u}$ is zero, is said to be *solenoidal*.) Equation (1.5) describes the constancy of ρ on individual fluid particles; we shall, however, interpret incompressibility as meaning $\rho =$ constant everywhere (which is clearly a solution of (1.5), and a very good model for fluids like water). Some of these basic ideas are explored in Q1.7–Q1.9.

1.1.2 The equation of motion: Euler's equation

We now turn our attention to the application of Newton's Second Law to a fluid, but a fluid which is assumed to be *inviscid*; that is, it has zero viscosity (internal friction). (The corresponding equation for a viscous fluid – the Navier–Stokes equation – is described in Appendix A.) Newton's Second Law requires us to balance the rate of change of (linear) momentum of the fluid against the resultant force acting on the fluid. First, therefore, we must find a representation of the forces acting on the fluid.

There are two types of force that are relevant in fluid mechanics: a *body force*, which is more or less the same for all particles and has its source exterior to the fluid, and a *local* (or *short-range*) *force*, which is the force exerted on a fluid element by other elements nearby. The body force which is almost always present is gravity, and this is certainly the case in the study of water waves. We define the general body force to be $\mathbf{F}(\mathbf{x}, t)$ per unit mass; if $\mathbf{F}$ is due solely to the (constant) acceleration of gravity (g) then we would write $\mathbf{F} \equiv (0, 0, -g)$ in both Cartesian and cylindrical coordinates (with z measured positive upwards). The local force is comprised of a pressure contribution together with any viscous forces that are present; in general, of course, this is conveniently represented by the stress tensor in the fluid: see Appendix A. Here we retain only the pressure (P), which produces a *normal* force acting *onto* any element of fluid.

To proceed we define (just as before) an imaginary volume V, bounded by the surface S, which is fixed in our frame of reference and totally occupied by the fluid. The total force (body + local) acting *on* the fluid in V is

$$\int_{\mathrm{V}} \rho \mathbf{F} \mathrm{d}v - \int_{\mathrm{S}} P\mathbf{n} \mathrm{d}s;$$

see Figure 1.2. (We remember that $\mathbf{n}$ is the *outward* unit normal on S.) Applying Gauss' theorem to the second integral (see Q1.2), we obtain the resultant force

$$\int_{\mathrm{V}} (\rho \mathbf{F} - \nabla P) \mathrm{d}v. \tag{1.7}$$

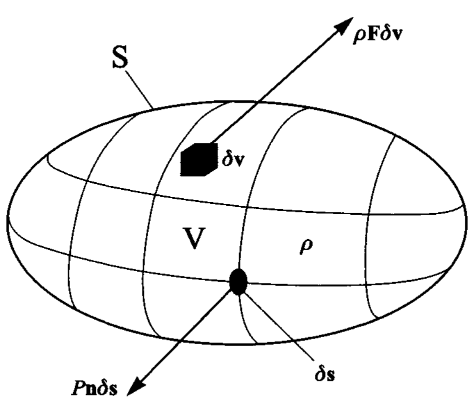


Figure 1.2. The volume V bounded by the surface S; the body force on an element is $\rho\mathbf{F}\delta v$ and the pressure force *on* an element of area is $-P\mathbf{n}\delta s$.

The rate of change of momentum of the fluid in V is simply

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho\,\mathbf{u}\,\mathrm{d}v\right), \tag{1.8}$$

and the rate of flow of momentum across S into V is

$$-\int_{\mathrm{S}} \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}s. \tag{1.9}$$

Now Newton's Second Law for the fluid in V (upon recalling that V is fixed in our coordinate frame) may be expressed as:

rate of change of momentum of fluid in V
= resultant force acting on fluid in V
+ rate of flow of momentum across S into V.

Thus from equations (1.7)–(1.9) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{V}} \rho\,\mathbf{u}\,\mathrm{d}v\right) = \int_{\mathrm{V}} (\rho\mathbf{F} - \nabla P)\,\mathrm{d}v - \int_{\mathrm{S}} \rho\,\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}s,$$

which is written more compactly by (a) taking $\mathrm{d}/\mathrm{d}t$ through the integral sign, (b) applying Gauss' theorem to each component of (1.9) (see Q1.3), and, (c), rearranging, to yield

$$\int_{\mathrm{V}} \left\{\frac{\partial}{\partial t}(\rho\,\mathbf{u}) + \rho\,\mathbf{u}(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho\mathbf{u}\right\}\mathrm{d}v = \int_{\mathrm{V}} (\rho\,\mathbf{F} - \nabla P)\,\mathrm{d}v. \tag{1.10}$$

We expand the integrand on the left side of this equation as

$$\int_{\mathrm{V}} \left\{\rho\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial\rho}{\partial t} + \rho\,\mathbf{u}(\nabla\cdot\mathbf{u}) + \mathbf{u}(\mathbf{u}\cdot\nabla)\rho + \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\right\}\mathrm{d}v = \int_{\mathrm{V}} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}v, \tag{1.11}$$

where we have used the equation of mass conservation, (1.4), and introduced the material derivative, (1.3). It is clear that, with sufficient understanding of the notion of the material derivative (see Q1.4–Q1.6), we could write (1.11) directly: it is the appropriate form of 'mass × acceleration' for all the fluid in V.

The equation (1.10), with (1.11), now becomes

$$\int_{\mathrm{V}} \left(\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{d}t} - \rho \mathbf{F} + \nabla P \right) \mathrm{d}v = \mathbf{0}$$

and, as before, for this to be valid for arbitrary V (and a continuous integrand) we must have

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla P + \mathbf{F}, \tag{1.12}$$

when written in its usual form. This is *Euler's equation*, which is the result of applying Newton's Second Law to an inviscid (that is, frictionless) fluid. (Notice that the pressure, P, may be defined relative to an arbitrary constant value without altering equation (1.12).)

It is convenient, particularly in view of our later work, to present the three components of Euler's equation, (1.12), and also the equation of mass conservation, in the two coordinate systems that we shall use. In rectangular Cartesian coordinates, $\mathbf{x} \equiv (x, y, z)$, with $\mathbf{u} \equiv (u, v, w)$ and $\mathbf{F} \equiv (0, 0, -g)$, and for constant density, equations (1.12) and (1.6) become, respectively,

$$\left.\begin{aligned} &\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial y}, \quad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \\ &\text{where} \\ &\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \end{aligned}\right\} \tag{1.13}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.14}$$

These same equations written in cylindrical coordinates, $\mathbf{x} \equiv (r, \theta, z)$, with $\mathbf{u} \equiv (u, v, w)$ (where the same notation for $\mathbf{u}$ in this system should not cause any confusion: it will be plain which coordinates are being used in a given calculation) are, again with $\mathbf{F} \equiv (0, 0, -g)$ and $\rho =$ constant,

$$\left.\begin{aligned} &\frac{\mathrm{D}u}{\mathrm{D}t} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \theta}, \\ &\qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \end{aligned}\right\} \tag{1.15}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z},$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \tag{1.16}$$

These equations, (1.13–1.16), will form the basis for the developments described in Chapters 2, 3, and 4, when coupled to the appropriate boundary conditions (Section 1.2) and – usually – after suitable simplification (Section 1.3). (The corresponding equations for a viscous fluid are presented in Appendix A, and are used in Chapter 5.)

1.1.3 Vorticity, streamlines and irrotational flow

A fundamental property of a fluid flow is the *curl* of the velocity field: $\nabla_{\wedge}\mathbf{u}$. This is called the *vorticity*, and it is conventionally represented by the vector $\boldsymbol{\omega}$; the vorticity measures the local spin or rotation of the fluid (that is, the rotational motion – as compared with the translational) of a fluid element (see Q1.12). In consequence, flows, or regions of flows, in which $\boldsymbol{\omega} \equiv \mathbf{0}$ are said to be *irrotational*; such flows can often be analysed by using particularly routine methods. Unfortunately, real flows are very rarely irrotational anywhere, but for many flows the vorticity is very small almost everywhere, and these may therefore be modelled by assuming irrotationality. Nevertheless, many important aspects of fluid flow require $\boldsymbol{\omega} \neq \mathbf{0}$ somewhere, and the study of such flows normally involves a consideration of the dynamics of vorticity and its properties. In water-wave problems, however, classical aspects of vorticity play a rather minor rôle, and so a deep knowledge of vorticity is not a prerequisite for a study of water waves. (Some small exploration of vorticity is offered in the exercises: see Q1.13–Q1.17.)

Now, before we make use of the vorticity vector in Euler's equation, we introduce a very powerful – but related – concept in the study of fluid motion: the *streamline*. Consider the family of (imaginary) curves which everywhere have the velocity vector as their tangent; these curves are the streamlines. If such a curve is described by $\mathbf{x} = \mathbf{x}(s; t)$ (at any instant in time), where s is the parameter which maps out the curve, then the streamlines are the solutions of

$$\frac{d\mathbf{x}}{ds} \propto \mathbf{u} \quad \text{or} \quad \frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t) \quad \text{(at fixed } t\text{)}. \tag{1.17}$$

In this second representation, the constant of proportionality has been absorbed into the definition of s. Then, for example, in rectangular Cartesian coordinates this vector equation becomes the three scalar equations

$$\frac{dx}{ds} = u, \quad \frac{dy}{ds} = v, \quad \frac{dz}{ds} = w,$$

or equivalently,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \tag{1.18}$$

for the streamlines. (The streamline should not be confused with the *path* of a particle; this is defined (see Q1.4 and also Q1.19) by

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t), \tag{1.19}$$

so particle paths and streamlines coincide, in general, only for steady flow; see Q1.19.) The streamlines provide a particularly effective way of describing a flow field: even a simple sketch of the streamlines for a flow often enables important characteristics to be recognised at a glance. (An associated concept, the *stream function*, is described in Q1.20–Q1.23.)

We now turn to a brief consideration of the results that can be obtained when the vorticity, $\boldsymbol{\omega}$, is introduced into Euler's equation, (1.12),

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{F}. \tag{1.20}$$

For our purposes we shall assume that $\rho =$ constant (but see Q1.18), and that the body force is represented by a *conservative* force field: $\mathbf{F} = -\nabla\Omega$ for some *potential function* $\Omega(\mathbf{x}, t)$, where the negative sign is a convenience. (This choice for $\mathbf{F}$ applies to most examples of interest; for our studies we shall use $\Omega = gz$ where g is the (constant) acceleration of gravity and z is measured positive upwards.) Equation (1.20) therefore becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\left(\frac{P}{\rho} + \Omega\right),$$

which is rewritten by introducing the identity (see Q1.1)

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) - \mathbf{u} \wedge (\nabla \wedge \mathbf{u})$$

where $\nabla \wedge \mathbf{u} = \boldsymbol{\omega}$, the vorticity. Thus we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{P}{\rho} + \Omega\right) = \mathbf{u} \wedge \boldsymbol{\omega}, \tag{1.21}$$

and there are two cases worthy of further examination.

The first is for steady flow, where $\mathbf{u}$, P, and Ω are all independent of time, t. Equation (1.21) therefore becomes

$$\nabla\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{P}{\rho} + \Omega\right) = \mathbf{u} \wedge \boldsymbol{\omega},$$

and a simple geometrical property enables us to make headway with this (apparently) intractable equation. (An alternative approach is to dot both sides with the vectors $\mathbf{u}$ and, separately, $\boldsymbol{\omega}$.) It is a familiar result that ∇f, the *gradient* of f, is a vector orthogonal to the surface $f(\mathbf{x}) = \text{constant}$; thus $\mathbf{u} \wedge \boldsymbol{\omega}$ is perpendicular to the surfaces

$$\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{P}{\rho} + \Omega = \text{constant}. \tag{1.22}$$

But $\mathbf{u} \wedge \boldsymbol{\omega}$ is also perpendicular to both the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$, so the surfaces (1.22) must contain lines which are everywhere parallel to $\mathbf{u}$ and $\boldsymbol{\omega}$. One such set of lines is the family of streamlines, (1.17) and (1.18). Thus equation (1.22), known as *Bernoulli's equation* (or *theorem*), applies on streamlines; it describes the conservation of energy (kinetic + work done by pressure forces + potential) for a steady (and inviscid) flow with vorticity. This is a fundamental and powerful result in the study of elementary flows. (Bernoulli's equation is also valid on the family of lines which has $\boldsymbol{\omega}$ as the tangent to the lines at every point, but these lines are not usually of much interest in this context.)

The second case, of some importance in water-wave problems, is for irrotational but unsteady flow. Now for irrotational flow we have $\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = \mathbf{0}$, and so $\mathbf{u} = \nabla\phi$ for a potential function $\phi(\mathbf{x}, t)$, the *velocity potential*; the study of irrotational flows reduces to the problem of determining ϕ; see Q1.24. Indeed, for irrotational and incompressible flow we have

$$\mathbf{u} = \nabla\phi \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0,$$

so ϕ satisfies *Laplace's equation*

$$\nabla^2\phi = 0. \tag{1.22}$$

Thus the *nonlinear* Euler's equation, (1.12), and the equation of mass conservation, (1.6), have been replaced by a classical *linear* second-order partial differential equation (provided that $\boldsymbol{\omega} = \mathbf{0}$ and $\mathbf{F} = -\nabla\Omega$). If we use $\mathbf{u} = \nabla\phi$ in equation (1.21), with $\boldsymbol{\omega} = \mathbf{0}$, then it follows directly that

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega\right) = \mathbf{0},$$

so

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega = f(t), \tag{1.23}$$

where $f(t)$ is an arbitrary function of integration. (It is always possible to redefine ϕ as $\phi + \int f(t)\mathrm{d}t$ and thereby remove $f(t)$ from equation (1.23); of course, this choice of ϕ does not affect the velocity field since $\nabla(\int f(t)\mathrm{d}t) = \mathbf{0}$.) Equation (1.23) is known by some authors as Bernoulli's equation (cf. equation (1.22)) or, at least more accurately, as the Bernoulli equation for unsteady flow. A less confusing name – unfortunately used rather rarely nowadays – is the *pressure equation*, which we prefer; this helps to avoid the possible problems of interpretation which we mention below. ('Pressure equation' is used to indicate that P is completely determined (to within initial data) once the velocity field is known through ϕ.)

If it is now assumed, in addition, that the flow is steady then equation (1.23) becomes

$$\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + \Omega = \text{ constant,} \tag{1.24}$$

which is equation (1.22) – or is it? Equations (1.23) and (1.24) describe the fluid *everywhere*; there is no reference to streamlines, as there is with equation (1.22). Equation (1.24) is associated with the *same* constant throughout the fluid, whereas equation (1.22) assigns *different* constants to *different* streamlines. This important distinction provides a contrast between irrotational and rotational steady flows.

We complete this section by quoting Laplace's equation, which is valid for incompressible, irrotational flow, in both rectangular Cartesian coordinates

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0, \tag{1.25}$$

and cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0. \tag{1.26}$$

The corresponding velocities are

$$\mathbf{u}\equiv\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

in rectangular Cartesian coordinates, and

$$\mathbf{u}\equiv\left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

in cylindrical coordinates.

1.2 The boundary conditions for water waves

The boundary conditions that define water-wave problems come in various forms. We first briefly describe these before we examine them in detail. At the surface, called a *free* surface because it is not defined by velocity conditions (as on a *rigid* boundary, for example), the atmosphere exerts stresses on the fluid surface. In general, these stresses will include a viscous component (which is particularly relevant if we wish to model the effects of a surface wind, for example). However, if the fluid may be reasonably modelled as inviscid, then the atmosphere exerts only a pressure on the surface. This pressure is often taken to be a constant – the atmospheric pressure – but it may vary in time and also from point to point on the surface. (The passage of a region of higher/lower pressure could be used to model the movement of a storm or other similar phenomena.) Further, any surface tension effects can also be included at a curved surface (in the presence of a wave, for example) giving rise to the maintenance of a pressure *difference* across the surface. We should comment that our philosophy here is to regard the conditions obtaining at the surface as prescribed. A more complete theory would couple the motion of the water surface and the air above it, but the small density of air compared with that of water makes our approach feasible. Nevertheless, one method – not discussed in this text – for studying ocean waves, for example, is to consider the exchange of momentum and energy between the air and the surface waves.

Another, perhaps less obvious, condition requires a statement that the (moving) surface is a surface of the fluid; that is, it is always composed of fluid particles. This is called the *kinematic condition*, since it does not involve the action of forces; the stress conditions described above (which obviously generate forces at the surface) are called the *dynamic conditions* (which reduce to just one condition for an inviscid fluid).

At the bottom of the fluid we shall assume, throughout our work, that the bed there is impermeable. Then, if the fluid is treated as viscous, we must impose the no-slip condition on this surface (so that fluid particles in contact with the surface move with that surface). Thus, for a fixed rigid boundary, the fluid velocity will be zero here. On the other hand, if the fluid is modelled as inviscid, then the bottom topography becomes a surface of the fluid, so that fluid particles in contact with the bed move in this surface. This therefore mirrors the kinematic condition at the free surface, except that the bottom is prescribed *a priori*. For many of our problems, the bottom surface will be fixed and rigid (but not necessarily a horizontal plane); however, it could move in a prescribed manner if we wished to model a marine earthquake, for example.

In most of our applications, the fluid will be assumed to extend to infinity in all horizontal directions. We might, rarely, encounter a boundary wall which will then provide the same type of boundary condition as the bottom topography.

Finally, we comment that the rôle of initial data is relatively unimportant in the type of water-wave problems that we shall discuss. Of course, the wave must be initiated in some fashion (by a suitable disturbance of the surface), but in most problems we shall assume that this has already occurred. Our main interest will be in following the evolution of the wave in many – and varied – situations.

We now turn to a careful formulation of these boundary conditions, based on the principles that we have just outlined, for an inviscid fluid. The corresponding results for a viscous fluid are briefly described and presented in Appendix B.

1.2.1 The kinematic condition

The free surface, whose determination is usually the primary objective in water-wave problems, will be represented by

$$z = h(\mathbf{x}_\perp, t), \tag{1.27}$$

where $\mathbf{x}_\perp$ denotes the two-vector which is perpendicular to the z-direction. In rectangular Cartesian coordinates we therefore have $\mathbf{x}_\perp \equiv (x, y)$, and in cylindrical coordinates this is $\mathbf{x}_\perp \equiv (r, \theta)$. Now a surface $F(\mathbf{x}, t) =$ constant which moves with the fluid, so that it always contains the same fluid particles, must satisfy

$$\frac{\mathrm{D}F}{\mathrm{D}t} = 0;$$

see Q1.5. The free surface, written in the form

$$z - h(\mathbf{x}_\perp, t) = 0,$$

must therefore satisfy this same condition:

$$\frac{\mathrm{D}}{\mathrm{D}t}\{z - h(\mathbf{x}_\perp, t)\} = 0,$$

the fluid particles being those that move in the surface. This yields, directly,

$$w - \{h_t + (\mathbf{u}_\perp \cdot \nabla_\perp)h\} = 0$$

(where the subscript in t denotes the time derivative), since

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp + w\frac{\partial}{\partial z}$$

where $\nabla_\perp$ is the grad operator perpendicular to the direction of the z-coordinate. (This symbol is usually pronounced 'del-perp'.) The velocity vector has been written as $\mathbf{u} \equiv (\mathbf{u}_\perp, w)$, although both $\mathbf{u}_\perp$ and $\nabla_\perp$ are, strictly, unnecessary notations here since $h = h(\mathbf{x}_\perp, t)$ only; we choose to use them in order to make quite clear the structure of the boundary condition. The kinematic condition is therefore

$$w = h_t + (\mathbf{u}_\perp \cdot \nabla_\perp)h \quad \text{on } z = h(\mathbf{x}_\perp, t), \tag{1.28}$$

and the evaluation of $z = h$ is needed to define the velocity field required here. (An alternative derivation of equation (1.28) is discussed in Q1.27.)

1.2.2 The dynamic condition

In the absence of viscous forces, the simplest dynamic condition merely requires that the pressure, P, is prescribed on $z = h(\mathbf{x}_\perp, t)$; the corresponding result for a viscous fluid is given in Appendix B. For most problems studied in the theory of water waves, it is usual to set

$P = P_a =$ constant, the pressure of the atmosphere. Of course, the simplicity of this boundary condition tends to obscure the fact that the evaluation is on the free surface ($z = h$) whose determination is part – often the most significant part – of the solution of the problem.

One special version of the dynamic boundary condition is offered by the case of an incompressible, irrotational, unsteady flow. From the pressure equation, (1.23), with $\Omega = gz$, we have

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P}{\rho} + gz = f(t)$$

everywhere. We consider the problem for which $P = P_a$ on $z = h(\mathbf{x}_\perp, t)$, then continuity of pressure requires that

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{P_a}{\rho} + gh = f(t) \quad \text{on } z = h.$$

Further, let us suppose that, somewhere (as $|\mathbf{x}_\perp| \to \infty$, for example), the fluid is stationary with $P = P_a$ and $h = h_0 =$ constant; then

$$f(t) = \frac{P_a}{\rho} + gh_0$$

so

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + g(h - h_0) = 0 \quad \text{on } z = h. \tag{1.29}$$

This equation, (1.29), constitutes one of the simplest descriptions of the surface-pressure condition. This is then one of the boundary conditions to be used in the construction of the relevant solution of Laplace's equation for ϕ.

For a rotational flow we cannot employ the pressure equation, and so we must solve Euler's equation with P given on $z = h$. Indeed, as we shall see, it turns out that there is very little to choose – even for irrotational flow – between solving Euler's equation with $P = P_a$ or Laplace's equation with (1.29), at least in the suitably approximate forms that we usually encounter.

Now we turn to the extension of this dynamic condition (for an inviscid fluid) which accommodates the effects of *surface tension* (which supports a pressure difference across a curved surface). The classical description of surface tension is represented by

$$\text{pressure difference} = \Delta P = \frac{\Gamma}{R}, \tag{1.30}$$

where $1/R$ is the mean curvature

$$\frac{1}{R} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2},$$

and κ_1, κ_2 are the principal radii of curvature. (The quantity $1/R$ is often called the *Gaussian curvature*.) The parameter Γ (> 0) is the coefficient of surface tension (force/unit length), and $\Delta P > 0$ if the surface is convex; see Figure 1.3. The fundamental equation, (1.30), is usually called *Laplace's formula* and, in general, Γ varies with temperature; here we shall treat Γ as a constant. The result of using this equation in the dynamic condition is to replace, for example, $P = P_a =$ constant at the fluid surface by

$$P = P_a - \frac{\Gamma}{R} \quad \text{on} \quad z = h(\mathbf{x}_\perp, t), \tag{1.31}$$

so that the pressure in the fluid at $z = h$ is increased if the surface is concave ($R < 0$).

It is clear that a complication in this formulation involves the precise description required for the curvature, $1/R$. Fairly elementary geometrical considerations lead, for the choice of rectangular Cartesian coordinates with $h = h(x, y, t)$, to

$$\frac{1}{R} = \frac{(1 + h_y^2)h_{xx} + (1 + h_x^2)h_{yy} - 2h_x h_y h_{xy}}{(1 + h_x^2 + h_y^2)^{3/2}}, \tag{1.32}$$

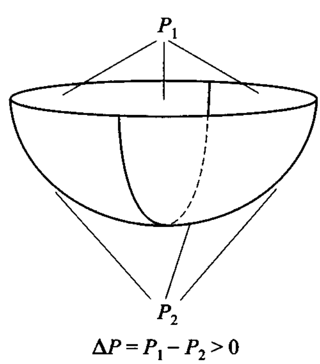


Figure 1.3. A convex surface with an 'internal' pressure P_1 and an 'external' pressure P_2, where $P_1 > P_2$.

where subscripts denote partial derivatives. A simple special case of this result recovers the well-known expression for curvature in only one direction:

$$\frac{1}{R}=\frac{h_{xx}}{(1+h_x^2)^{3/2}}, \quad h=h(x,t). \tag{1.33}$$

The corresponding representation in cylindrical coordinates, where $h=h(r,\theta,t)$, is

$$\frac{1}{R}=\frac{\left(1+\frac{1}{r^2}h_\theta^2\right)h_{rr}+\frac{1}{r^2}\left\{(1+h_r^2)(h_{\theta\theta}+rh_r)-2(h_{r\theta}-\frac{1}{r}h_\theta)h_r h_\theta\right\}}{\left(1+h_r^2+\frac{1}{r^2}h_\theta^2\right)^{3/2}}. \tag{1.34}$$

1.2.3 The bottom condition

For an inviscid fluid, the bottom constitutes – like the free surface – a boundary which is defined as a surface moving with the fluid. Let us represent the (impermeable) bed of the flow by

$$z=b(\mathbf{x}_\perp,t);$$

for this to be a fluid surface then

$$\frac{\mathrm{D}}{\mathrm{D}t}\{z-b(\mathbf{x}_\perp,t)\}=0.$$

Thus

$$w=b_t+(\mathbf{u}_\perp\cdot\nabla_\perp)b \quad \text{on} \quad z=b, \tag{1.35}$$

where $b(\mathbf{x}_\perp,t)$ will be prescribed in our problems. However, it should be mentioned that there are classes of problem (which we shall not discuss) where b is not known *a priori*; this situation can arise in the study of sediment movement, for example. Most of the calculations that we shall encounter in our work will involve a stationary bottom condition, so that equation (1.35) becomes

$$w=(\mathbf{u}_\perp\cdot\nabla_\perp)b \quad \text{on} \quad z=b. \tag{1.36}$$

(In the case of one-dimensional propagation, where $b=b(x)$ with $\mathbf{x}_\perp\equiv(x,0)$ and $\mathbf{u}_\perp\equiv(u,0)$, this reduces to the simple condition

$$w=u\frac{\mathrm{d}b}{\mathrm{d}x} \quad \text{on} \quad z=b(x),$$

which is readily understood from elementary considerations.)

1.2.4 An integrated mass conservation condition

Now that we have written down the general conditions that describe the kinematics of the motion at both the free surface and the bottom, we show how they can be combined with the equation of mass conservation. This produces a conservation condition for the whole motion, which will prove a useful result in some of our later work. First, the equation of mass conservation, (1.6), is written as

$$\nabla_\perp \cdot \mathbf{u}_\perp + w_z = 0,$$

which is then integrated in z over the depth of the fluid; that is, from $z = b(\mathbf{x}_\perp, t)$ to $z = h(\mathbf{x}_\perp, t)$. This yields

$$\int_b^h \nabla_\perp \cdot \mathbf{u}_\perp \mathrm{d}z + [w]_b^h = 0,$$

and then the conditions defining w on the bottom and the surface, (1.35) and (1.28), are introduced to give

$$\int_b^h \nabla_\perp \cdot \mathbf{u}_\perp \mathrm{d}z + h_t + (\mathbf{u}_{\perp \mathrm{s}} \cdot \nabla_\perp)h - \{b_t + (\mathbf{u}_{\perp \mathrm{b}} \cdot \nabla_\perp)b\} = 0. \qquad (1.37)$$

The subscripts s and b denote evaluations on the surface ($z = h$) and the bottom ($z = b$), respectively.

To proceed, it is necessary to interchange the differential and integral operations in the first term. This is accomplished by a careful application of the rule for 'differentiating under the integral sign'; see Q1.30. Here, this term becomes

$$\nabla_\perp \cdot \int_b^h \mathbf{u}_\perp \mathrm{d}z - (\mathbf{u}_{\perp \mathrm{s}} \cdot \nabla_\perp)h + (\mathbf{u}_{\perp \mathrm{b}} \cdot \nabla_\perp)b,$$

and so equation (1.37) can be written as

$$(h - b)_t + \nabla_\perp \cdot \int_b^h \mathbf{u}_\perp \mathrm{d}z = 0.$$

This equation is conveniently expressed as

$$d_t + \nabla_\perp \cdot \bar{\mathbf{u}}_\perp = 0, \qquad (1.38)$$

where $d = h - b$ is the (local) depth of the water, and

$$\bar{\mathbf{u}}_\perp = \int_b^h \mathbf{u}_\perp \mathrm{d}z, \tag{1.39}$$

so that $\bar{\mathbf{u}}_\perp/d$ is an average of the horizontal vector components describing the motion of the fluid. As a simple application of equation (1.38), consider motion in only one horizontal direction: let $\mathbf{u}_\perp \equiv (u, 0)$, for example; then we obtain

$$d_t + \bar{u}_x = 0.$$

If, further, we suppose that there is no motion at infinity (that is, $\bar{u} \to 0$ as $|x| \to \infty$), and that $b = b(x)$ with $h(x, t) = h_0 + H(x, t)$ where $H \to 0$ as $|x| \to \infty$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{-\infty}^{\infty} H(x, t)\mathrm{d}x\right\} = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} H(x, t)\mathrm{d}x = \text{constant}. \tag{1.40}$$

This latter condition means that, for all time and for all surface waves represented by $H(x, t)$, the mass of fluid associated with the wave (assumed finite here) is conserved – an otherwise obvious result. It is clear that this conclusion is true, no matter the solution for $H(x, t)$; indeed, it may prove impossible to obtain the form of $H(x, t)$ except in special cases, and then only approximately, but (1.40) will still hold precisely.

1.2.5 An energy equation and its integral

We have already introduced an energy equation – Bernoulli's equation, (1.22) – but we shall now present a more general result. This does not require the restriction to steady flow, for example, nor to the alternative choice of irrotational flow (which led to the pressure equation, (1.23)). The new equation is, in a sense, a global energy equation; it describes the consequences on general fluid motion of using Newton's Second Law: that is, Euler's equation, (1.12). Once we have derived this equation, we shall apply it to our water-wave problem by integrating it over the depth of the fluid (exactly as we did for the mass conservation equation in Section 1.2.4).

We start with equation (1.21),

$$\frac{\partial \mathbf{u}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{P}{\rho}+\Omega\right)=\mathbf{u}\wedge\boldsymbol{\omega}, \tag{1.41}$$

which is derived from Euler's equation for an incompressible fluid ($\rho=$ constant) and a conservative body force, $F=-\nabla\Omega$; we shall assume that $\Omega=\Omega(\mathbf{x})$, which applies to most situations of practical interest. To proceed, we take the scalar product of equation (1.41) with $\mathbf{u}$ to give

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)+(\mathbf{u}\cdot\nabla)\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{P}{\rho}+\Omega\right)=0, \tag{1.42}$$

since $\mathbf{u}\cdot(\mathbf{u}\wedge\boldsymbol{\omega})=0$ (two of the vectors are parallel). Because the fluid is incompressible, we have $\nabla\cdot\mathbf{u}=0$; we choose to add to equation (1.42) the expression

$$\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{P}{\rho}+\Omega\right)(\nabla\cdot\mathbf{u}) \quad (=0)$$

and hence we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)+\nabla\cdot\left\{\mathbf{u}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{P}{\rho}+\Omega\right)\right\}=0;$$

see Q1.1(a) for the relevant differential identity. It is convenient to add a further zero contribution, namely $\partial\Omega/\partial t$, to give

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\Omega\right)+\nabla\cdot\left\{\mathbf{u}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{P}{\rho}+\Omega\right)\right\}=0,$$

which is often rewritten (by multiplying throughout by ρ) as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho\Omega\right)+\nabla\cdot\left\{\mathbf{u}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho\Omega\right)\right\}=0. \tag{1.43}$$

This is an energy equation; we recognise the kinetic energy per unit volume ($\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}$) and the corresponding potential energy ($\rho\Omega$; for example, $\rho g z$). The equation represents the balance between the rate of change of the total (mechanical) energy and the energy flow carried by the velocity field, together with the contribution from the rate of working of the pressure forces. Clearly this energy equation is a general result in the theory of inviscid (and incompressible) fluids; we now apply it to the study of water waves.

Following the development presented in Section 1.2.4, we write equation (1.43), with $\Omega=gz$, in the form

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)+\nabla_\perp\left\{\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right\}$$
$$+\frac{\partial}{\partial z}\left\{w\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right\}=0$$

and then integrate over z, from $z=b(\mathbf{x}_\perp,t)$ to $z=h(\mathbf{x}_\perp,t)$; this yields

$$\int_b^h\left\{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)+\nabla_\perp\cdot\left[\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right]\right\}$$
$$+\left[w\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right]_b^h=0.$$

The evaluations at the surface (s), and the bottom (b), from equations (1.28) and (1.35), then give

$$\int_b^h\left\{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz)+\nabla_\perp\cdot\left[\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\right]\right\}\mathrm{d}z$$
$$+\{h_t+(\mathbf{u}_{\perp\mathrm{s}}\cdot\nabla_\perp)h\}\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{s}\cdot\mathbf{u}_\mathrm{s}+P_\mathrm{s}+\rho gh\right)$$
$$-\{b_t+(\mathbf{u}_{\perp\mathrm{b}}\cdot\nabla_\perp)b\}\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{b}\cdot\mathbf{u}_\mathrm{b}+P_\mathrm{b}+\rho gb\right)=0. \quad (1.44)$$

As before, it is necessary to interchange the differential and integral operations (see Q1.30); the first of these integrals (involving $\partial/\partial t$) gives

$$\frac{\partial}{\partial t}\left\{\int_b^h\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)\mathrm{d}z\right\}-\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{s}\cdot\mathbf{u}_\mathrm{s}+\rho gh\right)h_t$$
$$+\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{b}\cdot\mathbf{u}_\mathrm{b}+\rho gb\right)b_t. \quad (1.45)$$

The second integral (in $\nabla_\perp$) similarly becomes

$$\nabla_\perp\cdot\int_b^h\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\mathrm{d}z-\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{s}\cdot\mathbf{u}_\mathrm{s}+P_\mathrm{s}+\rho gh\right)(\mathbf{u}_{\perp\mathrm{s}}\cdot\nabla_\perp)h$$
$$+\left(\frac{1}{2}\rho\mathbf{u}_\mathrm{b}\cdot\mathbf{u}_\mathrm{b}+P_\mathrm{b}+\rho gb)(\mathbf{u}_{\perp\mathrm{b}}\cdot\nabla_\perp\right)b. \quad (1.46)$$

Upon using (1.45) and (1.46) in equation (1.44), this equation reduces to

$$\frac{\partial}{\partial t}\left\{\int_b^h\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)\mathrm{d}z\right\}+\nabla_\perp\cdot\int_b^h\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\mathrm{d}z$$
$$+P_\mathrm{s}h_t-P_\mathrm{b}b_t=0. \quad (1.47)$$

It is conventional to write

$$\mathscr{E}=\int_b^h\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho gz\right)\mathrm{d}z \quad (1.48)$$

and

$$\mathscr{F}=\int_b^h\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+P+\rho gz\right)\mathrm{d}z, \quad (1.49)$$

where $\mathscr{E}$ is the energy in the flow, per unit horizontal area, and $\mathscr{F}$ is the horizontal energy flux vector. The energy equation, (1.47), therefore becomes

$$\mathscr{E}_t+\nabla_\perp\cdot\mathscr{F}+\mathscr{P}=0, \quad (1.50)$$

where $\mathscr{P}=P_\mathrm{s}h_t-P_\mathrm{b}b_t$ is the net energy input due to the pressure forces doing work on the upper and lower boundaries of the fluid. In the case of a stationary bottom boundary, then $b_t=0$; further, if the pressure in the fluid at the surface (P_s) is constant, then we may assign $P_\mathrm{s}=0$; consequently $\mathscr{P}=0$ and so

$$\mathscr{E}_t+\nabla_\perp\cdot\mathscr{F}=0. \quad (1.51)$$

(If $P_\mathrm{s}=P_\mathrm{a}=$ constant, then we may redefine P in the fluid to be $P+P_\mathrm{a}$: the governing equations are unaltered. With this choice the surface pressure is now $P_\mathrm{s}=0$, but then the form of P used in (1.49) must be adjusted to accommodate this choice unless the P used in (1.49), and earlier, is measured relative to P_a. Of course, $P_\mathrm{s}=0$ is only possible if the coefficient of surface tension is set to zero; in general, the surface tension forces do work on the moving free surface.)

The energy equations presented here, particularly (1.51), can be used to describe the energy associated with a wave motion by averaging over a wavelength; see Section 2.1.2.

1.3 Nondimensionalisation and scaling

The governing equations and boundary conditions that have been described define a class of water-wave problems. For most of the discussions in this text, we shall be concerned with gravity waves propagating on the surface of an inviscid fluid. (This means that we shall often ignore the effects of surface tension, for example.) The arguments that suggest that such simplifying assumptions lead to problems worthy of consideration will be rehearsed later. However, we take this opportunity to emphasise that the main thrust of our work will be towards an understanding of the equations (and boundary conditions), and what they imply for wave propagation. It is not our purpose to provide an engineering or physical appraisal of the usefulness of these theories as they apply to the many and varied types of water waves that are encountered in nature. The importance of these considerations should not be underestimated though; they are paramount in the design of ships, offshore platforms, breakwaters, and dams, in the prediction and avoidance of catastrophes following earthquakes or storms, and a host of other areas of significance to mankind. Nevertheless, we shall extend our methods to some more obviously relevant and practical applications, such as flows with shear (rotational flows) and propagation over variable depth.

It is clear that our field of discussion will be somewhat restricted, but even so we shall still face immensely difficult mathematical problems that we wish to overcome. The most natural way forward is to develop a suitable – but systematic – approximation procedure. To this end we need to characterise problems in terms of the sizes of various fundamental parameters. These parameters are introduced by defining a set of nondimensional variables.

1.3.1 Nondimensionalisation

The nondimensionalisation that we adopt makes use of the length scales, time scales, etc., that naturally appear in the problem; this is altogether the obvious (and conventional) choice. First we introduce the appropriate length scales: we take h_0 to be a typical depth of the water and λ as the typical wavelength of the surface wave. (These and the other scales are depicted in Figure 1.4.) In order to define a time scale, we require a suitable velocity scale. Now, many of the problems that we shall consider involve the propagation of long waves, and the speed of these waves (as we shall demonstrate later) is approximately $\sqrt{gh_0}$; we make this choice

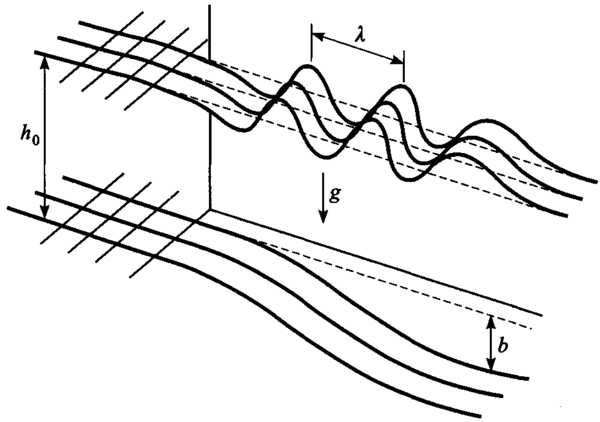


Figure 1.4. The scales for the water-wave problem: h_0 is the undisturbed or typical depth, λ is a typical wavelength, b is the bottom surface, and g is the acceleration of gravity.

for the speed scale. This choice is still useful even if we do not study, specifically, long gravity waves.

The characteristic speed, $\sqrt{gh_0}$, and the wavelength, λ, define a typical time associated with horizontal propagation, which is what interests us here; this is $\lambda/\sqrt{gh_0}$. We use $\sqrt{gh_0}$ to define the scale of the horizontal velocity components, but the vertical component (w) is treated differently. So that the equation of mass conservation makes good sense – and to be consistent with the boundary conditions – we must take this scale to be $h_0\sqrt{gh_0}/\lambda$. (One way to see this is to consider two-dimensional motion, for example

$$u_x + w_z = 0,$$

and then introduce the stream function, $\psi(x, z, t)$ (see Q1.20, Q1.34), so that

$$u = \psi_z \quad \text{and} \quad w = -\psi_x;$$

the scale of ψ is therefore $h_0\sqrt{gh_0}$, and that for w follows directly.)

The surface wave itself leads to the introduction of a further parameter: a typical (perhaps the maximum) amplitude of the wave. This is most conveniently done by writing the surface, $z = h(\mathbf{x}_\perp, t)$, as

$$h = h_0 + a\eta(\mathbf{x}_\perp, t) \qquad (1.52)$$

where a is this typical amplitude; the function η is therefore nondimensional. We are now able to define the set of nondimensional variables, which we first do for rectangular Cartesian coordinates. (The cylindrical version is very similar.) Rather than introduce a new notation for all our variables, we choose – where convenient – to write, for example, $x \to \lambda x$. This is to be read that x is replaced by λx, so that hereafter the symbol x will denote a nondimensional variable. With this understanding, we define

$$x \to \lambda x, \quad y \to \lambda y, \quad z \to h_0 z, \quad t \to (\lambda/\sqrt{gh_0})t, \tag{1.53}$$

$$u \to \sqrt{gh_0}u, \quad v \to \sqrt{gh_0}v, \quad w \to (h_0\sqrt{gh_0}/\lambda)w \tag{1.54}$$

with

$$h = h_0 + a\eta \quad \text{and} \quad b \to h_0 b. \tag{1.55}$$

Finally, the pressure is rewritten as

$$P = P_a + \rho g(h_0 - z) + \rho g h_0 p \tag{1.56}$$

where P_a is the (constant) pressure of the atmosphere, $\rho g(h_0 - z)$ the hydrostatic pressure distribution (see Q1.11) and the pressure scale, $\rho g h_0$, is based on the pressure at depth $z = h_0$. The pressure variable p introduced here, therefore measures the deviation from the hydrostatic pressure distribution; we shall find that $p \neq 0$ during the passage of a wave.

The Euler equation in component form, (1.13), and the equation of mass conservation, (1.14), now become

$$\left.\begin{aligned} &\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \quad \delta^2\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \\ &\text{where} \\ &\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \end{aligned}\right\} \tag{1.57}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.58}$$

These equations are written exclusively in terms of nondimensional variables, where $\delta = h_0/\lambda$ is the *long wavelength* or *shallowness* parameter; we shall have much to write about δ later.

The corresponding nondimensionalisation for cylindrical coordinates is precisely that given in (1.53)–(1.56), but with the transformations on x and y replaced by

$$r \rightarrow \lambda r. \tag{1.59}$$

The governing equations, (1.15) and (1.16), therefore become

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}, \quad \frac{\mathrm{D}v}{\mathrm{D}t}+\frac{uv}{r}=-\frac{1}{r}\frac{\partial p}{\partial \theta}, \quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{\partial p}{\partial z},\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial \theta}+w\frac{\partial}{\partial z}
\end{aligned}\right\} \tag{1.60}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0, \tag{1.61}$$

all expressed in nondimensional variables; δ is defined exactly as above: $\delta = h_0/\lambda$. Finally, in both versions, the upper and lower surfaces of the fluid are represented by

$$z = 1+\varepsilon\eta \quad \text{and} \quad z = b, \tag{1.62}$$

respectively. Here we have introduced the second important parameter in water-wave theory: $\varepsilon = a/h_0$, the *amplitude* parameter.

Now we turn to the boundary conditions, which are treated in precisely the same fashion. Thus we see that the surface kinematic condition, (1.28), becomes

$$w = \varepsilon\{\eta_t + (\mathbf{u}_\perp \cdot \nabla)\eta\} \quad \text{on} \quad z = 1+\varepsilon\eta, \tag{1.63}$$

in nondimensional variables. Similarly, the most general dynamic condition that we shall use in most of our work, (1.31) with (1.32) (for $h = h(x, y, t)$), yields

$$p-\varepsilon\eta = -\varepsilon\left(\frac{\Gamma}{\rho g \lambda^2}\right)\left\{\frac{(1+\varepsilon^2\delta^2\eta_y^2)\eta_{xx}+(1+\varepsilon^2\delta^2\eta_x^2)\eta_{yy}-2\varepsilon^2\delta^2\eta_x\eta_y\eta_{xy}}{(1+\varepsilon^2\delta^2\eta_x^2+\varepsilon^2\delta^2\eta_y^2)^{3/2}}\right\}$$
$$\text{on } z = 1+\varepsilon\eta, \tag{1.64}$$

where we write $\Gamma/(\rho g \lambda^2) = \delta^2 W$ with $W = \Gamma/(\rho g h_0^2)$, a *Weber number*. (It is usual to define this with respect to the appropriate (speed)$^2 = gh_0$, and the corresponding depth scale; sometimes, to avoid confusion, we shall write W_e for W.) This nondimensional parameter is used to measure the size of the surface tension contribution. A corresponding result is obtained in cylindrical coordinates, with $h = h(r, \theta, t)$; see Q1.36. An alternative dynamic condition is provided by the pressure equation, (1.29), for irrotational flow; this is discussed in Q1.37. Finally, the bottom boundary condition, (1.35), yields the unchanged form

$$w = b_t + (\mathbf{u}_\perp \cdot \nabla)b \quad \text{on} \quad z = b, \tag{1.65}$$

in nondimensional variables.

1.3.2 Scaling of the variables

We have described the nondimensionalisation of the governing equations, but another equally important transformation is also required. An examination of the surface boundary conditions, (1.63) and (1.64), yields the observation that both w and p (on $z = 1 + \varepsilon\eta$) are essentially proportional to ε; that is, proportional to the wave amplitude. This makes good sense, particularly as $\varepsilon \to 0$, for then $w \to 0$ and $p \to 0$: there is no disturbance of the free surface – it becomes a horizontal surface on which $w = 0 = p$. Thus we define a set of scaled variables, chosen to be consistent with the boundary conditions and governing equations; we write (again avoiding the introduction of a new notation)

$$p \to \varepsilon p, \quad w \to \varepsilon w, \quad (u, v) \to \varepsilon(u, v) \quad (\text{or } \mathbf{u}_\perp \to \varepsilon \mathbf{u}_\perp). \tag{1.66}$$

(The original, physical, variables are easily recovered from (1.66) and (1.53)–(1.56); for example, if w is the scaled variable from (1.66), then $\varepsilon(h_0\sqrt{gh_0}/\lambda)w$ is the original w.)

The equations (1.57) and (1.58) become

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z},\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\\
&\text{and}\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{aligned}\right\} \tag{1.67}$$

The equations in cylindrical coordinates, (1.60) and (1.61), are, correspondingly,

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t}-\frac{\varepsilon v^2}{r}=-\frac{\partial p}{\partial r},\quad \frac{\mathrm{D}v}{\mathrm{D}t}+\frac{\varepsilon uv}{r}=-\frac{1}{r}\frac{\partial p}{\partial \theta},\quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{\partial p}{\partial z},\\
&\text{where}\\
&\qquad\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+\varepsilon\left(u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial \theta}+w\frac{\partial}{\partial z}\right)\\
&\text{and}\\
&\qquad\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0.
\end{aligned}\right\}\quad(1.68)$$

The surface boundary conditions, (1.63) and (1.64), are now written as

$$\left.\begin{aligned}
w&=\eta_t+\varepsilon(\mathbf{u}_\perp\cdot\nabla_\perp)\eta\\
p&=\eta-\delta^2 W\left\{\frac{(1+\varepsilon^2\delta^2\eta_y^2)\eta_{xx}+(1+\varepsilon^2\delta^2\eta_x^2)\eta_{yy}-2\varepsilon^2\delta^2\eta_x\eta_y\eta_{xy}}{(1+\varepsilon^2\delta^2\eta_x^2+\varepsilon^2\delta^2\eta_y^2)^{3/2}}\right\}
\end{aligned}\right\}\quad(1.69)$$

both on $z=1+\varepsilon\eta$, and on the bottom (1.65) becomes

$$w=\varepsilon^{-1}b_t+(\mathbf{u}_\perp\cdot\nabla_\perp)b\quad\text{on}\quad z=b. \qquad (1.70)$$

For this last boundary condition we shall consider problems for which b_t is proportional to ε (or smaller); indeed, for almost all our discussions the bottom boundary will be stationary, so $b_t\equiv 0$. (The scaled dynamic conditions in cylindrical coordinates, and for irrotational flow, are given in Q1.36 and Q1.37, respectively.)

As we shall discuss in due course, scaling is not restricted to the dependent variables. Much of our later work (particularly in Chapters 3 and 4) relies on seeking solutions in appropriate scaled regions of space and time. So, for example, we might be interested in the solution when the depth variation is slow (for example, $b=b(\varepsilon\mathbf{x}_\perp)$), and then the transformation (scaling) $\mathbf{x}_\perp\to\varepsilon\mathbf{x}_\perp$ is likely to be required. This, and related ideas, will be described more fully in the brief introduction to asymptotics and multiple scales (Section 1.4), and when we need to develop the techniques needed to solve specific problems.

1.3.3 Approximate equations

The significance and usefulness of the nondimensionalisation and scaling presented above will now be made clear. The parameters, ε and δ, are used to define, in a rather precise manner, various approximate versions of the governing equations and boundary conditions. Similar ideas apply to the other parameters (such as W and R, the Weber and Reynolds numbers, respectively); we shall comment on these as it becomes necessary.

The two most commonly used – and useful – approximations are

(a) $\varepsilon \to 0$: the linearised problem;
(b) $\delta \to 0$: the long-wave (or shallow-water) problem.

The first of these, case (a), requires that the amplitude of the surface wave be small; then, in a first approximation, the equations become linear. For example, in rectangular Cartesian coordinates, equations (1.67), (1.69), and (1.70) simplify to

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}, \quad \delta^2 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\\
&\text{with}\\
&\qquad w = \eta_t \quad \text{and} \quad p = \eta - \delta^2 W(\eta_{xx} + \eta_{yy}) \quad \text{on} \quad z = 1\\
&\text{and}\\
&\qquad\qquad w = (\mathbf{u}_\perp \cdot \nabla_\perp) b \quad \text{on} \quad z = b(<1).
\end{aligned}\right\} \tag{1.71}$$

In these equations we have chosen $b_t \equiv 0$, and treated δ and W as fixed parameters as $\varepsilon \to 0$, as they clearly are. We note that, in particular, the evaluation on the (unknown) free surface has become an evaluation on the known surface, $z = 1$, even though the unknown free surface, η, still appears in the equations. The linear equations expressed in cylindrical coordinates take a similar form (from equations (1.68) and (1.70) and Q1.36). (The corresponding equations for irrotational flow are obtained in Q1.38.)

For case (b), the waves are long; that is, of long wavelength (or the water is shallow), in the sense that $\delta = h_0/\lambda$ is small. (Both descriptions are commonly used; we shall more often use the former – long waves – rather than the latter.) This time we keep ε and W fixed, and (with $b_t \equiv 0$) the approximation $\delta \to 0$ yields the problem

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \frac{\partial p}{\partial z} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\\
&\text{where}\\
&\qquad \frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)
\end{aligned}\right\} \tag{1.72}$$

with

$$w = \eta_t + \varepsilon(\mathbf{u}_\perp \cdot \nabla_\perp)\eta \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = (\mathbf{u}_\perp \cdot \nabla_\perp)b \quad \text{on} \quad z = b.$$

The equations which describe small amplitude *and* long waves (so $\varepsilon \to 0$ and $\delta \to 0$), are clearly consistent with both sets (1.71) and (1.72): the resulting equations are those of (1.71), but with

$$\frac{\partial p}{\partial z} = 0; \quad p = \eta \quad \text{on} \quad z = 1, \tag{1.73}$$

or (1.72) with $\varepsilon = 0$.

The solutions of these various approximate equations will form the basis for many of our descriptions in the selection of classical water-wave problems presented in Chapter 2.

1.4 The elements of wave propagation and asymptotic expansions

In this final section we describe the basic ideas that provide the essential background to any discussion of wave propagation. We shall present a brief overview of the mathematical description of elementary wave propagation: *d'Alembert's solution* of the wave equation, and the important properties of *dispersion*, *dissipation* and *nonlinearity*. Then we shall outline the concept of an asymptotic expansion, and show how this can be used to obtain appropriate asymptotic solutions of wave-like equations. This will introduce the important technique of rescaling the variables with respect to the (small) parameter(s) in the problem.

1.4.1 Elementary ideas in the theory of wave propagation

Wave propagation theories, at their simplest, usually involve the application of fundamental physical principles (to the motion of a stretched string, for example), leading to the classical one-dimensional *wave equation*

$$u_{tt} - c^2 u_{xx} = 0. \tag{1.74}$$

The function $u(x, t)$ represents the amplitude of the wave, c (> 0) is a constant, and the subscripts denote partial derivatives. This equation has the general solution

$$u(x, t) = f(x - ct) + g(x + ct), \tag{1.75}$$

written in terms of the *characteristic variables* $(x \pm ct)$; this solution, (1.75), is commonly described as *d'Alembert's solution*, where f and g are arbitrary functions. If, as is usual, x is a spatial coordinate and t a time coordinate, then c is a speed, so the solution represents right (f) and left (g) propagating waves. The functions f and g can be determined, for example, from suitable initial data, such as u and u_t prescribed at $t = 0$ (the *Cauchy problem*); see Q1.39.

The two wave components, f and g, propagate at constant speed (c) with unchanging form; they do not interact with themselves nor with each other. This is equivalent to the statement that the governing equation is *linear* which, of course, is precisely the form of (1.74). Each component is a separate and independent linear wave.

Now, for most of our work on water waves, we shall describe waves that propagate only in one direction (which usually will be to the right). One simple way to do this is simply to set $g \equiv 0$; an alternative is to suppose that the initial data is on *bounded* (or *compact*) *support*. Then, after an appropriate finite time, the two components (f and g) will move apart and no longer overlap (see Q1.40). In either event, it is then possible to follow just the one component. An equivalent approach is to restrict the discussion, *ab initio*, to waves propagating in one direction only; this is accomplished by working with the equation

$$u_t + cu_x = 0, \tag{1.76}$$

which has the general solution

$$u(x, t) = f(x - ct). \tag{1.77}$$

This is then completely determined, given the function $u(x, 0) = f(x)$.

Wave propagation equations, at least when derived from more complete physical models, are unlikely to be as simple as (1.74) or (1.76). More careful analyses, but with the restriction to unidirectional propagation, might lead to the linear equations

$$u_t + u_x + u_{xxx} = 0 \tag{1.78}$$

or

$$u_t + u_x - u_{xx} = 0. \tag{1.79}$$

(In these two equations, the coefficients have been normalised; this is always possible by redefining $x \to \alpha x$, $t \to \beta t$, for suitable constants

α, β.) One very familiar method for solving linear partial differential equations is to seek the *harmonic solution*

$$u(x, t) = e^{i(kx - \omega t)}, \tag{1.80}$$

where k is a real parameter. (A real solution for u can be constructed by taking the real or imaginary part, or by forming $A \exp\{i(kx - \omega t)\}$ + complex conjugate, where $A(k)$ is complex valued.) Upon substitution of (1.80) into (1.78) and (1.79), it follows that (1.80) is a solution of (1.78) if

$$\omega = k - k^3, \tag{1.81}$$

and of (1.79) if

$$\omega = k - ik^2. \tag{1.82}$$

In the case of (1.81), we see that

$$kx - \omega t = k\{x - (1 - k^2)t\},$$

so that the speed of propagation,

$$\frac{\omega}{k} = 1 - k^2, \tag{1.83}$$

is a function of k. Thus waves with different *wave number*, k, travel at different speeds (which, in this example, might be to the left or the right, depending on whether $k^2 > 1$ or $k^2 < 1$, respectively). This property of a wave is known as *dispersion*, and the wave is said to be *dispersive*; equation (1.78) is the simplest (unidirectional) dispersive wave equation and (1.81) is its *dispersion relation*. A solution of this equation, which is the sum of two components, each associated with different values of k, exhibits the property that each component will move at its own speed given by (1.83). Thus, if the solution is initially on compact support, the two components will move apart, or *disperse*. The separate components do *not* change shape, although the observed sum does give the appearance of a changing profile.

The speed, ω/k, is called the *phase speed* of the wave; this describes the motion of each individual component. However, as we shall discuss later, another speed, defined by $d\omega/dk$, describes the motion of a *group* of waves. This is called the *group speed* and, as we shall explain later, it is the speed at which energy is propagated.

A similar discussion for equation (1.79) yields

$$u(x, t) = \exp\{ik(x - t) - k^2 t\};$$

this describes a wave which propagates at a speed of unity (to the right) for all k, but which decays as $t \to +\infty$ (for $k \neq 0$). This phenomenon of a decaying wave is called *dissipation*; it usually arises from a physical system that incorporates some frictional behaviour, such as fluid viscosity. The values of the coefficients in equation (1.79) are unimportant, but the relative sign of the terms u_t and u_{xx} is; if this changes then the wave amplitude will grow without bound as $t \to +\infty$. (Further one-dimensional linear wave equations of this type can be constructed, with combinations of even and/or odd derivatives in x; see Q1.41.)

Finally, for us, a significant property of many of the waves that we shall encounter is that they are *nonlinear*. The simplest model equations usually involve linearisation (and so, perhaps, might lead to equations (1.78) or (1.79)), but a more careful analysis will often lead to a nonlinear equation, such as

$$u_t + (1 + u)u_x = 0. \tag{1.84}$$

The general solution of this equation is obtained directly from the *method of characteristics*:

$$u = \text{constant on lines } \frac{\mathrm{d}x}{\mathrm{d}t} = 1 + u.$$

Thus, supposing that we are given $u(x, 0) = f(x)$, the solution of (1.84) is

$$u(x, t) = f\{x - (1 + u)t\} \tag{1.85}$$

which, for general f, provides an *implicit* relation for $u(x, t)$. Only when f is particularly simple is it possible to solve for u explicitly; see Q1.43. Nevertheless, the solution can always be represented geometrically by using the information carried along the characteristic lines. Thus any point on a wave profile, at which u takes the value u_0, will propagate at the constant speed $1 + u_0$. Consequently, points of larger u_0 move faster than those of smaller u_0; this implies that a wave profile will change shape, as represented in Figure 1.5. This might result in a profile which becomes multivalued after a finite time, as our figure shows; this corresponds to the intersection of the characteristic lines. When this happens, it is usual to regard the solution as unacceptable, because we normally expect the solution-function to be single-valued (in x for any t). The solution can be made single-valued by the insertion of a discontinuity (or jump) which separates the characteristic lines and does not allow them to intersect; this is shown in Figure 1.6. (A discontinuous function is not, strictly, a proper solution of the equation (1.84), but it should be a

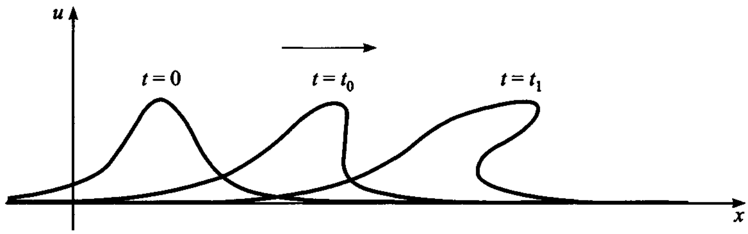


Figure 1.5. A breaking wave, according to equation (1.85); at $t = t_0$ the wave is about to break and at $t = t_1$ ($> t_0$) the wave has broken.

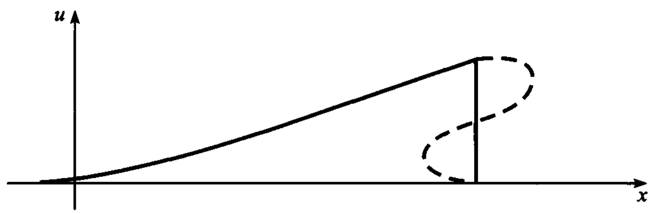


Figure 1.6. The insertion of a discontinuity (or jump) to make the solution single-valued.

solution of the underlying integral equation; see, for example, equations (1.1) and (1.2). A fuller discussion of discontinuous solutions will be given in Section 2.7.)

The form of the wave, for $t > t_0$ in Figure 1.5, is reminiscent of a wave breaking on a beach; indeed, this type of solution of a nonlinear equation is often called a *breaking wave*. However, this similarity is altogether superficial; waves that approach a beach, and then break, are described by a much more involved theory (which essentially requires the full water-wave equations). A related problem is described in Section 2.8.

1.4.2 Asymptotic expansions

Finally, we introduce the ideas that form the basis for handling the equations and problems that we encounter in water-wave theory, at least in the initial stages of much of the work. The technique that we adopt involves the construction of *asymptotic expansions*. This branch of mathematics has a reasonably long history, and one that has not been divorced from controversy (mainly over the interpretation of divergent series, which often appear in this work). The first systematic approach, to both the definition and use of asymptotic expansions, is due to Poincaré; we shall follow his lead.

First we require a little bit of notation: we write

$$f(x) = \mathrm{o}(g(x)); \quad f(x) = \mathrm{O}(g(x)); \quad f(x) \sim g(x),$$

as $x \to x_0$, if

$$\lim_{x \to x_0} [f(x)/g(x)]$$

is zero, finite non-zero, or unity, respectively. These are usually read as 'little oh', 'big oh' and 'varies as' (or 'asymptotically equal to'), respectively; the function $f(x)$ is the given function under discussion, and $g(x)$ is a suitable *gauge* function. We can then write in this notation

$$\begin{aligned} f(x) &= \frac{1}{2+x^2} = \mathrm{o}(x^{-1}) && \text{as } |x| \to \infty; \\ f(x) &= \frac{1}{2+x^2} = \mathrm{O}(1) && \text{as } x \to 0; \\ f(x) &= \sin 2x \sim 2x && \text{as } x \to 0, \end{aligned}$$

for example. It should be noted that the limit in which the behaviour occurs must be included in the statement of the behaviour.

This description of a function (in a limit) is now extended: we write

$$f(x) - \sum_{n=0}^{N-1} g_n(x) \sim g_N(x) \quad \text{as } x \to x_0,$$

for every $N \geq 1$, where $f(x) \sim g_0(x)$ as $x \to x_0$. It is then usual (and convenient) to express this property in the form

$$f(x) \sim \sum_{n=0}^{\infty} g_n(x) \quad \text{as } x \to x_0, \tag{1.86}$$

where N has been taken to infinity here; this 'series' is called an *asymptotic expansion* of $f(x)$, as $x \to x_0$. Of course, this is only a shorthand notation and does not imply any convergence (or otherwise) of the series in (1.86). In practice, asymptotic expansions are rarely taken beyond a few terms, but it must be possible – in principle – to find them all. This representation is merely a compact way of describing a sequence of limiting processes (as $x \to x_0$) on the functions $\{f(x)/g_0(x)\}$, $[\{f(x) - g_0(x)\}/g_1(x)]$, etc. However, the functions that we shall be working with involve one (or more) parameters; this is now introduced into our definition.

The asymptotic expansions that we require are defined with respect to a parameter, ε say, as $\varepsilon \to 0$, at fixed x. The asymptotic expansion of $f(x; \varepsilon)$ is then written as

$$f(x; \varepsilon) \sim \sum_{n=0}^{\infty} f_n(x; \varepsilon) \quad \text{as} \quad \varepsilon \to 0 \text{ at fixed } x \text{ (or } x = \mathrm{O}(1)), \tag{1.87}$$

where $f_{n+1}(x; \varepsilon) = \mathrm{o}\{f_n(x; \varepsilon)\}$ (as $\varepsilon \to 0$) for every $n \geq 0$. If this asymptotic expansion, (1.87), is defined for all x in the domain of the function, $f(x; \varepsilon)$, the expansion is said to be *uniformly valid*. However, if there is some x (in the domain), and some n for which

$$f_{n+1}(x; \varepsilon) \neq \mathrm{o}\{f_n(x; \varepsilon)\}$$

as $\varepsilon \to 0$, then the asymptotic expansion is said to *break down*, or to be *non-uniform*. Here we have written each term in the expansion as $f_n(x; \varepsilon)$, but quite often this occurs in the much simpler *separable* form: $f_n(x; \varepsilon) = \varepsilon^n a_n(x)$. Happily, this is usually the situation for our problems in water waves.

Briefly, we describe these ideas by considering the example

$$f(x; \varepsilon) = (1 + \varepsilon x + \mathrm{e}^{-x/\varepsilon})^{-1}, \quad 1 \leq x \leq 2, \tag{1.88}$$

for $\varepsilon \to 0^+$. For any $x = \mathrm{O}(1)$, in the given domain, we may therefore write

$$f(x; \varepsilon) \sim 1 - \varepsilon x + \varepsilon^2 x^2 \tag{1.89}$$

or

$$f(x; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n (-x)^n \tag{1.90}$$

or even

$$f(x; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n (-x)^n - \mathrm{e}^{-x/\varepsilon}. \tag{1.91}$$

(In this last case, it should be remembered that $\mathrm{e}^{-x/\varepsilon} = \mathrm{o}(\varepsilon^n)$ as $\varepsilon \to 0^+$, for every n, if $x > 0$.) Now let us take the same function, (1.88), but define the domain as $0 \leq x \leq 2$; the asymptotic expansions, (1.89)–(1.91), are clearly not uniformly valid when $x = \mathrm{O}(\varepsilon)$, for then $\mathrm{e}^{-x/\varepsilon} = \mathrm{O}(1)$. This choice of x is usually expressed by writing

$$x = \varepsilon X, \quad X = \mathrm{O}(1) \quad \text{as } \varepsilon \to 0;$$

the original function then becomes

$$f(\varepsilon X;\varepsilon) \equiv F(X;\varepsilon) = (1+\varepsilon^2 X + \mathrm{e}^{-X})^{-1},$$

so

$$F(X;\varepsilon) \sim \frac{1}{1+\mathrm{e}^{-X}} - \frac{\varepsilon^2 X}{1+\mathrm{e}^{-X}} \quad \text{as } \varepsilon \to 0. \tag{1.92}$$

Finally, if we further extend the domain to $0 \le x < \infty$, the asymptotic expansions, (1.89)–(1.91), are now not uniformly valid also for $x = \mathrm{O}(\varepsilon^{-1})$. For an x of this magnitude, we define

$$x = \chi/\varepsilon, \;\; \chi = \mathrm{O}(1) \quad \text{as } \varepsilon \to 0,$$

and then

$$f(\chi/\varepsilon;\varepsilon) \equiv \mathscr{F}(\chi;\varepsilon) = (1+\chi+\mathrm{e}^{-\chi/\varepsilon^2})^{-1}$$

which gives

$$\mathscr{F}(\chi;\varepsilon) \sim \frac{1}{1+\chi} \quad \text{as } \varepsilon \to 0. \tag{1.93}$$

These various asymptotic expansions also satisfy the *matching principle*. To demonstrate this, we consider the asymptotic expansions (1.89), (1.92), and (1.93). Thus

$$f \sim 1 - \varepsilon x + \varepsilon^2 x^2 = 1 - \varepsilon^2 X + \varepsilon^4 X^2 \sim 1 - \varepsilon^2 X \quad \text{as } \varepsilon \to 0^+, \;\; X = \mathrm{O}(1),$$

matches with

$$F \sim \frac{1}{1+\mathrm{e}^{-X}} - \frac{\varepsilon^2 X}{1+\mathrm{e}^{-X}} =\sim 1 - \varepsilon x \quad \text{as } \varepsilon \to 0^+, \;\; x = \mathrm{O}(1).$$

Similarly,

$$f \sim 1 - \varepsilon x + \varepsilon^2 x^2 = 1 - \chi + \chi^2 \sim 1 - \chi + \chi^2 \quad \text{as } \varepsilon \to 0^+, \;\; \chi = \mathrm{O}(1),$$

matches with

$$\mathscr{F} \sim \frac{1}{1+\chi} = \frac{1}{1+\varepsilon x} \sim 1 - \varepsilon x + \varepsilon^2 x^2 \quad \text{as } \varepsilon \to 0^+, \;\; x = \mathrm{O}(1).$$

Simple asymptotic expansions, and the matching principle, are briefly explored in Q1.45, Q1.46; the reader who requires a more expansive and comprehensive discussion of asymptotic expansions, the matching principle, etc., should consult the texts mentioned at the end of this chapter.

The application of asymptotic methods to the solution of differential equations, at least in the context of wave-like problems, is reasonably straightforward and requires no deep knowledge of the subject. The process is initiated – almost always – by assuming that a solution exists (for O(1) values of the independent variables) as a suitable asymptotic expansion with respect to the relevant small parameter. The form of this expansion is governed by the way in which the parameter appears in the equation and, perhaps, also how it appears in the boundary/initial conditions. Usually, a rather simple iterative construction will suggest how this expansion proceeds. In order to explain and describe how these ideas are relevant in theories of wave propagation (and, therefore, to our study of water waves), we consider the partial differential equation

$$u_{tt} - u_{xx} = \varepsilon(u^2 + u_{xx})_{xx}. \tag{1.94}$$

The small parameter, ε, in this equation (which here represents the characteristics of both small amplitude and long waves) suggests that we seek a solution in the form

$$u(x, t; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n(x, t) \quad \text{as } \varepsilon \to 0, \tag{1.95}$$

for $x = \mathrm{O}(1)$, $t = \mathrm{O}(1)$. We shall suppose that equation (1.94) is to be solved in $t \geq 0$ and for $-\infty < x < \infty$, with appropriate initial data being prescribed on $t = 0$ (that is, the *Cauchy problem*). The expansion (1.95) is then a solution of equation (1.94) if

$$u_{0tt} - u_{0xx} = 0; \quad u_{1tt} - u_{1xx} = (u_0^2 + u_{0xx})_{xx},$$

and so on. To obtain these, we simply collect together like powers of ε and set to zero each coefficient of ε^n.

We see immediately that the general solution of u_0 (d'Alembert's solution) is

$$u_0(x, t) = f(x - t) + g(x + t),$$

and we will suppose that the initial data is such as to generate only the right-going wave; for example

$$u(x, 0; \varepsilon) = f(x), \quad u_t(x, 0; \varepsilon) = -f'(x). \tag{1.96}$$

(This choice is not strictly necessary, even for our purposes; we could prescribe initial data on compact support as we have mentioned before (with $x = \mathrm{O}(1)$) and then, for large enough time (as we use below), the right- and left-going waves move apart and we may elect to follow just one component; see Q1.40.)

Now, with $u_0 = f(x-t)$, we see that

$$u_{1tt} - u_{1xx} = (f^2 + f'')'', \tag{1.97}$$

where the prime denotes the derivative with respect to $(x-t)$. To proceed, it is convenient to introduce the characteristic variables for this equation,

$$\xi = x - t, \quad \zeta = x + t,$$

so that equation (1.97) becomes

$$-4u_{1\xi\zeta} = (f^2 + f'')'',$$

and hence

$$u_1(\xi, \zeta) = -\frac{1}{4}\zeta(f^2 + f'')' + A(\xi) + B(\zeta),$$

where $f = f(\xi)$. The arbitrary functions, A and B, are determined from the initial data: if we use that choice given above, (1.96), then we require (for $u_1(x, t)$)

$$u_1(x, 0) = 0, \quad u_{1t}(x, 0) = 0$$

(since these data, (1.96), are independent of ε), so

$$u_1(\xi, \zeta) = \frac{1}{4}[(\xi - \zeta)\{f^2(\xi) + f''(\xi)\}' + f^2(\zeta) + f''(\zeta) - f^2(\xi) - f''(\xi)]$$

or

$$u_1(x, t) = -\frac{1}{2}tF'(x-t) + \frac{1}{4}\{F(x+t) - F(x-t)\},$$

where $F = f^2 + f''$. The asymptotic expansion, so far, is therefore

$$u(x, t; \varepsilon) \sim f(x-t) - \frac{\varepsilon}{4}\{2tF'(x-t) + F(x-t) - F(x+t)\}. \tag{1.98}$$

For $f(x)$ on compact support (and suitably differentiable), or at least for $f(x) \to 0$ (sufficiently rapidly) as $|x| \to \infty$, it is clear that the asymptotic expansion (1.98) is not uniformly valid for $\varepsilon t = O(1)$. Further, for our stated condition on $f(x)$, we need consider only $\xi = O(1)$ and thus we now examine the solution of equation (1.94) for

$$\xi = x - t = O(1), \quad \tau = \varepsilon t = O(1) \quad \text{as } \varepsilon \to 0. \tag{1.99}$$

(The asymptotic expansion, (1.98), will also be non-uniform at any values of ξ for which the first, second, or third derivatives of $f(\xi)$ are undefined; we do not normally countenance this possibility in these types of problem. From the above, we see that (1.98) is non-uniform in t no matter how well-behaved $f(\xi)$ might be – and we note that f = constant is of no practical interest!)

In wave-like problems, the region where a large time (or distance) variable is used (like τ in (1.99)) is usually called the *far-field*; the corresponding region for $t = O(1)$ is then referred to as the *near-field*. We note that, for $\xi = x - t = O(1)$, then $t = O(\varepsilon^{-1})$ implies that $x = O(\varepsilon^{-1})$; this relationship between the various asymptotic regions is made clear in Figure 1.7.

The transformation (1.99), applied to equation (1.94), makes use of the identities

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial t} \equiv \varepsilon\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi};$$

then the equation for $u(x, t; \varepsilon) \equiv U(\xi, \tau; \varepsilon)$ becomes

$$\varepsilon U_{\tau\tau} - 2U_{\tau\xi} = (U^2 + U_{\xi\xi})_{\xi\xi}. \tag{1.100}$$

An asymptotic solution of this equation is sought in the form

$$U(\xi, \tau; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n U_n(\xi, \tau), \quad \varepsilon \to 0, \tag{1.101}$$

for $\xi = O(1)$, $\tau = O(1)$, and then U_0 will satisfy the equation

$$2U_{0\tau\xi} + (U_0^2 + U_{0\xi\xi})_{\xi\xi} = 0,$$

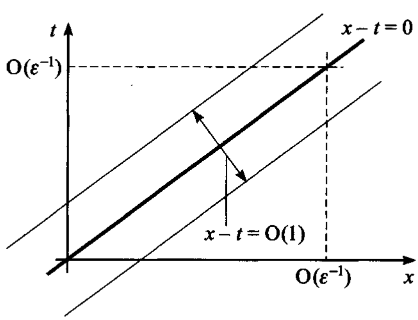


Figure 1.7. A schematic representation of the far-field, where $x = O(\varepsilon^{-1})$, $t = O(\varepsilon^{-1})$, with $x - t = O(1)$; the wavefront is $x - t = 0$.

or

$$2U_{0\tau} + 2U_0 U_{0\xi} + U_{0\xi\xi\xi} = 0, \qquad (1.102)$$

where we have invoked decay conditions as $|\xi| \to \infty$. This equation, (1.102), is a third-order nonlinear partial differential equation, which is one variant of a very famous equation: the *Korteweg–de Vries equation*, of which we shall write much (Chapter 3). It turns out that we can formulate the solution of this equation which satisfies (the matching condition)

$$U_0 \to f(\xi) \quad \text{as } \tau \to 0,$$

which corresponds to the initial-value problem for equation (1.102); this solution exists provided $f(\xi)$ decays sufficiently rapidly as $|\xi| \to \infty$. (The method of solution required here is at the heart of *inverse scattering transform* – or *soliton* – theory.) The solution thus obtained, for U_0, constitutes a one-term uniformly valid asymptotic expansion for $\tau \geq 0$ and $\tau = \mathrm{O}(1)$ (as $\varepsilon \to 0$). The next term in this expansion satisfies the equation

$$2U_{1\tau\xi} + 2(U_0 U_1)_{\xi\xi} + U_{1\xi\xi\xi\xi} = U_{0\tau\tau}$$

or

$$2U_{1\tau} + 2(U_0 U_1)_{\xi} + U_{1\xi\xi\xi} = -(U_0^2 + U_{0\xi\xi})_{\tau},$$

where we have used equation (1.102) for $U_{0\tau}$, and again imposed decay conditions as $|\xi| \to \infty$. The analysis hereafter is not particularly straightforward; the solution for U_1 is obtained by writing $U_1 = U_{0\xi}V(\xi, \tau)$, which can then be examined to see if the asymptotic expansion (1.101) is uniformly valid as $\tau \to \infty$. This involves very detailed and lengthy discussions, particularly if the general term (U_n) is to be included; such an analysis is altogether beyond the scope of our investigations. Suffice it to record that, for an $f(x)$ which is smooth enough and which decays rapidly (exponentially, for example) at infinity, the far-field expansion in problems of this type is usually uniformly valid. (For some problems, though, it is necessary to write the characteristic variable itself as an asymptotic expansion, a technique related to the familiar method known as the *method of strained coordinates*. That this might be required is easily seen if we attempt to find a representation of the exact characteristics of the original equation. Some of these ideas are touched on in the exercises; see Q1.47–Q1.49, Q1.53.)

Finally, we describe one other type of asymptotic formulation which is often used in wave-like problems; this is based on the *method of multiple scales*. As before, we explain the salient features by developing the ideas for a particular equation (which will be typical of some of our problems in water-wave theory). We consider the equation

$$u_{tt} - u_{xx} - u + \varepsilon(uu_x)_x = 0; \tag{1.103}$$

for $\varepsilon = 0$ this equation has a travelling-wave solution, expressed as a harmonic wave (see (1.80)),

$$u = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}, \tag{1.104}$$

which c.c. denotes the complex conjugate. This solution, (1.104), for an arbitrary complex constant A, leads to the dispersion relation (for $\varepsilon = 0$)

$$\omega^2 = k^2 - 1,$$

which possess real solutions for ω only if $|k| \geq 1$. We shall suppose that $k > 1$, and then there are two possible waves with speeds

$$c_p = \frac{\omega}{k} = \pm\sqrt{1 - k^{-2}}, \tag{1.105}$$

where the subscript p is used to denote the *phase speed*. Now, for a given k and one choice of c_p, we seek a harmonic-wave solution of equation (1.103) which *evolves slowly* on suitable scales. For these problems, a little investigation (or some experience) suggests that we should introduce *slow* variables

$$\zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t,$$

where the speed c_g is, in general, not equal to the phase speed, c_p, and is unknown at this stage. In addition, upon writing

$$\xi = x - c_p t,$$

the original equation, (1.103), is transformed according to

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \xi} + \varepsilon\frac{\partial}{\partial \zeta}; \quad \frac{\partial}{\partial t} \equiv -c_p\frac{\partial}{\partial \xi} - \varepsilon c_g\frac{\partial}{\partial \zeta} + \varepsilon^2\frac{\partial}{\partial \tau},$$

to yield (with $u(x, t; \varepsilon) \equiv U(\xi, \zeta, \tau; \varepsilon)$)

$$\begin{aligned}(c_p^2 - 1)U_{\xi\xi} - U + 2\varepsilon(c_p c_g - 1)U_{\xi\zeta} + \varepsilon^2\{(c_g^2 - 1)U_{\zeta\zeta} - 2c_p U_{\xi\tau}\} \\ + \varepsilon(UU_\xi)_\xi + \varepsilon^2\{(UU_\xi)_\zeta + (UU_\zeta)_\xi\} = \mathrm{O}(\varepsilon^3),\end{aligned}$$

where terms only as far as $\mathrm{O}(\varepsilon^2)$ have been written down. Thus the function $u(x, t; \varepsilon)$ is now treated as a function of the variables (ξ, ζ, τ):

this is the method of multiple scales (the scales here being O(1), O(ε^{-1}), O(ε^{-2}), respectively).

We seek a solution in the form of the asymptotic expansion

$$U(\xi, \zeta, \tau; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n U_n(\xi, \zeta, \tau) \quad \text{as } \varepsilon \to 0, \tag{1.106}$$

for ξ, ζ, τ all O(1). Thus

$$(c_p^2 - 1)U_{0\xi\xi} - U_0 = 0,$$

and we take the solution

$$U_0 = A_{01}(\zeta, \tau)\mathrm{e}^{ik\xi} + \text{c.c.} \quad \text{with } c_p^2 = 1 - \frac{1}{k^2} \ (k > 1),$$

where the first subscript in A_{01} denotes the term ε^0, and the second is associated with the choice

$$E^1 = \mathrm{e}^{ik\xi}.$$

At the next order, ε^1, we obtain the equation

$$\begin{aligned}(c_p^2 - 1)U_{1\xi\xi} - U_1 &= 2(1 - c_p c_g)U_{0\xi\zeta} + (U_0 U_{0\xi})_\xi \\ &= 2(1 - c_p c_g)(ikA_{01\zeta}E + \text{c.c.}) - (2k^2 A_{01}^2 E^2 + \text{c.c.}),\end{aligned}$$

and U_1 is a harmonic function only if

$$c_p c_g = 1, \tag{1.107}$$

which determines c_g. If this choice for c_g is *not* made, then U_1 will include a particular integral proportional to ξE which would lead to a non-uniformity in the asymptotic expansion, (1.106), as $|\xi| \to \infty$; terms like ξE are usually called *secular*, whereas uniformity in ξ is guaranteed only if terms *periodic* (harmonic) in ξ are allowed in U_1. The speed, c_g, which describes the motion of the amplitude A_{01}, is the *group speed* for this wave. To see this we start with the definition (see Section 1.4.1)

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}(kc_p),$$

and so we have

$$c_g = c_p + k\frac{\mathrm{d}c_p}{\mathrm{d}k} \quad \text{or} \quad c_p c_g = c_p^2 + \frac{1}{2}k\frac{\mathrm{d}c_p^2}{\mathrm{d}k}$$

which, from (1.105), yields

$$c_p c_g = 1 - \frac{1}{k^2} + k\left(\frac{1}{k^3}\right) = 1,$$

as required (see (1.107)). The solution for U_1 may therefore be written as

$$\begin{aligned} U_1 &= A_{11}(\zeta, \tau)E + \frac{2k^2 A_{01}^2}{4k^2(1 - c_p^2) - 1}E^2 + \text{c.c.} \\ &= A_{11}E + \frac{2}{3}k^2 A_{01}^2 E^2 + \text{c.c.} \end{aligned}$$

where A_{11} is (so far) unknown; this gives a correction (of $\mathrm{O}(\varepsilon)$) to the amplitude of the fundamental, E. We note that U_1 includes a higher harmonic, E^2 (and its complex conjugate, E^{-2}).

To proceed, the equation for U_2 is obtained, which, with (1.107) incorporated, is

$$(c_p^2 - 1)U_{2\xi\xi} - U_2 = (1 - c_g^2)U_{0\zeta\zeta} + 2c_p U_{0\xi\tau} - (U_0^2)_{\xi\zeta} - (U_0 U_1)_{\xi\xi}. \quad (1.108)$$

Again, we impose the condition that U_2 is to contain only terms periodic in ξ; to this end, any terms in E^1 which appear in the forcing terms in equation (1.108) must be removed. Such terms can arise only from

$$\begin{aligned} &(1 - c_g^2)U_{0\zeta\zeta} + 2c_p U_{0\xi\tau} - (U_0 U_1)_{\xi\xi} \\ &= (1 - c_g^2)(A_{01\zeta\zeta}E + \text{c.c.}) + 2c_p(A_{01\tau}ikE + \text{c.c.}) \\ &\quad - \frac{\partial^2}{\partial \xi^2}\left\{(A_{01}E + \bar{A}_{01}E^{-1})\left(\frac{2}{3}k^2 A_{01}^2 E^2 + \frac{2}{3}k^2 \bar{A}_{01}^2 E^{-2} + A_{11}E + \bar{A}_{11}E^{-1}\right)\right\}, \end{aligned}$$

where the overbar denotes the complex conjugate. In this expression, the coefficient of E which is to be set to zero (and, of course, its conjugate for terms E^{-1}) is

$$2ikc_p A_{01\tau} + (1 - c_g^2)A_{01\zeta\zeta} + \frac{2}{3}k^4 A_{01}|A_{01}|^2 = 0; \qquad (1.109)$$

all other terms generate higher harmonics in U_2, which is acceptable for uniform validity as $|\xi| \to \infty$. The equation which describes the evolution of the amplitude of the leading term, equation (1.109), is one version of another important and well-known equation: it is the *Nonlinear Schrödinger equation*, which we shall describe more fully later (Chapter 4). Other derivations of this type of equation are discussed in Q1.50 and Q1.54.

Further reading

This chapter, although it aims to provide a minimal base from which to explore the theory of water waves, cannot develop all the relevant topics to any depth. The following, therefore, referenced by the section numbers used in the chapter, is intended to present some useful – but not essential – additional reading.

1.1 There are many texts – and many good texts – on fluid mechanics; readers may have their favourites, but we list a few that can be recommended. A wide-ranging and well-written text is Batchelor (1967); more recent texts are Paterson (1983) and Acheson (1990), this latter including an introduction to waves in fluids. A more descriptive approach is provided by Lighthill (1986), and there are the classical texts: Lamb (1932), Schlichting (1960), Rosenhead (1964) and Landau & Lifschitz (1959).

1.2, 1.3 We shall provide many references to research papers and texts later, but two texts that can be mentioned at this stage are Stoker (1957) and Crapper (1984). A more general discussion of waves in fluids is given by Lighthill (1978).

1.4.1 For an excellent introduction to the theory of waves (including water waves), see Whitham (1974). An exploration of the concept of group velocity is given by Lighthill (1965). Of course, there is an extensive literature on the theory of partial differential equations; we mention as pre-eminent Garabedian (1964), and Bateman (1932) is also excellent, but good introductory texts are Haberman (1987), Sneddon (1957) and Weinberger (1965); two compact but wide-ranging texts are Vladimirov (1984) and Webster (1966). Finally, two excellent texts on general mathematical methods, including much work on partial differential equations, are Courant & Hilbert (1953) and Jeffreys & Jeffreys (1956).

1.4.2 The classical text, for applications to fluid mechanics, is van Dyke (1964). Introductory texts that cover a wide spectrum of applications, including examples on wave propagation, are Kevorkian & Cole (1985), Hinch (1991) and Bush (1992). More formal approaches to this material are given by Eckhaus (1979) and Smith (1985). The properties of divergent series are described in the excellent text by Hardy (1949), and their everyday use is described by Dingle (1973).

Exercises

Q1.1 *Some differential identities.* Given that $\phi(\mathbf{x})$ is a scalar function, and $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ are vector-valued functions, show that

(a) $\nabla \cdot (\phi \mathbf{u}) = (\mathbf{u} \cdot \nabla)\phi + \phi \nabla \cdot \mathbf{u}$;

(b) $\nabla \wedge (\phi \mathbf{u}) = (\nabla \phi) \wedge \mathbf{u} + \phi(\nabla \wedge \mathbf{u})$;

(c) $\mathbf{u} \wedge (\nabla \wedge \mathbf{u}) = \nabla(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u}$;

(d) $\nabla \wedge (\mathbf{u} \wedge \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u})$;

[A subscript notation, used together with the summation convention, is a very compact way to obtain these identities.]

Q1.2 *Two integral identities.* A volume V is bounded by the surface S on which there is defined the outward normal unit vector, $\mathbf{n}$. Given that $\phi(\mathbf{x})$ is a scalar function, use Gauss' theorem to show that

$$\int_V \nabla \phi \mathrm{d}v = \int_S \phi \mathbf{n} \mathrm{d}s,$$

and, for the vector function $\mathbf{u}$, that

$$\int_V \nabla \wedge \mathbf{u} \mathrm{d}v = \int_S \mathbf{n} \wedge \mathbf{u} \mathrm{d}s.$$

[It is convenient to introduce suitable arbitrary constant vectors into Gauss' theorem.]

Q1.3 *Another integral identity.* By considering, separately, each component of the vector $\mathbf{A}$, show that

$$\int_S \mathbf{A}(\mathbf{u} \cdot \mathbf{n})\mathrm{d}s = \int_V \{(\mathbf{u} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{u})\}\mathrm{d}v.$$

Q1.4 *Acceleration of a fluid particle.* The velocity vector which describes the motion of a particle (point) in a fluid is $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, so that the particle follows the path on which

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{U}(t) = \mathbf{u}\{\mathbf{x}(t), t\}.$$

Write $\mathbf{x} \equiv (x, y, z)$ and $\mathbf{u} \equiv (u, v, w)$ (in rectangular Cartesian coordinates), and hence show that the acceleration of the particle is

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \equiv \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t},$$

the material derivative.

Q1.5 *Material derivative.*

(a) A fluid moves so that its velocity is $\mathbf{u} \equiv (2xt, -yt, -zt)$, written in rectangular Cartesian coordinates. Show that the surface

$$F(x, y, z, t) = x^2 \exp(-2t^2) + (y^2 + 2z^2)\exp(t^2) = \text{constant}$$

moves with the fluid (so that it always contains the same fluid particles; that is, $\mathrm{D}F/\mathrm{D}t = 0$).

(b) Repeat (a) for

$$\mathbf{u} \equiv \left(-\frac{x}{t}, -\frac{y}{2t}, \frac{3z}{2t}\right) \quad \text{and} \quad F = t^2x^2 + ty^2 - \frac{z^2}{t^3}.$$

Q1.6 *Eulerian vs. Lagrangian description.* The Eulerian description of the motion is represented by $\mathbf{u}(\mathbf{x}, t)$: the velocity at any point and at any time. The Lagrangian description follows a given particle (point) in the fluid; the Lagrangian velocity is $\mathbf{u}(\mathbf{x}_0, t)$, where $\mathbf{x} = \mathbf{x}_0$ at $t = 0$ labels the particle.

A particle moves so that

$$\mathbf{x} \equiv \{x_0 \exp(2t^2), y_0 \exp(-t^2), z_0 \exp(-t^2)\},$$

written in rectangular Cartesian coordinates, where $\mathbf{x} = \mathbf{x}_0 \equiv (x_0, y_0, z_0)$ at $t = 0$.

(a) Find the velocity of the particle in terms of $\mathbf{x}_0$ and t (the Lagrangian description), and show that it can be written as

$$\mathbf{u} \equiv (4xt, -2yt, -2zt),$$

the Eulerian description.

(b) Now obtain the acceleration of the particle from the Lagrangian description.

(c) Also write down the Eulerian acceleration, $\partial\mathbf{u}/\partial t$, where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$.

(d) Show that the Lagrangian acceleration (that is, following a particle) is recovered from

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \quad \text{where} \quad \mathbf{u} = \mathbf{u}(\mathbf{x}, t).$$

Q1.7 *Incompressible flows.* Show that the velocity vectors introduced in Q1.5 (a), (b) and Q1.6 (a) all satisfy the condition for an incompressible flow, namely $\nabla \cdot \mathbf{u} = 0$.

Q1.8 *Steady, incompressible flow.* Show that the particle which moves according to

$$\mathbf{x} \equiv (x_0 e^{\alpha t}, y_0 e^{\beta t}, z_0 e^{\gamma t}),$$

written in rectangular Cartesian coordinates, where $\mathbf{x} \equiv (x_0, y_0, z_0)$ at $t = 0$ and α, β and γ are constants, is a steady flow (that is, $\mathbf{u} = \mathbf{u}(\mathbf{x})$). Find the condition which ensures that $\nabla \cdot \mathbf{u} = 0$.

Q1.9 *Another incompressible flow.* A velocity field is given by

$$\mathbf{u} = f(r)\mathbf{x}, \quad r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2},$$

written in rectangular Cartesian coordinates, where $f(r)$ is a scalar function. Find the most general form of $f(r)$ so that $\mathbf{u}$ represents an incompressible flow.

Q1.10 *A solution of Euler's equation.* Written in rectangular Cartesian coordinates, the velocity vector for a flow is

$$\mathbf{u} \equiv (xt, yt, -2zt) \quad \text{where} \quad \mathbf{x} \equiv (x, y, z);$$

show that $\nabla \cdot \mathbf{u} = 0$. Given, further, that the density is constant and that the body force is $\mathbf{F} \equiv (0, 0, -g)$, where g is a constant, find the pressure, $P(\mathbf{x}, t)$, in the fluid which satisfies $P = P_0(t)$ at $\mathbf{x} = \mathbf{0}$.

Q1.11 *Hydrostatic pressure law.* Consider a stationary fluid ($\mathbf{u} \equiv \mathbf{0}$) with $\rho =$ constant, and take $\mathbf{F} \equiv (0, 0, -g)$ with $g =$ constant. Find $P(z)$ which satisfies $P = P_a$ on $z = h_0$, where z is measured positive upwards. What is the pressure on $z = 0$?

Q1.12 *Vorticity.* Consider an imaginary circular disc, of radius R, whose arbitrary orientation is described by the unit vector, $\mathbf{n}$, perpendicular to the plane of the disc. Define the component, in the direction $\mathbf{n}$, of the angular velocity, $\boldsymbol{\Omega}$, at a point in the fluid by

$$\boldsymbol{\Omega} \cdot \mathbf{n} = \lim_{R \to 0} \left\{ \frac{1}{2\pi R^2} \oint_{\mathrm{C}} \mathbf{u} \cdot d\boldsymbol{\ell} \right\},$$

where C denotes the boundary (rim) of the disc. Use Stokes' theorem, and the arbitrariness of $\mathbf{n}$, to show that

$$\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\omega},$$

where $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ is the vorticity in the fluid at $R = 0$.

[This definition is based on a description applicable to the rotation of solid bodies. Confirm this by considering $\mathbf{u} = \mathbf{U} + \boldsymbol{\Omega} \wedge \mathbf{r}$, where $\mathbf{U}$ is the translational velocity of the body, $\boldsymbol{\Omega}$ is its angular velocity and $\mathbf{r}$ is the position vector of a point relative to a point on the axis of rotation.]

Q1.13 *A simple flow with vorticity.* Written in rectangular Cartesian coordinates, with $\mathbf{F} \equiv (0, 0, -g)$ where g is constant, show that $\mathbf{u} \equiv (U(z), 0, 0)$ with $P = P_0 - \rho g z$ (P_0 and ρ constants) is an exact solution of Euler's equation and the equation of mass conservation. What is the vorticity for this flow? Repeat this calculation for $U = U(y)$.

[A classical example is $U(z) = U_0 + (U_1 - U_0)H(z)$ where U_0 and U_1 are constants, and $H(z)$ is the *Heaviside step function*; this is called a *vortex sheet*.]

Q1.14 *Helmholtz's equation.* Given that $\rho =$ constant and $\mathbf{F} = -\nabla\Omega$, take the curl of Euler's equation to show that

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}.$$

Hence, for a flow that varies in only two spatial dimensions, show that $\boldsymbol{\omega} \cdot \nabla \equiv 0$ and so $\boldsymbol{\omega} = \mathbf{constant}$ on particles. (The vorticity then remains 'trapped' perpendicular to the plane of the flow; cf. Q1.13.)

Q1.15 *Helmholtz's equation for compressible flow.* Show, for a compressible flow (which satisfies the general equation of mass conservation, (1.4)) with $\mathbf{F} = -\nabla\Omega$, that

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left\{\left(\frac{\boldsymbol{\omega}}{\rho}\right) \cdot \nabla\right\}\mathbf{u} - \frac{1}{\rho}\left\{\nabla\left(\frac{1}{\rho}\right)\right\} \wedge \nabla P;$$

cf. Q1.14. Hence, given that the fluid is barotropic (see Q1.18) so that $P = P(\rho)$, show that this equation is that given in Q1.14 with $\boldsymbol{\omega}$ replaced by $\boldsymbol{\omega}/\rho$.

Q1.16 *Vorticity in cylindrical coordinates.* Given that the velocity vector for a flow is $\mathbf{u} \equiv (\theta u, u, \theta u)$, written in cylindrical coordinates (r, θ, z), find the vorticity when $u = u(r)$.

[The vorticity vector for (u, v, w) in cylindrical coordinates is

$$\left(\frac{1}{r}w_\theta - v_z,\ u_z - w_r,\ \frac{1}{r}(rv)_r - \frac{1}{r}u_\theta\right).]$$

Q1.17 *Rankine's vortex.* Find the vorticity for the velocity field

$$\mathbf{u} = \begin{cases} (0, \frac{1}{2}\omega r, 0), & 0 \le r \le a \\ (0, \frac{1}{2}\omega a^2/r, 0), & r > a, \end{cases}$$

written in cylindrical coordinates (see Q1.16), where ω is a constant. Confirm that this $\mathbf{u}$ describes an incompressible flow. With $\rho =$ constant and $\mathbf{F} = -\nabla\Omega$, use Euler's equation to find an expression for $(P/\rho) + \Omega$ that is continuous on $r = a$ and which satisfies $\{(P/\rho) + \Omega\} \to P_0/\rho$ as $r \to \infty$. What condition on P_0 ensures that $(P/\rho) + \Omega > 0$? (This condition is particularly relevant if $\Omega = 0$.)

Q1.18 *Barotropic fluid.* Given that a fluid is described by $P = P(\rho)$, show that

$$\frac{1}{\rho}\nabla P = \nabla\left(\int \frac{\mathrm{d}P}{\rho}\right).$$

[This generalises $\nabla(P/\rho)$ as used in equation (1.21); a barotropic fluid (Greek: βαρος, weight) is one in which lines of constant density coincide with lines of constant pressure.]

Q1.19 *Particle paths and streamlines.* For these flows, expressed in rectangular Cartesian coordinates, find the particle paths that pass through (x_0, y_0, z_0) at $t = 0$. In each case, also find the general equations describing the streamlines. Verify that each flow is incompressible.

(a) $\mathbf{u} \equiv (cx, -cy, 0)$; (b) $\mathbf{u} \equiv (2xt, -2yt, 0)$;
(c) $\mathbf{u} \equiv (x - t, -y, 0)$; (d) $\mathbf{u} \equiv \{cx^2, cy^2, -2c(x + y)z\}$,

where c is a constant.

Q1.20 *Stream functions.* The stream function, $\psi(x, y, t)$, satisfies the equation of mass conservation for incompressible flow

$$u_x + v_y = 0$$

with $u = \psi_y$ and $v = -\psi_x$. For each of the velocity fields given in Q1.19 (a), (b), and (c), find the stream function.

Q1.21 *The stream function.* For the two-dimensional flow field, $\mathbf{u} \equiv (u, v)$ with $\mathbf{x} \equiv (x, y)$, use the definition of the streamline (equation (1.17)) to show that $\psi =$ constant (at fixed t) on streamlines; see Q1.20.

Q1.22 *Stream function in polar coordinates.* Following Q1.20, define a stream function, $\psi(r, \theta, t)$, for the equation of mass conservation

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial \theta}=0.$$

Hence find the stream function for the flow with speed $U(t)$ along the x-axis; that is, along $\theta = 0$.

Q1.23 *Stream function in cylindrical polars.* Define a stream function for the equations of mass conservation

(a) $\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0$; (b) $\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0$;

see Q1.22.

Q1.24 *Irrotational flow.* Show that these velocity fields describe irrotational flows, and find the velocity potential in each case:
(a) $\mathbf{u} = (\mathbf{a}\cdot\mathbf{x})\mathbf{b} + (\mathbf{b}\cdot\mathbf{x})\mathbf{a}$ ($\mathbf{a}$, $\mathbf{b}$ arbitrary constant vectors);

(b) $\mathbf{u} \equiv \left\{\frac{-2xyz}{(x^2+y^2)^2}, \frac{(x^2-y^2)z}{(x^2+y^2)^2}, \frac{y}{x^2+y^2}\right\}$

in rectangular Cartesian coordinates.

Q1.25 *Complex potential.* An incompressible, irrotational flow in two dimensions, with $\mathbf{u} \equiv (u, v)$ and $\mathbf{x} \equiv (x, y)$, leads to the introduction of the stream function, ψ, and velocity potential, ϕ; see Q1.29 and Section 1.1.3. Show that ϕ and ψ satisfy the *Cauchy–Riemann relations*, and hence that there exists a function $w(z) = \phi + \mathrm{i}\psi$ (with $z = x + \mathrm{i}y$), the complex potential. Also demonstrate that

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - \mathrm{i}v,$$

the *complex velocity*. What flow is represented by the function

$$w(z) = U(t)\mathrm{e}^{\mathrm{i}\alpha}z,$$

where α (a constant) and $U(t)$ are both real?

Q1.26 *Vector potential.* Introduce the stream function, $\psi(x, y, t)$, for the incompressible flow field $\mathbf{u} \equiv (u, v)$ with $\mathbf{x} \equiv (x, y)$; see Q1.20. Define the vector potential $\boldsymbol{\Psi} \equiv (0, 0, \psi)$ and hence show that

$$\nabla \wedge \boldsymbol{\Psi} = \mathbf{u}.$$

Q1.27 *Kinematic condition.* Fluid particles move on the path, $\mathbf{x} = \mathbf{x}(t)$, in the free surface

$$z(t) = h\{\mathbf{x}_\perp(t), t\}.$$

Differentiate this equation with respect to t, and hence show that

$$w = h_t + (\mathbf{u}_\perp \cdot \nabla_\perp)h \quad \text{on} \quad z = h(\mathbf{x}_\perp, t).$$

Q1.28 *A two-dimensional bubble.* An incompressible fluid, $\rho =$ constant, is at rest on a flat horizontal surface and the surface tension causes it to form into a bubble. The pressure in the fluid satisfies the equation of hydrostatic equilibrium, and the free surface is in contact with the atmosphere at pressure $P = P_a =$ constant. Assuming that the 'bubble' exists only in the two-dimensional (x, z)-plane, for $-x_0 \le x \le x_0$ and $0 \le z \le h(x)$, write down the equation for $h(x)$.

Q1.29 *An axisymmetric bubble.* See Q1.28: now assume that the bubble is defined for $0 \le r \le r_0$, $0 \le z \le h(r)$, with $0 \le \theta \le 2\pi$, expressed in cylindrical coordinates. Write down the equation for $h(r)$. With the notation $\mathrm{h}(0) = \mathrm{h}_0$, define $R = r/r_0$ and $H(R) = h/h_0$; hence write your equation in terms of $H(R)$. Given that $\varepsilon = h_0/r_0 \ll 1$, show that an approximate solution exists which (for suitable parameter values) satisfies

$$H(0) = 1, \quad H'(0) = 0, \quad H(1) = 0,$$

provided $\alpha_0 < \alpha < \alpha_1$ where $\alpha = \rho g r_0^2/\Gamma$ (which uses the standard notation). Here, $\sqrt{\alpha_0}$ (> 0) is the first zero of the Bessel function J_0, and $\sqrt{\alpha_1}$ (> 0) is the second zero of J_1. Find $H'(1)$ and sketch the shape of the bubble.

Q1.30 *Differentiation under the integral sign.* Given

$$I(x) = \int_{a(x)}^{b(x)} f(x, y)\mathrm{d}y,$$

show that

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \int_a^b f_x(x, y)\,\mathrm{d}y + f(x, b)\frac{\mathrm{d}b}{\mathrm{d}x} - f(x, a)\frac{\mathrm{d}a}{\mathrm{d}x},$$

where the integral of f_x, and a' and b', are assumed to exist. Verify that this formula recovers a familiar and elementary result in the case: $f = f(y)$, $b(x) = x$, $a(x) =$ constant.

[You may find it helpful to introduce the *primitive* of $f(x, y)$ at fixed x: $g(x, y) = \int f(x, y)\,\mathrm{d}y$.]

Q1.31 *Differentiation under the integral sign: examples.* Use the formula given in Q1.30 to

(a) find an expression for $\mathrm{d}I/\mathrm{d}x$, where

$$I(x) = \int_x^{x^2} \frac{\mathrm{e}^{xy}}{y} \mathrm{d}y, \quad x > 0;$$

(b) show that

$$\phi(x, t) = \frac{1}{t^3} \int_{-t}^{t} (t^2 - y^2) g(x + y) \, \mathrm{d}y,$$

where g is a twice differentiable function, is a solution of the partial differential equation

$$\phi_{xx} - \phi_{tt} - \frac{n}{t}\phi_t = 0,$$

for a certain value of the positive integer, n, which should be determined. [Hint: integrate your expression for ϕ_{xx} by parts, twice.]

Q1.32 *An energy equation.* An incompressible, inviscid flow with $\mathbf{F} = -\nabla\Omega$ is described by Euler's equation. Take the scalar product of this equation with the velocity vector, $\mathbf{u}$. Integrate the resulting equation over the volume V, which is fixed in space, and hence show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathrm{V}} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \, \mathrm{d}v = - \int_{\mathrm{S}} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + P + \rho\Omega \right) \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s,$$

where S bounds V.

Q1.33 *Energy and a uniqueness theorem.*

(a) The kinetic energy of the fluid occupying the volume V is

$$T = \frac{1}{2} \int_{\mathrm{V}} \rho \mathbf{u} \cdot \mathbf{u} \, \mathrm{d}v;$$

see Q1.32. For an incompressible, irrotational flow, show that

$$T = \frac{1}{2} \int_{\mathrm{S}} \rho\phi \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s, \quad \mathbf{u} = \nabla\phi.$$

(b) The result in (a) can be used to provide a uniqueness theorem. Suppose that there are two possible flows, $\mathbf{u}_1 = \nabla\phi_1$ and $\mathbf{u}_2 = \nabla\phi_2$ both satisfying the same given conditions on S. Write $\mathbf{U} = \mathbf{u}_1 - \mathbf{u}_2$ and $\Phi = \phi_1 - \phi_2$, and show that

$$I = \int\limits_{\mathrm{V}} \mathbf{U}\cdot\mathbf{U}\,\mathrm{d}v = \int\limits_{\mathrm{S}} \Phi\mathbf{U}\cdot\mathbf{n}\,\mathrm{d}s,$$

and hence that $I = 0$ if either ϕ or $\mathbf{u}$ is prescribed on S. With these boundary conditions, deduce that $\mathbf{U} \equiv \mathbf{0}$, so $\mathbf{u}_1 \equiv \mathbf{u}_2$: the velocity field is unique.

Q1.34 *Elementary nondimensionalisation.* A two-dimensional flow, with $\mathbf{u} \equiv (u, w)$ and $\mathbf{x} \equiv (x, z)$, is both incompressible and irrotational. Nondimensionalise according to

$$u \to cu, \quad x \to \lambda x, \quad z \to hz,$$

and hence obtain the nondimensionalisation of the stream function, ψ, of the velocity potential, ϕ, and of w. Write down the nondimensional version of $w = \phi_z$.

Q1.35 *A Reynolds number.* Use the scheme described in Section 1.3 to nondimensionalise the Navier–Stokes equation (Appendix A), and hence obtain the nondimensional parameter which incorporates the viscosity, μ, and is based on the scales associated with the horizontal motion.

[The reciprocal of this parameter is called the *Reynolds number*; knowledge of its size is of fundamental importance in the study of fluid mechanics.]

Q1.36 *Dynamic condition in cylindrical coordinates.* Use the nondimensionalisation described in Section 1.3.1 to obtain the corresponding boundary condition written in cylindrical coordinates; cf. equation (1.64). Further, by suitably scaling the pressure in terms of ε, and by using an appropriate definition of the Weber number, rewrite this condition and then approximate it for $\varepsilon \to 0$.

Q1.37 *Nondimensionalisation of the pressure equation.* Use the nondimensionalisation described in Section 1.3.1, followed by the scaling adopted in Section 1.3.2, to obtain the appropriate form of the pressure boundary condition, equation (1.29). (This will require a nondimensionalisation and scaling of the velocity potential, ϕ; see Q1.34.)

Q1.38 *Irrotational flow: approximations.* Write down the non-dimensional, scaled equations for irrotational flow in the absence of surface tension (see Q1.37) and for a bottom boundary which is independent of time. Hence obtain the approximate form of these equations in the limit of small amplitude, $\varepsilon \to 0$. Also write down the corresponding approximate boundary condition when surface tension is included.

Q1.39 *The classical wave equation.* Use the method of characteristics to derive d'Alembert's solution of the wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

and hence obtain that solution which satisfies $u(x, 0) = p(x)$ and $u_t(x, 0) = q(x)$, $-\infty < x < \infty$.

Q1.40 *Data on compact support.* See Q1.39; now suppose that both $p(x)$ and $q(x)$ are zero for $x < 0$ and $x > x_0$ (> 0). Describe the form of the solution for $t > x_0/(2c)$.

Q1.41 *Dispersion relation.* Discuss the nature of the solution of the equation

$$u_t + u_x + u_{xxx} - u_{xx} = 0,$$

on the basis of its dispersion relation.

Q1.42 *Dispersion relations compared.* Compare the dispersion relations for the two equations

$$u_t + u_x + u_{xxx} = 0; \quad u_t + u_x - u_{xxt} = 0,$$

particularly for long waves ($k \to 0$) and short waves ($k \to \infty$).

Q1.43 *Nonlinear wave equation.* Obtain, explicitly, the solution of the equation

$$u_t + (1 + u)u_x = 0,$$

which satisfies

$$u(x, 0) = \begin{cases} \alpha x, & 0 \le x < 1 \\ \alpha(2 - x), & 1 \le x \le 2 \\ 0, & \text{otherwise}, \end{cases}$$

where α is a positive constant. Also, by using the characteristics, sketch this solution at various times, $t \ge 0$, and include $t = 1/\alpha$.

Q1.44 *An implicit solution.* Find the (implicit) solution of the equation

$$u_t + uu_x = 0,$$

which satisfies $u(x, 0) = \cos \pi x$. Show that $u(x, t)$ first has a point where u_x is infinite at time $t = \pi^{-1}$. What happens to this solution if it is allowed to develop beyond $t = \pi^{-1}$?

Q1.45 *Asymptotic expansions I.*

(a) Obtain the first two terms in the asymptotic expansion of

$$f(x; \varepsilon) = \left(1 + \varepsilon x - \frac{\varepsilon}{\varepsilon + x} + \mathrm{e}^{-x/\varepsilon}\right)^{-1}, \quad x \geq 0, \varepsilon > 0,$$

for $x = O(1)$ as $\varepsilon \to 0$. Also obtain the leading order terms in the expansions valid for (i) $x = O(\varepsilon)$, (ii) $x = O(\varepsilon^{-1})$. Show that your expansions satisfy the matching principle.

(b) Obtain the first three terms in an asymptotic expansion of

$$f(x; \varepsilon) = (1 + \varepsilon x + \varepsilon^2 x^4)^{-1/2}, \quad x \geq 0, \varepsilon > 0,$$

for $x = O(1)$ as $\varepsilon \to 0$. Show that your expansion is not uniformly valid as $x \to \infty$. In the *two* further asymptotic expansions that are required, find the first two terms in each and confirm that they satisfy the matching principle.

Q1.46 *Asymptotic expansions II.* The function

$$f(x; \varepsilon) = (1 - \varepsilon x - \varepsilon^4 x^3 - \mathrm{e}^{-x/\varepsilon})^{1/2} \quad \varepsilon > 0,$$

is real for $0 \leq x \leq x_0(\varepsilon)$. Construct asymptotic expansions of $f(x; \varepsilon)$, as $\varepsilon \to 0$, as follows:

(a) $x = O(1)$: first two terms algebraic in ε, first exponentially small;

(b) $x = O(\varepsilon)$: first two terms;

(c) $x = O(\varepsilon^{-1})$: first two terms.

Show that your expansions satisfy the matching principle and, from your expansion obtained in (c), deduce that $x_0(\varepsilon) \sim \varepsilon^{-1} - 1$ as $\varepsilon \to 0^+$.

Q1.47 *Long–distance scale.* For the propagation equation

$$u_{tt} - u_{xx} = \varepsilon(u^2 + u_{xx})_{xx},$$

introduce the characteristic variable $\xi = x - t$, and the long-distance variable $X = \varepsilon x$, and hence obtain the appropriate Korteweg–de Vries equation which describes the first approximation to u (as $\varepsilon \to 0$) in the far-field. (You may assume that $u \to 0$ as $|\xi| \to \infty$.)

Q1.48 *Left-going wave.* See Q1.47; for this equation, find the Korteweg–de Vries equation (as the first approximation for $\varepsilon \to 0$) in the far-field defined by $\zeta = x + t = O(1)$ and $\tau = \varepsilon t = O(1)$.

Q1.49 *Left- and right-going waves.* See Q1.47 (and also Q1.48); introduce the characteristic variables $\xi = x - t$, $\zeta = x + t$, and the long-time variable $\tau = \varepsilon t$. Seek a solution

$$u \sim f(\xi, \tau) + g(\zeta, \tau) + \varepsilon u_1(\xi, \zeta, \tau),$$

for ξ, ζ and τ all O(1) as $\varepsilon \to 0$, in which f and g separately satisfy appropriate Korteweg–de Vries equations. (You may assume that $f(g)$ decays as $|\xi| \to \infty$ (as $|\zeta| \to \infty$).) What, then, is the solution for u_1?

Q1.50 *Nonlinear Schrödinger equation.* A wave is described by the equation

$$u_{tt} - u_{xx} - u = \varepsilon\{(u_x)^2 - uu_{xx}\}.$$

Use the method of multiple scales with

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t,$$

and seek an asymptotic solution in the form

$$u \sim \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=0}^{n+1} A_{nm}(\zeta, \tau) E^m + \text{c.c.},$$

as $\varepsilon \to 0$, which is uniformly valid as $|\xi| \to \infty$. Here, $E = \exp(\mathrm{i}k\xi)$ and k (> 1) is a given (real) number. Find $c_p(k)$ and $c_g(k)$ (and confirm that $c_g = \mathrm{d}(kc_p)/\mathrm{d}k$), and show that

$$2\mathrm{i}kc_p \frac{\partial A_{01}}{\partial \tau} + (1 - c_g^2)\frac{\partial^2 A_{01}}{\partial \zeta^2} - 8k^4 A_{01}|A_{01}|^2 = 0.$$

Q1.51 *Wave hierarchies I.* A wave, which satisfies $u \to 0$ as $x \to +\infty$, is described by the multiwave speed equation

$$\left\{\frac{\partial}{\partial t} + (c_1 + \varepsilon^2 u)\frac{\partial}{\partial x} + \varepsilon^2 \frac{\partial u}{\partial x}\right\}\left\{\frac{\partial}{\partial t} + (c_2 + \varepsilon^2 u)\frac{\partial}{\partial x}\right\}u$$
$$+ \varepsilon^2\left\{\frac{\partial}{\partial t} + (c + \varepsilon u)\frac{\partial}{\partial x}\right\}u = 0,$$

where c_1, c_2 and c are constants. Show that, if $c_1 < c < c_2$, then on the time scale ε^{-2} the wave moving at speed c_1 decays exponentially in time, to leading order as $\varepsilon \to 0^+$. (To accomplish this, you will find it convenient to introduce

$\xi = x - c_1 t = O(1)$, $\tau = \varepsilon^2 t = O(1)$.) Now show that this same property is exhibited by the wave moving at speed c_2.

Q1.52 *Wave hierarchies II.* See Q1.51; show that, on the time scale ε^{-4}, the wave moving at speed c has diffused over a distance $O(\varepsilon^{-3})$ about its wavefront. In particular, show that this wave is described by an equation of the form

$$\phi_T + \phi\phi_X = \lambda\phi_{XX},$$

to leading order as $\varepsilon \to 0^+$. (Similar to Q1.51, it is useful to introduce $X = \varepsilon^3(x - ct) = O(1)$, $T = \varepsilon^4 t = O(1)$.) Determine the constant λ, and confirm that $\lambda > 0$ provided $c_1 < c < c_2$. Find the solution of this leading-order problem which describes a steady wave and which satisfies $u \to 0$ as $X \to \infty$, $u \to 1$ as $X \to -\infty$.

[This equation, here expressed in terms of ϕ, is a famous and important equation: it is the *Burgers equation*, which can be linearised by the *Hopf–Cole transformation* $\phi = -2\lambda\partial(\ln\theta)/\partial X$.]

Q1.53 *A nonlinear wave equation.* A wave motion is described by the equation

$$\left(\frac{\partial}{\partial t} + \varepsilon u\frac{\partial}{\partial x}\right)^2 u - \frac{\partial^2 u}{\partial x^2} = \varepsilon\frac{\partial^4 u}{\partial x^4}.$$

Introduce $\xi = x - t$ and $\tau = \varepsilon t$, and hence show that the leading approximation (as $\varepsilon \to 0$) satisfies the equation

$$2u_{\tau\xi} + 2uu_{\xi\xi} + u_\xi^2 + u_{\xi\xi\xi\xi} = 0, \qquad (*)$$

where $\xi = O(1)$, $\tau = O(1)$.

Now introduce *two* characteristic variables

$$\xi \sim x - t + \varepsilon f(x + t, \tau); \quad \eta \sim x + t + \varepsilon g(x - t, \tau),$$

and seek a solution

$$u = F(\xi, \tau) + G(\eta, \tau) + o(\varepsilon)$$

as $\varepsilon \to 0$, where F satisfies (*) and G satisfies the corresponding equation for left-running waves. Confirm that the results are consistent when only left- or right-running waves alone are present.

Q1.54 *Bretherton's equation.* A weakly nonlinear dispersive wave is described by

$$u_{tt} + u_{xx} + u_{xxxx} + u = \varepsilon u^3.$$

Introduce the variables $X = \varepsilon x$, $T = \varepsilon t$ and θ, where

$$\theta_x = k(X, T), \quad \theta_t = -\omega(X, T),$$

and seek an asymptotic solution

$$u \sim \sum_{n=0}^{\infty} \varepsilon^n U_n(\theta, X, T), \quad \varepsilon \to 0,$$

which is uniformly valid as $|\theta| \to \infty$. Write

$$U_0 = A(X, T)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.},$$

and obtain the equation for A which ensures that U_1 is periodic in θ. Introduce the dispersion relation, relating ω and k, and hence show that

$$A_T + \omega'(k)A_X = \frac{3\mathrm{i}}{2\omega}A|A|^2 - \frac{1}{2}k_X\omega''(k)A,$$

and then re-express this by writing $A = \alpha \mathrm{e}^{\mathrm{i}\beta}$ (for α, β real).

[This model equation for the weakly nonlinear interaction of dispersive waves was introduced by Bretherton (1964).]

Q1.55 *Steady travelling waves.* Seek a solution of each of these equations in the form $u(x, t) = f(x - ct)$, where c is a constant, satisfying the boundary conditions given:

(a) $u_t - 6uu_x + u_{xxx} = 0$ with $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$;

(b) $u_t + uu_x = u_{xx}$ with $u \to 0$ as $x \to \infty$, $u \to u_0$ (> 0) as $x \to -\infty$.

[The solution to (a) is the *solitary wave* of the Korteweg–de Vries equation, and (b) gives the *Taylor shock profile* of the Burgers equation.]

2

Some classical problems in water-wave theory

> Yet let us hence, and find or feel a way
> Thro' this blind haze
>
> *The Passing of Arthur*

The study of problems in water-wave theory, particularly under the umbrella of the linear approximation, goes back over 150 years. In the intervening time, many different problems – and extensions of standard problems – have been discussed by many authors. In a text such as ours, it is necessary to make a selection from this body of classical work; we cannot hope to describe all the various problems, nor all the subtle variants of standard problems. Our intention is, of course, to include the simplest and most fundamental results (such as, for example, the speed of waves over constant depth and the description of particle paths), but otherwise we choose those topics which contain some interesting and relevant mathematics. However, since we shall not present all that some readers might, perhaps, expect or prefer, we endeavour to remedy this by introducing additional examples through the exercises. The sufficiently dedicated reader is therefore directed to the exercises, particularly if a broader spectrum of water-wave theory is desired.

The material here is presented under two separate headings. The first is *linear problems*, where, apart from the elementary aspects mentioned above, we single out those topics that are attractive and which will prove relevant to some of our later discussions. Thus we describe *waves on sloping beaches*, as well as the phenomenon of *edge waves*. We shall also develop some rather general ideas associated with *ray theory*, and apply the results to *variable depth*, *ship waves*, and *waves on currents*. Under the second heading, *nonlinear problems*, we extend the application to waves on a sloping beach in order to include the effects of nonlinearity. We also describe the *Stokes expansion* (which produces higher approximations to the classical linear wave), and introduce the fully nonlinear *solitary wave* – a very famous wave. Other nonlinear waves that we shall describe include the *hydraulic jump* and *bore*, and we shall explain the analogy between nonlinear water waves and (nonlinear) *gas dynamics*; this leads us to introduce the notion of *simple waves* and the rôle of the *Riemann invariants*.

I Linear problems

Our hoard is little, but our hearts are great.

The Marriage of Geraint

The linear equations (defined by $\varepsilon \to 0$, keeping all other parameters fixed) have been described, for an inviscid fluid, in Section 1.3.1. These equations, expressed in Cartesian coordinates, are

$$\left.\begin{aligned} &u_t = -p_x; \quad v_t = -p_y; \quad \delta^2 w_t = -p_z; \quad u_x + v_y + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad \text{and} \quad p = \eta - \delta^2 W(\eta_{xx} + \eta_{yy}) \text{ on } z = 1 \\ &\text{and} \\ &w = u b_x + v b_y \text{ on } z = b. \end{aligned}\right\} \tag{2.1}$$

Correspondingly, written in cylindrical coordinates, these equations become

$$\left.\begin{aligned} &u_t = -p_r; \quad v_t = -\frac{1}{r} p_\theta; \quad \delta^2 w_t = -p_z; \quad \frac{1}{r}(ru)_r + \frac{1}{r} v_\theta + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad \text{and} \quad p = \eta - \delta^2 W\left(\eta_{rr} + \frac{1}{r}\eta_r + \frac{1}{r^2}\eta_{\theta\theta}\right) \text{ on } z = 1 \\ &\text{and} \\ &w = u b_r + \frac{v}{r} b_\theta \text{ on } z = b(r, \theta). \end{aligned}\right\} \tag{2.2}$$

Most of the problems that we present, however, will be based on rectangular Cartesian geometry.

2.1 Wave propagation for arbitrary depth and wavelength

We consider, first, the simplest problem of all: the propagation of a plane harmonic wave in the x-direction over constant depth. The depth, in nondimensional variables, is $1 - b$ (> 0), but we may choose $b = 0$

(since the actual depth is subsumed into the length scale in the z-direction; see Section 1.3.1). The governing equations, (2.1), therefore reduce to

$$\left.\begin{aligned} &u_t = -p_x; \quad \delta^2 w_t = -p_z; \quad u_x + w_z = 0 \\ \text{with}& \\ &w = \eta_t, \quad p = \eta - \delta^2 W \eta_{xx} \text{ on } z = 1; \quad w = 0 \text{ on } z = 0. \end{aligned}\right\} \tag{2.3}$$

The surface wave is described by

$$\eta = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}, \tag{2.4}$$

where A is a complex constant; this represents a wave whose initial form (at $t = 0$) is

$$\eta = A\mathrm{e}^{\mathrm{i}kx} + \text{c.c.},$$

where k is the (nondimensional) wave number.

It is convenient to write

$$E = \exp\{\mathrm{i}(kx - \omega t)\},$$

and then to seek a solution (upon the suppression of the complex conjugate) in the form

$$u = U(z)E, \quad w = W(z)E, \quad p = P(z)E. \tag{2.5}$$

To avoid the obvious confusion, the Weber number is rewritten here as W_{e}; the equations (2.3) now give, presented in the same order as above,

$$\frac{\omega}{k} U = P; \quad P' = \mathrm{i}\omega\delta^2 W; \quad W' + \mathrm{i}kU = 0 \tag{2.6}$$

(where the prime denotes the derivative with respect to z), with

$$W(1) = -\mathrm{i}\omega A; \quad P(1) = (1 + \delta^2 k^2 W_{\mathrm{e}})A; \quad W(0) = 0. \tag{2.7}$$

From equations (2.6) we see, directly, that

$$W'' = -\mathrm{i}kU' = -\frac{k^2}{\omega} P' = \delta^2 k^2 W,$$

so the general solution for $W(z)$ is

$$W = B\mathrm{e}^{\delta kz} + C\mathrm{e}^{-\delta kz},$$

where B and C are arbitrary constants. The two boundary conditions for $W(z)$ (given in (2.7)) then yield the solution

$$W = -\mathrm{i}\omega A\left(\frac{\sinh \delta kz}{\sinh \delta k}\right). \tag{2.8}$$

Also, equations (2.6) show that

$$P(1) = \frac{\omega}{k} U(1) = \frac{\mathrm{i}\omega}{k^2} W'(1)$$

and hence the boundary condition on P (in (2.7)) gives

$$1 + \delta^2 k^2 W_e = \frac{\delta\omega^2}{k} \frac{\cosh \delta k}{\sinh \delta k}$$

or

$$\left(\frac{\omega}{k}\right)^2 = c_p^2 = (1 + \delta^2 k^2 W_e) \frac{\tanh \delta k}{\delta k} \quad (> 0); \tag{2.9}$$

this is the *dispersion relation* for (plane) surface waves and so determines $\omega(k)$ (and hence the phase speed $c_p(k)$).

Thus for waves of any wave number, k, and with the surface tension contribution included, we can find the speed, c_p, of these waves. (We observe that (2.9) is an expression for c_p^2, so it is possible to have propagation both to the right ($c_p > 0$) and to the left ($c_p < 0$), as we would expect.) The dispersion relation is a function of $\delta k = h_0/\Lambda$, where $\Lambda = \lambda/k$ is the (physical) wavelength of the wave initiated at $t = 0$. We may now examine the special cases of $\delta k \to 0$ and $\delta k \to \infty$.

The first case, $\delta k \to 0$, which describes long waves (or shallow water), gives rise to the very simple result

$$c_p^2 \sim 1, \tag{2.10}$$

which, in original physical variables, produces the speeds of propagation

$$c_p \sim \pm\sqrt{gh_0}, \tag{2.11}$$

which is independent of the wave number, and so these waves are *non-dispersive*. (This speed of propagation, $\sqrt{gh_0}$, confirms the choice of scales adopted in Section 1.3.1.) The speeds given by (2.10) are also independent of the Weber number, but directly related to g, so waves that travel at these speeds are called *gravity waves* (see (2.11)). Indeed, the gravity wave describes an oscillatory balance between kinetic and potential energy, in the gravitational field.

On the other hand the limit $\delta k \to \infty$, which describes short waves (or deep water), yields

$$c_p^2 \sim \delta k W_e, \tag{2.12}$$

and waves moving at the speeds obtained from (2.12) are called *capillary waves* (or, sometimes, *ripples*). We comment that our preferred terminology is to emphasize the wavelength of the wave rather than the depth of the water (provided that this depth remains finite); we therefore discuss *long waves* or *short waves* as the limiting forms.

Now, if we consider an environment for which it is reasonable to ignore the effects of surface tension altogether (that is, W_e is always negligibly small), equation (2.9) becomes

$$c_p^2 = \frac{\tanh \delta k}{\delta k}, \tag{2.13}$$

for gravity waves of any wavelength. Then for short waves, where $\delta k \to \infty$, we obtain

$$c_p \sim \pm \frac{1}{\sqrt{\delta k}} \quad (\text{or } \pm\sqrt{g\Lambda} \text{ in dimensional variables});$$

this time the speed is not dependent on the depth. These various properties of the dispersion relation, expressed in terms of the phase speed c_p, are shown in Figure 2.1. It is evident that there is a minimum speed of propagation defined by equation (2.9); see Q2.1, Q2.2. Furthermore, at

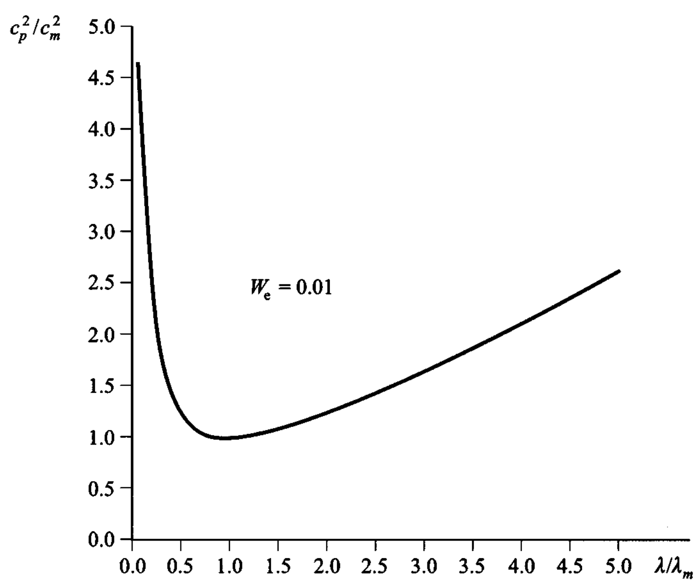


Figure 2.1. The wave speed obtained from equation (2.9), expressed as $(c_p/c_m)^2$ against λ/λ_m, where $\lambda = \delta k$, for W_e (or W) = 0.01; the subscript m denotes the value at the minimum point (see Q2.1 and Q2.2).

any given speed above this minimum, two waves – a gravity wave and a capillary wave – can coexist at the same speed. This is sometimes observed when capillary waves are seen 'riding on' gravity waves, both moving at essentially the same speed. However, a more dramatic phenomenon occurs if a disturbance is generated in a *moving* stream. Provided that the stream is moving faster than the minimum propagation speed, two sets of standing (stationary) waves can often be observed: one of rather long waves (gravity waves) behind the disturbance, the other of rather short waves (capillary waves) *ahead* of the disturbance; see Figure 2.2. (That some waves can propagate *forward* of the disturbance is, perhaps, rather surprising; this will be explained in due course.) The inclusion of a stream moving at a constant speed (for all x and z) is described in Q2.11.

Corresponding calculations are also possible in cylindrical geometry (and based, therefore, on equations (2.2)). One of the simplest cases arises for long waves ($\delta \to 0$) with $b = 0$; see Q2.17. The surface wave is then described by the classical wave equation, written in cylindrical coordinates

$$\eta_{tt} - \left(\eta_{rr} + \frac{1}{r}\eta_r + \frac{1}{r^2}\eta_{\theta\theta}\right) = 0. \tag{2.14}$$

This equation can be solved by using the conventional method of separation of variables, perhaps coupled with the use of an integral transform; see Q2.18 and Q2.19. Indeed, if we seek a solution for purely concentric waves, $\eta(r, t)$, and make use of the *Hankel transform*

$$\hat{y}(p) = \int_0^\infty r y(r) J_0(pr) \mathrm{d}r \quad (p > 0),$$

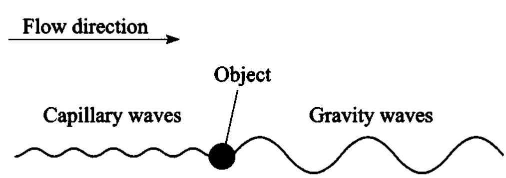


Figure 2.2. Schematic representation of the generation of capillary waves and gravity waves by a fixed object in the surface of a moving stream.

then the Hankel transform of $\eta(r, t)$ (written as $\hat{\eta}(t; p)$) satisfies

$$\hat{\eta}'' + p^2\hat{\eta} = 0;$$

see Q2.18. (This result does require the introduction of appropriate boundedness and decay conditions.) Then given, at $t = 0$, that

$$\eta = f(r) \quad \text{and} \quad \eta_t = 0,$$

we obtain

$$\hat{\eta} = \hat{f}(p) \cos pt,$$

where $\hat{f}(p)$ is the transform of $f(r)$; thus, using the inverse transform, we obtain

$$\eta(r, t) = \int_0^\infty p\hat{f}(p) \cos(tp)\, J_0(rp) \mathrm{d}p.$$

This type of solution, suitably adjusted for deep water (see Q2.19), will provide the basis for a brief description of the propagation of concentric waves in Section 2.1.3. (We comment that some authors prefer to use the symmetric version of the Hankel transform:

$$\hat{y}(p) = \int_0^\infty (pr)^{1/2} y(r) J_0(pr) \mathrm{d}r.)$$

2.1.1 Particle paths

An important consideration in any wave motion is to find what, if anything, is actually moved (presumably in the direction of propagation) as the wave progresses. This might involve, for example, mass or momentum or energy. In water waves, a first calculation of this type is to find the particle paths that describe the motion of the fluid particles on and below the surface. Then, for example, any motion that occurs near the bottom of the flow will provide the necessary source for the displacement of the sediment (if the bed of the flow is so comprised).

In our simple linear calculation, we have so far determined the vertical velocity component, from (2.8), and the horizontal velocity component (in Q2.3); these are

$$w = -\mathrm{i}\omega A \frac{\sinh \delta kz}{\sinh \delta k} E + \text{c.c.}; \quad u = \delta\omega A \frac{\cosh \delta kz}{\sinh \delta k} E + \text{c.c.},$$

respectively, where $E = \exp\{i(kx - \omega t)\}$. The particle paths are then defined by

$$\frac{dx}{dt} = \varepsilon u, \quad \frac{dz}{dt} = \varepsilon w, \tag{2.15}$$

and note the inclusion of the parameter ε, required since the particle paths are, in general,

$$\frac{d\mathbf{x}}{dt} = \mathbf{u} \quad \text{and} \quad \mathbf{u} \equiv \varepsilon(u, w) \text{ here.}$$

Thus equations (2.15) describe paths whose amplitude is $O(\varepsilon)$; it is therefore convenient to introduce

$$x = x_0 + \varepsilon X, \quad z = z_0 + \varepsilon Z,$$

where x_0 and z_0 are treated as fixed (and $O(1)$). The particle paths as $\varepsilon \to 0$ – the approximation used throughout this work on linear waves – are now described by

$$\frac{dX}{dt} \sim \delta\omega A \frac{\cosh \delta k z_0}{\sinh \delta k} E_0 + \text{c.c.}; \quad \frac{dZ}{dt} \sim -i\omega A \frac{\sinh \delta k z_0}{\sinh \delta k} E_0 + \text{c.c.},$$

where $E_0 = \exp\{i(kx_0 - \omega t)\}$. These may be integrated directly to give

$$X \sim i\delta A \frac{\cosh \delta k z_0}{\sinh \delta k} E_0 + \text{c.c.}, \quad Z \sim A \frac{\sinh \delta k z_0}{\sinh \delta k} E_0 + \text{c.c.},$$

where the arbitrary constant is set to zero in each case (so that $X = 0$ and $Z = 0$ when $A = 0$). This representation of the particle paths is usefully recast as

$$\left(\frac{X}{\delta \cosh \delta k z_0}\right)^2 + \left(\frac{Z}{\sinh \delta k z_0}\right)^2 = \frac{4|A|^2}{(\sinh \delta k)^2}, \quad 0 < z_0 \leq 1, \tag{2.16}$$

to leading order as $\varepsilon \to 0$.

The fluid particles, in the neighbourhood of the point (x_0, z_0), move on ellipses for which

$$\frac{\text{major axis}}{\text{minor axis}} = \delta \coth \delta k z_0,$$

(and which collapse to a point when $A = 0$, as one would expect). For long waves, $\delta \to 0$, the major and minor axes become, respectively,

$$4|A|/k \quad \text{and} \quad 4|A|z_0,$$

which describe different ellipses at different depths, and which approach the (degenerate) horizontal path as $z_0 \to 0$. On the other hand, for short waves ($\delta \to \infty$), the corresponding results are

$$4\delta|A|\mathrm{e}^{-\delta k(1-z_0)} \quad \text{and} \quad 4|A|\mathrm{e}^{-\delta k(1-z_0)},$$

whose ratio does not vary as z_0 varies. In this case the ellipses are all of the same eccentricity, but of decreasing size as z_0 decreases. (Indeed, in original physical variables, these trajectories become *circles* of decreasing radius as z_0 decreases; see Q2.4.)

We have found, therefore, that (in this first approximation) as the small-amplitude wave propagates on the surface, the fluid particles follow closed paths. Consequently there is no net transfer of material particles due to the passage of the wave (at least, at this order of approximation). In particular, near the bottom of the flow there is, predominantly, a horizontal *oscillatory* motion of the fluid as a long wave propagates overhead. Clearly, there is (at this order) no net flow of matter, but what of energy, for example?

2.1.2 Group velocity and the propagation of energy

We return to our first analysis in which we examined the solution initiated by a pure harmonic wave of fixed amplitude. This time, however, we construct the solution to equations (2.3) with the initial surface profile now given by

$$\eta = A(\alpha x)\mathrm{e}^{\mathrm{i}kx} + \text{c.c.},$$

where A is a complex-valued function. For $\alpha \to 0$, this describes (with k fixed) another pure harmonic wave, but here with a slowly varying amplitude; this is obviously an improvement on our simplest case. (Another generalisation is to allow for many – perhaps all – wave numbers, k; this choice is discussed in Q2.22.) The purpose is to obtain the appropriate solution of equations (2.3) which is uniformly valid as $\alpha \to 0$; see Section 1.4.2. The parameter, δ, is held fixed and, for simplicity, we consider only gravity waves (so the Weber number, W_e, is set to zero); the corresponding calculation for $W_\mathrm{e} \neq 0$ is described in Q2.26.

As before, it is convenient to introduce

$$E = \exp\{\mathrm{i}(kx - \omega t)\},$$

and then we seek a solution which also depends on the slow scales

$$X = \alpha x, \quad T = \alpha t. \tag{2.17}$$

The inclusion of T is a reasonable manoeuvre, since the solution is to be a wave that propagates in x and t, and the slow space scale ($X = \alpha x$, given in the initial datum) is therefore likely to have an associated slow time scale; of course, we lose nothing by including it (see also Q1.54). We seek a solution in the form

$$u = U(z, X, T; \alpha)E, \quad w = W(z, X, T; \alpha)E, \quad p = P(z, X, T; \alpha)E,$$

with

$$\eta = A(X, T; \alpha)E,$$

plus the complex conjugate in each case. The equations (2.3) yield

$$\left.\begin{array}{c} \mathrm{i}\omega U - \alpha U_T = \mathrm{i}kP + \alpha P_X; \quad \delta^2(\mathrm{i}\omega W - \alpha W_T) = P_z; \\ \mathrm{i}kU + \alpha U_X + W_z = 0, \end{array}\right\} \tag{2.18}$$

with

$$W(1, X, T; \alpha) = -\mathrm{i}\omega A + \alpha A_T; \quad P(1, X, T; \alpha) = A; \quad W(0, X, T, \alpha) = 0. \tag{2.19}$$

If an appropriate solution of these equations exists (at least, as $\alpha \to 0$), then uniform validity as $|kx - \omega t| \to \infty$ is guaranteed since the complete solution has been constructed with only E^1 (but then E^{-1} as well) included: no higher harmonics and secular terms can be generated (cf. equation (1.106) *et seq.*).

Directly from equations (2.18) we see that

$$\left(\mathrm{i}\omega - \alpha\frac{\partial}{\partial T}\right)U_z = \left(\mathrm{i}k + \alpha\frac{\partial}{\partial X}\right)P_z = \delta^2\left(\mathrm{i}k + \alpha\frac{\partial}{\partial X}\right)\left(\mathrm{i}\omega - \alpha\frac{\partial}{\partial T}\right)W,$$

and the relevant solution here satisfies

$$U_z = \delta^2\left(\mathrm{i}k + \alpha\frac{\partial}{\partial X}\right)W;$$

thus we obtain

$$W_{zz} + \delta^2\left(\mathrm{i}k + \alpha\frac{\partial}{\partial X}\right)^2 W = 0. \tag{2.20}$$

An asymptotic solution of the system (2.18) and (2.19) is sought in the form

$$Q \sim \sum_{n=0}^{\infty} \alpha^n Q_n, \quad \alpha \to 0. \tag{2.21}$$

where $Q \equiv U,\ W,\ P$, or A (and correspondingly for Q_n). Hence, with (2.21) used in (2.20), we obtain the equations

$$W_{0zz} - \delta^2 k^2 W_0 = 0; \quad W_{1zz} - \delta^2 k^2 W_1 = -2\mathrm{i}k\delta^2 W_{0X}, \tag{2.22}$$

and so on. From our previous calculation (Section 2.1), we have immediately that

$$W_0 = -\mathrm{i}\omega A_0 \left(\frac{\sinh \delta kz}{\sinh \delta k} \right), \tag{2.23}$$

where $c_p^2 = (\omega/k)^2 = (\tanh \delta k)/(\delta k)$; see equations (2.8) and (2.13). Now, for W_1, we obtain

$$W_{1zz} - \delta^2 k^2 W_1 = -2k\omega\delta^2 A_{0X} \left(\frac{\sinh \delta kz}{\sinh \delta k} \right) \tag{2.24}$$

which has the solution, for arbitrary $B_1(X, T)$,

$$W_1 = B_1 \sinh \delta kz - \delta\omega A_{0X} \frac{z \cosh \delta kz}{\sinh \delta k}, \tag{2.25}$$

which satisfies $W_1(0, X, T) = 0$. The other two boundary conditions at this order (see (2.19)) are

$$W_1 = -\mathrm{i}\omega A_1 + A_{0T} \quad \text{and} \quad P_1 = A_1 \text{ on } z = 1. \tag{2.26}$$

The first of these yields

$$\mathrm{i}\omega A_1 + A_{0T} = B_1 \sinh \delta k - \delta\omega A_{0X} \coth \delta k, \tag{2.27}$$

and the second uses (from equations (2.18))

$$\mathrm{i}kP_1 + P_{0X} = \mathrm{i}\omega U_1 - U_{0T} \quad \text{and} \quad \mathrm{i}kU_1 + U_{0X} + W_{1z} = 0 \text{ on } z = 1.$$

In Q2.3 we are led to the results

$$P_0 = \frac{\delta\omega^2}{k} A_0 \left(\frac{\cosh \delta kz}{\sinh \delta k} \right) \quad \text{and} \quad U_0 = \delta\omega A_0 \left(\frac{\cosh \delta kz}{\sinh \delta k} \right),$$

and so

$$\mathrm{i}kP_1 + \frac{\delta\omega^2}{k} A_{0X} \coth \delta k = -\delta\omega \left(\frac{\omega}{k} A_{0X} + A_{0T} \right) \coth \delta k - \frac{\omega}{k} W_{1z}$$

on $z = 1$. Hence from (2.25) and (2.26) we obtain

$$\mathrm{i}kA_1 + \delta\omega\left(2\frac{\omega}{k}A_{0X} + A_{0T}\right)\coth\delta k = -\,\delta\omega B_1\cosh\delta k + \frac{\delta\omega^2}{k}A_{0X}(\delta k + \coth\delta k) \qquad (2.28)$$

and upon the elimination of B_1 between equations (2.27) and (2.28) we finally have

$$A_{0T} + \frac{\omega}{2k}\{1 + \delta k(\coth\delta k - \tanh\delta k)\}A_{0X} = 0. \qquad (2.29)$$

We have derived the equation which describes the variation of the leading-order approximation to the amplitude, A_0; this equation does not involve A_1, because this is eliminated with B_1 when $\omega(k)$ is used. The general solution of (2.29) is

$$A_0 = F(X - c_g T),$$

where F is determined by the initial datum (on $T = 0$) and

$$c_g = \frac{\omega}{2k}\{1 + \delta k(\coth\delta k - \tanh\delta k)\}.$$

It is left as an exercise to confirm that this speed of propagation is indeed the group speed:

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} \quad \text{where} \quad \omega^2 = \frac{k}{\delta}\tanh\delta k.$$

In another context, we provide curves of c_p and c_g (for gravity waves) as functions of δk; see Figure 4.1.

Thus, although the individual waves move forward at the phase speed ($c_p = \omega/k$), the *envelope* or *group* moves at the group speed, c_g. Indeed, this general property of a wave is easily explained by a simple (but heuristic) argument involving two waves of the same amplitude but differing slightly in wave number; see Q2.28. (The reason for our rather lengthier approach, apart from presenting a more careful treatment, is to introduce the techniques that we shall require later. A neater approach, which avoids finding A_1, is described in Q2.30.) The inclusion of the surface tension leads to the corresponding result, but with c_g now the appropriate group speed deduced from the dispersion relation (2.9); see Q2.26.

The connection between the propagation of the group and the propagation of energy is now easily stated. It is a familiar result that the energy in a wave motion is proportional to the square of the amplitude of the wave; here, this implies that the energy is proportional to $|A_0|^2$. But we have just demonstrated that A_0 is a function of $(X - c_g T)$, and so the energy propagates at the group speed. There are many ways of presenting this argument in a more precise form, some of which are rehearsed in the exercises; here we describe one such method that uses the general notion of energy, as developed in Section 1.2.5. From equation (1.48), we have that the total energy (per unit horizontal area) in the flow is

$$\mathscr{E} = \int_b^h \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho g z \right) \mathrm{d}z,$$

written in physical variables. This is re-expressed using our nondimensional and scaled variables (described in Section 1.3) as

$$\mathscr{E} = \int_b^{1+\varepsilon\eta} \left\{ \frac{1}{2} \varepsilon^2 (\mathbf{u}_\perp \cdot \mathbf{u}_\perp + \delta^2 w^2) + z \right\} \mathrm{d}z,$$

where $\mathscr{E} \to \rho g h_0^2 \mathscr{E}$ describes the nondimensionalisation of $\mathscr{E}$. For our simple problem of one-dimensional wave propagation (with $b = 0$), this becomes

$$\mathscr{E} = \int_0^{1+\varepsilon\eta} \left\{ \frac{1}{2} \varepsilon^2 (u^2 + \delta^2 w^2) + z \right\} \mathrm{d}z,$$

where u and w are given by U_0 and W_0, respectively, to leading order as $\alpha \to 0$.

Our primary concern here is with the energy carried by, let us say, one period of the wave. Thus we first introduce

$$u \sim U_0 E + \bar{U}_0 E^{-1}, \quad w \sim W_0 E + \bar{W}_0 E^{-1},$$

where $E = \exp(\mathrm{i}k\xi)$, $\xi = x - c_p t$ and the overbar denotes the complex conjugate, and then we define the energy carried by just one period of the wave: this is

$$\int_0^{2\pi/k} \mathscr{E} \, \mathrm{d}\xi.$$

Consistent with the linearisation ($\varepsilon \to 0$) that we have so far adopted, we therefore obtain

$$\int_0^{2\pi/k} \mathscr{E}\,\mathrm{d}\xi$$

$$\sim \int_0^{2\pi/k}\int_0^1 \left\{\frac{1}{2}\varepsilon^2(U_0E+\bar{U}_0E^{-1})^2+\frac{1}{2}\varepsilon^2\delta^2(W_0E+\bar{W}_0E^{-1})^2+z\right\}\mathrm{d}z\,\mathrm{d}\xi,$$

where we have retained both the kinetic and potential contributions to the energy (although these are of different orders of magnitude as $\varepsilon \to 0$). It follows directly that we have

$$\int_0^{2\pi/k} \mathscr{E}\,\mathrm{d}\xi \sim \frac{2\pi}{k}\left\{\frac{1}{2}+\varepsilon^2\delta^2\omega^2|A_0|^2\int_0^1\left[\left(\frac{\cosh\delta kz}{\sinh\delta k}\right)^2+\left(\frac{\sinh\delta kz}{\sinh\delta k}\right)^2\right]\mathrm{d}z\right\},$$

where the term (π/k) represents the potential energy of the undisturbed fluid. (Because of our choice here of computing the energy in one period, this potential energy depends on the wavelength through k; it is quite usual, therefore, to define an *average energy* over one period:

$$\frac{k}{2\pi}\int_0^{2\pi/k} \mathscr{E}\,\mathrm{d}\xi.)$$

The second term is associated with the wave motion alone; because of our scaling, it is proportional to ε^2 (as $\varepsilon \to 0$) – which is to be expected – and it is also proportional to $|A_0|^2$, the required result.

Finally, we briefly describe the particular form that c_g takes for our water-wave problem, and what this implies for the propagation of waves. We already have (from equation (2.9)) the dispersion relation

$$\omega^2 = \left(\frac{k}{\delta}+\delta k^3 W_{\mathrm{e}}\right)\tanh\delta k.$$

It is left as an exercise (Q2.26) to show that the group speed may be written as

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}c_p\left\{\frac{1+3\delta^2k^2W_{\mathrm{e}}}{1+\delta^2k^2W_{\mathrm{e}}}+\frac{2\delta k}{\sinh 2\delta k}\right\},$$

where c_p is the phase speed. Then for long waves ($\delta k \to 0$) we see that $c_g \sim c_p$: the phase and group speeds are the same. On the other hand, for short waves ($\delta k \to \infty$), we see immediately that $c_g \sim 3c_p/2$: the group speed is *greater* than the phase speed. For the case of gravity waves only (so that $W_{\mathrm{e}} = 0$) we have

$$c_g = \frac{1}{2} c_p(1 + 2\delta k \operatorname{cosech} 2\delta k),$$

and hence $\frac{1}{2} < c_g/c_p < 1$; on the other hand, for infinitely deep water with surface tension (see Q2.27) we obtain

$$c_g = \frac{1}{2} c_p \left\{ \frac{1 + 3\delta^2 k^2 W_e}{1 + \delta^2 k^2 W_e} \right\};$$

that is, $\frac{1}{2} \leq c_g/c_p < \frac{3}{2}$ (where equality occurs when $W_e = 0$). These few observations are sufficient to explain, for example, the phenomenon represented earlier in Figure 2.2. Waves produced by a fixed disturbance in a moving stream can be stationary (provided that the speed of the stream is greater than the minimum speed of propagation of waves). The energy in the gravity component (the left-hand branch in Figure 2.1) is always propagated at a speed *less* than c_p, so these gravity waves appear *behind* the disturbance. The capillary waves, however, always have a group speed which is *greater* than c_p, and consequently the forward propagation of energy for this mode generates these waves *ahead* of the disturbance. (It turns out that the attenuation of gravity waves is much less than that for capillary waves – mainly because of their significantly different wavelengths; see Chapter 5 – so gravity waves are seen to extend much further behind the disturbance than capillary waves are seen ahead.)

2.1.3 Concentric waves on deep water

In Section 2.1 we mentioned some results that can be obtained for wave propagation, which is governed by the classical wave equation written in cylindrical coordinates. It is now our intention to describe the character of purely concentric gravity waves (initiated by a central disturbance) as they propagate over deep water. Of course, corresponding calculations are possible for any depth and with surface tension included, but it is sufficient, both to give a flavour of the results and also for our future work, to examine this one example. We start with the representation of the solution obtained from Q2.19:

$$\eta(r, t) = \int_0^\infty p\hat{f}(p) \cos\left(t\sqrt{\frac{p}{\delta}}\right) J_0(rp) \mathrm{d}p, \tag{2.30}$$

which satisfies $\eta(r, 0) = f(r)$ (with transform $\hat{f}(p)$) and $\eta_t(r, 0) = 0$. It is immediately evident that any useful description of the wave profile, η, based on the solution (2.30), requires some approach that will produce a simplification. To this end we choose to analyse the solution in the regions where $t^2/r \to \infty$, a choice that will look reasonable when we recast the problem so that we may invoke the method of stationary phase.

First, we express the Bessel function, $J_0(pr)$, in the familiar integral form

$$J_0(pr) = \frac{2}{\pi}\int_0^{\pi/2} \cos(pr\cos\theta)\mathrm{d}\theta, \tag{2.31}$$

and then write (2.30) as

$$\eta = \mathscr{R}\frac{1}{\pi}\int_0^{\infty}\int_0^{\pi/2} p\hat{f}(p)\left[\exp\left\{\mathrm{i}\left(t\sqrt{\frac{p}{\delta}} + pr\cos\theta\right)\right\}\right.$$
$$\left. + \exp\left\{\mathrm{i}\left(t\sqrt{\frac{p}{\delta}} - pr\cos\theta\right)\right\}\right]\mathrm{d}\theta\,\mathrm{d}p, \tag{2.32}$$

where $\mathscr{R}$ denotes the real part of the double integral. To proceed, we introduce a new integration variable (q) and a parameter (σ) defined by

$$q^2 = \frac{\delta r^2}{t^2}p \quad \text{and} \quad \sigma = \frac{t^2}{\delta r}, \tag{2.33}$$

respectively. The expression for the surface wave given in equation (2.32) may therefore be rewritten as

$$\eta = \mathscr{R}\frac{2t^4}{\pi\delta^2 r^4}\int_0^{\infty}\int_0^{\pi/2} q^3\hat{f}\left(\frac{q^2t^2}{\delta r^2}\right)[\exp\{\mathrm{i}\sigma(q + q^2\cos\theta)\}$$
$$+ \exp\{\mathrm{i}\sigma(q - q^2\cos\theta)\}]\mathrm{d}\theta\,\mathrm{d}q, \tag{2.34}$$

and we examine this by employing *Kelvin's method of stationary phase* to give the asymptotic behaviour of $\eta(r, t)$ as $\sigma \to \infty$ (that is, as $t^2/\delta r \to \infty$). This very powerful and widely used result states (see Q2.16) that

$$\phi(\sigma) = \int_{-\infty}^{\infty} f(q)\exp\{\mathrm{i}\sigma\alpha(q)\}\mathrm{d}q$$
$$\sim f(Q)\sqrt{\frac{2\pi}{\sigma|\alpha''(Q)|}}\exp\left\{\mathrm{i}\sigma\alpha(Q) + \mathrm{i}\frac{\pi}{4}\operatorname{sgn}\alpha''(Q)\right\} \tag{2.35}$$

as $\sigma \to +\infty$, where the point of *stationary phase* is defined by

$$\alpha'(Q) = 0;$$

the primes here denote derivatives with respect to Q, and sgn is the *signum* function (taking the values $+1$ or -1). Essentially the idea is that, for large σ, the integrand oscillates least rapidly near the point (or, perhaps, points) where $\alpha'(q) = 0$, and so this is where the dominant contribution will arise; elsewhere, rapid oscillations approximately cancel, although we might obtain a contribution from the end-points of the range since symmetry about these points is lost. The error in the behaviour given in (2.35) is $O(\sigma^{-1})$, in general, as $\sigma \to \infty$. This result is closely related to the *method of steepest descent*, and some standard references to these types of asymptotic evaluation are given in the section on Further Reading at the end of this chapter. We now apply (2.35) to the double integral (2.34), once in q and once in θ, and to both exponential terms.

First, in q, the points of stationary phase occur where

$$1 + 2q\cos\theta = 0; \quad 1 - 2q\cos\theta = 0,$$

which correspond, respectively, to the two exponental terms. However, since

$$0 \le q < \infty \quad \text{and} \quad 0 \le \theta \le \pi/2,$$

the dominant contribution will come only from the second term in the integral; that is, at

$$q = \frac{1}{2\cos\theta}.$$

The second derivative of the exponent, with respect to q, is then $-2\cos\theta$. Thus we have

$$\eta(r, t) \sim \mathscr{R}\frac{2t^4}{\pi\delta^2 r^4}\int_0^{\pi/2} (2\cos\theta)^{-3}\hat{f}\left(\frac{t^2}{4\delta r^2\cos^2\theta}\right)$$
$$\times\sqrt{\frac{\pi}{\sigma\cos\theta}}\exp\{\mathrm{i}(-\pi + \sigma/\cos\theta)/4\}\mathrm{d}\theta$$

which itself possesses a point of stationary phase where

$$\sin\theta = 0 \quad \text{or} \quad \theta = 0,$$

and so $q = \frac{1}{2}$. Since $\theta = 0$ occurs at the end of the range of integration, the method of stationary phase produces half the contribution represented in (2.35) (which there uses equal contributions from either side of $q = Q$).

The second derivative of the exponent, evaluated at $\theta = 0$, takes the value $\frac{1}{4}$ and so

$$\eta(r, t) \sim \mathscr{R}\frac{t^4}{8\pi\delta^2 r^4}\hat{f}\left(\frac{t^2}{4\delta r^2}\right)\sqrt{\frac{\pi}{\sigma}}\sqrt{\frac{8\pi}{\sigma}}e^{i\sigma/4} \quad (\text{as } \sigma \to \infty)$$
$$= \frac{t^2}{2\sqrt{2}\delta r^3}\hat{f}\left(\frac{t^2}{4\delta r^2}\right)\cos\left(\frac{t^2}{4\delta r}\right), \tag{2.36}$$

which is the asymptotic behaviour as $t^2/\delta r \to \infty$. How can we make use of this result?

Clearly, since the argument of the function $\hat{f}$ involves t^2/r^2, and $t^2/\delta r \to \infty$, we require to know the behaviour of $\hat{f}$ in order to describe $\eta(r, t)$. We consider the simple – and idealised – choice of initial disturbance given by

$$\eta(r, 0) = f(r) = \begin{cases} A, & 0 \le r < a \\ 0, & r \ge a, \end{cases}$$

where A is a constant; this describes a 'top-hat' profile which is used here to generate the outward propagating concentric wave. The transform of this function is

$$\hat{f}(p) = A\int_0^a rJ_0(pr)\mathrm{d}r = \frac{A}{p^2}\int_0^{pa} yJ_0(y)\mathrm{d}y$$
$$= \frac{A}{p^2}[yJ_1(y)]_0^{pa} = \frac{Aa}{p}J_1(pa),$$

where a standard identity between the J_0 and J_1 Bessel functions:

$$\frac{\mathrm{d}}{\mathrm{d}y}\{yJ_1(y)\} = yJ_0(y),$$

has been employed. Thus equation (2.36) becomes

$$\eta(r, t) \sim \sqrt{2}\frac{Aa}{r}J_1\left(\frac{at^2}{4\delta r^2}\right)\cos\left(\frac{t^2}{4\delta r}\right),$$

and we choose to interpret the limiting process $t^2/\delta r \to \infty$ as $t \to \infty$ at fixed r (and fixed δ).

Then, upon the use of the asymptotic behaviour

$$J_1(y) \sim \sqrt{\frac{2}{\pi y}}\cos\left(y - \frac{3}{4}\pi\right) \quad \text{as} \quad y \to +\infty,$$

we find that

$$\eta(r, t) \sim 4\frac{A\delta}{t}\sqrt{\frac{a}{\pi}}\cos\left(\frac{at^2}{4\delta r^2} - \frac{3}{4}\pi\right)\cos\left(\frac{t^2}{4\delta r}\right), \quad t \to \infty.$$

This describes, for example at fixed r, a wave whose amplitude decays like $1/t$ and for which the frequency increases; the general character of the wave is evident in Figure 2.3. This figure clearly demonstrates what is usually observed for cylindrical gravity waves: the wavelength decreases at any fixed radius or, equivalently, the wavelength increases outwards from the centre of the disturbance.

We have, thus far, presented only a rather brief introduction to the many elementary calculations that are possible in simple water-wave theory. As we mentioned earlier, other examples (such as waves on moving streams, standing waves and crossing waves) are explored in the exercises at the end of this chapter. We also take the opportunity there to expand on some of the results already discussed, and to describe alternative approaches to some of the standard problems. We now devote the rest of this section of the chapter to the study of a few slightly less routine calculations, but ones that begin to demonstrate the breadth and depth of what can be done even in the linear approximation. Furthermore, much of this will provide an excellent preparation for our work on the various nonlinear problems that we shall describe later.

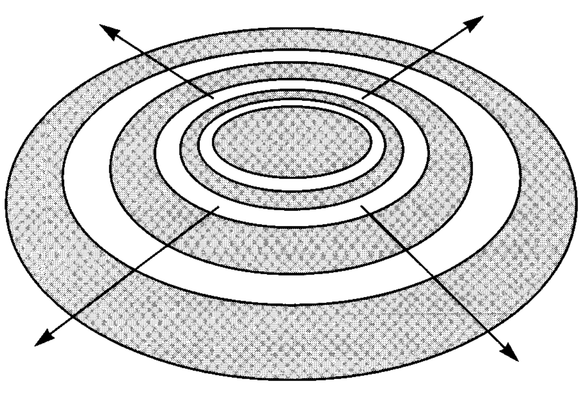


Figure 2.3. A representation of an outward propagating cylindrical (or concentric) gravity wave.

2.2 Wave propagation over variable depth

It is a matter of everyday experience that most water waves do not propagate over constant depth, whether they be in a man-made reservoir, a river, or the ocean. Therefore a useful extension to our studies is to examine the effects of incorporating variable depth. We start with the most straightforward problem of this type: a plane wave propagating in the x-direction, with a depth which also varies only in x (so that $b = b(x)$). From equations (2.1) we therefore obtain

$$u_t = -p_x, \quad \delta^2 w_t = -p_z, \quad u_x + w_z = 0,$$

with

$$w = \eta_t \quad \text{and} \quad p = \eta - \delta^2 W \eta_{xx} \quad \text{on } z = 1,$$

and

$$w = ub'(x) \quad \text{on } z = b(x).$$

In order to make the problem even more manageable, we simplify further by considering only long waves, so $\delta \to 0$, and then we are left with

$$\left.\begin{aligned} &u_t = -p_x, \quad p_z = 0, \quad u_x + w_z = 0, \\ &\text{with} \\ &w = \eta_t, \quad \text{and} \quad p = \eta \quad \text{on } z = 1, \\ &\text{and} \\ &w = ub'(x) \quad \text{on } z = b(x). \end{aligned}\right\} \tag{2.37}$$

These equations immediately yield

$$p = \eta, \quad b \le z \le 1,$$

and so

$$u_t + \eta_x = 0 \quad \text{with} \quad w = (1 - z)u_x + \eta_t,$$

since u is not a function of z (in this approximation). Finally, evaluating w on $z = b$, we obtain the pair of equations

$$u_t + \eta_x = 0; \quad \eta_t + (du)_x = 0, \tag{2.38}$$

where $d(x) = 1 - b(x)$ is the local depth. These equations, (2.38), are usually called the *linearised shallow water* equations and, upon the elimination of u, they give

$$\eta_{tt} - (d\eta_x)_x = 0, \tag{2.39}$$

the appropriate wave equation for the surface profile, $\eta(x, t)$. In some of our later work we shall examine the full governing equations, but incorporating a *slow* variation of the depth; for constant depth $d = 1$, we note that we recover the classical wave equation with propagation speeds ± 1 (cf. equation (2.10)). For our calculations with variable depth here we choose the example of propagation over a bed of constant slope. Thus we introduce

$$d(x) = \alpha(x_0 - x), \quad \alpha > 0, \quad x \leq x_0, \tag{2.41}$$

where the shoreline is to the right, at $x = x_0$ (in the absence of any surface disturbance); see Figure 2.4.

Before we proceed, however, we must add a word of caution: we cannot expect the results obtained in this calculation to be valid (or even meaningful) either as $d \to 0$ or as $d \to \infty$. Our original equations – the linearised shallow water equations, in particular – have been obtained under the assumption that d $(= 1 - b)$ is $O(1)$ (as $\varepsilon \to 0$ and $\delta \to 0$), and hence $d \to \infty$ is likely to be inadmissible, since this limit corresponds to short waves. Also, in this chapter, we are restricting the discussion to the linear approximation ($\varepsilon \to 0$), which is defined in terms of the ratio of a typical wave amplitude to a typical depth. At the shoreline, the depth decreases to zero, and so the (local) value of (amplitude/depth) will become large; this suggests that nonlinear terms cannot be

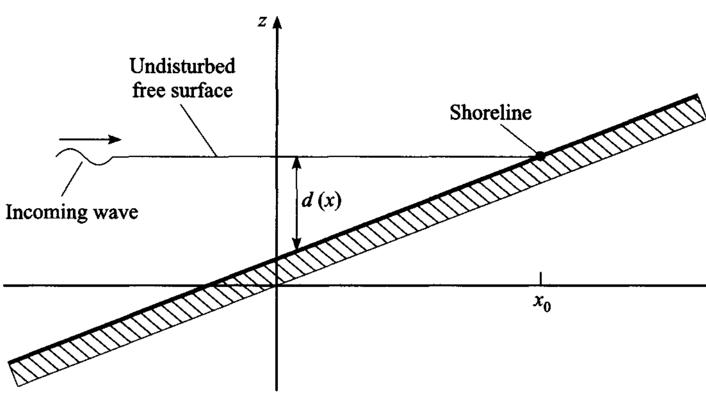


Figure 2.4. Defining sketch for a bed of constant slope; the shoreline is at $x = x_0$, and $d(x)$ is the depth below the undisturbed surface in $x < x_0$. The incoming wave from infinity moves from left to right.

ignored as $d \to 0$. With these provisos in mind, let us proceed with the analysis.

The wave equation for $\eta(x, t)$, (2.40), with $d(x)$ given by (2.41), becomes

$$\eta_{tt} - \alpha(x_0 - x)\eta_{xx} + \alpha\eta_x = 0, \tag{2.42}$$

and we seek a solution which is harmonic in t:

$$\eta = A(x)e^{-i\omega t} + \text{c.c.}, \tag{2.43}$$

where ω is a real constant (the frequency) and $A(x)$ is an amplitude function (which, in general, is complex). (As we have mentioned before, this type of solution can be used as the basis for more general solutions by introducing, for example, the Fourier transform.) Equation (2.42) therefore yields the differential equation for $A(x)$,

$$\alpha(x_0 - x)A'' - \alpha A' + \omega^2 A = 0,$$

which shows that we may take $A(x)$ to be real (but see below). It is convenient to treat A as a function of $x_0 - x = X$, say, so that

$$\alpha X A'' + \alpha A' + \omega^2 A = 0,$$

for $A(X)$, which we recognise is related to the Bessel equation. To confirm this, we now regard A as a function of

$$2\omega\sqrt{\frac{X}{\alpha}} = \chi,$$

and so we obtain

$$\chi A'' + A' + \chi A = 0, \tag{2.44}$$

where $A = A(\chi) = A(2\omega\sqrt{(x_0 - x)/\alpha})$. We observe that the shoreline is at $X = 0$ (so $\chi = 0$ there) and that the undisturbed water exists in $X > 0$. Equation (2.44) is the Bessel equation of zero order, and we now require the appropriate solution in $\chi > 0$.

The general solution for $A(\chi)$ is

$$A(\chi) = CJ_0(\chi) + DY_0(\chi),$$

where C and D are arbitrary (complex) constants. This solution contains a contribution (J_0) which is regular at the shoreline ($x = x_0$; that is, $\chi = 0$) since $J_0(\chi)$, as a power series, contains only even powers of χ (so only integer powers of $(x_0 - x)$ appear). On the other hand, the second part of the solution (Y_0) gives rise to a logarithmic singularity at $\chi = 0$, so we might expect that we should assign $D = 0$; we shall, however, retain this term for the present. The solution of our original equation, (2.42), therefore becomes

$$\eta(x,t)=\left\{CJ_0\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}\right)+DY_0\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}\right)\right\}\mathrm{e}^{-\mathrm{i}\omega t}+\text{c.c.},\quad(2.45)$$

and a natural next move is to examine the detailed character of this solution far away from the shore ($x_0-x\to\infty$) and close to the shoreline at $x=x_0$. (We observe that, in equation (2.45), it is quite acceptable to choose both C and D to be real, as we mentioned earlier.)

First, for large values of the argument χ, we use the standard results

$$(J_0,Y_0)\sim\sqrt{\frac{2}{\pi\chi}}(\cos(\chi-\pi/4),\sin(\chi-\pi/4))\quad\text{as }\chi\to+\infty,$$

and thus equation (2.45) yields

$$\eta(x,t)\sim\frac{1}{\sqrt{\pi\omega}}\left(\frac{\alpha}{x_0-x}\right)^{1/4}\left\{C\cos\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}-\frac{\pi}{4}\right)\right.$$
$$\left.+D\sin\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}-\frac{\pi}{4}\right)\right\}\mathrm{e}^{-\mathrm{i}\omega t}+\text{c.c.}$$

as $x_0-x\to+\infty$. This is usefully rewritten in the form

$$\eta(x,t)\sim\frac{1}{2\sqrt{\pi\omega}}\left(\frac{\alpha}{x_0-x}\right)^{1/4}\left[(C+D)\exp\left\{\mathrm{i}\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}-\omega t-\frac{\pi}{4}\right)\right\}\right.$$
$$\left.+(C-D)\exp\left\{-\mathrm{i}\left(2\omega\sqrt{\frac{x_0-x}{\alpha}}+\omega t-\frac{\pi}{4}\right)\right\}\right]+\text{c.c.}\quad(2.46)$$

which describes two wave components, one moving to the right and one to the left. The first exponential term (with the coefficient $C+D$) is a right-going wave; it is therefore the plane wave which is approaching the shoreline (see Figure 2.4). The second exponential term represents the left-going component, and this is therefore a wave which is *reflected from* the shoreline. In both components we observe that the speed of propagation is not constant. The speed can be determined by considering the lines of constant phase, defined by

$$2\omega\sqrt{\frac{x_0-x}{\alpha}}\pm\omega t=\text{constant}.$$

Along these lines we have

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\pm\sqrt{\alpha(x_0-x)}\quad(=\pm\sqrt{d(x)}),\qquad(2.47)$$

which shows that the characteristics for this propagation (drawn in (x, t)-space) are not straight lines; see Q2.33. We note, in passing, that the speed of propagation is the square root of the local depth (see equation (2.47)), which Q2.33 demonstrates is a general result. Furthermore, the wave decays as $x_0 - x \to \infty$; indeed, we see that the amplitude behaves like $(x_0 - x)^{-1/4}$ or $d^{-1/4}$ – a very famous result that we shall meet again later. (This is usually called *Green's law*; see Q2.34.) Finally, if we write the phase in the form

$$\frac{2\omega}{\sqrt{\alpha(x_0 - x)}}(x_0 - x) \pm \omega t,$$

we see that the wave number increases as $x \to x_0$, so the waves shorten as they approach the shore.

Near the shoreline $(x \to x_0)$ we make use of the familiar results

$$(J_0, Y_0) \sim (1, \frac{2}{\pi} \ln \chi) \quad \text{as } \chi \to 0^+,$$

and then equation (2.45) gives

$$\eta(x, t) \sim \left\{ C + \frac{2D}{\pi} \ln\left(2\omega \sqrt{\frac{x_0 - x}{\alpha}} \right) \right\} \mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}$$

as $x_0 - x \to 0^+$. As already mentioned, this exhibits the logarithmic behaviour at the shoreline; this singularity is removable only if $D = 0$ (and then the solution depends only on J_0). Apart from the presence of the singularity, the solution describes a wave which oscillates in time (t) at the shoreline $(x = x_0)$.

We may now collect together these various observations and hence describe the general nature of the solution (2.45) and, in particular, adumbrate its shortcomings. A reasonable problem in this context, we might suppose, is to prescribe an incoming wave at infinity which then moves towards the shoreline. To do this we must know the frequency of the wave and its amplitude; the frequency is no problem (it is ω), but at infinity its amplitude is zero – not what we want. This difficulty is associated with the inability of our shallow-water equations to describe accurately the effects of deep water – which is no surprise in view of the usual name for these equations! Furthermore, even if we are prepared to gloss over this problem, a reflected wave will also exist for all $x_0 - x$ unless we set $C = D$ $(\neq 0)$; see equation (2.46). But now the coefficient of Y_0 is non-zero, so we have a singularity at the shoreline. Of course, the presence of this singularity is presumably indicative of the failure of the linear equations to cope with the increase in amplitude near the shoreline. We would expect, based on everyday experience, that the incoming wave will (almost always) break at the shoreline; a linear wave theory cannot accommodate this phenomenon. If we do set $D = 0$ then the singularity is removed, but both incoming and reflected waves exist everywhere in $\chi > 0$; the shoreline has become a perfect reflector: no singularity is necessary in this solution to account for the difference in energy between the incoming and outgoing waves.

2.2.1 Linearised gravity waves of any wave number moving over a constant slope

In the previous calculation we simplified the problem by considering only long waves, so $\delta \to 0$; this led us to a form of the so-called shallow water equations. As we have seen, the solution in this case is not wholly satisfactory. We now consider the problem of plane gravity waves (as above) *without* invoking the long-wave assumption. Of course, we are still operating within the confines of the linear theory, so again we cannot expect to be able to cope with the singularity at the shoreline (unless, perhaps, we happen upon a special pure-reflecting solution, similar to that described earlier).

We take equations (2.1), with $W_e = 0$, $\eta = \eta(x, t)$ and $b = b(x)$; these are

$$\left.\begin{aligned} &u_t = -p_x; \quad \delta^2 w_t = -p_z; \quad u_x + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad \text{and} \quad p = \eta \quad \text{on } z = 1, \\ &\text{and} \\ &w = ub'(x) \quad \text{on } z = b(x). \end{aligned}\right\} \tag{2.48}$$

Again, for this particular calculation (with a constant slope), we require a depth variation which is linear in x but, for convenience, we translate the coordinates so that the shoreline is now along $x = 1/\alpha$, and so we write

$$d(x) = 1 - b(x) = 1 - \alpha x, \quad \alpha > 0, \quad x \leq 1/\alpha;$$

that is, $\alpha x_0 = 1$ in (2.41). We seek solutions (cf. equations (2.4), (2.5)) in the form

$$u = U(x, z)\mathrm{e}^{-\mathrm{i}\omega t}, \quad p = P(x, z)\mathrm{e}^{-\mathrm{i}\omega t}, \quad w = W(x, z)\mathrm{e}^{-\mathrm{i}\omega t}$$

with

$$\eta(x, t) = A(x)e^{-i\omega t},$$

plus the complex conjugate in each case. Equations (2.48) then become

$$i\omega U = P_x; \quad i\omega\delta^2 W = P_z; \quad U_x + W_z = 0, \tag{2.49}$$

with

$$W(x, 1) = -i\omega A; \quad P(x, 1) = A, \tag{2.50}$$

and

$$W(x, b(x)) = \alpha U(x, b(x)) \text{ where } b(x) = \alpha x. \tag{2.51}$$

We see immediately that

$$W_{zz} + U_{xz} = 0 \quad \text{and} \quad i\omega U_{xz} = P_{xxz} = i\omega\delta^2 W_{xx},$$

so $W(x, z)$ satisfies Laplace's equation

$$W_{zz} + \delta^2 W_{xx} = 0; \tag{2.52}$$

this corresponds directly to the alternative formulation of this problem in terms of the velocity potential ϕ (which then satisfies the same Laplace equation; see Q2.5).

On the basis of our previous experience (again see Q2.5), we seek a solution by the method of separation of variables:

$$W(x, z) = \sum_n X_n(x) Z_n(z). \tag{2.53}$$

Then, for each n, equation (2.52) is replaced by the pair of equations

$$Z_n'' - \lambda_n \delta^2 Z_n = 0; \quad X_n'' + \lambda_n X_n = 0, \tag{2.54}$$

where λ_n is a parameter (an *eigenvalue*) yet to be determined. Further, also based on our earlier work (see, for example, Section 2.1), a reasonable choice for λ_n is

$$\lambda_n = k_n^2 \ (> 0)$$

and then the solution for Z_n becomes

$$Z_n = C_n \exp(\delta k_n z) + D_n \exp(-\delta k_n z),$$

where C_n and D_n are arbitrary constants. However, the depth increases indefinitely (as $x \to -\infty$ where $z \to -\infty$), so a bounded solution is possible only if $D_n = 0$. The complete solution for the nth component of W, W_n say, is therefore

$$W_n(x, z) = \{A_n \exp(\mathrm{i}k_n x) + B_n \exp(-\mathrm{i}k_n x)\} \exp(\delta k_n z),$$

where C_n has been subsumed into the new arbitrary (complex) constants A_n and B_n.

If, for a moment, we retain only one term in the expansion (2.53) then we find (by integrating P_z in equations (2.49) with respect to z) that

$$P(x, z) = \frac{\mathrm{i}\omega\delta}{k_n}\{A_n \exp(\mathrm{i}k_n x) + B_n(-\mathrm{i}k_n x)\}\{\exp(\delta k_n z) - \exp(\delta k_n)\} + A(x). \tag{2.55}$$

The two boundary conditions on W require

$$\left.\begin{aligned} &\{A_n \exp(\mathrm{i}k_n x) + B_n \exp(-\mathrm{i}k_n x)\} \exp(\delta k_n) = -\mathrm{i}\omega A \\ \text{and} \\ &W(x, \alpha x) = \alpha U(x, \alpha x) = -\frac{\mathrm{i}\alpha}{\omega} P_x(x, \alpha x), \end{aligned}\right\} \tag{2.56}$$

where the partial derivative with respect to the first argument in $P(x, z)$ is implied, and P is given by (2.55). It should be clear that, upon eliminating $A(x)$ between the two equations in (2.56), we shall derive an identity involving terms

$$\exp(\mathrm{i}k_n x), \quad \exp(-\mathrm{i}k_n x) \quad \text{and} \quad \exp(\alpha\delta k_n x)$$

which cannot possibly be satisfied unless $A_n = B_n = 0$; we have apparently reached an impasse with this approach. However Hanson (1926), and others after him, made use of an important observation which allows some progress, at least for certain α.

At the heart of this discovery is the realisation that we might hope to satisfy all the boundary conditions if there can be arranged some appropriate symmetry between the x and z dependencies. In particular we seek a symmetry that will allow terms in x and z, when evaluated on the bottom ($z = b(x) = \alpha x$), to become essentially the same (but only for certain α). To demonstrate this idea, let us consider the simplest example of this type by using just two terms in the expansion (2.53). We write (cf. (2.54)), for $k > 0$,

$$Z_1'' - \delta^2 k^2 Z_1 = 0; \quad X_1'' + k^2 X_1 = 0,$$

so that we obtain the solution

$$W_1 = (A_1 \mathrm{e}^{\mathrm{i}kx} + B_1 \mathrm{e}^{-\mathrm{i}kx})\mathrm{e}^{\delta kz},$$

which is bounded as $z \to -\infty$. For the second component we write

$$Z_2'' + \delta^2 k^2 Z_2 = 0; \quad X_2'' - k^2 X_2 = 0,$$

and now we obtain

$$W_2 = (A_2 e^{i\delta k z} + B_2 e^{-i\delta k z}) e^{kx},$$

which is bounded as $x \to -\infty$. (Remember that the solution we seek is to be in $x \leq 1/\alpha$.) The solution

$$W = W_1 + W_2$$

therefore incorporates oscillatory structures in both x and z, which both decay (as $z \to -\infty$ and $x \to -\infty$, respectively).

As before, we first determine $P(x, z)$; this gives directly (as above, from equations (2.49))

$$\begin{aligned} P(x, z) = \frac{i\omega\delta}{k}(A_1 e^{ikx} + B_1 e^{-ikx})(e^{\delta k z} - e^{\delta k}) \\ + \frac{\omega\delta}{k}\{A_2(e^{i\delta k z} - e^{i\delta k}) - B_2(e^{-i\delta k z} - e^{-i\delta k})\}e^{kx} + A(x). \end{aligned}$$

The boundary conditions on W yield (from equations (2.50))

$$-i\omega A(x) = (A_1 e^{ikx} + B_1 e^{-ikx}) e^{\delta k} + (A_2 e^{i\delta k} + B_2 e^{-i\delta k}) e^{kx}, \tag{2.57}$$

and from equations (2.49) and (2.51):

$$W(x, \alpha x) = \alpha U(x, \alpha x) = -\frac{i\alpha}{\omega} P_x(x, \alpha x)$$

that is,

$$\begin{aligned} (A_1 e^{ikx} + B_1 e^{-ikx}) e^{\alpha\delta k x} + (A_2 e^{i\alpha\delta k x} + B_2 e^{-i\alpha\delta k x}) e^{kx} \\ = i\alpha\delta(A_1 e^{ikx} - B_1 e^{-ikx})(e^{\alpha\delta k x} - e^{\delta k}) \\ - i\alpha\delta\{A_2(e^{i\alpha\delta k x} - e^{i\delta k}) - B_2(e^{-i\alpha\delta k x} - e^{-i\delta k})\}e^{kx} - \frac{i\alpha}{\omega}A'(x). \end{aligned} \tag{2.58}$$

Finally, $A'(x)$ is obtained from equation (2.57). However, it is already clear that a consistent identity (for other than $A_1 = B_1 = A_2 = B_2 = 0$) is possible only if equation (2.58) involves terms $\exp(\pm ikx)$ and $\exp(kx)$, at most. This condition is satisfied if the slope of the bottom is such that $\alpha\delta = 1$. With this choice, then equation (2.58) (with A' from (2.57)) becomes

$$
\begin{aligned}
&(A_1E + B_1E^{-1} + A_2E + B_2E^{-1})\mathrm{e}^{kx} \\
&\quad = \mathrm{i}(A_1E - B_1E^{-1})(\mathrm{e}^{kx} - \mathrm{e}^{\delta k}) - \mathrm{i}\{A_2(E - \mathrm{e}^{\mathrm{i}\delta k}) - B_2(E^{-1} - \mathrm{e}^{-\mathrm{i}\delta k})\}\mathrm{e}^{kx} \\
&\qquad + \frac{\mathrm{i}k}{\delta\omega^2}(A_1E - B_1E^{-1})\mathrm{e}^{\delta k} + \frac{k}{\delta\omega^2}(A_2\mathrm{e}^{\mathrm{i}\delta k} + B_2\mathrm{e}^{-\mathrm{i}\delta k})\mathrm{e}^{kx}, \quad (2.59)
\end{aligned}
$$

where we have written $E = \exp(\mathrm{i}kx)$.

In order that equation (2.59) is an identity for arbitrary x, we require the following five equations to hold (each arising from the coefficient of the term given to the left); these are

$$
\begin{array}{ll}
E\mathrm{e}^{kx}: & A_1 + A_2 = \mathrm{i}(A_1 - A_2); \\[2ex]
E^{-1}\mathrm{e}^{kx}: & B_1 + B_2 = \mathrm{i}(B_2 - B_1); \\
E: & iA_1\left(\dfrac{k}{\delta\omega^2} - 1\right)\mathrm{e}^{\delta k} = 0; \\
E^{-1}: & \mathrm{i}B_1\left(\dfrac{k}{\delta\omega^2} - 1\right)\mathrm{e}^{\delta k} = 0; \\
\mathrm{e}^{kx}: & (1+\mathrm{i})A_2\mathrm{e}^{\mathrm{i}\delta k} = -(1-\mathrm{i})B_2\mathrm{e}^{-\mathrm{i}\delta k},
\end{array}
$$

for the five unknowns A_1, A_2, B_1, B_2, and $\omega(k)$. It is evident that the solution of this system is

$$\omega^2 = k/\delta \qquad (2.60)$$

together with

$$A_2 = \mathrm{i}A_1, \quad B_2 = -\mathrm{i}B_1, \quad \text{and} \quad B_1 = \mathrm{i}A_1\mathrm{e}^{2\mathrm{i}\delta k},$$

where we recognise (2.60) as the dispersion relation for short gravity waves (or deep water) (see equation (2.13) *et seq.*), giving the speeds of propagation $c_p = \pm 1/\sqrt{\delta k}$. The surface wave, from equation (2.57) with the factor $\exp(-\mathrm{i}\omega t)$ reintroduced, becomes

$$\eta(x, t) = A_0\{\mathrm{e}^{\mathrm{i}(kx - \omega t - \delta k)} + \mathrm{e}^{-\mathrm{i}(kx + \omega t - \delta k)} + (1+\mathrm{i})\mathrm{e}^{k(x-\delta) - \mathrm{i}\omega t}\} + \text{c.c.}, \quad (2.61)$$

where A_0 is an arbitrary (complex) constant, which plays the rôle of the A_1 used earlier. We see that this solution, (2.61), possesses a number of important and special properties. First, the solution is everywhere regular in $x \leq 1/\alpha$ ($= \delta$ since $\alpha\delta = 1$): there is no singularity at the shoreline, so there is some sort of perfect reflection here (cf. solution (2.45) with $D = 0$); indeed, at the shoreline ($x = \delta = 1/\alpha$) we have

$$\eta = (3 + \mathrm{i})A_0\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}$$

The incoming and outgoing wave components travel at a fixed speed (for given k), which is certainly at variance with our previous result (equation (2.47)). Finally, the wave at infinity ($x \to -\infty$) exhibits a nonvanishing amplitude in both components. Clearly the two ingredients in this solution which make it particularly distinctive are (a) that it contains a contribution which does *not* represent a travelling wave (the term $\exp\{k(x-\delta) - \mathrm{i}\omega t\}$) and (b) that the amplitudes of the two waves at infinity are nonzero (but proportional; cf. equation (2.46)). Nevertheless, even though we have described a very special – and intriguing – solution of the governing (linear) equations, this does provide the basis for constructing more general and useful solutions (which may be investigated through the references in the Further Reading at the end of this chapter).

As a final comment on this solution, we briefly return to the assumption that made all this possible: the choice of slope with $\alpha\delta = 1$. It is reasonable to ask whether other choices of α lead to similar – or at least analogous – results. In order to describe what can be done, it is convenient first to write the bottom boundary condition

$$w = ub'(x) \quad \text{on} \quad z = b(x) = \alpha x$$

in the form

$$w\delta\cos\beta - u\sin\beta = 0 \quad \text{on} \quad z\delta\cos\beta - x\sin\beta = 0$$

where $\alpha\delta = \tan\beta$. The case that we have presented then corresponds to $\beta = \pi/4$. The generalisation is to angles $\beta = \pi/2n$ ($n = 2, 3, \ldots$) where, for increasing n, there is a progressively increasing number of terms in the series (2.53), which are required to ensure that all the boundary conditions are satisfied; see Q2.36.

2.2.2 *Edge waves over a constant slope*

We now turn to a brief consideration of an altogether new phenomenon: the *edge wave*. It turns out that the linear equations (with or without the long-wave assumption) admit a solution which describes a wave which propagates *parallel* to the shoreline. In our notation, these waves propagate in the y-direction (sometimes called the *longshore* coordinate) and, as we shall demonstrate, their amplitude decays exponentially away from the shoreline (that is, as $x \to -\infty$); they are therefore usually called *trapped waves*.

We start with equation (2.1) but, as before, our interest is in gravity waves only, and so we set W (that is, W_e) = 0; then, with the long-wave assumption $\delta \to 0$ (used here for simplicity), we have

$$\left.\begin{aligned} &u_t = -p_x; \quad v_t = -p_y; \quad p_z = 0; \quad u_x + v_y + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1, \\ &\text{and} \\ &w = ub'(x) \quad \text{on} \quad z = b(x). \end{aligned}\right\} \tag{2.62}$$

We choose the same depth variation as used in Section 2.2.1, so $b(x) = \alpha x$ with $x \leq 1/\alpha$, where the undisturbed shoreline is along $x = 1/\alpha$. We seek harmonic waves that are propagating in the y-direction, and thus we set

$$\left.\begin{aligned} &u = U(x,z)E, \quad v = V(x,z)E, \quad w = W(x,z)E \\ &\text{with} \\ &p = P(x,z)E \quad \text{and} \quad \eta = A(x)E, \end{aligned}\right\} \tag{2.63}$$

where $E = \exp\{\mathrm{i}(ly - \omega t)\}$, plus the complex conjugate in each case. Equations (2.62) therefore become

$$\mathrm{i}\omega U = P_x; \quad \omega V = lP; \quad P_z = 0; \quad U_x + \mathrm{i}lV + W_z = 0,$$

with

$$W = -\mathrm{i}\omega A \quad \text{and} \quad P = A \quad \text{on} \quad z = 1,$$

and

$$W = \alpha U \quad \text{on} \quad z = \alpha x.$$

Consequently we have that

$$P(x,z) = A(x), \quad 1 \geq z \geq x, \quad x \leq 1/\alpha$$

and hence

$$U = -\frac{\mathrm{i}}{\omega}A', \quad V = \frac{l}{\omega}A;$$

thus

$$W = \frac{\mathrm{i}}{\omega}(l^2 A - A'')(1 - z) - \mathrm{i}\omega A.$$

The final boundary condition on W then yields

$$(1 - \alpha x)(A'' - l^2 A) - \alpha A' + \omega^2 A = 0$$

for $A(x)$. It is clearly convenient to regard $A = A(1 - \alpha x)$ and then, with $X = 1 - \alpha x$, we have

$$XA'' + A' + \left(\frac{\omega^2}{\alpha^2} - l^2 X\right)A = 0$$

which can be put into a standard form if we now write

$$A(X) = \mathrm{e}^{-lX} L(2lX). \tag{2.64}$$

The equation for $L(Y)$, with $Y = 2lX$, is therefore

$$YL'' + (1 - Y)L' + \gamma L = 0, \tag{2.65}$$

where

$$\gamma = \frac{1}{2}\left(\frac{\omega^2}{\alpha^2 l} - 1\right).$$

Now we recognise equation (2.65) as the equation that has as its solutions the *Laguerre polynomials*, $L_n(Y)$, whenever $\gamma = n$ $(n = 0, 1 \ldots)$. These are the only solutions of (2.65) which lead to a bounded solution for $A(X)$ in $x \leq 1/\alpha$; that is, for $X \geq 0$ (with $l > 0$). (In general, $A(X)$ is a linear combination of e^{-lX} and e^{lX} as $X \to \infty$; the Laguerre polynomials are those solutions for which the term e^{lX} is absent.) The dispersion relation for these waves is

$$\omega^2 = \alpha^2 l(2n + 1),$$

and we write the solution $L_n(Y)$ in the usual form

$$L_n(Y) = \mathrm{e}^Y \frac{\mathrm{d}^n}{\mathrm{d}Y^n}(Y^n \mathrm{e}^{-Y}), \quad n = 0, 1, 2 \ldots.$$

The problem of finding the edge waves has therefore been reduced to a familiar exercise in the theory of eigenmodes and orthogonal polynomials. The first three modes (for $\omega > 0$) are

$$\begin{aligned}
n = 0:&\quad \omega = \alpha\sqrt{l}, &\quad L_0 &= 1;\\
n = 1:&\quad \omega = \alpha\sqrt{3l}, &\quad L_1 &= 1 - Y = 1 - 2lX;\\
n = 2:&\quad \omega = \alpha\sqrt{5l}, &\quad L_2 &= 2 - 4Y + Y^2 = 2 - 8lX + 4l^2X^2,
\end{aligned}$$

and these then lead to surface profiles such as

$$\eta(x, t) = A_0 \mathrm{e}^{-l(1-\alpha x)} \mathrm{e}^{\mathrm{i}(ly - \alpha\sqrt{l}t)} + \text{c.c.} \quad (n = 0),$$

where A_0 is an arbitrary complex constant. By virtue of the general form exhibited in equation (2.64), where L is a Laguerre polynomial, all these modes decay exponentially as $x \to -\infty$.

To conclude, we make two observations. First, the dispersion relation is quite different from the others that we have encountered so far for gravity waves. We see that the frequency, ω, increases as the wave number (l) increases and, crucially, it also depends on the slope of the bottom (which, remember, is a slope in x – not y). Indeed, this dependence of ω on the slope (α) leads to the second point: if the bottom is flat, so that $\alpha = 0$, then $\omega = 0$ and no edge wave exists at all.

These waves are often generated by wind stresses (due to the passage of a storm, for example) if this disturbance moves parallel to the shoreline. They are of some significance because their largest amplitude occurs at the shoreline, and therefore they will contribute to the total *run-up* (the highest point reached by a wave on a beach).

2.3 Ray theory for a slowly varying environment

Many of the more general properties of water waves, some of which we have mentioned already, can be explored more fully if we examine propagation over a slowly varying depth or current. The restriction to a slowly varying environment – depth or current or both – enables us to exploit an asymptotic formulation without recourse to other assumptions (other than under the present umbrella of linearisation). Not surprisingly, water waves behave in a manner similar to light: the (slowly) varying conditions give rise to changes in wave number and phase speed, and so the waves, as they propagate, generally suffer refraction. It is possible to describe these and other phenomena in some detail; the results are usually collected together as *ray theory* (which is another name for the familiar *theory of geometrical optics*). In our presentation here we shall first describe the effects of a slowly varying depth, and then turn to a new area of study: slowly varying currents.

In contrast to much of our earlier work, we shall develop the theory of linear irrotational motion over a slowly varying depth from the point of view of Laplace's equation. We shall consider here only gravity waves (so we set the Weber number, W, to zero); then from equations (2.1), Q1.38 and Q2.5 we obtain

$$\phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0, \tag{2.66}$$

with

$$\phi_z = \delta^2 \eta_t \quad \text{and} \quad \phi_t + \eta = 0 \quad \text{on} \quad z = 1, \tag{2.67}$$

and

$$\phi_z = \delta^2 (\phi_x b_x + \phi_y b_y) \quad \text{on} \quad z = b(x, y). \tag{2.68}$$

(Q2.5 is the most useful guide to these equations, requiring only the addition of the second horizontal coordinate (y) and the variable depth.) It is convenient, first, to reduce the two boundary conditions on $z = 1$ to a single condition. Since these are evaluated on $z = 1$, we may take derivatives (as appropriate) in x, y, or t; in particular we can eliminate η altogether to give

$$\phi_z + \delta^2 \phi_{tt} = 0 \quad \text{on} \quad z = 1. \tag{2.69}$$

We then determine η at the end of the calculation as $(-\phi_t)$ on $z = 1$. Finally, the bottom topography is chosen to be

$$b(x, y) = B(\alpha x, \alpha y)$$

so that equation (2.68) now becomes

$$\phi_z = \alpha \delta^2 (\phi_x B_X + \phi_y B_Y) \quad \text{on} \quad z = B(X, Y), \tag{2.70}$$

where $X = \alpha x$, $Y = \alpha y$. The compact form of this problem (equations (2.66), (2.69), and (2.70)), coupled with the asymptotic approach that we introduce below, will confirm the usefulness of the Laplace formulation here.

The analysis that we now present is driven by the choice of depth variation for which $\alpha \to 0$. It is clear that we must use the variables

$$X = \alpha x, \quad Y = \alpha y, \quad T = \alpha t, \tag{2.71}$$

the scaled time (T) being required as we have seen before; cf. Q1.54 and equations (2.17). In addition, we shall need a suitable way of describing the harmonic wave which propagates – albeit with slowly varying parameters – on the O(1) time and space scales. The neatest device in this type of problem is to introduce a (real) phase function, θ, defined by

$$\left.\begin{aligned} &\nabla\theta = \mathbf{k}(X, Y, T), \text{ that is } (\theta_x, \theta_y) = \{k(X, Y, T), l(X, Y, T)\} \\ &\text{with} \\ &\theta_t = -\omega(X, Y, T), \end{aligned}\right\} \tag{2.72}$$

which is precisely the approach adopted in Q1.54. We therefore obtain the transformation

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \equiv \alpha\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right) + (k, l)\frac{\partial}{\partial \theta}; \quad \frac{\partial}{\partial t} \equiv \alpha\frac{\partial}{\partial T} - \omega\frac{\partial}{\partial \theta}, \tag{2.73}$$

and their use in equations (2.66), (2.69) and (2.70) yields

$$\phi_{zz} + \delta^2\{(k^2 + l^2)\phi_{\theta\theta} + 2\alpha(k\phi_{\theta X} + l\phi_{\theta Y}) \\ + \alpha(k_x + l_y)\phi_\theta + \alpha^2(\phi_{XX} + \phi_{YY})\} = 0,$$

with

$$\phi_z + \delta^2(\omega^2\phi_{\theta\theta} - 2\alpha\omega\phi_{\theta T} - \alpha\omega_T\phi_\theta + \alpha^2\phi_{TT}) = 0 \text{ on } z = 1;$$

and

$$\phi_z = \alpha\delta^2(kB_X + lB_Y)\phi_\theta + \alpha^2(B_X\phi_X + B_Y\phi_Y) \text{ on } z = B(X, Y),$$

where we now regard $\phi = \phi(\theta, X, Y, T, z; \alpha)$.

The solution that we seek takes the form of a single harmonic wave

$$\phi = a(X, Y, T, z; \alpha)\mathrm{e}^{i\theta} + \text{c.c.},$$

and so the problem for the amplitude function, a, becomes

$$a_{zz} + \delta^2\{-(k^2 + l^2)a + 2\mathrm{i}\alpha(ka_X + la_Y) \\ + \mathrm{i}\alpha(k_X + l_Y)a + \alpha^2(a_{XX} + a_{YY})\} = 0,$$

with

$$a_z + \delta^2(-\omega^2 a - 2\mathrm{i}\alpha\omega a_T - \mathrm{i}\alpha\omega_T a + \alpha^2 a_{TT}) = 0 \text{ on } z = 1,$$

and

$$a_z = \mathrm{i}\alpha\delta^2(kB_X + lB_Y)a + \alpha^2\delta^2(B_X a_X + B_Y a_Y) \text{ on } z = B(X, Y).$$

To proceed, we assume that a can be expressed as the asymptotic expansion

$$a \sim \sum_{n=0}^{\infty} \alpha^n a_n(X, Y, T, z) \quad \text{as } \alpha \to 0,$$

and then the problem for a_0 is simply

$$a_{0zz} - \delta^2(k^2 + l^2)a_0 = 0 \tag{2.74}$$

with

$$a_{0z} = \delta^2\omega^2 a_0 \text{ on } z = 1; \quad a_{0z} = 0 \text{ on } z = B. \tag{2.75}$$

We write

$$\sigma(X, Y, T) = \delta\sqrt{k^2 + \ell^2} \quad (> 0) \tag{2.76}$$

and then the solution for a_0 is immediately

$$a_0 = A_0 \cosh\{\sigma(z - B)\}, \tag{2.77}$$

where $A_0(X, Y, T)$ is, at this stage, arbitrary, and the dispersion relation is

$$\omega^2 = \frac{\sigma}{\delta^2}\tanh\{\sigma(1 - B)\}; \tag{2.78}$$

cf. equations (2.9) and (2.13), when we set $\sigma = \delta k$ (that is, $l = 0$) and $B = 0$ (constant depth).

The problem for a_1 is obtained from the equations that arise at $\mathrm{O}(\alpha)$; these are

$$a_{1zz} - \delta^2(k^2 + l^2)a_1 = -\mathrm{i}\delta^2\{2(ka_{0X} + la_{0Y}) + (k_X + l_Y)a_0\}, \tag{2.79}$$

with

$$a_{1z} - \delta^2\omega^2 a_1 = -\mathrm{i}\delta^2(2\omega a_{0T} + \omega_T a_0) \text{ on } z = 1 \tag{2.80}$$

and

$$a_{1z} = \mathrm{i}\delta^2(kB_X + lB_Y)a_0 \text{ on } z = B(X, Y). \tag{2.81}$$

Now, the main purpose in examining the solution for a_1 is in order to determine the A_0 (the amplitude function in equation (2.77)) which ensures uniform validity of the asymptotic expansion. This could be done by simply solving for a_1 directly and examining the nature of this solution (cf. Section 2.1.2), but since we do not require a_1 itself, here we develop the necessary condition on A_0 by finding the condition that a solution for a_1 exists; see Q2.30. To accomplish this we multiply equation (2.79) by a_0 and then integate over z (for $1 \geq z \geq B$), using the boundary conditions on a_0 and a_1 as required. (That such a condition must exist is related to an important idea in the theory of differential and integral equations: the *Fredholm alternative*. The particular method that we choose to use here can be interpreted as an application of *Green's formula*. However, knowledge of these two results is not a prerequisite to an understanding of the presentation that we now give.)

Equation (2.79) is multiplied by a_0 to give

$$a_0 a_{1zz} - \sigma^2 a_0 a_1 = -\mathrm{i}\delta^2\{2a_0(ka_{0X} + la_{0Y}) + (k_X + l_Y)a_0^2\}, \tag{2.82}$$

(where σ is given by equation (2.76)), which is then integrated with respect to z. The first term gives

$$\int_B^1 a_0 a_{1zz}\mathrm{d}z = [a_0 a_{1z}]_B^1 - \int_B^1 a_{0z}a_{1z}\mathrm{d}z = [a_0 a_{1z} - a_{0z}a_1]_B^1 + \int_B^1 a_{0zz}a_1\mathrm{d}z,$$

so the left-hand side of equation (2.82) becomes

$$[a_0\{\delta^2\omega^2 a_1 + \mathrm{i}\delta^2(2\omega a_{0T} + \omega_T a_0)\} - \delta^2\omega^2 a_0 a_1]_{z=1}$$
$$- [a_0 \mathrm{i}\delta^2(kB_X + lB_Y)a_0]_{z=B} + \int_B^1 (a_{0zz} - \sigma^2 a_0)a_1\mathrm{d}z,$$

where the boundary conditions (2.75), (2.80), and (2.81) have been used. Since a_0 is a solution of equation (2.74), equation (2.82) now reduces to

$$[\mathrm{i}\delta^2\frac{\partial}{\partial T}(\omega a_0^2)]_{z=1} - [\mathrm{i}\delta^2(kB_X + lB_Y)a_0^2]_{z=B}$$
$$= -\mathrm{i}\delta^2\left\{k\int_B^1 \frac{\partial(a_0^2)}{\partial X}\mathrm{d}z + l\int_B^1 \frac{\partial}{\partial Y}(a_0^2)\mathrm{d}z + (k_X + l_Y)\int_B^1 a_0^2\mathrm{d}z\right\}.$$

When we introduce the technique of differentiating under the integral sign (Q1.30), we see that this equation is written far more compactly as

$$\nabla\cdot(\mathbf{k}\int_B^1 a_0^2\mathrm{d}z) + \left[\frac{\partial}{\partial T}(\omega a_0^2)\right]_{z=1} = 0, \tag{2.83}$$

where

$$\nabla \equiv \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right) \quad \text{and} \quad \mathbf{k} \equiv (k, l).$$

Finally, we write a_0 from equation (2.77) so that

$$\int_B^1 a_0^2\mathrm{d}z = A_0^2\int_B^1 \cosh^2\{\sigma(z-B)\}\mathrm{d}z$$
$$= \frac{1}{2}A_0^2\int_B^1 [1 + \cosh\{2\sigma(z-B)\}]\mathrm{d}z$$
$$= \frac{1}{2}A_0^2\left\{D + \frac{1}{2\sigma}\sinh(2\sigma D)\right\}$$

where $D = 1 - B$ is the local depth. But we find from (2.76) and (2.78) – it is left as an exercise (cf. Q2.27) – that

$$\frac{\partial \omega}{\partial k} = \frac{\delta^2 k\omega}{2\sigma^2}\left(1 + \frac{2\sigma D}{\sinh 2\sigma D}\right), \tag{2.84}$$

and correspondingly for $\partial\omega/\partial l$, so

$$\mathbf{k}\int_B^1 a_0^2 \mathrm{d}z = (\omega A_0^2 \cosh^2 \sigma \mathrm{D})\mathbf{c}_g$$

where $\mathbf{c}_g \equiv (\partial\omega/\partial k, \partial\omega/\partial l)$, the group velocity; see. Q2.38. Now the amplitude of the surface wave (obtained as $(-\phi_t)$ evaluated on $z = 1$) is, to leading order as $\alpha \to 0$, $\omega A_0 \cosh \sigma D$; let us write

$$E = \frac{1}{2}\omega^2 A_0^2 \cosh^2 \sigma D \tag{2.85}$$

to denote the energy associated with the wave (cf. Q2.31). Then equation (2.83) can be written as

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega}\right) + \nabla\cdot\left(\frac{E}{\omega}\mathbf{c}_g\right) = 0 \tag{2.86}$$

where the term in $\partial/\partial T$ follows directly from the expression for a_0 given in (2.77).

It is not unusual, in the study of oscillators, to call the ratio of energy to frequency the *action*; in the context of these wave-like problems, therefore, we call E/ω the *wave action*. This quantity turns out to be more fundamental than energy in that, as the wave properties slowly change, so in general E (the energy) and ω both change, but (E/ω) is conserved as it is transported at the group velocity. Equation (2.86), for the wave action, is the main result of our calculations and, as we shall see, it plays an important rôle in the development and interpretation of the properties of wave propagation. However, there is at least one other important result that we shall require in due course: an expression for the lines of constant phase – the wavefronts (or wave crests) – which are defined by $\theta =$ constant.

From equations (2.72) we see, first, that (provided the appropriate derivatives exist)

$$\theta_{xt} = \alpha k_T = -\alpha\omega_X; \quad \theta_{yt} = \alpha l_T = -\alpha\omega_Y$$

and so

$$\nabla\omega + \frac{\partial \mathbf{k}}{\partial T} = \mathbf{0} \quad \left(\nabla \equiv \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right)\right), \tag{2.87}$$

which is the two-dimensional version of the consistency condition described in Q2.29. The relevant equation for θ follows directly from

$$\theta_x = k \quad \text{and} \quad \theta_y = l$$

for then

$$(\theta_x)^2 + (\theta_y)^2 = k^2 + l^2 = |\mathbf{k}|^2, \tag{2.88}$$

which is called the *eikonal* equation (from the Greek εικον, meaning *image* or *form*). This is more naturally expressed as

$$\Theta_X^2 + \Theta_Y^2 = |\mathbf{k}|^2 = \left(\frac{\sigma}{\delta}\right)^2, \tag{2.89}$$

where $\theta = \Theta/\alpha$ is one way to represent the fast phase variable as compared with the slow evolution of the wave parameters. This equation, (2.89), is an equation for Θ, given $\sigma(X, Y, T)$; its solution is a fairly standard exercise in the method of characteristics. We also have

$$\theta_{xy} = \alpha k_Y \quad \text{and} \quad \theta_{yx} = \alpha l_X, \quad \text{that is } k_Y = l_X,$$

so that the vector $\mathbf{k}$ can be treated as 'irrotational'.

Lines which everywhere have the group velocity vector, $\mathbf{c}_g$, as their tangent are called *rays*; these lines are therefore defined by

$$\frac{\mathrm{d}\mathbf{x}_\perp}{\mathrm{d}t} = \mathbf{c}_g.$$

Further, since $\mathbf{c}_g$ and $\mathbf{k}$ are parallel (see above and Q2.32), and the waves propagate in the $\mathbf{k}$-direction, we see that *rays are orthogonal to the wavefronts*. (We shall find that this is no longer true if a current is present; see Section 2.3.3.) Also, by virtue of equation (2.86), we see that the wave action (E/ω) is conserved along rays as it propagates at the group velocity.

We now explore these ideas by examining a few specific examples which, in particular, make use of equations (2.89) and (2.86). This will enable us to describe how the surface waves refract as the depth varies and, via the wave action, how the amplitude varies along rays. However, before we present these particular calculations let us confirm that our equations recover the usual results for steady propagation over *constant* depth. In this case, equation (2.89) becomes

$$\Theta_X^2 + \Theta_Y^2 = \left(\frac{\sigma}{\delta}\right)^2 = \text{constant},$$

and the relevant solution (at fixed T) takes the form

$$\Theta = f(X + \lambda Y), \quad \lambda = \text{constant}. \tag{2.91}$$

Thus

$$(1 + \lambda^2)(f')^2 = (\sigma/\delta)^2$$

and so

$$\Theta = f = \pm \frac{(\sigma/\delta)}{\sqrt{1+\lambda^2}}(X + \lambda Y) + G(T)$$

where $G(T)$ is arbitrary – the arbitrary 'constant' of integration; the lines $\theta =$ constant are therefore

$$\theta = \pm \frac{(\sigma/\delta)}{\sqrt{1+\lambda^2}}(x + \lambda y) + g(t) = \text{constant},$$

where $g(t) = G(T)/\alpha$. But from equations (2.72)

$$\theta_x = k \left(= \pm \frac{(\sigma/\delta)}{\sqrt{1+\lambda^2}} \right) \quad \text{and} \quad \theta_y = l \left(= \pm \frac{(\sigma/\delta)\lambda}{\sqrt{1+\lambda^2}} \right)$$

with $\theta_t = g'(t) = -\omega$ and hence, as expected,

$$\theta = kx + ly - \omega t = \text{constant}$$

describes the wavefronts; cf. Q2.7. Finally, equation (2.86) for the wave action gives $E/\omega =$ constant with all the parameters (such as ω) constant, and so the amplitude of the wave also remains constant (again, as expected).

2.3.1 *Steady, oblique plane waves over variable depth*

Let us consider the case of a depth variation which depends only on X: $1 - B = D(X)$. A steady, oblique plane wave is propagating on the surface. (The restriction to steady motion is in order to simplify the calculation; this assumption implies that, over constant depth, the wave parameters will remain constant.) For steady propagation,

$$\frac{\partial \mathbf{k}}{\partial T} = \mathbf{0},$$

and then equation (2.87) shows us that

$$\nabla\omega = \mathbf{0};$$

that is, $\omega =$ constant (since $\omega = \omega(X, Y)$, at most, for steady motion), and so, from equation (2.78),

$$\sigma \tanh(\sigma D) = \text{constant}. \tag{2.92}$$

Thus, with $D = D(X)$, we have that $\sigma = \sigma(X)$ and, further, as D decreases so σ increases, and vice versa. (We shall be more precise about this relationship later; also see Q2.39.) The eikonal equation, (2.89), is therefore

$$\Theta_X^2 + \Theta_Y^2 = \left(\frac{\sigma(X)}{\delta}\right)^2, \tag{2.93}$$

which possesses the solution (cf. equation (2.91) *et seq.*)

$$\Theta = f(X) + \lambda Y - \omega T, \quad \lambda = \text{constant},$$

where

$$(f')^2 + \lambda^2 = (\sigma/\delta)^2.$$

The solution for the phase, Θ, is therefore

$$\Theta = \frac{1}{\delta}\left\{\mu Y \pm \int^X \sqrt{\sigma^2(X) - \mu^2}\,\mathrm{d}X\right\} - \omega T, \tag{2.94}$$

where we have written $\lambda = \mu/\delta$; the wavefronts are then represented by $\Theta =$ constant. Correspondingly, the rays (which are orthogonal to the wavefronts) are given (at fixed T) by

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \pm\frac{1}{\mu}\frac{1}{\sqrt{\sigma^2 - \mu^2}}$$

or

$$\mu Y \mp \int^X \frac{\mathrm{d}X}{\sqrt{\sigma^2(X) - \mu^2}} = \text{constant}. \tag{2.95}$$

As a first example, consider a plane wave which propagates from a region of constant depth ($D = D_0$ in $X \leq X_0$) into a region which contains a submerged ridge. Let the wave have a phase, where $D = D_0$, given by

$$\Theta = k_0 X + l_0 Y - \omega T,$$

so that $\mu/\delta = l_0$ (that is, $l = l_0$ for $\forall X$) and $\sigma^2(X) = k_0^2 + \mu^2$ (in $X \le X_0$). In this situation, equation (2.92) can be written

$$\sigma \tanh(\sigma D) = \sigma_0 \tanh(\sigma_0 D_0) \tag{2.96}$$

where $\sigma_0 = \delta\sqrt{k_0^2 + l_0^2}$ (and then $\omega^2 = (\sigma_0/\delta^2)\tanh(\sigma_0 D_0)$). The slope of the wavefronts (at fixed T) is

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = -\frac{1}{l_0}\sqrt{\sigma^2(X) - \mu^2} \quad (= -k_0/l_0 \text{ for } X \le X_0), \tag{2.97}$$

and as the wave passes over the ridge, $D(X)$ first decreases and then increases; consequently $\sigma^2 - \mu^2$ increases and then decreases (eventually returning to its value of k_0^2, we will suppose). Thus the slope, $\mathrm{d}Y/\mathrm{d}X$, decreases and then increases, resulting in the wavefront turning more in-line with the ridge, and then away from it; see Figure 2.5, where a typical set of wavefronts and rays is depicted.

An extension of this problem arises if the depth decreases to zero, thereby producing a shoreline. In this case $\sigma \to +\infty$ as $D \to 0^+$, so $\mathrm{d}Y/\mathrm{d}X \to -\infty$: the wavefronts turn so that, in the limit (at the shoreline), they all become *parallel* to the shoreline. This explains the very familiar observation that virtually all ocean waves arrive at a beach parallel to one another and to the shoreline itself; see Figure 2.6.

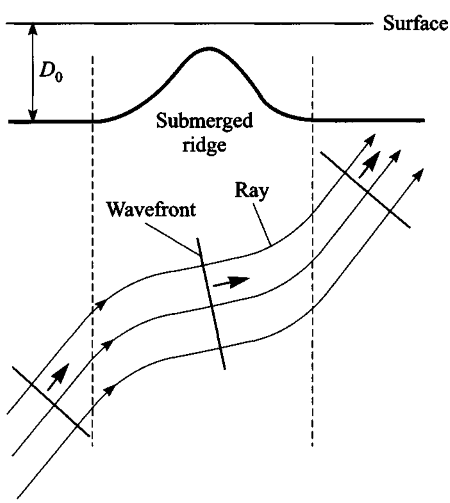


Figure 2.5. The rays and wavefronts for oblique plane waves passing over a submerged ridge; the undisturbed depth is $D = D_0$.

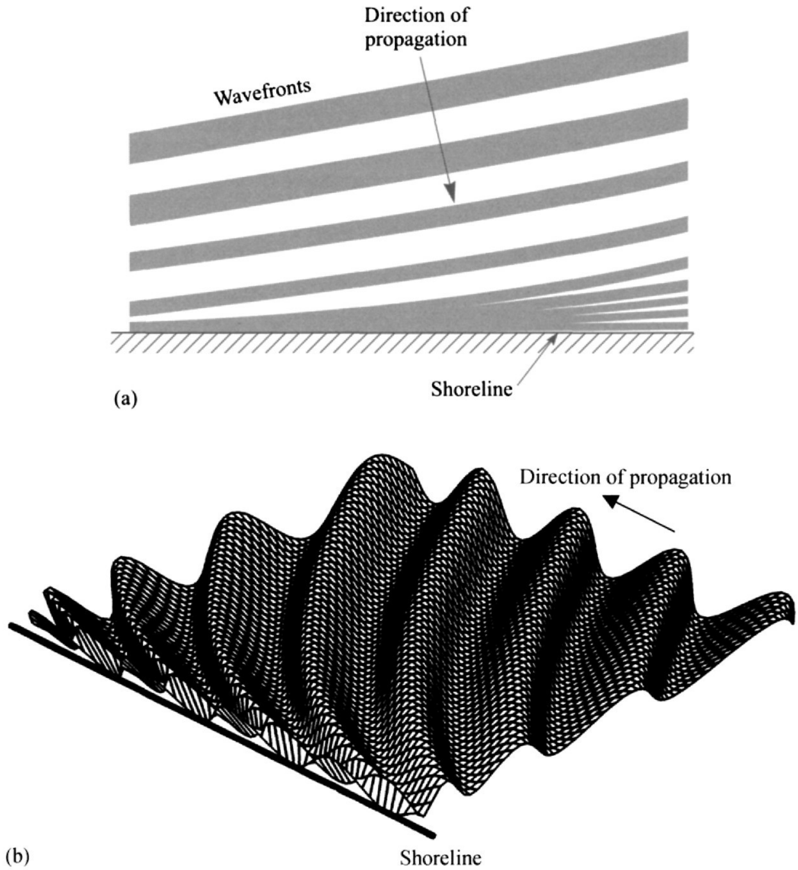


Figure 2.6. A representation of waves approaching a beach; (a) viewed from above, (b) seen as a surface in 3-space.

Further, we can also examine how the amplitude varies as the shoreline is approached. Along the rays we have, from equation (2.86),

$$\frac{\partial}{\partial X}\left(A^2\frac{\partial\omega}{\partial k}\right)+\frac{\partial}{\partial Y}\left(A^2\frac{\partial\omega}{\partial l}\right)=0,$$

and so

$$A^2\frac{\partial\omega}{\partial k}=\text{constant} \tag{2.98}$$

since there is no variation in Y. Near the shoreline we have $D \to 0$ and $\sigma \to \infty$, but with $\sigma D \to 0$, as we see from equation (2.96), which then gives

$$\sigma^2 D \to \sigma_0 \tanh(\sigma_0 D_0) = \delta^2 \omega^2$$

and

$$\frac{\partial \omega}{\partial k} \sim \frac{\delta^2 k \omega}{\sigma^2} \sim \frac{kD}{\omega} \sim \frac{\sigma D}{\delta \omega} \sim \sqrt{D}$$

since $k \sim \sigma/\delta$ as $\sigma \to \infty$ (from $\sigma = \delta\sqrt{k^2 + l_0^2}$). Hence, using equation (2.98), we see that

$$A = \mathrm{O}(D^{-1/4}) \text{ as } D \to 0$$

which is Green's law again (see equation (2.46), *et seq*., and Q2.34). Also, since $k \to \infty$ as $D \to 0$, the waves approaching the shoreline get shorter, as we have already discussed in Section 2.2; this phenomenon is included in Figure 2.6. (We should recall the warnings given in Section 2.2 concerning the dubious validity of the linear equations as the depth decreases to zero.)

Finally, we consider a wave which is propagating in a region where the depth is $D = D_0$, for $0 \le X \le X_0$ let us say. For $X < 0$ and $X > X_0$ the depth increases (so that $D = D_0$, $0 \le X \le X_0$, describes a submerged ridge); as before, we then have $\mu/\delta = l_0$ and $\sigma^2(X) = \delta^2 k_0^2 + \mu^2 = \delta^2(k_0^2 + l_0^2)$, given that $\mathbf{k} \equiv (k_0, l_0)$ in $0 \le X \le X_0$. As the depth increases so σ decreases, and if it drops sufficiently so that $\sigma^2 < \mu^2 = \delta^2 l_0^2$ then equation (2.97) shows that the wavefronts no longer exist. Of course, exactly the same can be said of the rays. Indeed, at the points where $\sigma^2 = \mu^2$ the slope of the rays becomes infinite and this will happen for all rays; the lines along which $\sigma^2 = \mu^2$ are called *caustics* (and are, perhaps, familiar from the theory of geometrical optics). The caustic is therefore the *envelope* of the rays. The continuation of a ray, beyond the point where $\mathrm{d}Y/\mathrm{d}X$ on it becomes infinite, is possible by switching to the other sign in the equation of the ray, (2.95), and producing it back into the region where the depth decreases. If this phenomenon occurs in both $X < 0$ and $X > X_0$, then the surface wave over depth $D = D_0$ remains *trapped* in a region containing the ridge; it is called a *trapped wave*, and this is depicted in Figure 2.7.

The caustic is where $\sigma^2 - \mu^2 = \delta^2 k^2 \to 0$, and so $\partial\omega/\partial k \to 0$; see equation (2.84) and remember that ω is constant and that both σ and D approach finite (nonzero) values at the caustic. Hence equation (2.98)

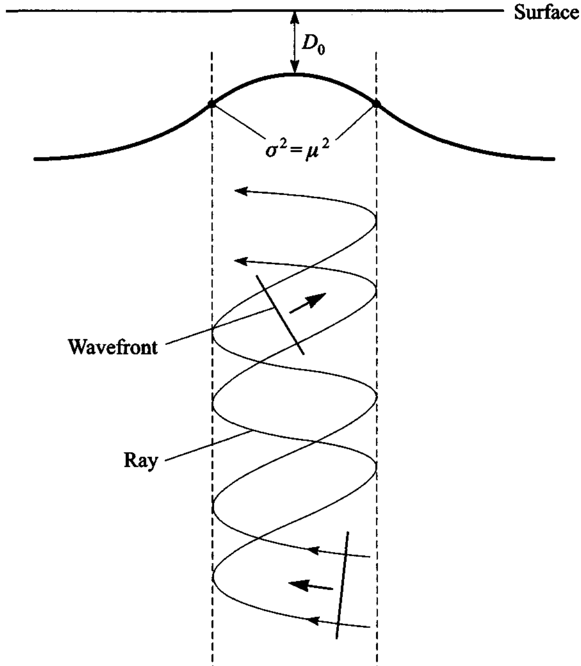


Figure 2.7. The rays and wavefronts for waves trapped between caustics (which are represented by the dashed lines).

shows that the amplitude of the wave diverges at the caustic, so our simple linear theory is no longer adequate. Some appropriate higher-order effects must be invoked in the neighbourhood of the caustic in order to produce a theory in which the wave amplitude remains finite. This more detailed discussion is not pursued here, but some further reading in this direction is mentioned at the end of the chapter.

2.3.2 Ray theory in cylindrical geometry

The equations and examples that we have presented so far have been written in rectangular Cartesian coordinates. However, problems that involve cylindrical surface waves or circular depth contours are clearly best discussed in cylindrical polar coordinates. Rather than derive the relevant equations from first principles, we follow the far simpler route of merely transforming the equations that we already have, according to

$$X = R\cos\theta, \quad Y = R\sin\theta.$$

Here, R is the radial coordinate suitably scaled like both X and Y, that is, $R = \alpha r$; the angular variable, θ, obviously requires no scaling. The phase function then satisfies

$$\Theta_R = |\mathbf{k}|\cos(\chi - \theta), \quad \frac{1}{R}\Theta_\theta = |\mathbf{k}|\sin(\chi - \theta) \tag{2.99}$$

since, in the constant state, Θ takes the form

$$\begin{aligned}\Theta &= |\mathbf{k}|R\cos(\chi - \theta) - \omega T \\ &= (|\mathbf{k}|\cos\chi)R\cos\theta + (|\mathbf{k}|\sin\chi)R\sin\theta - \omega T,\end{aligned}$$

where χ is a constant and the phase function is written as Θ, and only this form will be used here in order to avoid the obvious confusion with the angular variable θ. The wave-number vector in cylindrical polars is written using the same notation as earlier, so

$$\mathbf{k} \equiv (k, l) \text{ [} = |\mathbf{k}|\{\cos(\chi - \theta), \sin(\chi - \theta)\} \text{ from above].}$$

Thus we obtain

$$\Theta_R^2 + \frac{1}{R^2}\Theta_\theta^2 = k^2 + l^2 = |\mathbf{k}|^2, \tag{2.100}$$

which is obviously the polar form of equation (2.89). The corresponding equation for the action is, from equation (2.86),

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{E}{\omega}Rc_{g1}\right) + \frac{1}{R}\frac{\partial}{\partial \theta}\left(\frac{E}{\omega}c_{g2}\right) = 0, \tag{2.101}$$

where the group velocity is written as $\mathbf{c}_g \equiv (c_{g1}, c_{g2})$ in cylindrical polars.

Similar to our discussion in Section 2.3.1, let us consider steady wave propagation over a depth variation which depends only on R, so that $D = D(R)$. The dispersion relation is unchanged:

$$\omega^2 = \frac{\sigma}{\delta^2}\tanh(\sigma D) \quad \text{with} \quad \sigma = \delta\sqrt{k^2 + l^2},$$

since the derivation leading to these (given earlier as equations (2.78) and (2.76)) does not involve derivatives with respect to the slow scales. (Remember that we have used the same notation here for the wave-number vector, and so $|\mathbf{k}|^2 = k^2 + l^2$ is the relevant expression.) For the analogue of a submerged ridge (which was discussed above), we now have a shoal with cylindrical symmetry (with the origin of coordinates chosen so that $R = 0$ is the centre of the shoal); the minimum depth occurs at the centre of the shoal. The wavefronts are described by $\Theta = \text{constant}$, where

$$\Theta = \frac{1}{\delta}\left\{\mu\theta \pm \int^{R} \sqrt{R^2\sigma^2(R) - \mu^2}\,\frac{\mathrm{d}R}{R}\right\} - \omega T; \qquad (2.102)$$

cf. equation (2.94). Correspondingly, the equations for the rays (at fixed T) become

$$\mu\theta \mp \int^{R} \frac{\mathrm{d}R}{R\sqrt{R^2\sigma^2(R) - \mu^2}} = \text{constant}; \qquad (2.103)$$

cf. equation (2.95), and remember that the orthogonality of two curves $(r = f(\theta), r = g(\theta))$, written in polar coordinates, requires that

$$f'(\theta)g'(\theta) = -r^2.$$

Let us suppose that $R\sigma(R)$ is monotonic in $R \geq 0$ (which will certainly describe a class of circular shoals). On the rays we have

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = \pm\mu R\sqrt{R^2\sigma^2 - \mu^2} \qquad (2.104)$$

and a ray approaching the shoal must have either $\mathrm{d}R/\mathrm{d}\theta > 0$ or $\mathrm{d}R/\mathrm{d}\theta < 0$; then R decreases as either θ decreases or increases, respectively. This obtains until the ray reaches a minimum distance from $R = 0$, which occurs where

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = 0 \quad \text{or} \quad R^2\sigma^2 = \mu^2$$

on the ray; thereafter R increases, which is accommodated by switching to the other sign in equation (2.104). This type of solution, for two different rays (one with $\mathrm{d}R/\mathrm{d}\theta > 0$ initially, that is, to the left, the other with $\mathrm{d}R/\mathrm{d}\theta < 0$) is shown in Figure 2.8. Two important observations can now be made: first, as the depth decreases, so the rays turn towards the centre of the shoal until they reach a minimum distance from $R = 0$, and then they turn away. This general description is what we should expect, based on the corresponding problems with $D = D(X)$ given in Section 2.3.1. Second, we see that a consequence of the bending of the rays is that, in the lee of the shoal (that is, 'behind'), the rays – and wavefronts, of course – cross. Where these waves intersect there may be either a constructive or a destructive interaction; a peak plus a peak (or trough plus trough) is constructive, but a peak plus a trough will – at least

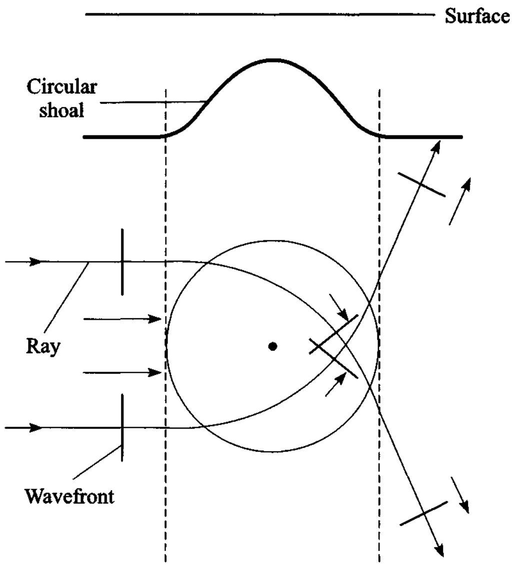


Figure 2.8. Two typical rays and wavefronts for propagation over a circular shoal.

in part – cancel. Obviously, which case arises will depend on the phases of the individual waves.

Other calculations for different choices of $D(R)$ are clearly possible. Indeed, for example, corresponding to our discussion in Section 2.3.1 for straight contours, we may construct a shoreline problem; this becomes a circular island when $D = D(R)$. Similarly, waves trapped on a straight ridge translates into the problem of a circular ridge, for which it is then possible to find conditions which ensure that the waves remain trapped on the ridge. Simple examples of these types of problem, and others, will be found in the exercises (Q2.47, Q2.48).

2.3.3 Steady plane waves on a current

Our second example of a slowly varying environment arises when the surfaces waves propagate in the presence of a (slowly varying) current. (Of course, the effect of both variable depth and a varying current could be studied together, but we opt for the simplification which treats these two problems separately.) A current is the movement, in horizontal directions, of a body of water (but, in order to maintain the condition of mass conservation, some vertical motion may also be present); we shall treat the current as a prescribed ambient state which is then perturbed by the surface waves. These motions are, in general, rotational, and so we must use the Euler equation (rather than Laplace's equation). In Cartesian geometry, we therefore start with the problem described by the equations (1.57) and (1.63)–(1.65), namely

$$\frac{\mathrm{D}u}{\mathrm{D}t}=-\frac{\partial p}{\partial x},\quad \frac{\mathrm{D}v}{\mathrm{D}t}=-\frac{\partial p}{\partial y},\quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{\partial p}{\partial z},$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

with

$$w=H_t+uH_x+vH_y \quad \text{and} \quad p=H \text{ on } z=1+H;$$

and

$$w=0 \text{ on } z=0.$$

Here we have chosen to consider only gravity waves (so that the Weber number, W, is set to zero) and the bottom is $z=b=0$; we have written $\varepsilon\eta=H$. It is seen that we have not quoted the corresponding scaled equations ((1.67), (1.69), (1.70)) for these cannot accommodate the imposed current; cf. Q2.11 and Q2.12.

The relevant form of the governing equations required here – that is, linearised about the ambient state – is obtained (cf. Section 1.3.3) by transforming

$$\mathbf{u}_\perp \to \mathbf{U}_\perp+\varepsilon\mathbf{u}_\perp,\quad w\to W+\varepsilon w$$

where $(\mathbf{U}_\perp, W)$ represents the current; this state must satisfy the equations with $\varepsilon=0$, so we also require

$$p\to P+\varepsilon p,\quad H\to H+\varepsilon\eta.$$

Thus, with $\mathbf{u}_\perp\equiv(U, V)$, we have

$$\left.\begin{aligned}
&\frac{\mathrm{D}U}{\mathrm{D}t}=-\frac{\partial P}{\partial x},\quad \frac{\mathrm{D}V}{\mathrm{D}t}=-\frac{\partial P}{\partial y},\quad \delta^2\frac{\mathrm{D}W}{\mathrm{D}t}=-\frac{\partial P}{\partial z},\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}+W\frac{\partial}{\partial z},\\
&\text{with}
\end{aligned}\right\} \qquad (2.105)$$

$$\left.\begin{aligned}\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0,\end{aligned}\right.$$

and

$$\left.\begin{aligned}W=H_t+UH_x+VH_y, \quad P=H \text{ on } z=1+H;\\ W=0 \text{ on } z=0,\end{aligned}\right\}$$

for the current alone. Note that, in general, in the presence of a current, the undisturbed surface is not a $z=$ constant surface. We now restrict consideration to a current which is steady but which slowly varies in the horizontal directions; thus we regard $\mathbf{U}_\perp$, W, P, and H as functions of $(\alpha x, \alpha y)$, and z as appropriate, with $\alpha \to 0$ (which corresponds to the choice made in Section 2.3.1). It is clear from the equation of mass conservation (in (2.105)) that $W = \mathrm{O}(\alpha)$; consequently any upwelling or down-currents are weak (although necessarily present, in general).

The linearised equations for the surface wave are now obtained by collecting the leading-order terms as $\varepsilon \to 0$ (from the governing equations), but after we have satisfied equations (2.105) for the current. This leads to the set of equations

$$\frac{\mathrm{D}u}{\mathrm{D}t}+(\mathbf{u}\cdot\nabla)U=-\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t}+(\mathbf{u}\cdot\nabla)V=-\frac{\partial p}{\partial y},$$

$$\delta^2\left\{\frac{\mathrm{D}w}{\mathrm{D}t}+(\mathbf{u}\cdot\nabla)W\right\}=-\frac{\partial p}{\partial z},$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}+W\frac{\partial}{\partial z}$$

with

$$\left.\begin{aligned}w+\eta W_z&=\eta_t+U\eta_x+V\eta_y+uH_x+vH_y\\ p+\eta P_z&=\eta\end{aligned}\right\}\text{ on } z=1+H$$

and

$$w=0 \text{ on } z=0.$$

We seek a solution of this linearised problem in the form of asymptotic expansions valid as $\alpha \to 0$, just as we did for the case of variable depth. To this end (cf. equations (2.71)–(2.73)) we write

$$X=\alpha x, \quad Y=\alpha y, \quad T=\alpha t,$$

and

$$(\theta_x, \theta_y) = \{k(X, Y, T),\ l(X, Y, T)\}, \quad \theta_t = -\omega(X, Y, T)$$

which produces the set

$$\begin{aligned}
&\alpha u_T - \omega u_\theta + U(\alpha u_X + k u_\theta) + V(\alpha u_Y + l u_\theta) \\
&\qquad + W u_z + \alpha(u U_X + v U_Y) + w U_z = -(\alpha p_X + k p_\theta); \\
&\alpha v_T - \omega v_\theta + U(\alpha v_X + k v_\theta) + V(\alpha v_Y + l v_\theta) \\
&\qquad + W v_z + \alpha(u V_X + v V_Y) + w V_z = -(\alpha p_Y + l p_\theta); \\
&\delta^2\{\alpha w_T - \omega w_\theta + U(\alpha w_X + k w_\theta) + V(\alpha w_Y + l w_\theta) \\
&\qquad + W w_z + \alpha(u W_X + v W_Y) + w W_z\} = -p_z; \\
&\qquad k u_\theta + l v_\theta + w_z + \alpha(u_X + v_Y) = 0
\end{aligned}$$

with

$$\left.\begin{aligned}
w + \eta W_z &= \alpha\eta_T - \omega\eta_\theta + U(\alpha\eta_X + k\eta_\theta) \\
&\qquad + V(\alpha\eta_Y + l\eta_\theta) + \alpha(u H_X + v H_Y) \\
p &= \eta - \eta P_z
\end{aligned}\right\} \text{ on } z = 1 + H$$

and

$$w = 0 \text{ on } z = 0.$$

Further, in order to make the problem a little more manageable, we shall assume that the horizontal components of the current depend only on (X, Y), at least to $O(\alpha^2)$, and therefore not z. Thus, from equations (2.105), we obtain

$$W = -\alpha z(U_X + V_Y) + O(\alpha^3)$$

so that $H(X, Y; \alpha)$ satisfies

$$\{(1 + H)U\}_X + \{(1 + H)V\}_Y = O(\alpha^2)$$

and then

$$P = H + O(\alpha^2) \text{ for } 0 \le z \le 1 + H.$$

The expression for W describes the upwelling (or down current) associated with the current; it is absent at this order only if the current satisfies $U_X + V_Y = 0$.

The solution that we seek comprises a single harmonic component, and so we write

$$Q \sim (Q_0 + \alpha Q_1)e^{i\theta} + \text{c.c.} + O(\alpha^2),$$

where Q represents each of u, v, w, p, and η. The leading-order problem then becomes

$$\begin{aligned}-\omega u_0 + Uku_0 + Vlu_0 &= -kp_0;\\ -\omega v_0 + Ukv_0 + Vlv_0 &= -lp_0;\\ \mathrm{i}\delta^2(-\omega w_0 + Ukw_0 + Vlw_0) &= -p_{0z};\\ \mathrm{i}(ku_0 + lv_0) + w_{0z} &= 0,\end{aligned}$$

with

$$\left.\begin{aligned}w_0 &= -\mathrm{i}\omega\eta_0 + \mathrm{i}kU\eta_0 + \mathrm{i}lV\eta_0\\ p_0 &= \eta_0\end{aligned}\right\} \text{ on } z = 1 + H$$

and

$$w_0 = 0 \text{ on } z = 0.$$

Thus

$$(k^2 + l^2)p_0 = \Omega(ku_0 + lv_0),$$

where we have written

$$\Omega = \omega - kU - lV \tag{2.106}$$

and then

$$\frac{\mathrm{i}}{\Omega}(k^2 + l^2)p_0 + w_{0z} = 0; \quad \mathrm{i}\delta^2\Omega w_0 = p_{0z}$$

so that

$$w_{0zz} - \delta^2(k^2 + l^2)w_0 = 0;$$

cf. equation (2.74). The boundary conditions for w_0 are

$$w_0 = -\mathrm{i}\Omega\eta_0 \text{ on } z = 1 + H; \quad w_0 = 0 \text{ on } z = 0$$

which require

$$w_0 = -\mathrm{i}\Omega\eta_0 \frac{\sinh(\sigma z)}{\sinh\{\sigma(1 + H)\}}$$

where

$$\sigma(X, Y, T) = \delta\sqrt{k^2 + l^2},$$

exactly as before (equations (2.76)). Finally, from

$$p_0 = \frac{i\Omega}{k^2+l^2} w_{0z} = \frac{\Omega^2 \delta \eta_0}{\sqrt{k^2+l^2}} \frac{\cosh(\alpha z)}{\sinh\{\sigma(1+H)\}}$$

and the boundary condition on p_0, we obtain the familiar dispersion relation

$$\Omega^2 = (\omega - kU - lV)^2 = \frac{\sigma}{\delta^2} \tanh\{\sigma(1+H)\}, \qquad (2.107)$$

where Ω replaces ω; see equation (2.78) and Q2.12.

We now proceed to the next order, but we shall describe the calculation in outline only. The technique follows precisely that presented for the case of slowly varying depth (given in equations (2.79)–(2.81) *et seq.*). Furthermore, the results are essentially identical to those obtained previously, the difference arising from the appearance of Ω (for ω), for example. The equations at O(α) take the form

$$kp_1 = \Omega u_1 + F_1; \quad lp_1 = \Omega v_1 + F_2; \quad p_{1z} = \mathrm{i}\delta^2 \Omega w_1 + F_3;$$
$$w_{1z} + \mathrm{i}(ku_1 + lv_1) = F_4,$$

with

$$w_1 = -\mathrm{i}\Omega \eta_1 + F_5 \text{ and } p_1 = \eta_1 \text{ on } z = 1 + H,$$

and

$$w_1 = 0 \text{ on } z = 0,$$

where the forcing terms, F_i ($i = 1, \ldots, 5$), depend on the leading-order solution. These produce an equation for w_1, of the form

$$w_{1zz} - \delta^2(k^2 + l^2) w_1 = G,$$

where G depends on the F_i. Rather than solve for w_1, we multiply by w_0 and integrate with respect to z, $0 \le z \le 1 + H$; cf. equation (2.82) *et seq.*, to see how this will produce the equation for $\eta_0(X, Y, T)$ (which is the first term in the asymptotic expansion of the (complex) amplitude of the surface wave).

As before, this equation for η_0 is far more usefully written in terms of the energy, E; cf. equation (2.85). We then find directly (although the details are rather tiresome and are left as an exercise) that, corresponding to equation (2.86), E satisfies

$$\frac{\partial}{\partial T}\left(\frac{E}{\Omega}\right) + \nabla \cdot \left((\mathbf{U}_\perp + \mathbf{c}_g)\frac{E}{\Omega}\right) = 0. \qquad (2.108)$$

Here, $\mathbf{c}_g$ is the group velocity relative to the current, so $\mathbf{c}_g \equiv (\partial\Omega/\partial k, \partial\Omega/\partial l)$. In other words, equation (2.108) does indeed correspond precisely to equation (2.86), provided due account is taken of motion *relative* to the current; thus $\omega \to \Omega$ and $\mathbf{c}_g \to \mathbf{U}_\perp + \mathbf{c}_g$. The eikonal equation for the phase function, θ (or $\Theta = \alpha\theta$), is exactly as before (equation (2.89)):

$$\Theta_X^2 + \Theta_Y^2 = |\mathbf{k}|^2.$$

It is, perhaps, no surprise to learn that equation (2.108) for the wave action (E/Ω) also arises when both a variable depth and a varying current occur together (defined on the same slow scales, of course). Indeed, the conservation of wave action as it is transported at the group velocity (relative to the current and then plus the current velocity here) is a very general result. It appears in many (nondissipative) physical systems that incorporate a slowly varying background on which small amplitude waves are superimposed. (We have shown how this equation comes about in a very direct manner, but a far more elegant approach is to use the average-Lagrangian methods developed by Whitham; these ideas are beyond the scope of this text, but additional reading in this direction is indicated at the end of the chapter.)

Finally, we describe a few consequences for waves that propagate on a slowly varying current. First a general result: the rays are now defined by

$$\frac{d\mathbf{x}_\perp}{dt} = \mathbf{U}_\perp + \mathbf{c}_g$$

and only $\mathbf{c}_g$ is in the direction of the wave-number vector, $\mathbf{k}$. Hence the wavefronts, which are orthogonal to $\mathbf{k}$, are *no longer orthogonal* to the rays (in the presence of a general current); cf. variable depth only, as discussed in Section 2.3. The eikonal equation for θ (or $\Theta = \alpha\theta$) is unchanged; see equations (2.88) and (2.89). Thus the methods for finding the wave fronts, $\Theta =$ constant, are the same no matter whether we have a slowly varying depth or current (or, indeed, the two phenomena combined). Now let us briefly examine two particular examples of waves on a current.

First we consider one of the simplest problems of this type: a steady wave propagating in the x-direction (so $\mathbf{k} \equiv (k, 0)$), with a current $\mathbf{U}_\perp \equiv (U(X), 0)$. (We note that $W \neq 0$ for this solution; see equations (2.105).) As we saw in Section 2.3.1, for steady propagation we obtain

$$\omega = \text{constant},$$

and so

$$\Omega + kU = \text{constant } (= \omega) \tag{2.109}$$

(since, here, $V = 0$ and $l = 0$), where Ω is determined from

$$\Omega^2 = \frac{\sigma}{\delta^2}\tanh(\sigma \mathrm{D}), \quad \sigma = \delta k,$$

and D $(= 1 + H)$ is the local depth. Just to make the calculation a little more transparent – but this does not alter the essential character of the problem – let us suppose that we have short waves so that $\delta k \to \infty$ (which, as we have seen in Section 2.1, is equivalent to having deep water). In this case we may write $\Omega \sim \sqrt{k/\delta}$, and then the speed of the wave relative to the current in $\Omega/k \sim \sqrt{1/\delta k}$. Hereafter, we therefore choose to write

$$c = \frac{\Omega}{k} = \frac{1}{\sqrt{\delta k}} \quad \text{or} \quad k = \frac{1}{\delta c^2} \tag{2.110}$$

so equation (2.109) becomes

$$\omega = kc + kU = \frac{1}{\delta c} = \frac{U}{\delta c^2}$$

or

$$(\delta\omega)c^2 - c - U = 0,$$

a quadratic equation for the speed c, given the constant $\delta\omega$, and the current $U(X)$. It is convenient to introduce the phase speed, c_p, of the waves in the absence of any current; that is, $c_p = 1/\delta\omega$, and then

$$c = \frac{1}{2}c_p(1 \pm \sqrt{1 + 4U/c_p}); \tag{2.111}$$

clearly only the positive sign is meaningful, for then $c = c_p$ when $U = 0$ (as we have just prescribed). The negative sign yields $c = 0$ when $U = 0$, which is plainly inconsistent.

This surprisingly simple solution, (2.111), yields important results, only some of which might have been anticipated. For example, a current moving in the same direction as the wave (that is, $U > 0$), produces a (local) phase speed $c > c_p$ with a decreased k (from equations (2.111)): the waves travel faster in the presence of a current, but are longer. On the other hand, if the current opposes the waves, so that $U < 0$, then $c < c_p$ and the waves are now shorter. However, the significant prediction from equation (2.111) is that c does not exist (as a meaningful wave speed) if

$U < -c_p/4$. What has happened? The explanation is readily obtained from the equation for the wave action, (2.108).

For our problem, equation (2.108) reduces to

$$\left(U + \frac{\partial \Omega}{\partial k}\right) \frac{E}{\Omega} = \text{constant},$$

where E is proportional to the square of the wave amplitude, A say, and $c_g = \partial\Omega/\partial k$ is the group speed relative to the current. For $\delta k \to \infty$ (or deep water) we have that $c_g = c/2$ (see Q2.27) and thus we obtain

$$\left(U + \frac{1}{2}c\right) cA^2 = \text{constant};$$

consequently, as $U \to -c_p/4$, we then have $c \to c_p/2$ and so $U + c/2 \to 0$; that is, $A \to \infty$. This is exactly the phenomenon associated with a caustic, as described in Section 2.3.1; our solution is inadmissible close to any region where the current is such that $U \to -c_p/4$. As our theory stands, the caustic constitutes a line across which the wave cannot cross; the waves approaching the caustic line produce a build-up of energy (and amplitude) there.

For our second example we describe another classical problem, namely that of a steadily propagating oblique wave moving across the current $\mathbf{U}_\perp \equiv (0, V(X))$. In this case $U_X + V_Y = 0$, so no up-welling is required to maintain the ambient state (and, indeed, there then exists a solution $H = 0$; see equation (2.105)). As above, for steady propagation, we have

$$\omega = \text{constant}$$

so

$$\Omega + lV = \text{constant } (= \omega)$$

where the wave number is $\mathbf{k} \equiv (k, l)$. Further, since there is no variation of $\mathbf{k}$ in Y, we have that $l_X = 0$ (from the irrotationality of $\mathbf{k}$), so $l = \text{constant } (= l_0, \text{ say})$. Thus we obtain

$$l_0 V + \frac{1}{\delta}\sqrt{\sigma \tanh(\sigma \mathrm{D})} = \omega, \quad \sigma = \delta\sqrt{k^2 + l_0^2},$$

which becomes the equation for $k(X)$, given $V(X)$; here, we may write $D = 1$ if $H = 0$. The short-wave approximation (as used above) then leads to the simplified equation

$$\omega = \frac{1}{\delta c} + l_0 V,$$

where $c = \Omega/|\mathbf{k}|$ and $|\mathbf{k}| = 1/\delta c^2$. This time we have a linear equation for c, where

$$c = \frac{1}{\delta(\omega - l_0 V)},$$

with

$$|\mathbf{k}| = \sqrt{k^2 + l_0^2} = \delta(\omega - l_0 V)^2, \tag{2.112}$$

and (essentially as before) the conservation of wave action moving with the group yields

$$\frac{\partial}{\partial X}\left(\frac{\partial \Omega}{\partial k}\frac{E}{\Omega}\right) = 0 \quad \text{or} \quad \frac{\partial \Omega}{\partial k}\frac{E}{\Omega} = \text{constant},$$

where the group speed is

$$\frac{\partial \Omega}{\partial k} = \frac{1}{2}\frac{k}{\sqrt{\delta}}(k^2 + l_0^2)^{-3/4} = \frac{1}{2}c\frac{k}{|\mathbf{k}|}.$$

Now from equation (2.112) we see that, as $V(X)$ increases, so $k(X)$ decreases (since we take $\omega > 0$ and $l_0 > 0$). Eventually we shall reach the condition $k = 0$; the wavefront has turned so that it is perpendicular to the direction of the current. From the equation for the wave action we have that

$$c\frac{k}{|\mathbf{k}|}\frac{E}{\Omega} = \frac{k}{|\mathbf{k}|^2}E = \text{constant},$$

and as $k \to 0$ so $E \to \infty$: again, the amplitude of the wave grows without bound (in this theory) when the caustic associated with $k = 0$ is encountered. Typical of other results like this – some of which we described earlier – we must expect that waves cannot cross the caustic (but a reflection may occur). This situation is represented in Figure 2.9.

2.4 The ship-wave pattern

One of the most intriguing, and often spectacular sights when viewed from a distance, is the pattern produced by a moving object on the surface of water. Surprisingly, this pattern is essentially the same no matter whether it is a moorhen or an aircraft carrier that is the source of the disturbance. (However, even from a photograph, the scale can often be

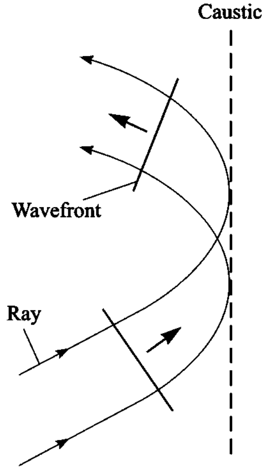


Figure 2.9. The reflection of rays and wavefronts at a caustic formed by the presence of a current.

judged – in the absence of the source – if capillary waves are also present.) A typical wave pattern is depicted in Figure 2.10.

It is our intention here to give an explanation and description of this pattern, an explanation which was first presented by Lord Kelvin. In fact, it was the solution of this problem (in 1887) which led him, first, to develop his method of stationary phase; see Section 2.1.3 and Q2.16. He realised that the salient features of the pattern can be extracted from an otherwise intractable integral (which itself is based on an idealised model of the phenomenon) provided that a suitable limit is taken. Of course, the fine detail of the wave pattern, and particularly a precise estimate of the energy lost in generating the waves, are very significant problems in naval design. The sophisticated analysis required to accomplish this is far beyond the scope of the material presented here; we shall concern ourselves only with the classical – and simple – problem posed by the idealisation introduced by Kelvin.

This method of solution proposes that the disturbance caused by the object (be it the moorhen or the ship) is replaced by a moving *point impulse* at the surface. This impulse – an impulsive pressure, analogous to the impulse in elementary mechanics – at the instant it is applied, causes no displacement, but does impart a vertical velocity to the surface. That is, the surface wave (η) satisfies

$$\eta = 0 \quad \text{with} \quad \eta_t \neq 0$$

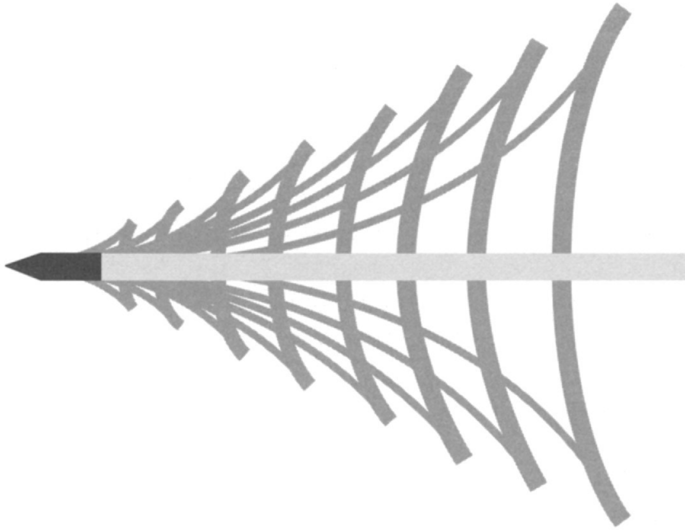


Figure 2.10. A representation of the ship-wave pattern generated by an object moving on the surface of water.

at $t = 0$, when the impulse is applied at time $t = 0$ (cf. Q2.18). Of course, our model here requires this process to occur continuously as the point (the ship) moves and so continuously disturbs the surface; this is the essence of Kelvin's wonderfully perceptive view of the problem. Further, the restriction to a point disturbance indicates that the results we obtain are valid at, perhaps, moderate and, probably, large distances from the centre of the disturbance. We must not expect to produce a description of the waves which is accurate close to a specific object. Indeed, the picture that we shall present corresponds closely to the typical observer's view: the waves are seen, and are well-defined, a reasonable distance from the ship (or whatever) and extend to far distances. Kelvin's approximation to a moving point models the (finite) dimensions of the initiating disturbance, when viewed on a scale that is large; that is, from far away. It is for this reason, primarily, that Kelvin's theory for the ship-wave pattern is independent of the scale – bird or ship – of the moving object.

We shall first present a development based on Kelvin's approach (but within the framework of our earlier discussions, and we shall also need to recall some of the exercises). Then, second, we shall recover some of the main features of the wave pattern by a far simpler approach: we shall invoke the ideas of ray theory.

2.4.1 Kelvin's theory

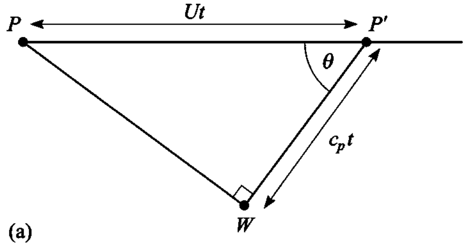
Before we describe the details of the solution, which follows that obtained by Kelvin, we first demonstrate that the region of the wave pattern is easily characterised by the application of simple principles. Indeed, this was one part of the important explanations given by Kelvin.

We consider, in order readily to familiarise ourselves with the idea, a ship (or any object) moving at a constant speed, U, in a straight line on stationary water. The only ingredient that we require which is especially pertinent to this problem is the observation that the waves, when viewed relative to the ship, are stationary; this we shall assume is a given property here. Because of this, the natural way to present the wave field is also relative to the ship; this requires the water to be regarded as moving at speed U opposite to the ship. Let the ship be at P, with the water flowing from left to right. We examine the contribution to the wave profile which was generated at point P' at a time t earlier; see Figure 2.11(a). It is clear that the distance $P'P$ is Ut. Now consider a wave front (at W) which travels at a (constant) speed c_p away from P' in the direction θ (measured with respect to $P'P$); let this wave have wave number k (so that $c_p = c_p(k)$).

Now, this is to be stationary in our frame of reference, and thus we must have

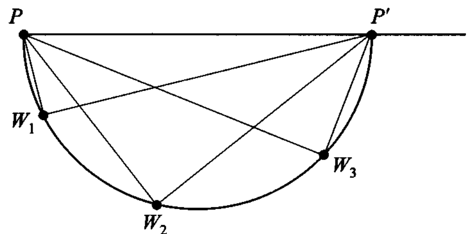
$$c_p = U\cos\theta, \tag{2.113}$$

which therefore describes how k must vary with θ (for fixed U, and a known dispersion relation yielding $c_p = \omega(k)/k$). But equation (2.113) implies that $P'WP$ is a right-angled triangle, with the angle $\pi/2$ at W. Thus, for fixed P' but different angles θ, all wavefronts emanating from P' must lie on a semicircle with diameter $P'P$; see Figure 2.11(b). In Figure 2.11(c) we show the result of including some waves that have been generated *after* the ship has passed P'. Presumably the complete picture is now obtained by combining all such waves (which is, in essence, what we shall do later), and then the envelope of these waves will describe the region inside which the wave pattern is observed. This is, in principle, correct, but an important property of these waves has so far been omitted.



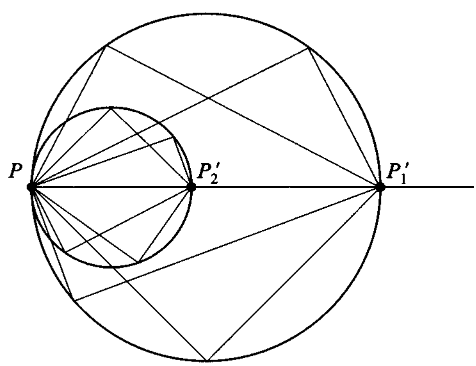
(a)

Figure 2.11(a). The ship is at P, which was at P' at a time t earlier; the wavefront has reached the point W.



(b)

Figure 2.11(b). Three different wavefronts (W_1, W_2, W_3) all emanating from P'.



(c)

Figure 2.11(c). Wavefronts generated when the ship was at P_1' and then at P_2'.

We know that the energy of water waves does not propagate at the phase speed, c_p, but at the group speed c_g. Waves that are observed can have an appreciable amplitude only near where the energy has reached (see Section 2.1.2). For gravity waves we found that $c_g/c_p < 1$ and in particular, for infinitely deep water, $c_g = c_p/2$; let us suppose that our ship creates gravity waves and is moving in deep water (because this is the simplest choice, and it will correspond to most – but, of course, not all – wave patterns that are observed). The propagation of the relevant (energy-carrying) fronts is now only half as far as that supposed earlier; we now have Figure 2.12 where $U = c_p$ on $\theta = 0$. We see that the waves are restricted to a wedge-shaped region, with the ship at the vertex. The semi-angle of the wedge, Θ, is then $\arcsin(1/3)$ ($\approx \pi/9$) (which some readers may recognise as the Mach angle associated with a supersonic flow at Mach number 3).

Of course, this simple analysis cannot supply any predictions for the wave pattern itself; it tells us only where to expect to see the main disturbance (and this is easily confirmed, by observation, to be essentially correct). Also, we have described the case for deep water; as the depth decreases, so c_g approaches c_p and the wedge angle increases (see Q2.49). Now we turn to a far more detailed and careful analysis, following the route laid down by Kelvin.

We consider stationary water of infinite depth, over which a point moves on a prescribed path (which need not be a straight line). Since (as for many of our calculations) we are concerned only with the generation of gravity waves, we set the Weber number to zero. (Of course, we can always retain the effects of surface tension; indeed, capillary ship-waves – no gravity at all – provide an amusing exercise; see Q2.54. Generalisations to finite depth, as we mentioned earlier, are also possible.)

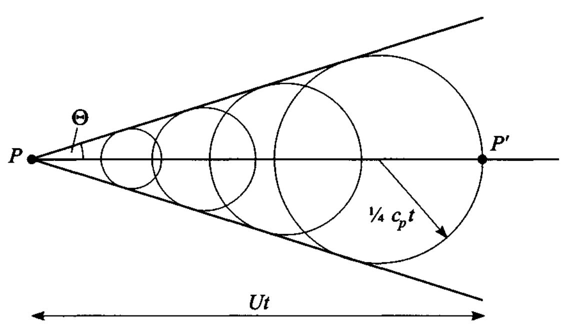


Figure 2.12. The wedge-shaped region inside which the ship-wave pattern is evident, for the case of deep water.

The first stage in this calculation is to obtain the relevant concentric surface wave, $\eta(r, t)$ which is produced by a point impulse. To this end, we recall the analysis for concentric waves on deep water (Section 2.1.3); here, however, we require the solution (written via the Hankel transform) which satisfies

$$\eta(r, 0) = 0 \quad \text{and} \quad \eta_t(r, 0) \neq 0.$$

However, it is not clear what form we should choose for $\eta_t(r, 0)$. Of course, the main idea here – Kelvin's – is to impose a point impulse, so that is what we use in order to make headway.

To see how the impulse is introduced, it is convenient to call upon the pressure equation evaluated at the surface, where $P = P_s$ on $z = h$:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{P_s}{\rho} + gh = f(t),$$

which is written here in physical (dimensional) variables; see Section 1.2.2. As before, let us suppose that somewhere $h = h_0$ (= constant) and $P_s = P_a$ (= constant atmospheric pressure) with no motion, then

$$f(t) = \frac{P_a}{\rho} + gh_0,$$

and hence

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \frac{1}{\rho}(P_s - P_a) + g(h - h_0) = 0.$$

In the present context, we are analysing a certain class of linear waves, so it is the linearised version of this equation that we require: we have, approximately,

$$\frac{\partial \phi}{\partial t} + \frac{1}{\rho}(P_s - P_a) + g\eta = 0 \text{ on } z = h_0,$$

where $h - h_0 = \eta$. The impulse is obtained by integrating this equation over the time interval $(0, T)$ and then letting $T \to 0$. Performing this integration yields

$$\phi(\mathbf{x}_\perp, h_0, T) + \frac{1}{\rho}\int_0^T (P_s - P_a)\mathrm{d}t + g\int_0^T \eta \, \mathrm{d}t = 0$$

where we have set $\phi(\mathbf{x}_\perp, h_0, 0) = 0$, as we may always do. Now, for a finite-amplitude surface wave we must have

$$\int_0^T \eta \, \mathrm{d}t \to 0 \quad \text{as} \quad T \to 0^+,$$

but for an impulse we require

$$\int_0^T (P_s - P_a)\mathrm{d}t$$

to have a finite and nonzero limit as $T \to 0^+$; this is the *impulsive pressure*. Hence the required initial condition (for the concentric wave) is that $\phi(\mathbf{x}_\perp, h_0, 0)$ is to be specified. This condition is to be incorporated into the determination of $\eta(r, t)$ (where $\mathbf{x}_\perp \equiv (r, \theta)$, and there is no dependence here on θ).

It is clear that $\phi(r, 1, t)$ satisfies the same wave equation as $\eta(r, t)$, equation (2.14), essentially by virtue of the boundary condition

$$\phi_t + \eta = 0 \quad \text{on} \quad z = 1;$$

see equation (2.67) (and note that we have reverted to our non-dimensional variables). Thus, immediately, we have the appropriate solution for ϕ (cf. equation (2.30)) as

$$\phi(r, 1, t) = \int_0^\infty p\hat{f}(p) \cos\left(t\sqrt{\frac{p}{\delta}}\right) J_0(rp) \, \mathrm{d}p$$

where

$$\phi(r, 1, 0) = \int_0^\infty p\hat{f}(p) J_0(rp) \, \mathrm{d}p = f(r),$$

say, and $\eta(r, 0) = -\phi_t(r, 1, 0) = 0$. The impulse that we use (a point impulse) is modelled by

$$f(r) = \begin{cases} I, & 0 \le r \le a \\ 0, & r > a \end{cases}$$

with $a \to 0$, so that

$$\hat{f}(p) = I \int_0^a rJ_0(pr)\,\mathrm{d}r$$

$$\to Ia^2 \int_0^1 y\mathrm{d}y = \frac{1}{2}Ia^2 \quad \text{as} \quad a \to 0,$$

if Ia^2 is fixed. (We have written $r = ay$ here and used the familiar result $J_0(x) \to 1$ as $x \to 0$.) Thus, with $\hat{f}(p) = \frac{1}{2}Ia^2 = \beta$, say, we obtain

$$\eta(r, t) = -\phi_t(r, 1, t) = \frac{\beta}{\sqrt{\delta}} \int_0^\infty p^{3/2} \sin\left(t\sqrt{\frac{p}{\delta}}\right) J_0(rp)\mathrm{d}p \tag{2.114}$$

(from which we can determine the corresponding form taken by $\eta_t(r, 0)$; cf. Q2.19)). It is this solution, (2.114), for concentric waves on deep water generated by a point impulse, which we now examine.

This first stage of the calculation follows precisely that presented in Section 2.1.3. We introduce the integral representation of J_0 (see equation (2.31)) and then write η as the real part of the sum of two integrals (as in equation (2.32). This is transformed according to

$$p = \frac{t^2}{\delta r^2} q^2, \quad \sigma = \frac{t^2}{\delta r},$$

and then we use Kelvin's method of stationary phase for $\sigma \to \infty$ to give

$$\eta(r, t) \sim \frac{\beta}{8\sqrt{2}} \frac{1}{\delta^2} \frac{t^3}{r^4} \sin\left(\frac{1}{4} \frac{t^2}{\delta r}\right); \tag{2.115}$$

cf. equation (2.36). This calculation, which parallels that described in Section 2.1.3, is left as an exercise (Q2.50). Our task now is to incorporate this result into a description of the waves generated when the point impulse moves along a prescribed path.

The point impulse – a ship, perhaps – moves on the surface of the water along a path Γ, described in Cartesian coordinates by

$$\mathbf{x}_\perp \equiv (X(t), Y(t)),$$

where t is the time elapsed since the ship (let us call it that) was at the point P', namely at (X, Y); the ship is now at P, the origin of coordinates. The path is assumed smooth (so that both $X(t)$ and $Y(t)$ are (at least) once-differentiable functions) and then the X-axis is chosen to be tangent to Γ at P; all this is summarised in Figure 2.13. The ship, as it passes through P', initiates a disturbance there that propagates outwards and, in the direction defined by θ and at distance r from P', which has reached W. (The angle θ is measured relative to the (backwards) tangent to Γ at P'.) The disturbance at W is a distance r from P' (in the direction θ), and it has taken a time t to reach there; we shall assume, for the purposes of the following discussion, that the elevation of the wave at the time t, and distance r from P', where the impulse was applied at $t = 0$, is given by

$$\eta(r, t) = A\frac{t^3}{r^4}\sin\left(\frac{1}{4}\frac{t^2}{\delta r}\right), \tag{2.116}$$

where A is a constant; cf. equation (2.115). But the total disturbance at W will have contributions from all points along the path, to a greater or lesser extent (depending on the position of W). We therefore require the sum of all contributions like (2.116) over all time; however, it is clear that the integral of (2.116) in t over $[0, \infty)$ does not exist. We circumvent this difficulty by positing that the ship has been moving only for a finite time, T, say. Thus the total effect of all impulses along the path produces the amplitude

$$H(x, y) = A\int_0^T \frac{t^3}{r^4}\sin\left(\frac{1}{4}\frac{t^2}{\delta r}\right)\mathrm{d}t \tag{2.117}$$

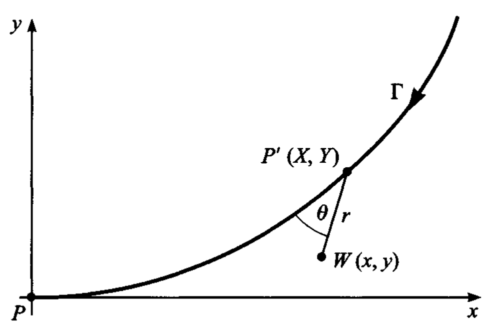


Figure 2.13. The path of the ship is Γ; the ship is now at P (the origin) and it was at P' at a time t earlier.

at W, where $r^2 = (x - X(t))^2 + (y - Y(t))^2$. (It is clear that this integral also does not exist for points *on* the ship's path, where $r = 0$; however, the method of stationary phase that led to (2.115) has already been interpreted only for points away from $r = 0$: we are seeking the wave pattern as seen some distance from the ship.)

To proceed, we express (2.117) in the form

$$H(x, y) = \mathscr{I}\left\{A \int_0^T \frac{t^3}{r^4} \exp(\mathrm{i}t^2/4\delta r)\mathrm{d}t\right\}, \tag{2.118}$$

and we have previously used $\sigma = t^2/\delta r \to \infty$; thus we may apply the method of stationary phase yet once more! The point(s) of stationary phase occur where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{r}\right) = 0 \quad \text{for} \quad r = r(t),$$

at fixed x, y: thus

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2r}{t}. \tag{2.119}$$

(The fact that we are treating $\sigma = \sigma(t)$, and $\sigma \to \infty$ is required for the method of stationary phase, is irrelevant in the application of the method.) But from

$$r^2 = (x - X(t))^2 + (y - Y(t))^2$$

we have (at fixed (x, y))

$$\begin{aligned} r\frac{\mathrm{d}r}{\mathrm{d}t} &= -\left\{(x - X)\frac{\mathrm{d}X}{\mathrm{d}t} + (y - Y)\frac{\mathrm{d}Y}{\mathrm{d}t}\right\} \\ &= -(x - X, y - Y) \cdot \left(\frac{\mathrm{d}X}{\mathrm{d}t}, \frac{\mathrm{d}Y}{\mathrm{d}t}\right) \\ &= rU(t)\cos\theta, \end{aligned}$$

where $U(t)$ $(= \sqrt{(\mathrm{d}X/\mathrm{d}t)^2 + (\mathrm{d}Y/\mathrm{d}t)^2})$ is the speed of the ship, so (since $r \neq 0$)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = U\cos\theta. \tag{2.120}$$

Thus the condition of stationary phase, (2.119), becomes

$$r = \frac{1}{2}Ut\cos\theta, \tag{2.121}$$

which represents (in plane polar coordinates) a circle of diameter $\frac{1}{2}Ut$, with the end of a diameter tangent to Γ at P' and the circle orientated from P$'$ towards P; see Figure 2.14. (If the ship is moving on a straight-line path – the x-axis – at constant speed, then this construction immediately recovers Figure 2.12.) It is only points on this circle that correspond to the points of stationary phase and therefore provide the dominant contribution to the wave amplitude. All points P$'$ which contribute, in this sense, to the disturbance at a given point W are usually called the *influence points* of W. A question that we might pose at this stage is: how many influence points are there, for a given W? For example, for constant speed, straight-line motion, it is a simple exercise to show that there are just two influence points (in general); see Q2.51 and Q2.52. This suggests that there are two families of curves that contribute to the ship-wave pattern.

The wave pattern, whose determination is our main goal, is obtained by constructing the lines of constant phase, consistent with the condition of stationary phase. We shall describe this calculation for the simple case of constant speed, straight-line motion; the corresponding problem for a circular course is set as an exercise (see Q2.53). The phase (see (2.118)) is proportional to t^2/r; it is convenient to introduce

$$\lambda = \frac{U^2t^2}{2r},$$

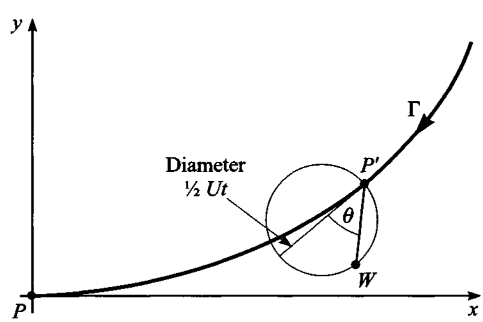


Figure 2.14. The position of the points of stationary phase (points W on the circle) for the disturbance initiated at P' at a time t earlier; the ship is now at P.

where U is the constant speed of the ship, and then λ = constant yields the curves of constant phase. But the condition of stationary phase, from (2.121), yields

$$rUt = \frac{1}{2}U^2t^2\cos\theta = \lambda r\cos\theta$$

so

$$\left.\begin{aligned} Ut &= \lambda\cos\theta, \\ &\text{and then} \\ r &= \frac{1}{2}Ut\cos\theta = \frac{1}{2}\lambda\cos^2\theta. \end{aligned}\right\} \tag{2.122}$$

(In fact, the equations (2.122) are valid for any path and, indeed, in the construction of these equations we may allow $U = U(t)$.) The path here is simply

$$X = Ut, \quad Y = 0,$$

and then any point W is

$$x = Ut - r\cos\theta, \quad y = -r\sin\theta,$$

where r and θ are shown in Figure 2.13. Thus, using equations (2.122), we obtain directly

$$x = \lambda(\cos\theta - \frac{1}{2}\cos^3\theta), \quad y = -\frac{1}{2}\lambda\cos^2\theta\sin\theta, \tag{2.123}$$

which are the parametric equations (parameter θ) for the dominant contribution to the wave pattern, each wave crest/trough being associated with a fixed value of λ. The pattern of wave crests (or troughs) is shown in Figure 2.15, which closely resembles the wave pattern produced in nature; compare this figure with Figure 2.10. Note that in this figure we have included points $r = 0$ (which do exist on the curve (2.123)) for completeness only.

Before we leave our discussion of this pattern, we comment that Figure 2.15 plainly shows two families of curves (exactly as we observe) which meet on the boundary of the region. Where these two families meet is quite significant; consider the derivatives obtained from equations (2.123):

$$\frac{dx}{d\theta} = \lambda(-\sin\theta + \frac{3}{2}\cos^2\theta\sin\theta) = \frac{1}{2}\lambda(1 - 3\sin^2\theta)\sin\theta$$

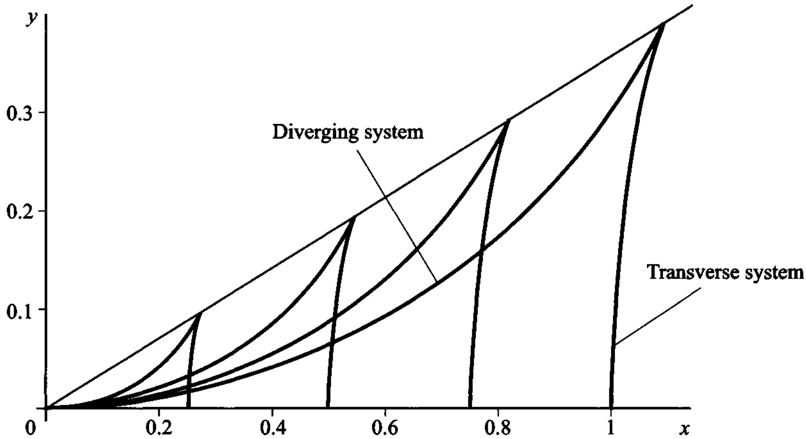


Figure 2.15. The ship-wave pattern as obtained from equations (2.123), for various values of λ (= 0.5, 1, 1.5, 2); the fine line denotes the boundary of the wedge which contains the dominant contributions.

and

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\frac{1}{2}\lambda(\cos^3\theta - 2\cos\theta\sin^2\theta) = -\frac{1}{2}\lambda(1 - 3\sin^2\theta)\cos\theta.$$

It is clear that $\mathrm{d}y/\mathrm{d}x$ is singular at $\theta = 0$ (which is where all curves meet at P) and also where $3\sin^2\theta - 1 = 0$; this defines the angle θ_0 that is attained where the two influence points coincide (and we note that this is the same for all waves, since it is independent of λ). This and other relevant points are included in Figure 2.15; in particular we see that the two families are defined by $0 \le \theta \le \theta_0$ and $\theta_0 \le \theta \le \pi/2$, respectively.

Finally, we make full use of Kelvin's method of stationary phase in order to provide an estimate for the wave amplitude along the lines of constant phase where the dominant contributions occur. To this end, we use the general result given in equation (2.35) and apply it to the integral in (2.118). Thus we require

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{t^2}{r}\right)$$

evaluated at the points of stationary phase; first we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{t^2}{r}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2t}{r}-\frac{t^2}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right)=\frac{2}{r}-\frac{4t}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}+\frac{2t^2}{r^3}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2-\frac{t^2}{r^2}\frac{\mathrm{d}^2r}{\mathrm{d}t^2}$$

which, on lines $\mathrm{d}r/\mathrm{d}t=2r/t$ (equation (2.119), for stationary phase), yields the expression

$$\frac{1}{r}\left(2-\frac{t^2}{r}\frac{\mathrm{d}^2r}{\mathrm{d}t^2}\right). \tag{2.124}$$

But on these lines we also have, (2.120),

$$\frac{\mathrm{d}r}{\mathrm{d}t}=U\cos\theta,$$

which gives

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=-U\frac{\mathrm{d}\theta}{\mathrm{d}t}\sin\theta$$

since $U=$ constant; now we must find $\mathrm{d}\theta/\mathrm{d}t$.

For the straight-line course (along $y=Y=0$), at constant speed U, we see that (cf. Figure 2.13)

$$\theta+\arctan\left(\frac{y}{Ut-x}\right)=\pi$$

with $X=Ut$. Thus, at fixed (x,y), we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}-\frac{yU}{(x-Ut)^2+y^2}=0$$

and we also have $\sin(\pi-\theta)=-y/r$ where $r^2=(x-Ut)^2+y^2$; hence

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{U}{r}\sin\theta$$

and so

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=\frac{U^2}{r}\sin^2\theta.$$

The expression (2.124) therefore becomes

$$\begin{aligned}\frac{1}{r}\left(2-\frac{U^2t^2}{r^2}\sin^2\theta\right)&=\frac{2}{r}(1-2\tan^2\theta)\\&=\frac{2}{r}(1-3\sin^2\theta)/\cos^2\theta,\end{aligned}\tag{2.125}$$

which we observe is zero at $\theta = \theta_0$ $(= \pm\arcsin(1/\sqrt{3}))$ where the two families of wave crests/troughs meet. Further, we must include two dominant contributions to the wave amplitude – one from each family (although, perhaps, we may find that one of these dominates the other). Since the two families are generated (in $y > 0$) by $0 \le \theta \le \theta_0$ and $\theta_0 \le \theta \le \pi/2$, respectively, we see from (2.125) that

$$\frac{d^2}{dt^2}\left(\frac{t^2}{r}\right) > 0 \quad \text{for} \quad 0 \le \theta < \theta_0$$

and

$$\frac{d^2}{dt^2}\left(\frac{t^2}{r}\right) < 0 \quad \text{for} \quad \theta_0 < \theta < \pi/2.$$

For a given point, W, we let the contribution in the range $0 \le \theta < \theta_0$ – usually called the *transverse* wave system – be designated by the subscript t, and for the other contribution, usually called the *diverging* system, we shall write the subscript d; this terminology is used in Figure 2.15.

The two terms that provide the dominant asymptotic behaviour (as $t^2/\delta r \to \infty$), according to Kelvin's result (2.35), therefore yield (after a little manipulation)

$$H \sim \mathscr{I}\left\{A\sqrt{\frac{\pi}{r}}\frac{\cos\theta_t \exp\{i(r/\delta\alpha_t^2 + \pi/4)\}}{\alpha_t^3\sqrt{1 - 3\sin^2\theta_t}}\right.$$
$$\left. + A\sqrt{\frac{\pi}{r}}\frac{\cos\theta_d \exp\{i(r/\delta\alpha_d^2 - \pi/4)\}}{\alpha_d^3\sqrt{3\sin^2\theta_d - 1}}\right\}. \qquad (2.126)$$

Here, we have substituted for time t from (2.121) and written

$$\alpha_q = \frac{1}{2}U\cos\theta_q \quad (q \equiv t, d).$$

The solution expressed by (2.126) is the final result that we present in this section. We see that both contributions are of the same order, that the amplitude decays like $r^{-1/2}$ away from the ship's path, but that the amplitude is undefined where the two families meet (at $\theta_t = \theta_d = \theta_0 = \arcsin(1/\sqrt{3})$). (The amplitude is also undefined where $\theta_d = \pi/2$, but this is at P, the origin, and is to be expected because of the nature of our point impulse model.) An analysis can be performed, by taking Kelvin's method of stationary phase to the next order, for the case $\theta_t = \theta_d = \theta_0$;

the wave amplitude can then be shown to be finite but now it behaves like $r^{-1/3}$ away from the ship's path. This goes some way to explaining why the waves near the edge of the wedge-shaped region are observed so readily: they are larger than those nearby and on each of the families separately. Finally, we note that the two systems of waves, transverse and diverging, have a phase difference of $\pi/2$ even when $\alpha_t \approx \alpha_d$. Thus we anticipate that near the edge of the wedge, where α_t and α_d are nearly equal, a phase difference will be evident; this is, indeed, seen in well-defined ship waves (and just hinted at in our Figure 2.10). The effect of the phase difference is to produce transverse and diverging systems that do not meet with a common tangent at $\theta = \theta_0$; this phenomenon is shown in Figure 2.16.

In summary, we have seen how the application of Kelvin's method of stationary phase – ultimately three times – enables us to provide a surprisingly accurate description of the ship-wave pattern. This approach is based on the point impulse model for the moving object (bird or ship) and, importantly, on the limiting process $t^2/r \to \infty$ (provided $r \neq 0$). We conclude by observing that this requires, first, that we are not at points on the ship's path. (Indeed, in practice, this region is the one that is significantly disturbed by the propulsion system, be it a screw-propeller or paddling feet.) Second, since $r \neq 0$, the limit must be interpreted as

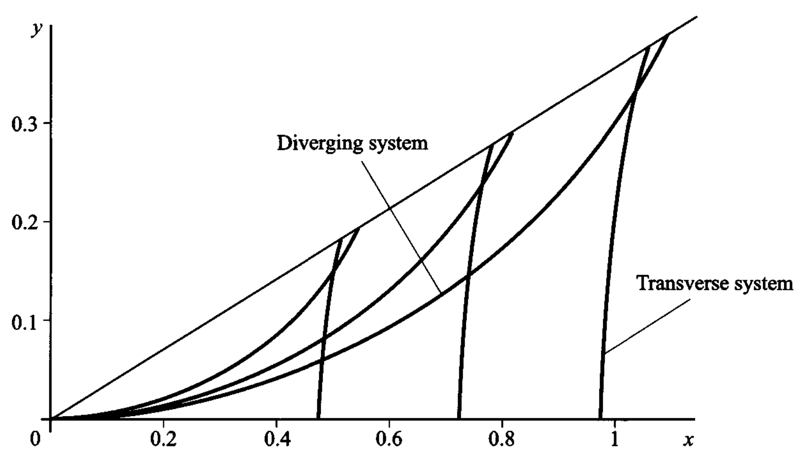


Figure 2.16. A more accurate version of the ship-wave pattern, with the phase difference between the transverse and diverging wave systems now evident (particularly near the edge of the region).

$t \to \infty$ sufficiently rapidly; that is, we are seeing the pattern well after the passing of the ship (and consequently well behind and far away from the ship). Of course this means, as mentioned earlier, that the precise nature of the object producing the waves will play no part in this theory.

2.4.2 Ray theory

In the previous section we described, with some care and in some detail, the important predictions first developed by Lord Kelvin. We now demonstrate how the salient features can be obtained directly from ray theory (Section 2.3).

We invoke ray theory by treating the problem as a stationary object (the ship), at the origin of the horizontal coordinate system, in the presence of a current. The current is, of course, just that required to bring the ship to a halt (and, as before, we suppose that the ship is moving in stationary water). Let the (steady) current be

$$\mathbf{U}_{\perp} \equiv (U(X, Y),\ V(X, Y)),$$

at least to $O(\alpha^2)$; cf. the discussion in Section 2.3.3. We seek waves that are steady, so $\omega =$ constant, where

$$\Omega + kU + lV = \omega$$

with

$$\Omega = -\frac{1}{\delta}\sqrt{\sigma \tanh\{\sigma(1+H)\}}, \qquad \sigma = \delta|\mathbf{k}|;$$

see equation (2.107). (We have chosen the sign of the square root to correspond to waves behind the ship in $X > 0$.)

We restrict the calculation to the case of deep water (equivalently, that is, for short waves), and so hereafter we write

$$\Omega = -\sqrt{|\mathbf{k}|/\delta}.$$

Further, since the ship waves are stationary – do not change with time – in the frame of reference fixed relative to the ship, we have $\omega = 0$. Thus

$$\sqrt{|\mathbf{k}|/\delta} = kU + lV,$$

which describes the relation between k and l (given U and V) for stationary waves to exist. Indeed, for $U =$ constant and $V = 0$, we obtain

$$c_p = \frac{1}{\sqrt{\delta|\mathbf{k}|}} = \frac{k}{|\mathbf{k}|}U = U\cos\theta \qquad (2.127)$$

exactly as in Section 2.4.1 (equation (2.113) and Figure 2.11(a)). The rays are described by

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}_\perp}{\mathrm{d}t} &= \mathbf{U} + \mathbf{c}_g \\ &\equiv (U + \partial\Omega/\partial k,\ V + \partial\Omega/\partial l) \\ &= \left(U - \frac{1}{2}c_p\frac{k}{|\mathbf{k}|},\ V - \frac{1}{2}c_p\frac{l}{|\mathbf{k}|}\right)\end{aligned}$$

or

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{V - \frac{1}{2}c_p l/|\mathbf{k}|}{U - \frac{1}{2}c_p k/|\mathbf{k}|};$$

see Figure 2.17. But from equation (2.127) we see that for $U =$ constant and $V = 0$ we may write this in the form

$$\tan\phi = \frac{\frac{1}{2}\sin\theta\cos\theta}{1 - \frac{1}{2}\cos^2\theta}, \qquad (2.128)$$

which determines ϕ in terms of θ; conversely, this equation can be rewritten as

$$2\tan\phi\tan^2\theta - \tan\theta + \tan\phi = 0.$$

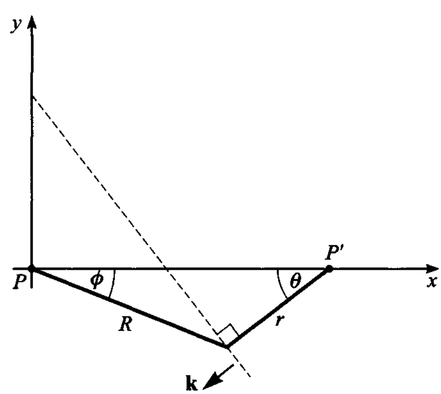


Figure 2.17. A wavefront, with wave number vector **k**, emanating from the point P'; the ray measured from P is at a distance R from P, and at an angle ϕ to the X-axis (measured in the negative sense, to be consistent with the direction in which θ is measured).

Thus the disturbance at any point on a given ray is contributed to by two waves, in general, determined by two values of θ – which, of course, correspond to the two influence points that we introduced earlier.

The solution for $\tan\theta$ is immediately

$$\tan\theta = \frac{1}{4}\cot\phi\left(1 \pm \sqrt{1 - 8\tan^2\phi}\right),$$

so two solutions exist for $\tan^2\phi < 1/8$, but no (real) solutions exist for $\tan^2\phi > 1/8$. The two wave systems (to use the terminology of the previous analysis) coincide where

$$\tan^2\phi = 1/8 \quad \text{or} \quad \sin^2\phi = 1/9,$$

which recovers our result for the angle of the wedge inside which the dominant disturbance occurs.

We now turn to the determination of the lines of constant phase, $\Theta =$ constant. This requires us to find the appropriate solution of the eikonal equation

$$\Theta_X^2 + \Theta_Y^2 = |\mathbf{k}|^2,$$

where $|\mathbf{k}|^2 = \sec^4\theta/\delta^2 U^4$ (from equation (2.127) with $U =$ constant). It is convenient to express this equation in polar coordinates defined at the origin of (X, Y), which here we write as (R, ϕ); see Figure 2.17. Thus we have

$$\Theta_R^2 + \frac{1}{R^2}\Theta_\phi^2 = \frac{\sec^4\theta}{\delta^2 U^4};$$

cf. equation (2.100). The relevant wavefronts are obtained by mapping the rays that correspond to the lines of constant phase, and the rays are radial lines ($\phi =$ constant) out from the origin. But, on the rays, θ and ϕ are related by equation (2.128) and so we seek a solution

$$\Theta = Rf(\theta);$$

thus

$$f^2 + (f')^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2 = \frac{\sec^4\theta}{\delta^2 U^4} \tag{2.129}$$

where

$$\frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{4 - 3\cos^2\theta}{3\cos^2\theta - 2}$$

(which follows directly from equation (2.128)).

It is a fairly straightforward exercise to show that equation (2.129) has a solution which is proportional to

$$\left(\cos\theta\sqrt{4-3\cos^2\theta}\right)^{-1};$$

the verification of this result is left as an exercise (and you may wish to find the constant of proportionality in this solution, but its precise form is irrelevant here). Thus the lines $\Theta =$ constant become

$$\frac{R}{\cos\theta\sqrt{4-3\cos^2\theta}} = \text{constant} = \frac{1}{2}\lambda, \text{ say.} \tag{2.130}$$

We revert to Cartesian coordinates in order to present the lines of constant phase, where we use

$$X = R\cos\phi = \frac{R(2-\cos^2\theta)}{\sqrt{4-3\cos^2\theta}} \quad (>0)$$

and

$$Y = -R\sin\phi = -\frac{R\sin\theta\cos\theta}{\sqrt{4-3\cos^2\theta}} \qquad (<0 \text{ for } 0<\phi<\pi).$$

(Again, these follow directly from equation (2.128), and we have chosen the signs of the square roots to be consistent with our definitions.) Inserting the expression for R from equation (2.130), we obtain

$$X = \lambda\cos\theta(1-\frac{1}{2}\cos^2\theta), \quad Y = -\frac{1}{2}\lambda\cos^2\theta\sin\theta$$

which is precisely the parametric form obtained in Section 2.4.1 (equation (2.123)). It is clear, however, that ray theory does not contain sufficient information to describe the phase difference along the edge of the wedge (which Kelvin's more complete wave theory produced). Finally, we comment that the equation for the wave action can be used to show that the amplitude of the dominant wave decays like $r^{-1/2}$ away from the ship's path (as previously given in equation (2.126)).

This concludes our presentations of various linear problems in the theory of water waves. As we have mentioned earlier, the exercises may be used to discover and investigate other interesting problems – but even these do not claim to be exhaustive. Additional material can be found in the books listed in the further reading at the end of this chapter.

II Nonlinear problems

A higher height, a deeper deep.
In Memoriam A.H.H. LXII

Our discussion thus far has been restricted to various problems in linear theory. These have been chosen for their mathematical content, and to give a flavour of the breadth of results that is available. We now turn to the more demanding arena that is the study of nonlinear wave propagation. As before, we shall continue our philosophy of selecting problems which contain interesting mathematical elements and which, for the most part, lay the foundations for our later presentations.

Most – but by no means all – of our earlier analyses have considered the case of gravity waves (which are, after all, the most relevant waves for the engineer involved in the design of ships, offshore platforms, or sea walls, to mention but three). Here, for all our work on nonlinear phenomena, we shall limit ourselves to the description of gravity waves. Thus, for the inviscid model with no surface tension ($W = 0$), we have (equations (1.67), (1.69) and (1.70))

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\\
&\text{and}\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\\
&\text{with}\\
&w = \eta_t + \varepsilon(u\eta_x + v\eta_y) \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta;
\end{aligned}\right\} \qquad (2.131)$$

$$w = ub_x + vb_y \quad \text{on } z = b,$$

written in Cartesian coordinates. Correspondingly, for irrotational flow (Q1.38), we have

$$\left.\begin{aligned}
&\phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0 \\
&\text{with} \\
&\left.\begin{aligned}
&\phi_z = \delta^2\{\eta_t + \varepsilon(\phi_x\eta_x + \phi_y\eta_y)\}; \\
&\phi_t + \eta + \frac{1}{2}\varepsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2 + \phi_y^2\right) = 0
\end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta, \\
&\text{and} \\
&\phi_z = \delta^2(\phi_x b_x + \phi_y b_y) \text{ on } z = b.
\end{aligned}\right\} \qquad (2.132)$$

(In both these sets of equations we have taken the bed of the flow to be steady; that is, $z = b(x, y)$.)

For what we present here, we shall no longer consider the approximate equations obtained by setting $\varepsilon = 0$: of course, the inclusion of nonlinearity requires $\varepsilon \neq 0$. 'Full' nonlinearity occurs for ε fixed, that is, O(1), even if $\delta \to 0$; indeed, in this case, we may just as well set $\varepsilon = 1$. (This is equivalent to scaling the wave on the typical undisturbed depth of the water.) The consequences of allowing the strongest possible contribution from the nonlinearity will be the basis for much of what we shall present here, but we start with a simpler problem: $\varepsilon \to 0$. This is the problem first discussed by G. G. Stokes in 1847, and aims to produce higher approximations to the (linear) oscillatory wave (given in Section 2.1). This approach, as we shall see, is in the spirit of much that we shall present in later chapters.

2.5 The Stokes wave

The problem that we present here is that of determining the solution of equations (2.132) (which describe irrotational flow) as an asymptotic solution for $\varepsilon \to 0$ (at fixed δ). This is to be compared with the analysis given in Section 2.1 where only the first approximation was obtained (and there we worked from Euler's equation with the effects of surface tension retained). Here, we shall seek a solution in the form

$$Q \sim \sum_{n=0}^{\infty} \varepsilon^n Q_n,$$

where Q (and correspondingly Q_n) represents each of ϕ and η; these expansions, or more precisely, those for the velocity components (ϕ_x, ϕ_z), and for η, are to be uniformly valid as $t \to \infty$ and as $|x| \to \infty$. We shall restrict the discussion to plane harmonic waves that travel in the x-direction. The undisturbed water is stationary and the depth is constant (so we set $b = 0$).

The procedure is, in principle, altogether straightforward, but complications do arise, not least because of the nature of the surface boundary conditions. To be consistent with our assumed form of solution (in powers of ε) we must expand these boundary conditions about $z=1$. (Strictly, any requirement on the convergence of the series that we generate is unnecessary: our solution need only satisfy the conditions laid down for *asymptotic* validity as $\varepsilon \to 0$.) In addition, to describe the harmonic wave we introduce the phase variable

$$\theta = kx - \omega t$$

where we shall regard the wave number, k, as prescribed. Equations (2.132) now become

$$\phi_{zz} + \delta^2 k^2 \phi_{\theta\theta} = 0,$$

with

$$\phi_z + \varepsilon\eta\phi_{zz} + \frac{1}{2}\varepsilon^2\eta^2\phi_{zzz} = -\delta^2\omega\eta_\theta + \varepsilon\delta^2k^2\eta_\theta(\phi_\theta + \varepsilon\eta\phi_{\theta z}) + \mathrm{O}(\varepsilon^3);$$

$$\eta + \alpha - \omega(\phi_\theta + \varepsilon\eta\phi_{\theta z} + \frac{1}{2}\varepsilon^2\eta^2\phi_{\theta zz})$$
$$+\frac{1}{2}\varepsilon\left\{\frac{1}{\delta^2}\left(\phi_z^2 + 2\varepsilon\eta\phi_z\phi_{zz}\right) + k^2(\phi_\theta^2 + 2\varepsilon\eta\phi_\theta\phi_{\theta z})\right\} = \mathrm{O}(\varepsilon^3)$$

both on $z=1$, and

$$\phi_z = 0 \quad \text{on} \quad z = 0,$$

where we have incorporated the convenient shift $\phi \to \alpha t + \phi$ (which we shall discuss below; also cf. equation (1.23)). It will transpire that, in order to find a uniform solution, we must expand the frequency in terms of ε; thus we write

$$\omega \sim \sum_{n=0}^{\infty} \varepsilon^n \omega_n,$$

where the ω_n are constants, which may depend on k. This dependence of ω on ε, essentially the amplitude, is a very significant result, as we shall see.

The expansions for ϕ, η, and the constant α (treated like ω) are used in the above equations; the leading-order problem is then

$$\phi_{0zz} + \delta^2 k^2 \phi_{0\theta\theta} = 0$$

with

$$\phi_{0z} = -\omega_0 \delta^2 \eta_{0\theta} \quad \text{and} \quad \eta_0 + \alpha_0 - \omega_0 \phi_{0\theta} = 0 \quad \text{on} \quad z = 1$$

and

$$\phi_{0z} = 0 \quad \text{on} \quad z = 0.$$

This is the familiar and standard problem; see Section 2.1 and Q2.5. The solution for a single harmonic wave is

$$\left.\begin{aligned} \eta_0 &= AE + \text{c.c.}, \quad \alpha_0 = 0, \\ \phi_0 &= -\frac{\mathrm{i}A}{\omega_0}\frac{\cosh(\delta k z)}{\cosh(\delta k)}E + \text{c.c.}, \end{aligned}\right\} \tag{2.133}$$

where $E = \exp(\mathrm{i}\theta)$, A is a complex constant and c.c. denotes the complex conjugate. We see that this solution does not require a contribution from α, but it exists only if

$$\omega_0^2 = \frac{k}{\delta}\tanh(\delta k); \tag{2.134}$$

cf. equations (2.9) and (2.13).

At the next order we obtain the equations

$$\phi_{1zz} + \delta^2 k^2 \phi_{1\theta\theta} = 0$$

with

$$\left.\begin{aligned} \phi_{1z} + \eta_0 \phi_{0zz} &= -\delta^2(\omega_1 \eta_{0\theta} + \omega_0 \eta_{1\theta}) + \delta^2 k^2 \eta_{0\theta}\phi_{0\theta}; \\ \eta_1 + \alpha_1 - (\omega_1 \phi_{0\theta} + \omega_0 \phi_{1\theta}) - \omega_0 \eta_0 \phi_{0\theta z} &+ \frac{1}{2}\left(\frac{1}{\delta^2}\phi_{0z}^2 + k^2 \phi_{0\theta}^2\right) = 0 \end{aligned}\right\} \text{ on } z = 1$$

and

$$\phi_{1z} = 0 \quad \text{on} \quad z = 0.$$

We could include a term in the solution of this problem which contributes to the first harmonic ($E^{\pm 1}$), but we choose not to do so; the amplitude of the first harmonic (to this order) is taken to be A. However, the surface boundary conditions *do* include terms $E^{\pm 1}$, but these are completely eliminated if we choose $\omega_1 = 0$; thus we set $\omega_1 = 0$. Now we seek a solution

$$\eta_1 = A_1 E^2 + \text{c.c.}, \quad \phi_1 = B_1 \cosh(2\delta k z)E^2 + \text{c.c.},$$

and α_1 (a real constant) will be required here to remove the non-periodic term E^0 (which is generated by the product E^1E^{-1}). The corresponding terms in the first boundary condition exactly cancel.

Our solution for ϕ_1 satisfies Laplace's equation and the bottom boundary condition; the other two boundary conditions (on $z = 1$) yield

$$kB_1 \sinh(2\delta k) + \mathrm{i}\delta\omega_0 A_1 = \mathrm{i}\frac{\delta k^2}{\omega_0}A^2;$$
$$A_1 - 2\mathrm{i}\omega_0 B_1 \cosh(2\delta k) = \delta k A^2 \tanh(\delta k) - \delta k A^2 \operatorname{cosech}(2\delta k),$$

with

$$\alpha_1 = -2\delta k|A|^2 \operatorname{cosech}(2\delta k).$$

We see that the term α_1 is needed here; it can be associated with the arbitrary function, $f(t)$, that appears in the pressure equation, (1.23). It might be thought that such a term could not appear after we have introduced appropriate conditions at infinity; see equation (1.29). However, once we have fixed the undisturbed surface level at $\eta = 0$, the constant pressure condition has to be maintained in this way if the nonlinearity is also included. There is, nevertheless, an alternative which allows $\alpha_1 = 0$: this is to redefine the undisturbed water level as

$$\eta \sim -2\varepsilon\delta k|A|^2 \operatorname{cosech}(2\delta k),$$

which hydraulic engineers usually call the *set-down*. In any event, we see that the term αt does not contribute to the velocity components (ϕ_x, ϕ_z).

To proceed, we solve for A_1 and B_1 and simplify, to give

$$A_1 = A^2 \delta k \coth(\delta k)\left\{1 + \frac{3}{2}\operatorname{cosech}^2(\delta k)\right\}, \quad B_1 = -\mathrm{i}A^2 \frac{3}{4}\delta^2 \omega_0 \operatorname{cosech}^4(\delta k).$$

Thus we have, so far, the asymptotic solution

$$\eta \sim AE + \varepsilon A^2 E^2 \delta k \coth(\delta k)\left\{1 + \frac{3}{2}\operatorname{cosech}^2(\delta k)\right\} + \text{c.c.} \tag{2.135}$$

and

$$\phi \sim -\frac{\mathrm{i}A}{\omega_0} E \operatorname{sech}(\delta k) \cosh(\delta k z) - 2\varepsilon\delta k|A|^2 t \operatorname{cosech}(2\delta k)$$
$$- \mathrm{i}\varepsilon A^2 \frac{3}{4}\delta^2 \omega_0 E^2 \operatorname{cosech}^2(\delta k) \cosh(2\delta k z) + \text{c.c.}, \tag{2.136}$$

both as $\varepsilon \to 0$. The non-uniformity implied by the contribution from αt, as $t \to \infty$, appears only in the expansion of ϕ; the relevant asymptotic expansions are for ϕ_x (that is, ϕ_θ) and ϕ_z, which do not contain this term. We now examine the next order, but only to demonstrate the rôle of ω_2 and how we determine its value.

The terms at $O(\varepsilon^2)$ yield the equations

$$\phi_{2zz} + \delta^2 k^2 \phi_{2\theta\theta} = 0,$$

with

$$\phi_{2z} + \eta_0\phi_{1zz} + \eta_1\phi_{0zz} + \frac{1}{2}\eta_0^2\phi_{0zzz}$$
$$= -\delta^2(\omega_0\eta_{2\theta} + \omega_2\eta_{0\theta}) + \delta^2k^2(\eta_{0\theta}\phi_{1\theta} + \eta_{1\theta}\phi_{0\theta} + \eta_0\eta_{0\theta}\phi_{0\theta z});$$
$$\eta_2 + \alpha_2 - \omega_0(\phi_{2\theta} + \eta_0\phi_{1\theta z} + \eta_1\phi_{0\theta z} + \frac{1}{2}\eta_0^2\phi_{0\theta zz}) - \omega_2\phi_{0\theta}$$
$$+ \frac{1}{\delta^2}(\phi_{0z}\phi_{1z} + \eta_0\phi_{0z}\phi_{0zz}) + k^2(\phi_{0\theta}\phi_{1\theta} + \eta_0\phi_{0\theta}\phi_{0\theta z}) = 0$$

both on $z = 1$, and

$$\phi_{2z} = 0 \quad \text{on} \quad z = 0.$$

To find ω_2 we must be more circumspect in our treatment of these equations than we were for ω_1. Here, the boundary conditions on $z = 1$ include terms $E^{\pm 1}$ which cannot be eliminated; thus our solution for ϕ and η must include these terms. But we know that the combinations $\phi_{2z} + \delta^2\omega_0\eta_{2\theta}$ and $\eta_2 - \omega_0\phi_{2\theta}$ (evaluated on $z = 1$) are essentially identical when evaluated from terms in $E^{\pm 1}$ and the expression for ω_0 is invoked. (This was how we determined ω_0 in the first place.) To be consistent, the same property must obtain for all the terms in $E^{\pm 1}$; this is possible only for one choice of ω_2. Let us now fill in some of the details in this calculation.

If we write

$$\eta_2 = A_2E + \text{c.c.}, \qquad \phi_2 = B_2\cosh(\delta kz)E + \text{c.c.}$$

(which would constitute one part of the complete solution for η_2 and ϕ_2), then, on $z = 1$,

$$\begin{aligned}\phi_{2z} + \delta^2\omega_0\eta_{2\theta} &= \delta kB_2\sinh(\delta k)E + \mathrm{i}A_2\delta^2\omega_0E + \text{c.c.}\\ &= \mathrm{i}\delta^2\omega_0\left(A_2E - \mathrm{i}\frac{k}{\delta}\frac{B_2}{\omega_0}\sinh(\delta k)E\right) + \text{c.c.}\\ &= \mathrm{i}\delta^2\omega_0(A_2E - \mathrm{i}\omega_0B_2\cosh(\delta k)E) + \text{c.c.}\\ &= \mathrm{i}\delta^2\omega_0(\eta_2 - \omega_0\phi_{2\theta})\end{aligned}$$

where we have used equation (2.134) for ω_0^2. Therefore we form $\mathrm{i}\delta^2\omega_0 \times$ (second boundary condition on $z = 1$) and subtract the first boundary condition, but we retain only the terms in E^1 (which can arise here from the products E^2E^{-1} and E^1E^0); these terms are to be absent from the combined boundary conditions, thereby fixing ω_2. After some rather tedious algebra, we find that the appropriate choice is

$$\omega_2 = \frac{1}{4}\delta^2k^2\omega_0|A|^2\{8\coth^2(\delta k) + 9\,\mathrm{cosech}^4(\delta k)\},$$

so the dispersion function becomes

$$\omega \sim \omega_0 + \frac{\varepsilon^2}{4}\delta^2k^2\omega_0|A|^2\{8\coth^2(\delta k) + 9\,\mathrm{cosech}^4(\delta k)\} \tag{2.137}$$

where ω_0 is obtained from equation (2.134).

The significant result embodied in equation (2.137), and first described by Stokes, is that the frequency (and hence the phase speed) now depends on the amplitude of the wave. This is a fundamental property of nonlinear waves, and has no counterpart in linear theory (but remember that, in linear theory, water waves are dispersive, so their speed does still depend on the wave number). In particular we see that

$$c_p \sim c_{p0}\left\{1 + \frac{\varepsilon^2}{4}\delta^3k^2|A|^2[8\coth^2(\delta k) + 9\,\mathrm{cosech}^4(\delta k)]\right\},$$

where $c_{p0} = \omega_0/k$ is the speed of linear waves; here, waves of larger amplitude travel faster (although we are still restricted by the small-amplitude assumption implied by $\varepsilon \to 0$).

Furthermore, the inclusion of higher-order terms in the representation of the surface profile (equation (2.135)) distorts its shape away from the (linear) sinusoidal curve. The effects of the nonlinearity are to make peaks narrower (sharper) and the troughs flatter; this tendency is depicted in Figure 2.18. The resulting profile more accurately portrays the gravity waves that are observed in nature. Later (Section 2.9) we shall describe more fully the characteristics of certain nonlinear waves for which the Stokes expansion can give only a hint.

Before we leave the Stokes expansion, we make two observations. First, we have presented the results for arbitrary wavelength (or depth); clearly, we may approximate further for long waves (or shallow water) and for short waves (or deep water). For example, from (2.137) and (2.135), we obtain

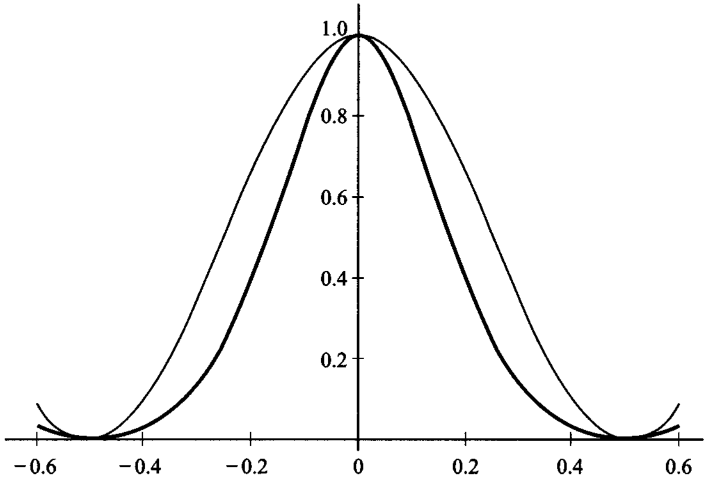


Figure 2.18. A nonlinear wave (——) and a corresponding linear wave (——) for comparison; the waves have been drawn with the same amplitude and the same period.

$$\omega \sim k\left\{1 - \frac{1}{6}\delta^2k^2 + \frac{9}{4}\frac{\varepsilon^2|A|^2}{\delta^2k^2}\right\} \quad \text{as} \quad \delta \to 0,$$

and

$$\omega \sim \sqrt{\frac{k}{\delta}}\left\{1 + \frac{1}{2}\varepsilon^2|A|\delta^2k^2\right\} \quad \text{as} \quad \delta \to \infty,$$

provided we also have $(\varepsilon/\delta) \to 0$ in the former, and $(\varepsilon\delta) \to 0$ in the latter. (These simple derivations are left as an exercise.) Now, second, we may use our more complete results to compute, for example, the correct average mass flux in the water as the wave propagates; see Q2.32. Previously we calculated as far as $O(\varepsilon^2)$, yet the solution to this order was unknown. We have

$$\mathscr{F} = \frac{\varepsilon}{2\pi}\int_0^{2\pi}\int_0^{1+\varepsilon\eta} u \,\mathrm{d}z\,\mathrm{d}\theta$$

where

$$\eta \sim AE + \varepsilon A_1 E^2 + \text{c.c.}$$

and

$$u \sim \frac{AE}{\omega_0} \frac{\cosh(\delta kz)}{\cosh(\delta k)} + \varepsilon 2\mathrm{i}B_1 E^2 \cosh(2\delta kz) + \text{c.c.};$$

see equations (2.133), (2.135) and (2.136). Thus we obtain

$$\mathscr{F} \sim \frac{\varepsilon}{2\pi} \int_0^{2\pi} \left\{ \frac{AE}{\delta k\omega_0} \frac{\sinh[\delta k(1+\varepsilon AE)]}{\cosh(\delta k)} + \frac{\varepsilon \mathrm{i}B_1 E^2}{\delta k} \sinh(2\delta k) + \text{c.c.} \right\} \mathrm{d}\theta$$

from which it is clear that the term at $O(\varepsilon)$ in the expression for u does not contribute at $O(\varepsilon^2)$ (because it is periodic in θ). The non-periodic term arising from the expansion of $\sinh[\delta k(1+\varepsilon AE)]$, exactly as in Q2.32, provides the $O(\varepsilon^2)$ term in $\mathscr{F}$. The conclusion we reached in Q2.32, it turns out, is correct: there is a mass flux of $O(\varepsilon^2)$ generated by the passage of the $O(\varepsilon)$ surface wave. (This is discussed further in Q4.4.)

2.6 Nonlinear long waves

We now undertake our first examination of a set of equations that describe fully nonlinear wave propagation. To simplify matters, we restrict the discussion to waves that are propagating in only one (spatial) dimension and, most importantly, we shall invoke the condition for long waves. From equations (2.131), for propagation in the x-direction and with the bed fixed at $z = 0$, we obtain

$$\begin{aligned} u_t + \varepsilon(uu_x + wu_z) &= -p_x; \\ \delta^2\{w_t + \varepsilon(uw_x + ww_z)\} &= -p_z; \\ u_x + w_z &= 0, \end{aligned}$$

with

$$w = \eta_t + \varepsilon u\eta_x \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

Then for long waves (or shallow water) we impose the condition $\delta \to 0$, so

$$p_z = O(\delta^2),$$

and the first approximation for p requires that

$$p = \eta,$$

everywhere. The equations are now reduced to

$$u_t + \varepsilon(uu_x + wu_z) = -\eta_x; \quad u_x + w_z = 0, \tag{2.138}$$

with

$$w = \eta_t + \varepsilon u\eta_x \text{ on } z = 1 + \varepsilon\eta; \quad w = 0 \text{ on } z = 0,$$

to leading order as $\delta \to 0$.

These equations admit a solution for which $u = u(x, t)$ (which is the only solution if, somewhere, u is independent of z, for then it will remain so); thus w_z $(= -u_x)$ is independent of z, so

$$w = \left(\frac{\eta_t + \varepsilon u\eta_x}{1 + \varepsilon\eta}\right)z,$$

where the boundary conditions have been used. The two equations in (2.138) therefore become

$$u_t + \varepsilon uu_x + \eta_x = 0;$$
$$(1 + \varepsilon\eta)u_x + \eta_t + \varepsilon u\eta_x = 0,$$

where we have made no assumption about the size of ε. We wish to retain 'full' nonlinearity, so that $\varepsilon = \mathrm{O}(1)$ as $\delta \to 0$; it is therefore convenient to set $\varepsilon = 1$ and to write the surface as

$$1 + \eta(x, t) = h(x, t).$$

Our pair of equations are then expressed as

$$u_t + uu_x + h_x = 0; \quad h_t + (hu)_x = 0; \tag{2.139}$$

these are often called the *shallow water* equations (for obvious reasons). The important simplifying assumption that leads to these equations is, of course, $\delta \to 0$; this, in turn, implies that $p = \eta$ (to leading order), which means that the pressure is everywhere dominated by the hydrostatic pressure distribution (see Q1.11). The higher-order corrections to the pressure, as the wave propagates, are ignored in this model.

An interesting observation about our equations (2.139) is made if we write

$$h(u_t + uu_x) = -hh_x = -\left(\frac{1}{2}h^2\right)_x,$$

for then the shallow water equations take the form

$$u_t + uu_x = -\frac{1}{\rho}p_x, \quad \rho_t + (\rho u)_x = 0, \quad p = \frac{1}{2}\rho^2, \tag{2.140}$$

where ρ is written for h. Equations (2.140) are the equations of one-dimensional gas dynamics, for the *adiabatic law* $P \propto \rho^2$, that is, $p \propto \rho^\gamma$, $\gamma = 2$; see equations (1.2) and (1.12). Of course, for a real gas, $\gamma = 2$ cannot be realised; nevertheless, our equations (2.139) are identical in structure to the appropriate equations of gas dynamics (and, as such, are the basis for demonstrating some steady gas-dynamic flow phenomena on a water table). All this means that we may take over much of the analysis and discussion pertinent to gas dynamics. This we shall now do, at least in part, but we shall provide all the relevant information and derivations.

2.6.1 The method of characteristics

We have already found (Section 2.1), for long waves, that the speed of propagation of small amplitude waves is $\sqrt{gh_0}$ (in dimensional variables; see equation (2.11)). Here we are also working with long waves, so we might hope that a similar result obtains. Of course, our equations (2.139) are fully nonlinear, which could lead to some doubt about the validity of this proposition. To see that there is a connection, we introduce the definition

$$c(x,t) = \sqrt{h}, \tag{2.141}$$

which is the nondimensional equivalent of $\sqrt{gh_0}$ (and which also avoids the restriction to small amplitude waves). We note that h is the total depth, and so $h \geq 0$. Equations (2.139) then become

$$u_t + uu_x + 2cc_x = 0;$$

$$2cc_t + c^2u_x + 2ucc_x = 0 \quad \text{or} \quad (2c)_t + u(2c)_x + cu_x = 0,$$

which are added to give

$$(u + 2c)_t + u(u + 2c)_x + 2cc_x + cu_x = 0$$

and subtracted to give

$$(u - 2c)_t + u(u - 2c)_x + 2cc_x - cu_x = 0.$$

This pair of equations is rewritten in the form

$$\left\{\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right\}(u+2c)=0;$$

$$\left\{\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x}\right\}(u-2c)=0,$$

which can be solved directly (cf. equation (1.84)) to give

$$\left.\begin{aligned} u+2c=&\ \text{constant on lines } C^+\text{: } \frac{\mathrm{d}x}{\mathrm{d}t}=u+c;\\ u-2c=&\ \text{constant on lines } C^-\text{: } \frac{\mathrm{d}x}{\mathrm{d}t}=u-c,\end{aligned}\right\} \tag{2.142}$$

by the method of characteristics.

The lines (C^+, C^-) are the two families of characteristic lines, and the functions $(u \pm 2c)$, which are constant on their respective lines, are usually called the *Riemann invariants*. We see that these characteristic lines describe propagation at a speed $(\mathrm{d}x/\mathrm{d}t)$ that is either upstream or downstream $(\mp c)$ relative to the flow speed (u). The (implicit) solution can be expressed in the form

$$\left.\begin{aligned} u+2c=f(\alpha),&\quad \alpha \text{ constant on lines } \frac{\mathrm{d}x}{\mathrm{d}t}=u+c;\\ u-2c=g(\beta),&\quad \beta \text{ constant on lines } \frac{\mathrm{d}x}{\mathrm{d}t}=u-c,\end{aligned}\right\} \tag{2.143}$$

where f and g are arbitrary functions. The problem is then completely described if we are given, for example, the initial $(t=0)$ distribution (as a function of x) of both u and c (that is, h); this will prescribe both $f(\cdot)$ and $g(\cdot)$.

A particularly important and special class of solutions is obtained when one of the Riemann invariants (f or g) is constant *everywhere* (or at least constant where we seek a solution). These special types of solution are called *simple waves*. As an example, let us consider the propagation of a wave moving only rightwards into stationary water of constant depth $h=h_0$. All the C^- characteristics emanate from the undisturbed region (see Figure 2.19), so

$$u-2c=g=-2c_0,$$

since $u=0$ here and we have written $c_0=\sqrt{h_0}$. Now, $u-2c$ is constant everywhere and $u+2c$ is constant on C^+ characteristics, so u and c are constant on these C^+ lines; hence

$$x-(u+c)t=\alpha \quad \text{and then} \quad u+2c=f\{x-(u+c)t\}.$$

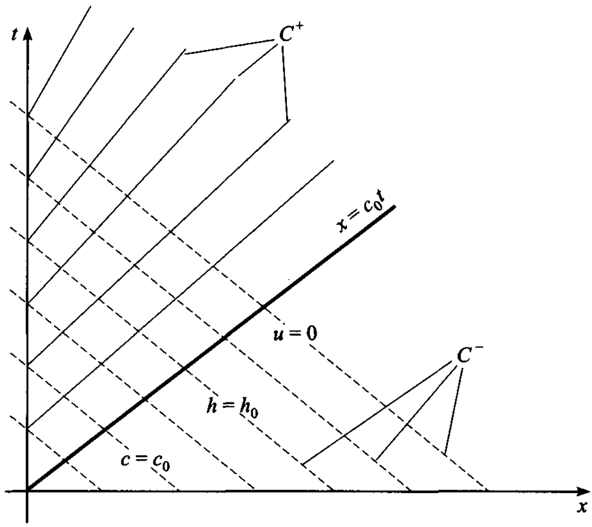


Figure 2.19. The characteristic lines, C^+ and C^-, for a wave moving rightwards into stationary water ($u = 0$) of constant depth ($h = h_0$) in $x > 0$.

On $t = 0$ we prescribe

$$h = H(x)$$

and so

$$f(x) = u + 2c = 4c - 2c_0 = 4\sqrt{H(x)} - 2\sqrt{h_0}.$$

Thus we have

$$u + 2c = 4\sqrt{H\{x - (u + c)t\}} - 2\sqrt{h_0},$$

and so

$$h(x, t) = H\{x - (u + \sqrt{h})t\}$$

where

$$u(x, t) = 3\{\sqrt{h(x, t)} - \sqrt{h_0}\}$$

which means that we can write, finally,

$$h(x, t) = H\{x - (3\sqrt{h} - 2\sqrt{h_0})t\} \tag{2.144}$$

the implicit solution for $h(x, t)$, given $H(x)$ and h_0. If the initial profile, $H(x)$, incorporates any wave of elevation (that is, $H(x) > 0$ for some x), then the solution given by (2.144) will eventually 'break' (in the sense that the characteristic lines then cross; cf. equation (1.85) *et seq.*, and Figures 1.5 and 1.6).

A second example, which is very much a classical one, is the problem of the 'dam break'. Although much is lost in the use of our shallow water equations in modelling this situation, these equations do capture the essential features of the resulting flow. Furthermore, this does prove to be an interesting – and surprisingly simple – application of equations (2.142) (or (2.143)). At time $t = 0$ the dam is broken, and therefore at this instant we suppose that $u = 0$ everywhere and that

$$h(x) = \begin{cases} h_0, & x < 0 \\ 0, & x > 0, \end{cases}$$

where h_0 (> 0) is constant. This represents (at $t = 0$) a vertical wall of water behind which the water is at rest at a constant depth. Our problem is therefore modelled by the instantaneous removal of the vertical retaining wall: hence the dam break problem.

Now, on the C^+ characteristics which emanate from the region $x < 0$ (where the water is situated at $t = 0$), we have that $u = 0$ and $c = \sqrt{h_0}$ there, and so

$$u + 2c = 2\sqrt{h_0} = \text{constant}$$

everywhere in the flow. Further, it is clear that an infinity of characteristic lines, each of different slope, will emerge from the origin $x = t = 0$ because of the step in $h(x)$ at $t = 0$. (That is, at $x = t = 0$, h must take all values $0 \le h \le h_0$ and each h determines the slope of a characteristic line.) To accommodate this phenomenon we require a degenerate form of the characteristic solution.

The C^- characteristics are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u - c$$

on which $u - 2c = $ constant; but $u + 2c$ is the same constant ($= 2\sqrt{h_0}$) everywhere (the simple wave condition), and so, corresponding to the first example, on C^- lines u, c, and then $u - c$ are constant. Hence the C^- characteristics are

$$x = (u - c)t + \text{constant} = (u - c)t$$

since all these lines pass through (0, 0); this pattern of characteristic lines is usually called an *expansion fan* (see Figure 2.20). Thus we have

$$u + 2c = 2\sqrt{h_0}, \quad u - c = x/t, \quad (c = \sqrt{h}),$$

which is the solution, for we now obtain

$$\sqrt{h} = \frac{1}{3}(2\sqrt{h_0} - x/t); \quad u = \frac{2}{3}(\sqrt{h_0} + x/t). \tag{2.145}$$

This solution is defined in the wedge (in (x, t)-space) from where $h = h_0$ to where $h = 0$, namely

$$-\sqrt{h_0} \leq x/t \leq 2\sqrt{h_0},$$

since

$$x/t = u - \sqrt{h} = 2\sqrt{h_0} - 3\sqrt{h}.$$

This solution describes an evolving surface profile, which is represented by the parabola

$$h(x, t) = \frac{1}{9}(2\sqrt{h_0} - \frac{x}{t})^2, \quad -\sqrt{h_0} \leq \frac{x}{t} \leq 2\sqrt{h_0},$$

at any fixed $t > 0$. In particular, at $x/t = 2\sqrt{h_0}$, we have $h = 0$: the wave front moves forward at a speed $2\sqrt{h_0}$. Correspondingly at $x/t = -\sqrt{h_0}$, where $h = h_0$, the uppermost point of the collapsing wall of water moves

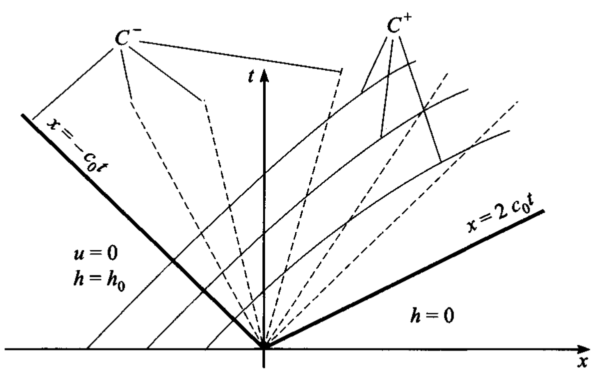


Figure 2.20. The characteristic lines, C^+ and C^-, for the dam-break problem; at $t = 0$ the water exists only in $x < 0$, where it is stationary ($u = 0$) and of constant depth ($h = h_0$).

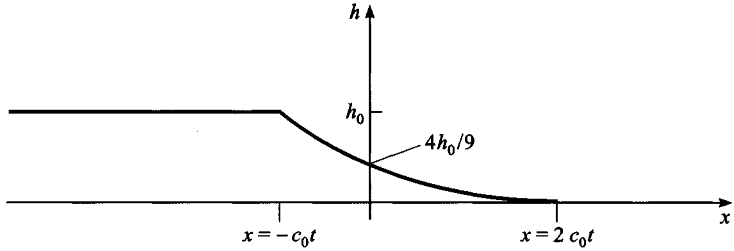


Figure 2.21. The surface profile at a time t after the dam has broken.

backwards at a speed $\sqrt{h_0}$; the profile is shown in Figure 2.21. Finally, we observe from solution (2.145) that, at $x = 0$ (which marks the initial position of the dam wall), the depth of the water remains at the constant value $4h_0/9$ for $t > 0$; indeed, as $t \to \infty$, the depth approaches this same constant value $(4h_0/9)$ everywhere.

These two examples that we have described are particularly straightforward because we have been able to incorporate the idea of a simple wave; some other related problems will be found in Q2.55–2.57. Of course, not all problems can be treated in this manner; certainly, if nontrivial (that is, variable) information is carried by both sets of characteristics then a more general approach must be adopted. This is what we now describe.

2.6.2 The hodograph transformation

A technique that is sometimes employed in the solution of ordinary differential equations is to interchange the rôles of the dependent and independent variables. So, for example, the equation

$$\{xf(y) + g(y)\}\frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$

which is, in general, nonlinear, nonseparable, and nonhomogeneous, can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}y} - xf(y) = g(y).$$

This equation is linear in x; thus standard methods can be employed to find the solution $x = x(y)$. This same idea – to interchange the dependent and independent variables – provides a powerful method in the solution of certain types of partial differential equation. A particular example is our pair of shallow water equations, (2.139).

The method was first developed for the corresponding problem in gas dynamics, and it has retained its name used in this context: the *hodograph transformation*. (The word 'hodograph' is based on the Greek ὁδος, which means *way* or *road*, and is used to describe the (graphical) representation of a motion which uses as coordinates the components of the velocity vector rather than of the position vector.) As before, it is convenient to introduce $c = \sqrt{h}$, so we obtain from equations (2.139)

$$\left.\begin{aligned} u_t + uu_x + 2cc_x &= 0; \\ c_t + uc_x + \frac{1}{2}cu_x &= 0, \end{aligned}\right\} \tag{2.146}$$

where the coefficients of the derivative terms depend only on u and c, and otherwise all terms are first partial derivatives. We introduce the hodograph transformation

$$x = x(u, c), \quad t = t(u, c);$$

differentiating each of these with respect to x yields

$$1 = x_u u_x + x_c c_x; \quad 0 = t_u u_x + t_c c_x$$

and so

$$u_x = t_c / J, \quad c_x = -t_u / J \tag{2.147}$$

where

$$J = \frac{\partial(x, t)}{\partial(u, c)} = x_u t_c - x_c t_u \tag{2.148}$$

is the *Jacobian* of the transformation. Similarly, by differentiating with respect to t, we obtain two equations for u_t and c_t which yield

$$u_t = -x_c / J, \quad c_t = x_u / J, \tag{2.149}$$

and clearly these transformations of the derivatives require $J \neq 0$.

We now substitute from equations (2.147) and (2.149) into equations (2.146), to obtain

$$\begin{aligned} x_c - ut_c + 2ct_u &= 0; \\ x_u - ut_u + \frac{1}{2}ct_c &= 0, \end{aligned}$$

which are *linear* equations in x and t. Furthermore, the two equations involve only either x_c or x_u; thus we may form x_{uc} from both and thereby eliminate x. Thus we have

$$\frac{\partial}{\partial u}(ut_c - 2ct_u) = \frac{\partial}{\partial c}(ut_u - \frac{1}{2}ct_c)$$

which simplifies to give

$$4ct_{uu} - ct_{cc} = 3t_c, \tag{2.150}$$

a linear second-order partial differential equation which can be solved by standard methods. Indeed, the characteristic variables for this equation, (2.150), are

$$\xi = u - 2c, \quad \eta = u + 2c$$

(combinations that we recognise from equations (2.142)), and then we obtain

$$2(\eta - \xi)t_{\xi\eta} = 3(t_\eta - t_\xi). \tag{2.151}$$

The solution is then completely determined by imposing appropriate boundary conditions, but these must (for equation (2.151)) describe t in the (ξ, η)–plane, a prescription that may not be straightforward. This is a difficulty that is often encountered in the hodograph method: interchanging the dependent and independent variables simplifies the governing equation(s), but complicates the boundary/initial conditions. A further inconvenience is that the simple-wave solutions cannot be accessed through the hodograph method, since the transformation is singular in this case. We can see this directly if we calculate

$$J^{-1} = u_x c_t - u_t c_x;$$

a simple wave exists when $u - 2c$ or $u + 2c$ is constant, and then clearly $J^{-1} = 0$. The transformation from $\{u(x, t), c(x, t)\}$ to $\{x(u, c), t(u, c)\}$, and back again, requires J (and therefore J^{-1}) to be finite and nonzero everywhere. Nevertheless, because equation (2.151) is linear, its solution can be approached by standard techniques (such as the separation of variables or integral transforms). Indeed, a more useful result in this respect is obtained from equation (2.150) by writing

$$t = \frac{1}{c}\frac{\partial T}{\partial c} \quad \text{where} \quad T = T(u/2, c),$$

for then (2.150) becomes

$$T_{cvv} - T_{ccc} + \frac{2}{c}T_{cc} - \frac{2}{c^2}T_c = 3\left(\frac{1}{c}T_{cc} - \frac{1}{c^2}T_c\right),$$

where $v = u/2$. This equation is clearly

$$(T_{vv} - T_{cc})_c - (T_c/c)_c = 0$$

and so

$$T_{vv} = T_{cc} + \frac{1}{c}T_c + F(v)$$

which is the (inhomogeneous) cylindrical wave equation. If, finally, we map $T \to T + G(v)$ where $G'' = F$, we are left with

$$T_{vv} = T_{cc} + \frac{1}{c}T_c,$$

for which the application of the method of separation of variables, for example, is a familiar exercise; see Section 2.1, Q2.20 and Q2.21. The problem of finding solutions of the equation for T is addressed in Q2.58 and Q2.59.

2.7 Hydraulic jump and bore

A familiar phenomenon, observed particularly below weirs or dams, is the *hydraulic jump*. This is a relatively rapid increase in the depth of the water (essentially across the whole width of the river). The depth increase is often associated with a very turbulent mixing of the water, producing a significant energy loss there. (A similar jump can be seen when water from a tap hits a horizontal surface. In this case there is a (roughly) circular region of fast-flowing water moving radially outwards in a thin layer. This region is terminated by a sudden increase in depth: the circular hydraulic jump, Q2.62.) The hydraulic jump is stationary with respect to the riverbank; when this same phenomenon moves along a river it is called a *bore*. The most famous bore in Britain is the one that appears periodically on the River Severn, athough there are other rivers in other parts of the world that can boast much larger bores with depth changes of many feet.

For either the hydraulic jump or the bore, the change in depth can be a few metres but this will occur in, typically, a distance of only a metre or two. In other words, it might be reasonable to model this change as an abrupt jump or *discontinuity*; this is what we shall now investigate. A sketch of a section through an hydraulic jump (or bore) is shown in Figure 2.22. We have already mentioned the analogy between the water-wave equations and the equations of gas dynamics; the corresponding jump in gas dynamics is, of course, the shock wave associated with supersonic flow. In this case the jump is far narrower, and far more dramatic, and in consequence is more readily modelled as a discontinuity.

The hydraulic jump is formed when a wave has fully broken, and as such can also be observed at a shoreline after a wave has completely broken and is in the final stage of its run-up. The jump is what replaces the breaking of our nonlinear waves, which in that context corresponds to the crossing of the characteristics. The mathematical device we then adopt is to replace the region where the solution is multivalued by a line which separates the two sets of characteristics and, therefore, across which there will be a jump in value; see Section 1.4.1. This certainly enables us to produce a solution that is meaningful after breaking has occurred. (We recall that the accurate representation of a real breaking wave – at a shoreline, for example – requires a far more sophisticated theory than we are working with here.) However, as we mention in Section 1.4.1, a discontinuity cannot be regarded as a solution of our partial differential equations (for which continuity and some differentiability is needed). Our main task now is to describe how to overcome this mathematical difficulty, and then we shall be able to present some properties of the hydraulic jump (or bore) as based on our model.

The differential equations (the Euler equation and mass conservation equation) are not valid for discontinuous solutions, but the integral form from which they have been obtained (Sections 1.1.1 and 1.1.2) do admit

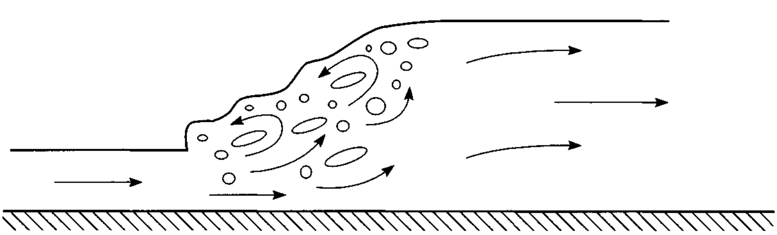


Figure 2.22. Sketch of an hydraulic jump, where the flow is from left to right. (The equivalent bore over stationary water moves to the left at the speed of the oncoming flow.)

such solutions. Indeed, it is the integral form of the governing equations which should be regarded as the fundamental equations, and it is to these that we must turn. Now, rather than quote the general equations from Chapter 1, we choose to construct the appropriate forms from the equations (2.131). But to simplify the problem still further we shall incorporate, *ab initio*, the long-wave assumption, so that p and u (for one-dimensional motion) are independent of z. Thus we start from equations (2.139),

$$h_t + (hu)_x = 0; \quad u_t + uu_x + h_x = 0. \tag{2.152}$$

The first of these, the equation that describes the conservation of mass, is already in the form that we obtain by integrating in z, namely

$$\int_0^h u_x \mathrm{d}z + [w]_0^h = 0$$

so

$$\int_0^h u_x \mathrm{d}z + h_t + uh_x = 0,$$

which immediately gives the above equation (since $u = u(x, t)$). Thus the integral form of the equation we require is recovered if we integrate in x, between constants a and b, say; thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_a^b h\mathrm{d}x\right\} + [hu]_a^b = 0. \tag{2.153}$$

Let h (and u) be discontinuous at $x = X(t)$, so that we may accommodate either the hydraulic jump or the bore, and such that $a < X < b$. Then we may write (2.153) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_a^{X^-} h\,\mathrm{d}x + \int_{X^+}^b h\,\mathrm{d}x\right\} + [hu]_a^b = 0$$

where the superscripts $-/+$ denote evaluation as $x \to X^-/x \to X^+$, in the usual way. Upon differentiating under the integral signs (Q1.30), we obtain

$$\int_a^b h_t \mathrm{d}x + h^- \frac{\mathrm{d}X}{\mathrm{d}t} - h^+ \frac{\mathrm{d}X}{\mathrm{d}t} + [hu]_a^b = 0,$$

where we have assumed that the *path* of the discontinuity, $x = X(t)$, is differentiable. Finally, we find the *jump condition* that must be satisfied across the discontinuity by taking the limit $a \to b$, which yields

$$-U[\![h]\!] + [\![hu]\!] = 0, \tag{2.154}$$

where $U(t) = \mathrm{d}X/\mathrm{d}t$ and $[\![y]\!] = y^+ - y^-$, the jump in value across $x = X(t)$. Equation (2.154) is the first jump condition, which, particularly in the context of the gas-dynamic shock wave, is usually called a *Rankine–Hugoniot condition*. We see that, if the discontinuity is stationary (the hydraulic jump), then $U = 0$, and so hu is conserved across the discontinuity. Indeed, we can write (2.154) as

$$[\![h(u - U)]\!] = 0,$$

since $U(t)$ is continuous, which states the otherwise obvious condition that mass (volume per unit width here) is conserved relative to the jump: what goes in from one side must come out the other.

The second equation in (2.152) is clearly the appropriate x-momentum equation based on Euler's equation, and hence this must be integrated in both z and x. First we have

$$\int_0^h (u_t + uu_x + h_x)\mathrm{d}z = 0$$

which yields immediately

$$hu_t + huu_x + hh_x = 0;$$

we rewrite this as

$$hu_t + uh_t + (huu)_x + hh_x = 0$$

by incorporating equation (2.152a). This is integrated in x, from a to b as above, to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_a^b hu \, \mathrm{d}x\right\} + \left[hu^2 + \frac{1}{2}h^2\right]_a^b = 0$$

and then

$$\int_a^b (hu)_t \mathrm{d}x + (hu)^- \frac{\mathrm{d}X}{\mathrm{d}t} - (hu)^+ \frac{\mathrm{d}X}{\mathrm{d}t} + \left[hu^2 + \frac{1}{2}h^2\right]_a^b = 0.$$

When we take $a \to b$ we obtain the second jump condition

$$-U[\![hu]\!] + [\![hu^2 + \frac{1}{2}h^2]\!] = 0, \qquad (2.155)$$

which describes the conservation of momentum across $x = X(t)$. This is obviously interpreted as: the total momentum change across the moving front ($-U[\![hu]\!]$) is produced by the difference in momentum on either side ($[\![hu^2]\!]$) plus the difference in the pressure forces ($[\![\frac{1}{2}h^2]\!]$).

In summary, we have the pair of jump (Rankine–Hugoniot) conditions

$$-U[\![h]\!] + [\![hu]\!] = 0; \quad -U[\![hu]\!] + \left[\!\!\left[hu^2 + \frac{1}{2}h^2\right]\!\!\right] = 0 \qquad (2.156)$$

which can be regarded as two equations for h^- and u^-, say, given h^+, u^+ and U. That is, given the speed of the bore (which may be zero – the hydraulic jump), and the conditions on one side, equations (2.156) determine the conditions on the other side. However, it is reasonable to ask whether there is a third jump condition that has been overlooked, namely an energy condition. This possibility we shall now investigate.

The appropriate energy integral (an integration in z) is equation (1.47), which here becomes

$$\frac{\partial}{\partial t}\left\{\int_0^h \left(\frac{1}{2}u^2 + z\right)\mathrm{d}z\right\} + \frac{\partial}{\partial x}\left\{\int_0^h u\left(\frac{1}{2}u^2 + h\right)\mathrm{d}z\right\} = 0 \qquad (2.157)$$

with $P = P_a + \rho g(h - z)$ and we have used our familiar nondimensionalisation (with $\varepsilon = 1$). Thus, since $u = u(x, t)$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}hu^2 + \frac{1}{2}h^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}hu^3 + uh^2\right) = 0,$$

and integrating in x across the jump $x = X(t)$ we find that

$$-U\left[\!\!\left[\frac{1}{2}hu^2 + \frac{1}{2}h^2\right]\!\!\right] + \left[\!\!\left[\frac{1}{2}hu^3 + uh^2\right]\!\!\right] = 0. \qquad (2.158)$$

(This can be written down directly if we observe that, to obtain equations (2.156), we merely use the correspondence

$$\frac{\partial}{\partial x}(\alpha) \to [\![\alpha]\!], \quad \frac{\partial}{\partial t}(\beta) \to -U[\![\beta]\!].)$$

We now have, apparently, a third equation relating $h^{\pm}, u^{\pm}$, and U, but this is unreasonable, since we would expect to be able to determine the conditions on one side given the conditions on the other (and given U) – and two equations are sufficient for this. In order to investigate the rôle of equation (2.158), let us consider the simple case of $U = 0$ (the hydraulic jump); then we obtain

$$\left[\!\!\left[\frac{1}{2}hu^3 + uh^2\right]\!\!\right] = 0.$$

But equations (2.156) imply, after a little manipulation, that

$$\begin{aligned}\left[\!\!\left[\frac{1}{2}hu^3 + uh^2\right]\!\!\right] &= \frac{1}{2}m[\![u^2]\!] + m[\![h]\!]\\ &= \frac{1}{4}(u^+ + u^-)[\![-h^2]\!] + m[\![h]\!]\\ &= \frac{\mathrm{m}}{4h^+h^-}(h^+ - h^-)^3, \end{aligned} \tag{2.159}$$

where $m = (uh)^+ = (uh)^-$. Clearly the expression in (2.159) will be zero only if $h^+ = h^-$ (since $m \neq 0$): there is no jump. Consequently, if there *is* a jump, then we cannot impose the energy conservation condition, (2.158). Indeed, solving equations (2.156) for a jump, we may use the expressions in equation (2.158) (that is, (2.159) if $U = 0$) to determine the appropriate sense of the transition. The flow through the jump is taken to correspond to energy *loss*, this loss normally occurring (as we mentioned at the outset) because of the turbulent nature of the conditions in the neighbourhood of the jump. When we impose this energy-loss condition we find that, relative to the jump, the flow must enter from the faster and shallower side (that is, $u^- > u^+$ and $h^- < h^+$), and then the expression in (2.159) is negative (energy loss). (The alternative ($h^- > h^+$) requires an energy input and no mechanism in nature exists for providing an energy source.)

Finally, we briefly examine the consequences of using equations (2.156) for the hydraulic jump (so again $U = 0$). Suppose that we are given the conditions to the left, u^- and h^-; then we write

$$u^+ = \frac{u^-h^-}{h^+} \quad \text{and} \quad \frac{1}{2}(h^{+2} - h^{-2}) = h^-u^{-2} - h^+u^{+2};$$

thus

$$\frac{1}{2}(h^{+2} - h^{-2}) = h^{-}u^{-2}\left(1 - \frac{h^{-}}{h^{+}}\right).$$

It is convenient to introduce

$$H = \frac{h^{+}}{h^{-}} \quad \text{and} \quad F = \frac{u^{-}}{\sqrt{h^{-}}},$$

then we obtain

$$H^2 - 1 = 2F^2\left(1 - \frac{1}{H}\right),$$

which has a root $H = 1$ (of no interest since this corresponds to no change) and otherwise

$$H = \frac{1}{2}\left(-1 \pm \sqrt{1 + 8F^2}\right).$$

A physically meaningful solution is possible only for the positive sign, and then $H > 1$ only if $F > 1$. The parameter F is called the *Froude number* (which in dimensional variables is usually written $u/\sqrt{gh_0}$); this parameter corresponds to the *Mach number* for the flow of a compressible gas. There can be a jump in water depth only if the flow upstream is *supercritical* ($F > 1$) (sometimes called *shooting flow*); if the flow is *subcritical* or *tranquil* ($F < 1$) then no hydraulic jump is possible.

We have commented that the energy loss at the hydraulic jump or bore is by virtue of the dissipation of this energy through the turbulent motion in the neighbourhood of the jump; see Figure 2.22. However, if the energy loss is not too great (typically, if $1 < F \lesssim 1.2$) then the required energy loss can be *transported away* by a train of waves on the downstream side of the jump. This gives rise to the so-called *undular bore*, which is a form of the bore that sometimes occurs on the River Severn. A more detailed discussion of this phenomenon, together with descriptions of how it may be modelled, will be given in Chapter 5.

2.8 Nonlinear waves on a sloping beach

In Section 2.2 we presented the theory of linearised long waves moving over a bed of constant slope, and in Section 2.5 we developed some of the ideas involved in the theory of nonlinear long waves. We now turn to a brief discussion of a mathematically interesting problem that combines these two phenomena, namely nonlinearity and variable depth. From Section 2.5, and following that development, we consider long waves ($\delta \to 0$) and 'full' nonlinearity ($\varepsilon = 1$) so that the governing equations are

$$u_t + uu_x + wu_z = -p_x, \quad p_z = 0, \quad u_x + w_z = 0$$

with

$$w = \eta_t + u\eta_x \quad \text{and} \quad p = \eta \quad \text{on} \quad z = 1 + \eta$$

and

$$w = ub'(x) \quad \text{on} \quad z = b(x).$$

Thus $p = \eta$ for all z and, as before, we take $u = u(x, t)$ so that

$$u_t + uu_x + \eta_x = 0 \quad \text{and} \quad w = \left(\frac{\eta_t + u\eta_x - ub'}{1 + \eta - b}\right)(z - b) + ub'$$

and then $u_x + w_z = 0$ yields

$$(1 + \eta - b)u_x + \eta_t + u\eta_x - ub' = 0.$$

It is convenient to introduce

$$d(x, t) = 1 + \eta(x, t) - b(x),$$

the local depth of the water, to give

$$u_t + uu_x + d_x - b'(x) = 0; \quad d_t + (du)_x = 0, \tag{2.160}$$

which are to be compared with equations (2.139). The important difference is, of course, the appearance of the term in $b'(x)$ in equations (2.160); for general $b(x)$ this makes the methods used earlier essentially inapplicable. However, one special case can be successfully explored, as Carrier and Greenspan (1958) first showed.

We choose $b'(x)$ to be a constant, so the bed is of constant slope; following equation (2.41) we write

$$b(x) = 1 - \alpha(x_0 - x), \quad \alpha > 0,$$

so that $b'(x) = \alpha$. Our equations (2.160) therefore become

$$u_t + uu_x + d_x - \alpha = 0; \quad d_t + (du)_x = 0. \tag{2.161}$$

We saw in Section 2.5.1 that $c = \sqrt{h}$ was a useful change of variable, and the same applies here; we introduce $c = \sqrt{d}$ to give

$$u_t + uu_x + 2cc_x - \alpha = 0$$

and

$$2c_t + 2uc_x + cu_x = 0.$$

Again, we combine these to produce the equations written in characteristic form

$$(u+2c)_t + u(u+2c)_x + c(u+2c)_x - \alpha = 0$$

and

$$(u-2c)_t + u(u-2c)_x - c(u-2c)_x - \alpha = 0,$$

which can be expressed as

$$\left.\begin{aligned} \left\{\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right\}(u + 2c - \alpha t) = 0; \\ \left\{\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right\}(u - 2c - \alpha t) = 0, \end{aligned}\right\} \tag{2.162}$$

by interpreting α as $\partial(\alpha t)/\partial t$. Thus (cf. equations (2.142)) we have

$$\left.\begin{aligned} u + 2c - \alpha t = \text{ constant on lines } C^+ \colon \frac{\mathrm{d}x}{\mathrm{d}t} = u + c; \\ u - 2c - \alpha t = \text{ constant on lines } C^- \colon \frac{\mathrm{d}x}{\mathrm{d}t} = u - c, \end{aligned}\right\} \tag{2.163}$$

so the method of characteristics again results in a particularly simple structure.

The important realisation described by Carrier and Greenspan was that this problem, like that with $\alpha = 0$, can be linearised by an appropriate hodograph transformation. It is far from obvious that this is a possibility, since the method described earlier certainly requires the cancellation of the Jacobian (J) throughout the equation, which apparently cannot happen with $\alpha \neq 0$. To proceed, we recall that the neatest form of our earlier calculation involved $\xi = u - 2c$ and $\eta = u + 2c$, which led to equation (2.151). Here, we define corresponding variables

$$\xi = u - 2c - \alpha t, \quad \eta = u + 2c - \alpha t,$$

for use in the hodograph method, and transform

$$(x, t) \to (\xi, \eta).$$

This gives, after differentiating with respect to x,

$$1 = x_\xi(u_x - 2c_x) + x_\eta(u_x + 2c_x); \quad 0 = t_\xi(u_x - 2c_x) + t_\eta(u_x + 2c_x)$$

and so

$$u_x = \frac{1}{2}(t_\eta - t_\xi)/J, \quad c_x = -\frac{1}{4}(t_\xi + t_\eta)/J$$

where, here, the Jacobian is

$$J = \frac{\partial(x, t)}{\partial(\xi, \eta)} = x_\xi t_\eta - x_\eta t_\xi.$$

Similarly, the derivatives with respect to t yield

$$u_t = \alpha + \frac{1}{2}(x_\xi - x_\eta)/J, \quad c_t = \frac{1}{4}(x_\xi + x_\eta)/J.$$

Equations (2.162) therefore become

$$x_\xi - \frac{1}{4}(\xi + 3\eta + 4\alpha t)t_\xi = 0;$$
$$x_\eta - \frac{1}{4}(3\xi + \eta + 4\alpha t)t_\eta = 0,$$

which are *nonlinear* in t; this is bad news but exactly what we would have expected. However, when we form $x_{\xi\eta}$, and eliminate this term between these two equations, we also eliminate the nonlinear term – this is the crucial observation presented in Carrier and Greenspan (1958). Thus we finally obtain

$$2(\eta - \xi)t_{\xi\eta} = 3(t_\eta - t_\xi),$$

the same linear equation for $t(\xi, \eta)$ that we found for the nonlinear problem with *constant* depth, equation (2.151). The reduction of this equation to the cylindrical wave equation then follows (much as described in Section 2.6.2). Simple solutions of this standard equation can now be used to describe the behaviour of a nonlinear wave as it runs up a beach, for example; cf. Section 2.2. This particular application is addressed through the exercises (Q2.58 and Q2.59).

2.9 The solitary wave

At this stage in our investigations it would not be unreasonable to suppose that the fully nonlinear (inviscid) equations of motion admit travelling-wave solutions of permanent form: that is, waves that propagate at constant speed without change of shape (see Q1.55). We have previously (Section 2.4) obtained approximations to the periodic waves of this type – the Stokes wave – where the wave profile is a distortion of the sine wave and the speed is dependent on both the wave number and the amplitude. The appearance of the amplitude here is indicative of the rôle of the nonlinear terms, and also suggests that waves of larger amplitude might be possible (even if we cannot express them in closed form).

It is a matter of observation that gravity waves of permanent form, and of considerable amplitude, can propagate on the surface of water. Indeed, this can occur whether the water is stationary or is moving with some velocity distribution below the surface. In particular, it is sometimes observed that single waves can be generated. These have a profile which is a symmetrical hump of water which drops smoothly back to the undisturbed surface level far ahead and far behind the wave; the wave propagates at a constant speed. This wave was first observed and described by J. Scott Russell, an engineer, naval architect and Victorian man of affairs. In 1834 he was observing the motion of a boat on the Edinburgh–Glasgow canal and the waves that it generated. Russell's description of what he saw is now much-quoted, but it still evokes the era and the man; we make no apologies for reproducing it here. In his 'Report on Waves' to the British Association meeting (at York) in 1844, he writes:

> I believe I shall best introduce the phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

After his initial observations, Russell performed a number of laboratory experiments to investigate the nature of what he called 'the great wave of translation', but which soon came to be known as the *solitary wave*. The most significant experiment involved the dropping of a weight at one end of a water channel (see Figure 2.23). He found that the volume of water displaced was the volume of water in the wave and, by careful measurement, that the wave moved at a speed, c, where

$$c^2 = g(h_0 + a),$$

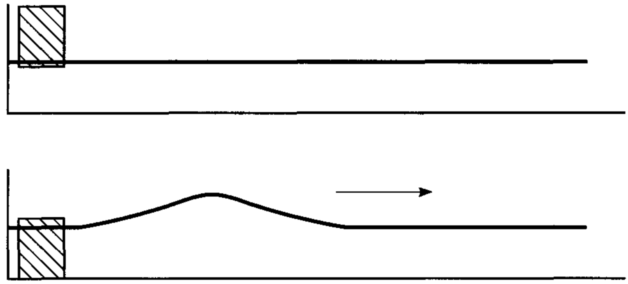


Figure 2.23. J. Scott Russell's experiment in which a weight is dropped at one end of the channel; the displaced water is propagated away as a solitary wave.

where h_0 is the undisturbed depth of the water and a is the amplitude of the wave. We see that we recover the wave speed of small-amplitude long waves ($c = \pm\sqrt{gh_0}$, equation (2.11)). Furthemore, it is clear that higher waves (that is, larger a) travel faster (cf. equation (2.137) *et seq.*). Here we have described a wave of *elevation*; the corresponding wave of depression does not exist, for it immediately collapses into a train of oscillatory waves.

Early attempts were made by Boussinesq (1871) and Rayleigh (1876) to find a mathematical description of the solitary wave. On the basis that the wave is long ($\delta \to 0$ in our terminology), they were able to confirm Russell's formula for the speed of the wave, and also to show that the profile is accurately represented by the sech^2 function (although this requires the additional assumption of small amplitude). (In the early days, the existence of this wave excited some controversy; in fact, both Airy and Stokes were initially of the opinion that it could not exist.) Much of the mathematical detail in this description will be developed here and in the later chapters. Indeed, it is the mathematical investigations that were initiated by Russell's observations that have eventually led to the extensive and modern ideas in nonlinear wave propagation, and in water-wave theory in particular, that we shall describe in the following chapters.

We begin our study of the solitary wave by treating the flow as irrotational, with the wave propagating (as a plane wave) in the x-direction. Thus, from equations (2.132), we have

$$\left.\begin{aligned}
&\phi_{zz} + \delta^2\phi_{xx} = 0,\\
&\text{with}\\
&\left.\begin{aligned}
&\phi_z = \delta^2(\eta_t + \varepsilon\phi_x\eta_x);\\
&\phi_t + \eta + \frac{1}{2}\varepsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2\right) = 0,
\end{aligned}\right\}\ \text{on } z = 1 + \varepsilon\eta\\
&\text{and}\\
&\phi_z = 0 \quad \text{on} \quad z = 0,
\end{aligned}\right\} \tag{2.164}$$

where the bed is taken to be fixed and horizontal ($b = 0$). The general solitary-wave solution is associated with arbitrary values of ε and δ; they are not assumed to be small. It is convenient (as we have done previously) to set $\varepsilon = 1$, but retain the parameter δ in our formulation. We are seeking a travelling-wave solution, and so we treat $\phi = \phi(\xi, z)$ and $\eta = \eta(\xi)$ where $\xi = x - ct$, and c is the (nondimensional) speed of the wave. Then from equations (2.164) we obtain

$$\left.\begin{aligned}
&\phi_{zz} + \delta^2\phi_{\xi\xi} = 0,\\
&\text{with}\\
&\left.\begin{aligned}
&\phi_z = \delta^2(\phi_\xi - c)\eta_\xi;\\
&-c\phi_\xi + \eta + \frac{1}{2}\left(\frac{1}{\delta^2}\phi_z^2 + \phi_\xi^2\right) = 0,
\end{aligned}\right\}\ \text{on } z = 1 + \eta\\
&\text{and}\\
&\phi_z = 0 \quad \text{on} \quad z = 0.
\end{aligned}\right\} \tag{2.165}$$

We first see if these equations admit a solution that represents a profile which decays exponentially as $|\xi| \to \infty$. Thus we write

$$\eta \sim a\mathrm{e}^{-\alpha|\xi|}, \quad \phi \sim \psi(z)\mathrm{e}^{-\alpha|\xi|}, \quad |\xi| \to \infty,$$

where α (> 0) is the exponent; it is clear that both η and ϕ must have the same exponential behaviour in order to satisfy the surface boundary conditions. Laplace's equation (in (2.165)) then requires that

$$\psi'' + \alpha^2\delta^2\psi = 0$$

so

$$\psi = A\cos(\alpha\delta z),$$

when the boundary condition on $z = 0$ is invoked; A is an arbitrary constant. The leading-order balance from the boundary conditions on $z \sim 1$ gives (for $\xi > 0$, say)

$$-A\alpha\delta\sin(\alpha\delta) = ca\alpha\delta^2, \quad cA\alpha\cos(\alpha\delta) + a = 0$$

so

$$c^2 = \frac{\tan(\alpha\delta)}{\alpha\delta}. \tag{2.166}$$

A solution with the required behaviour does therefore exist provided c (the speed, which here is the same as the Froude number since the non-dimensionalisation uses $\sqrt{gh_0}$; see Section 2.7) and α (the exponent) are related by equation (2.166), a result first found by Stokes (1880). All solitary waves exhibit exponential decay in their tails and all satisfy the relation (2.166).

Another important and general question addressed by Stokes in 1880 concerned the notion of a *highest wave*; that is, to examine what might limit the amplitude of the solitary wave and then what conditions obtain when this occurs. We consider a wave of permanent form travelling at the speed c in the positive x-direction over water of constant depth which is stationary at infinity. It is convenient to use a coordinate which is moving at the speed c, so that in this frame the wave is stationary; see Figure 2.24. We introduce $\Phi(\xi, z)$

$$\Phi = \phi - c\xi \quad \text{so that} \quad \Phi_\xi = \phi_\xi - c;$$

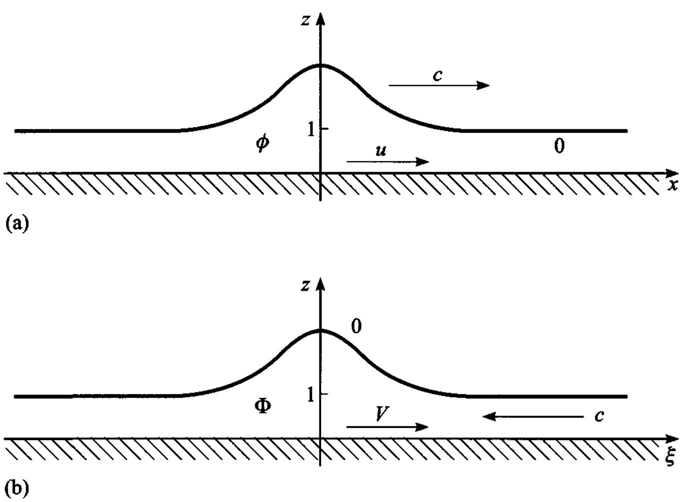


Figure 2.24. (a) The physical frame of reference for the solitary wave moving at speed c to the right into stationary water. (b) The frame of reference moving at speed c to the right.

that is, $V = u - c$, since $u = \phi_x = \phi_\xi$, where $\xi = x - ct$. Equations (2.165) then become

$$\left.\begin{aligned} \Phi_{zz} + \delta^2\Phi_{\xi\xi} &= 0; \\ \Phi_z = \delta^2\Phi_\xi\eta_\xi \text{ and } 2\eta - c^2 + \frac{1}{\delta^2}\Phi_z^2 + \Phi_\xi^2 &= 0 \text{ on } z = 1 + \eta; \\ \Phi_z = 0 \text{ on } z = 0. & \end{aligned}\right\} \qquad (2.167)$$

Stokes argued that the highest wave will be attained when the fluid-particle speed at the peak of the wave is equal to the speed of the wave. For waves of small amplitude, the particle speed is certainly less than the wave speed; if the particle speed exceeds the wave speed then the wave will be breaking and so cannot be steady; that is, not of permanent form. Thus $V = \Phi_\xi = 0$ at the peak, where $\eta = \eta_0$ say (so $c^2 = 2\eta_0$); further, we shift the origin of the (z, ξ)-coordinates to the peak of the wave, so that the peak is now at $z = 0 = \xi$ (which is how we have presented Figure 2.24).

The most direct route is to invoke the approach based on analytic functions of a complex variable; we therefore write

$$\Phi + \mathrm{i}\Psi = F(Z), \quad Z = \xi + \mathrm{i}\delta z,$$

where $\Psi(\xi, z)$ is the stream function for the flow (see Q1.25). In the neighbourhood of $Z = 0$ we seek a solution in the form

$$F(Z) \sim AZ^m, \quad |Z| \to 0,$$

and at the surface

$$\eta \sim -H|\xi|^n, \quad \xi \to 0, \ (H > 0)$$

where this η is relative to the peak at $\eta = \eta_0$; we expect $n > 0$ and $m > 1$ for physically reasonable behaviours near the peak. The kinematic surface condition in equations (2.167) then yields

$$\mathscr{R}\{\mathrm{i}\delta AmZ_0^{m-1}\} \sim -\delta^2 nH\xi^{n-1}\,\mathscr{R}\{AmZ_0^{m-1}\}, \quad Z_0 \sim \xi - \mathrm{i}\delta H\xi^n,$$

(for $\xi > 0$, say) which requires that $n = 1$. Thus the surface, in this limiting case, has a peak which is not smooth: it has a sharp crest. The dynamic (pressure) boundary condition now gives (again for $\xi > 0$)

$$-2H\xi \sim \xi^{2(m-1)}\left(\frac{1}{\delta^2}\left[\mathscr{R}\{\mathrm{i}\delta Am(1 - \mathrm{i}\delta H)^{m-1}\}\right]^2 + \left[\mathscr{R}\{Am(1 - \mathrm{i}\delta H)^{m-1}\}\right]^2\right)$$

which requires $m = 3/2$.

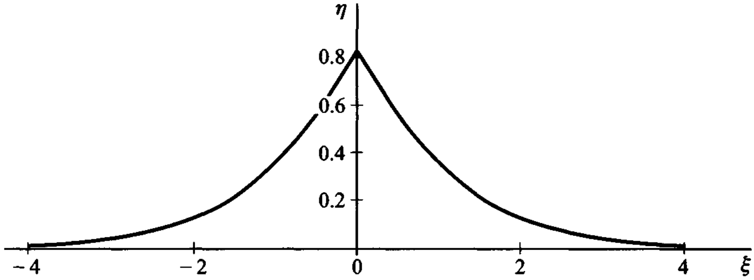


Figure 2.25. The highest wave of Stokes, based on the calculations described in Q2.63; here we have approximated the wave by two exponentials.

Finally, the angle of the wedge that forms the sharp crest is determined directly by this value of m. The solution near the crest is described by

$$\Phi + \mathrm{i}\Psi = F(Z) \sim AZ^{3/2}, \quad |Z| \to 0,$$

and this represents flow in a wedge of angle $\theta = 2\pi/3$; that is, $Z^{3/2}$ is the complex potential for a flow with boundaries $\theta = 0$, $\theta = 2\pi/3$ (and also $\theta = 4\pi/3$). For our problem, the requirement for symmetry about $\xi = 0$ leads to a choice of the complex constant A that, when combined with $Z^{3/2}$, implies a rotation of these boundaries. Thus, near $Z = 0$, the sharp crest is represented by the lines $\theta = 7\pi/6$, $\theta = 11\pi/6$: the crest includes an angle of 120°, the result first found by Stokes. We must emphasise that neither large-amplitude solitary waves, nor the sharp-crested highest wave, can be represented by a mathematical expression of closed form. This wave, based on a numerical approximation for its shape, is shown in Figure 2.25; see Q2.63. Nevertheless, the work initiated by Longuet-Higgins has enabled very accurate numerical representations of these waves and their properties to be obtained; see the section on Further Reading at the end of this chapter (and also Section 2.9.2).

2.9.1 *The sech² solitary wave*

In our discussion of the solitary wave thus far we have described various exact results that provide some useful information about the nature of this wave. What we cannot do is to present a complete solution of the governing equations, for arbitrary amplitude, which would then give us a mathematical representation of the solitary wave. Nevertheless, much that we have described can be incorporated in very accurate numerical solutions of these equations (and employed with great success by Longuet-Higgins and his co-workers). We therefore return to the approach that was first developed by Boussinesq and Rayleigh, which we mentioned earlier. We shall now see how we can proceed with an appropriate approximation of the equations; this eventually leads to a fundamental equation that provides the starting point for the work in the next chapter.

The equations are those given in (2.164), and we examine these for the case of long waves and small amplitude. The solitary wave extends from $-\infty$ to $+\infty$, so its length scale is certainly much greater than any (finite) depth of water. The assumption of long waves ($\delta \to 0$) should therefore be appropriate for the solitary wave. The restriction to small amplitude ($\varepsilon \to 0$) is necessary because we cannot otherwise make headway. In the initial stages of the calculation we shall treat these two parameters as independent.

Laplace's equation, from equation (2.164), is

$$\phi_{zz} + \delta^2 \phi_{xx} = 0$$

which, for small δ, clearly has the asymptotic solution

$$\phi(x, t, z; \delta) \sim \sum_{n=0}^{\infty} \delta^{2n} \phi_n(x, t, z), \quad \delta \to 0,$$

where

$$\phi_0 = \theta_0(x, t)$$

in order to satisfy the bottom boundary condition; θ_0 is an arbitrary function. The higher-order terms are given by

$$\phi_{n+1zz} = -\phi_{nxx}, \quad n = 0, 1, 2, \ldots .$$

We therefore obtain

$$\phi_1 = -\frac{1}{2} z^2 \theta_{0xx} + \theta_1(x, t);$$
$$\phi_2 = \frac{1}{24} z^4 \theta_{0xxxx} - \frac{1}{2} z^2 \theta_{1xx} + \theta_2(x, t),$$

and so on, where each θ_n is an arbitrary function and each ϕ_n satisfies the boundary condition

$$\phi_{nz} = 0 \quad \text{on} \quad z = 0.$$

The expansion for ϕ is used in the two surface boundary conditions (2.164), which involve evaluation on $z = 1 + \varepsilon\eta$. The first of these gives

$$-(1+\varepsilon\eta)\theta_{0xx}+\delta^2\left\{\frac{1}{6}(1+\varepsilon\eta)^3\theta_{0xxxx}-(1+\varepsilon\eta)\theta_{1xx}\right\}+\ldots$$
$$\sim \eta_t+\varepsilon\eta_x\left\{\theta_{0x}+\delta^2\left[\theta_{1x}-\frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxx}\right]+\ldots\right\}, \quad (2.168)$$

and the second becomes

$$\theta_{0t}+\delta^2\left\{-\frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxt}+\theta_{1t}\right\}+\ldots+\eta$$
$$\sim -\frac{1}{2}\varepsilon\delta^2\{-(1+\varepsilon\eta)\theta_{0xx}+\ldots\}^2$$
$$-\frac{1}{2}\varepsilon\left\{\theta_{0x}+\delta^2\left[\theta_{1x}-\frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxx}\right]+\ldots\right\}^2. \quad (2.169)$$

Now, for $\varepsilon \to 0$ and $\delta \to 0$, we see that the leading order terms yield

$$-\theta_{0xx} \sim \eta_t \quad \text{and} \quad \theta_{0t} \sim -\eta \quad (2.170)$$

and so

$$\theta_{0xx} \sim \theta_{ott}.$$

Thus we seek a solution which depends on $\xi = x - t$ (for right-running waves). This means that the wave will propagate, at this order of approximation, at the (nondimensional) speed of unity, which is completely consistent with our earlier work on long waves (see equations (2.10), (2.137), *et seq.*). We therefore treat both θ and η, in their dependence on x and t, as functions of $\xi = x - t$ and t; equations (2.170) then become

$$-\theta_{0zz} \sim \eta_t - \eta_\xi \quad \text{and} \quad \theta_{0t} - \theta_{0\xi} \sim -\eta$$

which imply

$$2\theta_{0t\xi} \sim \theta_{0tt}.$$

But when these terms are balanced against the others in the boundary conditions (2.168) and (2.169) we see that derivatives in t are small; cf. equation (1.99) and Q1.47–Q1.54. Thus we proceed with

$$\xi = x - t \quad \text{and} \quad \tau = \Delta t, \quad \Delta \to 0,$$

and we shall choose Δ later.

Our two surface boundary conditions (2.168, 2.169) now become, upon retaining only terms as small as $O(\varepsilon)$, $O(\delta^2)$ and $O(\Delta)$,

$$-(1+\varepsilon\eta)\theta_{0\xi\xi}+\delta^2\left(\frac{1}{6}\theta_{0\xi\xi\xi\xi}-\theta_{1\xi\xi}\right) \sim \Delta\eta_\tau-\eta_\xi+\varepsilon\eta\theta_{0\xi}$$

and

$$\Delta\theta_{0\tau} - \theta_{0\xi} + \delta^2\left(\frac{1}{2}\theta_{0\xi\xi\xi} - \theta_{1\xi}\right) + \eta \sim -\frac{1}{2}\varepsilon(\theta_{0\xi})^2.$$

The second of these is differentiated with respect to ξ and subtracted from the first, thereby eliminating the terms $-\theta_{0\xi\xi} + \eta_\xi$; this produces

$$-\varepsilon\eta\theta_{0\xi\xi} + \delta^2\left(\frac{1}{6}\theta_{0\xi\xi\xi\xi} - \theta_{1\xi\xi}\right) - \Delta\theta_{0\tau\xi} - \delta^2\left(\frac{1}{2}\theta_{0\xi\xi\xi\xi} - \theta_{1\xi\xi}\right)$$
$$\sim \Delta\eta_\tau + \varepsilon\eta\theta_{0\xi} + \varepsilon\theta_{0\xi}\theta_{0\xi\xi}, \quad (2.171)$$

and we see that the terms in θ_1 cancel identically. Finally, from the second equation in (2.170), we have that

$$\eta = \theta_{0\xi} + \mathrm{O}(\Delta)$$

so (2.171) is rewritten as

$$2\Delta\eta_\tau + 3\varepsilon\eta\eta_\xi \sim -\frac{\delta^2}{3}\eta_{\xi\xi\xi}.$$

Let us choose $\varepsilon = \mathrm{O}(\delta^2)$ and $\Delta = \varepsilon$, and write $\delta^2 = K\varepsilon$, then the leading-order equation for the surface profile is

$$2\eta_\tau + 3\eta\eta_\xi + \frac{K}{3}\eta_{\xi\xi\xi} = 0, \quad (2.172)$$

the Korteweg–de Vries (KdV) equation (Korteweg and de Vries, 1895); cf. equation (1.102), Q1.47–Q1.49 and Q1.55. This equation describes a balance between nonlinearity ($\eta\eta_\xi$), which tends to steepen the wave profile, and dispersion (by virtue of $\eta_{\xi\xi\xi}$) which works the other way. The solitary wave is that wave of permanent form for which this balance is precisely maintained. To see how this happens we seek the travelling-wave solution of equation (2.172) by writing $\eta = f(\xi - c\tau)$, for some constant c, then

$$-2cf' + 3ff' + \frac{K}{3}f''' = 0; \quad (2.173)$$

see Q1.55. The solution of this equation (see Q2.64) which satisfies

$$f, f', f'' \to 0 \quad \text{as} \quad |\xi - c\tau| \to \infty$$

is

$$f = 2c\,\mathrm{sech}^2\left\{\sqrt{\frac{3c}{2K}}(\xi - c\tau)\right\}$$

or

$$\varepsilon\eta \sim \varepsilon a \operatorname{sech}^2\left[\sqrt{\frac{3a}{4K}}\{x - (1 + \frac{1}{2}\varepsilon a)t\}\right], \tag{2.174}$$

where $\varepsilon\eta$ is the surface wave and its amplitude is εa ($= 2\varepsilon c$). This is the *sech² solitary wave*, which is the small-amplitude version of the classical solitary wave. We see that the speed of the wave $(1 + \frac{1}{2}\varepsilon a)$ increases as εa increases; indeed, solution (2.174) is defined for all $\varepsilon a > 0$ (but remember that it is a solution of the governing equations only for small ε, since we have used $\varepsilon = \mathrm{O}(\delta^2)$ and $\delta \to 0$). The wave speed agrees with the early observations of Russell for, in nondimensional variables, the speed is

$$\sqrt{1 + \varepsilon a} \sim 1 + \frac{1}{2}\varepsilon a \quad \text{as} \quad \varepsilon \to 0.$$

Finally, we observe that the 'width' of this solitary wave (defined as the distance between points of height $\frac{1}{2}\varepsilon a$, say) is inversely proportional to $\sqrt{a}$. This means that taller waves not only travel faster but are also narrower; see Figure 2.26. The behaviour of the exponential tails should also satisfy the general result given by equation (2.166); see Q2.64.

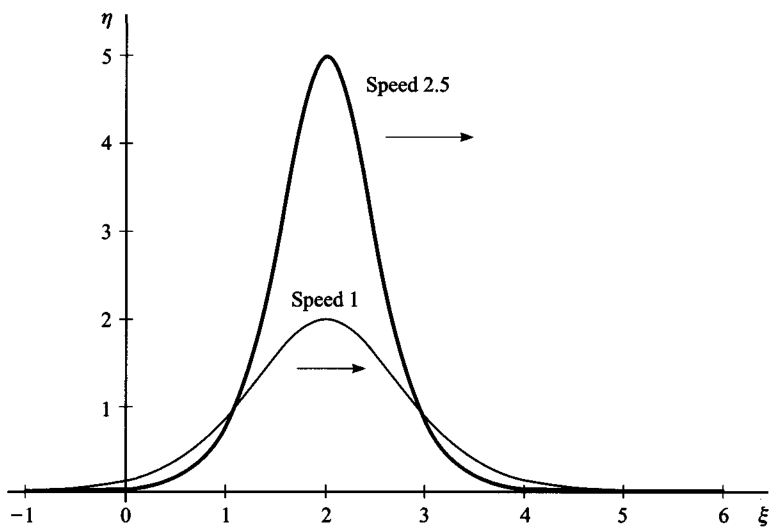


Figure 2.26. Two sech² solitary waves, each drawn in the frame $\xi = x - ct$ for $c = 1$ and $c = 2.5$.

In conclusion, two comments: the first addresses a general observation about a crucial assumption underlying the calculation that we have just presented. It would appear that we can obtain the sech^2 solitary wave (via the KdV equation) only if a special balance of parameter values arises, namely $\varepsilon = O(\delta^2)$. (The choice of the time-scale, Δ, is at our disposal; this merely tells us when and where to look for the wave.) This requirement for the balance would suggest that the solitary wave is a rare occurrence, rather than a familiar object. Certainly single such waves may be rather rare, but their counterparts in many-wave interactions, or perhaps as periodic waves, are often observed. It will be shown in the next chapter that a minor adjustment to our formulation enables us to show that the results described here are more widely applicable.

The second point picks up the comment just made about periodic solutions. The KdV equation for travelling waves, (2.173), admits periodic solutions of permanent form. That such solutions exist is easily demonstrated by integrating this equation twice, but without the use of decay conditions at infinity; this gives

$$\frac{K}{6}(f')^2 = cf^2 - \frac{1}{2}f^3 + Af + B = F(f),$$

where A and B are arbitrary constants. In the case where the cubic $F(f)$ has three distinct zeros, the solution can be expressed in terms of the Jacobian elliptic function, *cn*, giving rise to the Korteweg and de Vries *cnoidal wave*, which they first named. This description, and some related properties of the Jacobian elliptic functions, are explored through Q2.65–Q2.67.

2.9.2 Integral relations for the solitary wave

We conclude this chapter of classical results by briefly returning to the general solitary wave (Section 2.8). In work that dates back to McCowan (1891), and taken much further by Longuet-Higgins over the last twenty years or so, some exact identities for the solitary wave have been obtained. In recent times these have proved very powerful in the development of numerical methods for describing large-amplitude solitary waves (including the highest wave) and for laying the foundations for calculations that allow a study of breaking waves; much of this work has been pioneered by Longuet-Higgins and his co-workers. Here we shall give a brief introduction to these ideas, and a few are taken further in the exercises. The interested reader may also explore this material through the references given in the further reading at the end of this chapter.

We consider a wave of permanent form, moving at the speed c, which decays for $|\xi| \to \infty$; this is described by the equations (2.165):

$$\phi_{zz} + \delta^2 \phi_{\xi\xi} = 0$$

with

$$\left.\begin{aligned} \phi_z &= \delta^2(\phi_\xi - c)\eta_\xi; \\ -c\phi_\xi + \eta + \frac{1}{2}\left(\frac{1}{\delta^2}\phi_z^2 + \phi_\xi^2\right) &= 0 \end{aligned}\right\} \text{ on } z = 1 + \eta$$

and

$$\phi_z = 0 \quad \text{on} \quad z = 0.$$

We define a number of properties of the wave and its motion. These are the mass associated with the wave

$$M = \int_{-\infty}^{\infty} \eta \, \mathrm{d}\xi, \tag{2.175}$$

the total momentum (or impulse) of the motion of the fluid

$$I = \int_{-\infty}^{\infty} \int_{0}^{1+\eta} \phi_\xi \, \mathrm{d}z \, \mathrm{d}\xi, \tag{2.176}$$

the total kinetic energy of the motion

$$T = \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{1+\eta} \left(\frac{1}{\delta^2}\phi_z^2 + \phi_\xi^2\right) \mathrm{d}z \, \mathrm{d}\xi \tag{2.177}$$

and the potential energy of the wave

$$V = \frac{1}{2} \int_{-\infty}^{\infty} \eta^2 \, \mathrm{d}\xi. \tag{2.178}$$

In addition we define a *circulation* for the motion,

$$C = \int_{-\infty}^{\infty} \mathbf{u} \cdot \mathrm{d}\mathbf{s} = [\phi]_{-\infty}^{\infty}, \tag{2.179}$$

where the integral is taken along any streamline. These forms of these fundamental quantities are all defined here as the nondimensional counterparts of their physical equivalents.

First, from the equation of mass conservation,

$$u_\xi + w_z = 0,$$

and, in particular, since we are in the frame moving with the wave we write

$$(u - c)_\xi + w_z = 0,$$

and then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\int_0^{1+\eta}(u-c)\mathrm{d}z\right\} = 0;$$

cf. equation (1.40). Thus

$$\int_0^{1+\eta}(u-c)\mathrm{d}z = \text{constant} = \int_0^1(-c)\mathrm{d}z = -c$$

since both $u = \phi_\xi$ and η tend to zero as $|\xi| \to \infty$; hence

$$\int_0^{1+\eta} u\mathrm{d}z \left(= \int_0^{1+\eta}\phi_\xi \mathrm{d}z\right) = c\eta,$$

and then

$$\int_{-\infty}^{\infty}\int_0^{1+\eta}\phi_\xi \mathrm{d}z\,\mathrm{d}\xi = c\int_{-\infty}^{\infty}\eta\mathrm{d}\xi$$

or

$$I = cM. \tag{2.180}$$

This is an identity first obtained by Starr (1947).

Next we use Green's theorem in the form

$$\int_V\{(\nabla u)\cdot(\nabla v) + u\nabla^2 v\}\,\mathrm{d}V = \int_S u(\nabla v)\cdot\mathrm{d}\mathbf{S},$$

per unit length in the y-direction (so $\mathrm{d}V = 1 \times \mathrm{d}s$, $\mathrm{d}\mathbf{S} = \mathbf{n}(1 \times \mathrm{d}l)$, and choose

$$u = v = \Phi = \phi - c\xi \quad \text{and} \quad \nabla \equiv \left(\frac{\partial}{\partial \xi}, \frac{1}{\delta}\frac{\partial}{\partial z}\right).$$

The resulting plane region for the integration is bounded by a curve (Γ) which is taken to be

$$z = 1 + \eta \quad \text{and} \quad z = 0 \quad \text{for} \quad -\xi_0 \le \xi \le \xi_0$$

and $\xi = \pm\xi_0$, $\xi_0 > 0$; see Figure 2.27. We may think of ξ_0 as large for, eventually, we shall impose $\xi_0 \to \infty$. Now, since

$$\nabla^2\Phi = \nabla^2\phi = \phi_{\xi\xi} + \frac{1}{\delta^2}\phi_{zz} = 0,$$

we obtain Green's theorem in the form

$$\int_{-\xi_0}^{\xi_0} \int_0^{1+\eta} \left\{\frac{1}{\delta^2}\phi_z^2 + (\phi_\xi - c)^2\right\} \mathrm{d}z\, \mathrm{d}\xi = \int_\Gamma \Phi\left(m\Phi_\xi + \frac{n}{\delta}\Phi_z\right) \mathrm{d}l, \qquad (2.181)$$

where $\mathbf{n} \equiv (m, n)$ is the outward unit normal vector on Γ. Note that, because we are using the coordinates (ξ, z), the surface wave is stationary in our frame; also, across $\xi = \pm\,\xi_0$, there is (approximately) a uniform stream of speed c in the negative ξ-direction.

To proceed, we evaluate the various contributions in equation (2.181). The left-hand side becomes

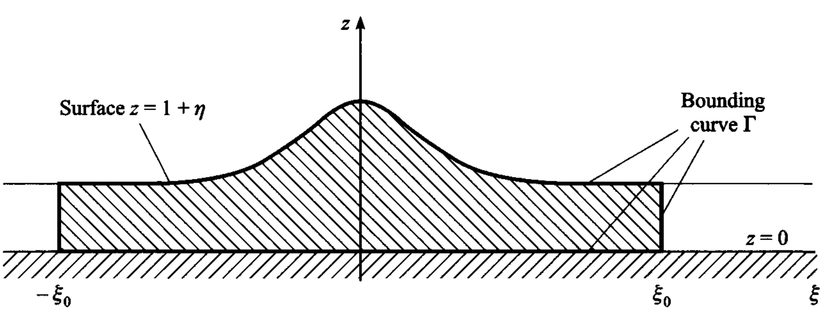


Figure 2.27. The region (whose boundary is designated by Γ) used in the application of Green's theorem to find one of the identities satisfied by the solitary wave.

$$2\hat{T} - 2c\int_{-\xi_0}^{\xi_0}\int_0^{1+\eta} \phi_\xi \,\mathrm{d}z\,\mathrm{d}\xi + c^2\int_{-\xi_0}^{\xi_0}\int_0^{1+\eta} \mathrm{d}z\,\mathrm{d}\xi = 2\hat{T} - 2c\hat{I} + 2c^2\xi_0 + c^2\hat{M}$$

where $\hat{T} \to T$, $\hat{I} \to I$ and $\hat{M} \to M$ as $\xi_0 \to \infty$; see equations (2.177), (2.176) and (2.175). For the right-hand side we find that

$$\begin{array}{lll} \text{on } z = 0: & m = 0, n = -1, & \Phi_z = \phi_z = 0; \\ \text{on } \xi = \xi_0: & m = 1, n = 0, & \Phi_\xi = -\hat{c}; \\ \text{on } \xi = -\xi_0: & m = -1, n = 0, & \Phi_\xi = -\hat{c}; \end{array}$$

and on $z = 1 + \eta$, which is a streamline (or, rather, a stream surface), $\mathbf{n}$ is here normal to $\nabla\Phi$; we have introduced $\hat{c}$ where $\hat{c} \to c$ as $\xi_0 \to \infty$. Thus equation (2.181) becomes

$$2\hat{T} - 2c\hat{I} + 2c^2\xi_0 + c^2\hat{M} = -\int_0^{1+\eta} \Phi_+\hat{c}\mathrm{d}z + \int_0^{1+\eta} \Phi_-\hat{c}\mathrm{d}z \qquad (2.182)$$

where $\Phi_\pm$ denotes Φ evaluated on $\xi = \pm\xi_0$. It is simplest, at this stage, to allow $\xi_0 \to \infty$ (so that $\hat{c} \to c$ and $\eta \to 0$) and hence obtain for the right-hand side of (2.182)

$$-\int_0^{1+\eta} \Phi_+\hat{c}\mathrm{d}z + \int_0^{1+\eta} \Phi_-\hat{c}\mathrm{d}z \sim -c\Phi_+ + c\Phi_- \sim -c[\phi]_{-\infty}^{\infty} + 2c^2\xi_0.$$

Thus equation (2.182) produces, in the limit $\xi_0 \to \infty$, the identity

$$2T - 2cI + c^2M = -cC$$

or

$$2T = c(I - C) \qquad (2.183)$$

after we introduce equation (2.180). The relation (2.183) was first derived by McCowan (1891).

A third useful identity introduces the potential energy, V, and takes the form

$$3V = (c^2 - 1)M;$$

a derivation of this result can be found in Longuet-Higgins (1974). Other identities (involving these quantities or for the surface profile itself) have been obtained by Longuet-Higgins, and used very successfully in numerical investigations of the large-amplitude solitary wave. These three integral identities are examined, for the approximate sech^2 profile, in Q2.69.

Further reading

This chapter has introduced a number of classical problems in both linear and nonlinear water-wave theory. Similar material will be found in many of the classical texts and, in some cases, the presentation in these will go beyond the topics developed here or use a different approach to that adopted here. General texts that the reader may find useful are Stoker (1957), Crapper (1984), Mei (1989) and the more recent publication Debnath (1994). In addition, some important aspects of water-wave theory are developed in Whitham (1974). A more engineering-oriented approach is to be found in Dean & Dalrymple (1984). All these references are particularly relevant to the fundamental ideas described in Sections 2.1–2.1.3.

2.1.3 The method of stationary phase, and of steepest descents, is nicely described in Copson (1967). A far more thorough and expansive treatment will be found in the excellent text by Olver (1974).

2.2 A neat discussion of waves over variable depth, and in particular building on the work of Hanson (1926), will be found in Whitham (1979). This monograph also includes some work on edge waves, as does the text by Mei (1989).

2.3 A fairly complete description of ray theory, with some applications to variable depth and to variable currents, is given by Mei (1989). Ray theory is also mentioned in Crapper (1984) and in Whitham (1974), and an introduction to Whitham's averaged Lagrangian will also be found in this latter text.

2.4 Stoker (1957) provides an extensive presentation of many aspects of ship waves; the elements can also be found in Crapper (1984). A text which incorporates more practical aspects of ship waves and ship hydrodynamics is Timman, Hermans & Hsiao (1985).

2.5 A description of the Stokes wave can be found in many texts on fluid mechanics. In the context of books on water waves, the reader is directed to Mei (1989), Dean & Dalrymple (1984), Whitham (1974) and Crapper (1984).

2.6 and 2.7 Excellent descriptions of the method of characteristics, Riemann invariants, discontinuous solutions and the hodograph transformation can be found in Stoker (1957) and Courant & Friedrichs (1967). Presented from the viewpoint of the theory of partial differential equations, there is no better text than Garabedian (1964).

2.8 The work that was first described by Carrier & Greenspan (1958) is given a careful treatment in Whitham (1979), and is also mentioned in Mei (1989) and Debnath (1994).

2.9 The classical (small-amplitude) solitary wave is described in Stoker (1957), as well as in numerous other texts on fluids or nonlinear waves (especially those that touch on 'soliton' theory, for example Drazin & Johnson (1993)). The more modern treatments on the large-amplitude solitary wave, and on breaking waves, are best addressed through some of Longuet-Higgins' papers, which are listed in the references, in particular Longuet–Higgins (1974, 1975), Longuet-Higgins & Fenton (1974) and Longuet-Higgins & Cokelet (1976).

Our text does not incorporate photographs of surface waves. Although the quality of some of the pictures does vary considerably, the readers who wish to add to their own observations are directed, for example, to Stoker (1957) and Crapper (1984); a few useful pictures appear in Lighthill (1978). A fine collection of early photographs, with extensive descriptions, will be found in Cornish (1910).

Exercises

Q2.1 *Minimum of c_p*. Write $\delta k = \lambda$ in the expression for c_p^2 (equation (2.9)), and show that c_p has a single minimum (in $0 < \lambda < \infty$). Also describe the behaviour of $c_p(\lambda)$ as $\lambda \to \infty$.

Q2.2 *Simplified form of c_p*. Show, for moderate values of $\lambda = \delta k$, that c_p^2 may be written (approximately) as a linear combination of λ and λ^{-1}; cf. Q2.27. Hence find the minimum of c_p $(= c_m)$, at $\lambda = \lambda_m$ $(0 < \lambda_m < \infty)$, and find the expression for $(c_p/c_m)^2$ in terms of $l = \lambda/\lambda_m$.

[All these results are to be compared with those obtained in Q2.1; it is clear that this simplified (approximate) form of c_p is much easier to work with, and it is often used because of this.]

Q2.3 *Plane harmonic wave.* Extend the problem described in Section 2.1, to obtain the functions $U(z)$ and $P(z)$ that correspond to $W(z)$ (given by solution (2.8)).

Q2.4 *Particle paths.* Show, when written in original physical variables, that the particle paths described by equation (2.16) are circles in the short-wave limit.

Q2.5 *Laplace's equation and separation of variables.* Recover the results presented in Section 2.1, for the case of gravity waves only, by first formulating the problem in terms of the velocity potential, ϕ; see Q1.38. To proceed, construct the solution of Laplace's equation (for ϕ) by the method of separation of variables; in particular show that ϕ takes the form

$$\phi = \{A(t)\cos kx + B(t)\sin kx\}\cosh \delta kz,$$

for any given value of $k\,(\neq 0)$, where A and B are both general solutions of

$$\frac{\mathrm{d}^2 F}{\mathrm{d}t^2} + \omega^2 F = 0, \quad \omega^2 = \frac{k}{\delta}\tanh \delta k.$$

Q2.6 *Standing waves.* Take, as a special case of the result obtained from Q2.5, a choice of $A(t)$ and $B(t)$ which describes a solution for $\eta(x, t)$ which is a single separable function of x and t. In this solution, at a given position (x), the surface oscillates vertically between its maximum and minimum values; the maximum (or minimum) value does not propagate. This is therefore a standing wave. Use your solution to show how this wave can be interpreted as two propagating waves.

Q2.7 *Oblique plane waves.* Follow the presentation given in Section 2.1, but for a surface wave described by

$$\eta = A\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)} + \text{c.c.};$$

see equation (2.4). Find the dispersion relation, and show that this is equation (2.9) with k^2 replaced by $k^2 + l^2$. Confirm that the wave propagates in the direction of the wave-number vector $\mathbf{k} \equiv (k, l)$.

Q2.8 *Waves along a rectangular channel.* A channel, $-\infty < x < \infty$ with $0 \leq y \leq l$, contains water ($0 \leq z \leq 1$ when undisturbed) on the surface of which a gravity wave propagates in the (positive) x-direction. Show that there is a solution of the governing linear equations (cf. Q2.5) for which

$$\eta = A\cos(\alpha y)\cos(kx - \omega t).$$

Determine the constant α and the dispersion function ω.

Q2.9 *Sloshing in a rectangular container*. A rectangular tank, $0 \le x \le l$ and $0 \le y \le L$, contains a liquid ($0 \le z \le 1$ when undisturbed) whose surface is described by the standing gravity wave

$$\eta = A\cos(\alpha x)\cos(\beta y)\cos(\omega t),$$

for suitable constants α and β; A is the fixed amplitude of the wave. Seek an appropriate solution of Laplace's equation, (2.66), which satisfies the surface (2.67) and bottom conditions (2.68) (with $b = 0$), as well as the conditions on the side walls; that is, $\phi_x = 0$ on $x = 0, l$; $\phi_y = 0$ on $y = 0, L$. (Note that our formulation here is in terms of nondimensional variables.) Find α, β and the dispersion function ω.

[It is the *standing wave* which constitutes the *sloshing mode* in a container.]

Q2.10 *Short-crested waves*. Follow the formulation described in Q2.5, but retain the dependence on y; cf. Q2.7. Seek a solution for ϕ, by using an appropriate separation variables, that will allow the surface wave to take the form

$$\eta(x, y, t) = A\cos(mx + ny)\cos(kx + ly - \omega t),$$

where A, m, n, k and l are constants; ω is the dispersion function. Find ω and the relation that must exist between the wave numbers m, n, k and l for this type of solution to exist. Interpret this condition geometrically. Describe your solution, and find the speed and direction of propagation of the wave.

[These waves are called *short-crested* to differentiate them from plane waves, which are *non-oscillatory* along their wavefronts.]

Q2.11 *Waves on a uniform stream*. Consider the propagation of plane harmonic waves in the x-direction, on the surface of a fluid (of constant depth, $b = 0$) which moves at constant speed, $u = u_0$, also in the x-direction. (The relevant equations are obtained from (1.57), (1.58), (1.63), (1.64) and (1.65), with $\varepsilon u \to u_0 + \varepsilon u$; otherwise follow the method that leads to equations (2.1).) Show that the dispersion relation corresponding to equation (2.9) is exactly (2.9), but with ω replaced by $\omega - u_0 k$ (where k is the wave number).

[This result describes the familiar *Doppler shift*.]

Q2.12 *Oblique waves on a uniform stream.* See Q2.11; repeat this calculation for the constant uniform flow $\mathbf{u} \equiv (u_0, v_0)$ and for a plane wave with a wave-number vector $\mathbf{k} \equiv (k, l)$. Show that, now, ω is replaced by $\omega - u_0 k - v_0 l = \Omega$, and that the wave crests move forward at the velocity $(u_0 + \Omega\hat{k}, v_0 + \Omega\hat{l})$, where $(\hat{k}, \hat{l}) = (k, l)/(k^2 + l^2)$. Hence describe the condition for which the waves are stationary in the physical frame of reference.

Q2.13 *Gravity waves over a step.* Stationary water of constant depth, h_-, is in $x < 0$, and of constant depth, h_+, in $x > 0$; there is a step at $x = 0$. Small amplitude gravity waves ($W_e = 0$), of wave number k and amplitude A, approach the step from $-\infty$. The step generates, in general, a transmitted wave which propagates towards $+\infty$ and a reflected wave which moves back to $-\infty$. Follow the development given in Section 2.1 and, at $x = 0$, impose the conditions of (a) continuity of wave amplitude; (b) conservation of mass flux across $x = 0$. Find the amplitudes of the transmitted and reflected waves.

Q2.14 *Kelvin–Helmholtz instability.* An incompressible fluid of density $\lambda\,(< 1)$ exists in $z > 0$ and is moving at a constant speed, U, in the (positive) x-direction. Another incompressible fluid of density 1, which is stationary in its undisturbed state, is in $z < 0$; at $z = 0$ there exists an interface on which a small amplitude harmonic wave propagates (also in the x-direction). (This problem is presented in nondimensional variables, using the properties of the lower fluid for the purposes of nondimensionalisation; thus λ = (density of upper fluid)/(density of lower fluid). Because the fluids are of infinite depth, it is convenient to choose the vertical scale to be the same as the horizintal; thus set $\delta = 1$.)

Formulate the problem in terms of Laplace's equation for each of the fluids with an interfacial wave

$$\eta = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.},$$

where A is a complex constant. Use the kinematic condition on $z = 0$ for both fluids, and impose the continuity of pressure across $z = 0$ (and include the effects of surface tension). Show that the dispersion function is

$$(1+\lambda)\omega^2 - 2\lambda kU\omega - k(1-\lambda) + \lambda k^2 U^2 - k^3 W = 0,$$

and hence deduce that harmonic waves are *stable* if

$$\lambda U^2 \leq (1+\lambda)\{(1-\lambda)/k + kW\}$$

for all $k\,(>0)$.

[This exercise describes the simplest model for wind blowing over the surface of water. Notice that the expression on the right in the inequality has a minimum in $k > 0$; what is it?]

Q2.15 *Rayleigh–Taylor instability*. See Q2.14; now set $U = 0$ (so that both fluids are stationary in the undisturbed state) but consider $\lambda > 1$, so that the *heavier* fluid is above the lighter. Show that the wave with number k is *stable* if

$$k^2 W \geq \lambda - 1.$$

[This demonstrates the property that surface tension tends to stabilise the system: it is certainly *unstable* if W is small enough, for any $k \neq 0$.]

Q2.16 *Method of stationary phase*. Consider the integral

$$I(\sigma) = \int_a^b f(x) \mathrm{e}^{\mathrm{i}\sigma\alpha(x)} \mathrm{d}x,$$

for $\sigma \to \infty$, where the path of integration is taken along the real axis with a and b independent of the parameter σ.

(a) Suppose that $\alpha'(x)$ does not vanish for any $x \in [a, b]$; then show by integration by parts that

$$I(\sigma) = \frac{\mathrm{i}}{\sigma}\left\{\frac{f(a)}{\alpha'(a)}\mathrm{e}^{\mathrm{i}\sigma\alpha(a)} - \frac{f(b)}{\alpha'(b)}\mathrm{e}^{\mathrm{i}\sigma\alpha(b)}\right\} + \mathrm{O}(\sigma^{-2}),$$

provided that $f(a)$ and $f(b)$ are not both zero.

(b) This time suppose that $\alpha'(a) = 0$, with $\alpha''(a) > 0$, and $\alpha'(x) \neq 0$ for all $x \in (a, b]$. Write the interval (a, b) as $(a, a+\varepsilon)$ plus $(a+\varepsilon, b)$, where we can use the calculation in (a) for the latter interval. In the former interval, write

$$\alpha(x) = \alpha(a) + u^2, \quad x = \alpha + \sum_{n=1}^{\infty} b_n u^n, \quad u \in [0, \hat{u}]$$

where $\hat{u} = \sqrt{\alpha(a+\varepsilon) - \alpha(a)}$; further, it is convenient to introduce

$$f(x)\frac{\mathrm{d}x}{\mathrm{d}u} = \sum_{n=0}^{\infty} c_n u^n = c_0 + uF(u), \quad c_0 = b_1 f(a),$$

where $F(u)$ is regular for $u \in [0, \hat{u}]$. Hence show that

$$I(\sigma) = \left\{\frac{\pi}{2\sigma\alpha''(a)}\right\}^{1/2} f(\alpha)\exp\{i[\sigma\alpha(a) + \pi/4]\} + O(\sigma^{-1}).$$

[More details of this calculation, and of related problems, can be found in Copson (1967), Olver (1974).]

Q2.17 *Cylindrical coordinates.* Use equations (2.2) to obtain, for long waves ($\delta \to 0$) and with $b = 0$, the equation (2.14) for the surface waves written in cylindrical coordinates.

Q2.18 *Concentric waves I.* See Q2.17; now consider waves that are purely concentric (so that $\eta = \eta(r, t)$ only). Use the Hankel transform to obtain the solution which satisfies

$$\eta(r, 0) = f(r), \quad \eta_t(r, 0) = 0,$$

for which $\eta(0, t)$ and $\eta_r(0, t)$ are bounded and $\eta(r, t) \to 0$ as $r \to \infty$ for $0 < t < \infty$.

Q2.19 *Concentric waves II.* Repeat the calculations described in Q2.18, but now for the linear water-wave problem which represents propagation on infinitely deep water in the absence of surface-tension effects. (This requires starting from equations (2.2), with $W = 0$, $\partial/\partial\theta \equiv 0$ and $w \to 0$ as $z \to -\infty$.) What is the corresponding solution which satisfies the initial data

$$\eta(r, 0) = 0, \quad \eta_t(r, 0) = f(t)?$$

Q2.20 *Sloshing in a cylindrical tank.* A cylindrical tank, $0 \le r \le a$, contains a liquid ($0 \le z \le 1$ when undisturbed) which is in motion due to the presence of a small-amplitude standing gravity wave. Show that there is a solution which takes the form

$$\eta = AJ_n(\sigma r)\cos(n\theta)\sin(\omega t)$$

for suitable σ and ω; n (≥ 0) is an integer and J_n is the Bessel function of the first kind, of order n. (See equations (2.66)–(2.68), Q1.38 and Q2.17.) What is special about the choice $n = 0$?

Q2.21 *Wave propagation in a cylindrical tank.* See Q2.20; now seek a solution

$$\eta = AJ_n(\sigma r)\sin(n\theta - \omega t),$$

which describes a wave propagating around the tank. Find σ and ω, and compare all your results with those obtained in Q2.20 (and, in particular, check agreement for $n = 0$).

Q2.22 *General initial-value problem.* Use the Fourier transform to write down the solution described in Section 2.1 which satisfies $\eta(x, 0) = f(x)$ and $\eta_t(x, 0) = 0$; to do this you must allow the possibility that waves may propagate in both directions. In the special case where $f(x) = A\delta(x)$, where $\delta(x)$ is the Dirac delta function and A is a constant, find a wholly real expression (that is, $\mathrm{i} = \sqrt{-1}$ appears nowhere) for $\eta(x, t)$.

Q2.23 *Simple linear dispersion.* Write down the dispersion relation for gravity waves moving over water of arbitrary depth. Consider waves propagating only to the right and approximate $\omega(k)$ as $\delta k \to 0$, retaining terms as far as $\mathrm{O}(\delta^2 k^2)$. Hence write down a simple linear partial differential equation which has your approximate dispersion relation as its (exact) dispersion relation; cf. equation (1.78).

Q2.24 *Behaviour near a wavefront.* See Q2.22; consider the component of $\eta(x, t)$ (for a general initial profile) which is propagating to the right, and examine the approximate form of this solution for long gravity waves. (This is the relevant approximation near a wavefront.) To accomplish this, retain terms as far as $\mathrm{O}(k^3)$ in $\omega(k)$ (cf. Q2.23) and retain just the first term in the expansion (as $\delta k \to 0$) of the Fourier transform of $\eta(x, 0)$. (You should assume that $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ is finite and nonzero.) Now express this solution in terms of the Airy function, Ai, and hence describe the behaviour of $\eta(x, t)$ (a) ahead of the wavefront; (b) behind the wavefront; (c) at the wavefront, as a function of t.

Q2.25 *Complex variable method.* Consider the problem described by equations (2.3) and Q2.5 (but include the Weber number, labelled W_e here, in this latter problem). Introduce the complex potential

$$W(Z, t) = \phi + \mathrm{i}\psi,$$

in the usual notation, where $Z = x + \mathrm{i}\delta z$. Let the bottom, $z = 0$, correspond to the streamline $\psi = 0$ and hence deduce that

$$\bar{W} = W(\bar{Z}, t) = \phi - \mathrm{i}\psi,$$

where the overbar denotes the complex conjugate. Show that the problem reduces to finding an appropriate solution of

$$\delta \frac{\partial^2}{\partial t^2}(W + \bar{W}) + \mathrm{i}\left(\frac{\partial}{\partial x} - \delta^2 W_\mathrm{e} \frac{\partial^3}{\partial x^3}\right)(W - \bar{W}) = 0$$

on $z = 1$. Confirm that the dispersion relation (2.9) is recovered if we seek a solution $W = A\cos(kZ - \omega t)$, where A, k, and ω are real constants.

Q2.26 *Group speed for general water waves.* Repeat the calculation described in Section 2.1.2, but now retain the effects of surface tension. Show that the amplitude of the wave, which is prescribed as a function of $X = \alpha x$ at $t = 0$, propagates at the group speed

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} \quad \text{where} \quad \omega^2 = \left(\frac{k}{\delta} + \delta k^3 W_\mathrm{e}\right)\tanh \delta k.$$

Further, show that c_g may be written as

$$c_g = \frac{1}{2}c_p\left\{\frac{1 + 3\delta^2 k^2 W_\mathrm{e}}{1 + \delta^2 k^2 W_\mathrm{e}} + \frac{2\delta k}{\sinh 2\delta k}\right\} \text{ where } c_p = \frac{\omega}{k}.$$

Q2.27 *Propagation over infinitely deep water.* A plane wave propagates in the x-direction over water of infinite depth. Follow the calculation described in Section 2.1, starting from equations (2.1) but with $w \to 0$ as $z \to -\infty$, and hence obtain the dispersion relation; cf. Q2.2. What is the group speed?

[Observe that this solution describes a disturbance which decays exponentially with depth.]

Q2.28 *Group speed: general argument I.* A wave motion is described by the sum of two components

$$\eta(x, t) = A_0 \exp\{\mathrm{i}(kx - \omega(k)t\} + A_0 \exp\{\mathrm{i}(lx - \omega(l)t)\} + \text{c.c.},$$

based on two different wave numbers (k and l), but one dispersion relation, $\omega = \omega(k)$; both components have the same amplitude, A_0. Now suppose that $l = k(1 + \alpha)$ with $\alpha \to 0$ (so that the wave numbers differ by $\mathrm{O}(\alpha)$); for x and t fixed, as $\alpha \to 0$, show that

$$\eta \sim A(X, T)\exp\{\mathrm{i}(kx - \omega(k)t)\},$$

where $\alpha x = X$, $\alpha t = T$. Further, confirm that A is a wave which propagates at the speed $\omega'(k) = c_g$.

Q2.29 *Group speed: general argument II.* A wave motion depends on the phase variable θ, such that

$$\frac{\partial\theta}{\partial x} = k, \quad \frac{\partial\theta}{\partial t} = -\omega.$$

Confirm that $\theta = kx - \omega t +$ constant if both k and ω are constants. We now suppose that the wave evolves so that both k and ω change; deduce that

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0,$$

and explain how this can be interpreted as a conservation of waves. (It is usual to regard k and ω as *slowly* evolving, in the sense that they are functions of $X = \alpha x$ and $T = \alpha t$ as $\alpha \to 0$; see Section 2.1.1 and Q2.28.) Given, further, that $\omega = \omega(k)$ and that the energy is represented by $E = E(k)$, deduce that E propagates at the group speed, $\omega'(k)$.

Q2.30 *Group speed: orthogonality approach.* Derive equation (2.29), for $A_0(X, T)$, directly from the equation for W_1, (2.24). To accomplish this, multiply this equation by W_0 and then integrate it in z, from $z = 0$ to $z = 1$. Use integration by parts to form the term W_{0zz}, and use the equation defining W_0 together with boundary conditions for both W_0 and W_1.

[Since the equation for W_1 is an inhomogeneous version of W_0, with corresponding boundary conditions, a solution for W_1 exists only if an *orthogonality condition* is satisfied. This condition is the equation for A_0.]

Q2.31 *Energy and energy flux I.* Consider the solution developed in Q2.10, and choose the case of plane oblique waves (that is, $m = n = 0$). Use the details derived in this calculation to find the energy, $\mathscr{E}$, of the plane waves; see equation (1.48) and Section 2.1.2. Write your expression for $\mathscr{E}$ with error $O(\varepsilon^3)$ as $\varepsilon \to 0$. Now obtain the corresponding expression for the energy flux, $\mathscr{F}$ (given by equation (1.49) with P measured relative to P_a), but written in nondimensional form.

As mentioned in Section 2.1.2, it is convenient to compute the average values of $\mathscr{E}$ and $\mathscr{F}$ taken over one period; do this by integrating in θ from 0 to 2π (where $\theta = kx + ly - \omega t$) and dividing by 2π. Show that these average values (denoted by the overbar) are

$$\bar{\mathscr{E}} = \bar{\mathscr{E}}_0 + \frac{1}{2}\varepsilon^2 A^2 + O(\varepsilon^3),$$

where $\mathscr{E}_0$ is the contribution to the potential energy in the absence of the wave, and

$$\bar{\mathscr{F}} \equiv \frac{1}{2}\varepsilon^2 A^2\left(\frac{1}{\omega} + \frac{c_g}{|\mathbf{k}|}\right)\mathbf{k} + \mathrm{O}(\varepsilon^3)$$

where $\mathbf{k} \equiv (k, l)$. (You may also confirm that the contribution from the wave to $\bar{\mathscr{E}}, \frac{1}{2}\varepsilon^2 A^2$, is comprised of two equal parts: the average kinetic and potential energies of the wave; this equality always arises in linear problems.)

Q2.32 *Energy and energy flux II.* See Q2.31; now show that the first term in $\bar{\mathscr{F}}, \frac{1}{2}(\varepsilon^2 A^2/\omega)\mathbf{k}$, arises from a contribution from the mass flux $\int_0^h \rho \mathbf{u}_\perp \mathrm{d}z$ (written here in dimensional variables). We write this contribution as $\mathscr{F}_0$, and then set

$$\bar{\mathscr{E}} = \mathscr{E}_0 + \mathscr{E}_\mathrm{w} + \mathrm{O}(\varepsilon^3), \quad \bar{\mathscr{F}} = \mathscr{F}_0 + \mathscr{F}_\mathrm{w} + \mathrm{O}(\varepsilon^3),$$

where the subscript w denotes the contribution from the wave motion. Hence deduce that

$$\mathscr{F}_\mathrm{w} = c_g \mathscr{E}_\mathrm{w}\mathbf{k}/|\mathbf{k}|,$$

and so the energy flux is in the direction of the wave-number vector, and the energy moves at the speed c_g.

[The term $\mathscr{F}_0$ shows that there is a mass flux of $\mathrm{O}(\varepsilon^2)$, even for particle paths that are *closed* at $\mathrm{O}(\varepsilon)$; this is usually called the *Stokes mean drift*, and it is explored further in Q4.4.]

Q2.33 *Characteristics for variable depth.* The equation for variable depth, (2.40), is

$$\eta_{tt} - (d\eta_x)_x = 0;$$

rewrite this equation in terms of the characteristic variables $\xi = \int_0^x \mathrm{d}x/\sqrt{d} - t$ and $\zeta = \int_0^x \mathrm{d}x/\sqrt{d} + t$. Sketch the characteristic lines for $d(x) = \alpha(x_0 - x)$, where $\alpha > 0$ is a constant and x_0 is fixed.

Q2.34 *Green's law.* See Q2.33; now seek a solution in the form

$$\eta = d^{-1/4} H(\xi, \zeta),$$

and obtain the equation for H (which will include coefficients that depend on $d(x)$; these could be written in terms of ξ and ζ, but there is no need to do that here). Describe the special forms that H takes in the two cases

$$\text{(a) } d(x) = (\alpha x + \beta)^{4/3}; \quad \text{(b) } d(x) = (\alpha x + \beta)^2,$$

where α and β are arbitrary constants.

[The amplitude factor, $d^{-1/4}$, is the property usually associated with Green's law, although a more precise statement of the law usually includes the requirement that the horizontal velocity component also be proportional to $d^{-3/4}$.]

Q2.35 *Laplace's equation and waves over a constant slope.* State the problem described by equations (2.48), with $b(x) = \alpha x$, in terms of Laplace's equation; cf. Q2.5. For the choice $\alpha\delta = 1$, recover Hanson's solution given by equation (2.61). [Hint: see also equations (2.66)–(2.69).]

Q2.36 *Waves over a constant slope with $\alpha\delta = 1/\sqrt{3}$.* Repeat the calculation of Q2.35, but now for the case $\alpha\delta = 1/\sqrt{3}$. Show that a consistent solution is obtained, following the approach introduced by Hanson, if *three* sets of terms are now introduced: one oscillatory in x, with wave number $k\ (>0)$, and two oscillatory in z with (complex) wave numbers $\frac{1}{2}(\sqrt{3} \pm \mathrm{i})k$.

[See Whitham (1979) for a further exploration of these ideas, and Hanson (1926) for applications to a variety of wave problems.]

Q2.37 *Oblique-cum-edge waves.* See Q2.35 and Q2.7; seek a solution of these equations with $b(x) = \alpha x$ and $\alpha\delta = 1$, in the form

$$\phi = F(x, z)\mathrm{e}^{\mathrm{i}(ly-\omega t)} + \text{c.c.},$$

which is bounded as $x \to -\infty$, $z \to -\infty$. Show that your solution represents an oblique wave at infinity (with both incoming and outgoing components), as well as an edge-wave structure in a neighbourhood of the shoreline.

Q2.38 *Group velocity for slow depth change.* From equations (2.76) and (2.78), with $D = 1 - B$, obtain an expression for the group velocity $\mathbf{c}_g \equiv (\partial\omega/\partial k, \partial\omega/\partial l)$; see equation (2.84).

Q2.39 *Dispersion relation for steady waves.* Describe the variation of σ with D (for $0 < D < \infty$), as given by $\sigma \tanh(\sigma D) = \text{constant}$; see equation (2.92).

Q2.40 *Eikonal equation.* Use the method of characteristics to obtain the solutions of

$$\Theta_X^2 + \Theta_Y^2 = c^2,$$

where $c > 0$ is a constant, in the two cases

(a) $\Theta = kcs$ on the line $X = s$, $Y = s$ (where $k \neq \pm\sqrt{2}$ is a constant);

(b) $\Theta = \sqrt{2}cs$ on the line $X = s$, $Y = s$.

Q2.41 *Ray theory for propagation over a ridge.* See equations (2.95) and (2.97); obtain the equations, for both the rays and the wavefronts, for a depth variation which gives $\sigma^2(X) - \mu^2 = \sigma_0^2 \tanh^2 \beta X$, where σ_0 and β are positive constants.

Q2.42 *Ray theory with a shoreline.* See Q2.41; repeat this calculation for a depth variation which gives rise to $\sigma^2(X) - \mu^2 = -\beta/X$ for $X < 0$, where $\beta > 0$ is a constant. Also determine how the amplitude, $A(X)$, varies (cf. equation (2.98) *et seq.*).

Q2.43 *Trapped waves.* Obtain the equation for the rays in the case where the depth variation is such that $\sigma^2(X) - \mu^2 = \beta X(X_0 - X)$, where β and X_0 are positive constants.

Q2.44 *Differential equation for the rays.* Consider the eikonal equation given in Q2.40, but now with $c = c(X, Y)$. Write down the equations that define the solution using the method of characteristics. (These are equations for X, Y, Θ, Θ_X, Θ_Y in terms of a parameter.) Treat the ray as a curve $Y = Y(X)$ and, by eliminating Θ_X and Θ_Y between your equations, show that $Y(X)$ satisfies

$$c\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} + \left(c_X \frac{\mathrm{d}Y}{\mathrm{d}X} - c_Y\right)\left\{1 + \left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2\right\} = 0.$$

Hence describe the rays for

(a) $c =$ constant; (b) $c = c(X)$ only.

Q2.45 *Fermat's principle.* This states that light travels between any two points along a path which minimises the time. If the path is represented by $Y = Y(X)$, and the speed of light at any point is $1/c(X, Y)$, show that the Euler–Lagrange equation for this problem in the calculus of variations recovers the equation given in Q2.44. (The speed is written in this form, rather than simply $c(X, Y)$, in order to correspond to the particular choice of eikonal equation used in Q2.40.)

Q2.46 *Snell's Law.* Suppose that $c = c(X)$; show that the equation for the rays may be integrated once to yield

$$cY'/\sqrt{1 + (Y')^2} = \text{ constant},$$

where $Y' = \mathrm{d}Y/\mathrm{d}X$; see Q2.44(b). On the ray, let $Y'(X) = \tan \alpha(X)$ and deduce that

$$c(X) \sin \alpha(X) = \text{ constant},$$

which is Snell's law of refraction.

Q2.47 *A circular shoal.* In cylindrical geometry, suppose that the depth varies so that $R^2\sigma^2(R) = \beta R$, where β is a positive constant; see equation (2.104). Obtain the equation for the rays that approach from infinity, and describe their behaviour.

Q2.48 *A circular island.* See Q2.47; the depth now varies so that $R^2\sigma^2(R) - \mu^2 = \beta R^2/(R - R_0)$, where β and R_0 are positive constants. Find the equation for the rays and describe their behaviour as $R \to R_0$ (which is the shoreline).

Q2.49 *Ship waves: the wedge angle.* Describe the behaviour of the angle of the wedge inside which the dominant ship-wave pattern is observed as the depth is decreased. (You should consider only constant speed, straight-line motion.) What is the wedge angle if $c_g = 3c_p/4$?

Q2.50 *Ship waves: method of stationary phase.* Use Kelvin's method of stationary phase to show that the dominant asymptotic behaviour of

$$\int_0^\infty p^{3/2} \sin\left(t\sqrt{\frac{p}{\delta}}\right) J_0(rp)\mathrm{d}p,$$

as $t^2/\delta r \to \infty$, is

$$\frac{1}{8\delta\sqrt{2\delta}}\frac{t^3}{r^4}\sin\left(\frac{1}{4}\frac{t^2}{\delta r}\right).$$

Q2.51 *Influence points I.* A simple geometrical construction enables us to show that there are just two influence points. Consider the motion of a point (ship) moving at constant speed in a straight line; the ship is at P, and W is any point behind the ship and off the ship's path. Draw PW, the mid-point of PW at M and the circle with diameter MW; identify the points (where they exist) where this circle intersects the path of the ship. Hence deduce that, at most, only two influence points exist.

Q2.52 *Influence points II.* Reconstruct the argument used in Q2.51 by an algebraic method. (For example, show that there are, at most, only two instants in time before $t = 0$ at which disturbances could have been initiated and which contribute to any given point W). Repeat this calculation for a ship moving on a circular course at constant speed.

Q2.53 *Ship on a circular path.* A ship is moving on a circular path, of radius R_0, at a constant speed U. Find the parametric representation of the curves that describe the dominant wave pattern, equivalent to equations (2.123). [You may also show how equations (2.123) are recovered from your equations derived here.]

Q2.54 *Ship waves: capillary-wave limit.* Repeat the analysis described in Section 2.4.2, but now take the dispersion relation for Ω to be that which describes capillary waves in the absence of gravity waves (and approximated for long waves). Show that

$$2\tan\phi\tan^2\theta - 3\tan\theta - \tan\phi = 0,$$

and deduce that solutions exist for all ϕ (and so the dominant waves are no longer confined to a wedge-shaped region).

Q2.55 *Simple waves: wave-maker problem.* Use the method of characteristics (Section 2.6.1) to solve the problem of flow in $x > 0$, over constant depth, given $h(0, t) = H(t)$ for $t > 0$ with $u(x, 0) = 0$ and $h(x, 0) = h_0 =$ constant. What is the corresponding solution if $u(0, t) = U(t)$, $t > 0$, is given?

Q2.56 *Simple waves: piston problem.* See Q2.55; repeat this calculation but now in $x > X(t)$, $t > 0$; that is, the 'end wall' is moved according to $x = X(t)$. The water in $x > 0$ is of constant depth (h_0), and it is stationary, at $t = 0$.

[If $X'(t) < 0$, then we generate an expansion fan.]

Q2.57 *Simple waves with a shear structure.* Use the governing equations discussed in Section 2.8, but for constant depth $b(x) = 0$, and show that we may seek a solution in the form

$$\eta = H(\xi), \quad u = U(\xi, z), \quad w = \hat{W}(\xi, z)\frac{\partial \xi}{\partial x},$$

for suitable functions H, U, and $\hat{W}$, where $\xi = x - ct$ and $c = c(H)$. Further, simplify the problem by writing $U = U(H, z)$ and $\hat{W} = H'W(H, z)$; show that

$$\int_0^{1+H} \frac{\mathrm{d}z}{(U - c)^2} = 1 \quad \text{(the Burns condition)}$$

and that

$$I_{zH} + II_{zz} = 2I_z(I_z \pm c'\sqrt{|I_z|})$$

with $I(H, 1+H) = 1$, $I(H, 0) = 0$, where

$$I(H, z) = \int_0^z \frac{\mathrm{d}z}{(U-c)^2}.$$

(The complete formulation of this problem requires the 'initial' condition: $I(H, z)$ given at some H; $c(H)$ also must be known, which is determined from the Burns condition, with either $U - c > 0$ or $U - c < 0$.)

Finally, rewrite this problem in terms of the similarity variable $Z = z/(1+H)$, so that now $I = I(H, Z)$. Confirm that the choice $I = kZ$ and $c'\sqrt{1+H} = \mp 3/2$ recovers the simple-wave solution (in the absence of shear) described in Section 2.6.1.

[More information about this problem can be found in Freeman (1972) and Blythe, Kazakia & Varley (1972); the Burns condition is described in Burns (1953) and in Thompson (1949). We shall provide a discussion involving some properties of the Burns condition in Chapter 3.]

Q2.58 *Nonlinear wave run-up.* See Section 2.8; reduce the equation for $t(\xi, \eta)$ to the cylindrical wave equation in $T(\xi+\eta, \eta-\xi)$ (cf. Section 2.6.2), and find expressions for u, c, t and x in terms of T. Hence use the method of separation of variables to find a solution for T which is bounded at the shoreline. Use your results to find: (a) the maximum run-up; (b) the behaviour of the solution far from the shoreline (cf. Section 2.2).

Q2.59 *Wave breaking.* See Q2.58; the condition for the breaking of the wave corresponds to where the Jacobian in the hodograph transformation is first zero. Use the results obtained in Q2.58 to show that breaking first occurs at the shoreline (as we would expect).

[This problem requires the introduction of some identities involving the Bessel functions J_0, J_1 and J_2; a description of this problem is to be found in Whitham (1979) and Mei (1989).]

Q2.60 *Hydraulic jump and bore.* Extend the analysis of Section 2.7 for the case of the hydraulic jump ($U = 0$), and find the speed of the flow behind the jump (u^+) and verify that $u^+ < u^-$ if $F > 1$. In this case, show that the Froude number for the flow behind the jump is less than unity.

Use the results obtained here, and in Section 2.7, to describe the characteristics of the flow associated with a bore which moves at a speed U into stationary water.

Q2.61 *Reflection of a bore from a wall.* Stationary water of depth $h = h_0$ is in $x < 0$, and there is a vertical wall at $x = 0$. A bore moving at a constant speed U approaches the wall from $-\infty$ and is reflected by it and returns to $-\infty$. Find expressions for the speed of the returning bore and the depth of water behind it.

[The solution of this problem reduces to the solution of a cubic equation, a problem which need not be pursued.]

Q2.62 *Circular hydraulic jump.* It is readily observed that water flowing from a tap into a sink almost always spreads out in a thin, fast-moving layer over the surface of the sink. Furthermore, this layer is often approximately circular in shape being terminated by a narrow region (a jump) where the depth and speed change dramatically; thereafter the water makes its way down the plughole.

Consider the problem of the circular hydraulic jump and, from equations (2.2) without the effects of surface tension and without any variation in θ, follow the methods of Section 2.7 to find the jump conditions across the circular hydraulic jump.

[Incorporate the long-wave assumption, exactly as in Section 2.7; a discussion of this problem, with the inclusion of many realistic physical properties, will be found in Watson (1964).]

Q2.63 *Modelling the highest wave.* Stokes' highest wave can be modelled by satisfying some appropriate conditions, but not all of them. The simplest model is obtained by writing

$$\eta = a\mathrm{e}^{-\alpha|\xi|}$$

and then satisfying the conditions that prescribe $\eta(0)$ and $\eta'(0)$, together with equation (2.166). What is the value of c in this case?

An improvement is to write

$$\eta = a\mathrm{e}^{-\alpha|\xi|} + b\mathrm{e}^{-2\alpha|\xi|},$$

to impose the same conditions as above and, in addition, to ensure that

$$3V = (c^2 - 1)M$$

is satisfied. What now is the value of c?

[The value of c for the highest wave is, based on numerical evidence, about 1.286; see Longuet-Higgins & Fenton (1974) for more details, where it is shown that a wave exists for which $c \approx 1.294$!]

Q2.64 *The sech² solitary wave.* Verify, or by direct integration show, that

$$f = 2c\,\mathrm{sech}^2\left(\sqrt{\frac{3c}{2K}}\zeta\right)$$

is a solution of equation (2.173), where $\zeta = \xi - c\tau$. Confirm that the behaviour of this solution, as $|\zeta| \to \infty$, satisfies the condition (2.166). Explain the connection between the two expressions for c^2: (2.13) and (2.166).

Q2.65 *Jacobian elliptic functions.* Define the integral

$$u = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}},$$

where m $(0 \le m \le 1)$ is a parameter (called the *modulus*). Then we write

$$\mathrm{sn}\,u = \sin\phi, \qquad \mathrm{cn}\,u = \cos\phi,$$

the Jacobian elliptic functions; these are sometimes written as $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$. Show that

(a) $\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1$;

(b) $\mathrm{cn}\,u = \cos u$ if $m = 0$; $\mathrm{cn}\,u = \mathrm{sech}\,u$ if $m = 1$;

(c) $\dfrac{\mathrm{d}}{\mathrm{d}u}(\mathrm{cn}\,u) = -\mathrm{sn}\,u\,\mathrm{dn}\,u$ where $\mathrm{dn}\,u = \sqrt{1 - m\sin^2\phi}$, and find the corresponding results for the derivatives of $\mathrm{sn}\,u$ and $\mathrm{dn}\,u$.

Q2.66 *Complete elliptic integral.* Define the integral

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - \mathrm{m}\sin^2\theta}},$$

the complete elliptic integral of the first kind. Deduce that the period of the elliptic functions sn and cn is $4K(m)$, $0 \le m < 1$. Show that

(a) $K(0) = \pi/2$;

(b) $K(m) = \dfrac{\pi}{2}F(\dfrac{1}{2}, \dfrac{1}{2};\ 1;\ m)$, where $F(a, b; c; z)$ is the hypergeometric function;

(c) $K(m) \sim \frac{1}{2}\log\{16/(1 - m)\}$ as $m \to 1^-$.

[Hint: in (c) write $\mathrm{d}\theta = (1 - \sqrt{m}\sin\theta + \sqrt{m}\sin\theta)\,\mathrm{d}\theta$.]

Q2.67 *Cnoidal-wave solution.* Verify that

$$f(\zeta) = a + b\,\mathrm{cn}^2\{\alpha(\zeta - \zeta_0)|m\},$$

where $\zeta = \xi - c\tau$, is a solution of equation (2.173) for suitable relations between the constants a, b, α and m; the phase shift, ζ_0, is an arbitrary constant.

[For this solution to exist, the cubic, $F(f)$, in Section 2.9.1 has three real, distinct zeros; indeed, it is convenient to write

$$F(f) = -\frac{1}{2}(f - f_1)(f - f_2)(f - f_3),$$

where f_i, $i = 1, 2, 3$, are the three roots.]

Q2.68 *Circulation associated with a solitary wave.* Show that

$$C = \int_{-\infty}^{\infty} \mathbf{u} \cdot \mathrm{d}\mathbf{l} = [\phi]_{-\infty}^{\infty}$$

by evaluating (a) along the bottom streamline; (b) along the surface streamline (and Stokes' theorem may be invoked).

Q2.69 *Integral identities for the sech^2 profile.* Examine the three integral identities, relating T, V, I, C and M (discussed in Section 2.9.2), when the solitary wave is approximated by the sech^2 profile of small amplitude; that is, η is written as $\varepsilon\eta$, $\varepsilon \to 0$ and $\eta \propto \mathrm{sech}^2$ (see Section 2.9.1).

Q2.70 *Variational principle for water waves.* Show that the equations for gravity waves on stationary water over a rigid impermeable surface (Q1.38) are obtained from the variational principle

$$\delta \iint_D L\,\mathrm{d}x\,\mathrm{d}t = 0,$$

where the Lagrangian is

$$L = \int_{b(x,y)}^{\eta(x,y,t)} [\phi_t + \frac{1}{2}(\nabla\phi)^2 + z]\mathrm{d}z.$$

The region D, assumed to contain fluid, is arbitrary; the variations of ϕ and η are zero on the boundaries of D. (The nondimensional parameters, ε and δ, are not included in this formulation.)

[These ideas are developed in Luke (1967) and Whitham (1965, 1974).]

3

Weakly nonlinear dispersive waves

The old order changeth, yielding place to new

The Passing of Arthur

In Chapter 2 we presented some classical ideas in the theory of water waves. One particular concept that we introduced was the phenomenon of a balance between nonlinearity and dispersion, leading to the existence of the solitary wave, for example. Further, under suitable assumptions, this wave can be approximated by the sech^2 function, which is an exact solution of the Korteweg–de Vries (KdV) equation; see Section 2.9.1. We shall now use this result as the starting point for a discussion of the equations, and of the properties of corresponding solutions, that arise when we invoke the assumptions of small amplitude and long wavelength. In the modern theories of nonlinear wave propagation – and certainly not restricted only to water waves – this has proved to be an exceptionally fruitful area of study.

The results that have been obtained, and the mathematical techniques that have been developed, have led to altogether novel, important and deep concepts in the theory of wave propagation. Starting from the general method of solution for the initial value problem for the KdV equation, a vast arena of equations, solutions and mathematical ideas has evolved. At the heart of this panoply is the *soliton*, which has caused much excitement in the mathematical and physical communities over the last 30 years or so. It is our intention to describe some of these results, and their relevance to the theory of water waves, where, indeed, they first arose.

3.1 Introduction

The existence of a steadily propagating nonlinear wave of permanent shape, such as the solitary wave, probably seems altogether likely. On the other hand, that somewhat similar objects could exist in pairs (or

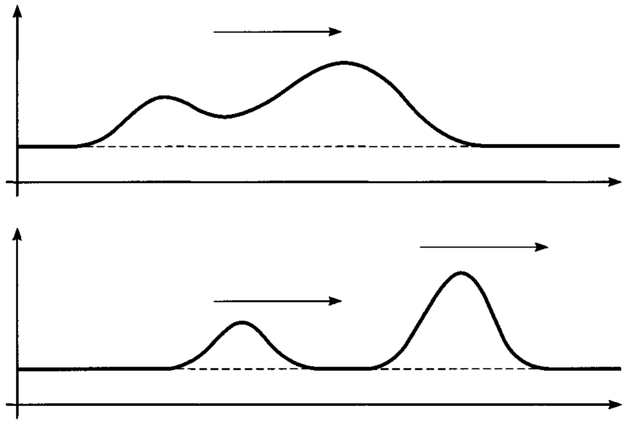


Figure 3.1. A sketch of J. Scott Russell's *compound* wave which 'represents the genesis by a large low column of fluid of a compound or double wave . . . the greater moving faster and altogether leaving the smaller'.

larger numbers) of different amplitude, interact nonlinearly and yet not destroy each other – indeed, retain their identities – would seem rather unlikely. However, precisely this phenomenon does occur for solutions of the KdV equation, and for the many other so-called *completely integrable* equations.

This very special type of interaction was first observed and described by Russell (1844); the essentials of his plate XLVII are shown in Figure 3.1. An obvious interpretation of the development represented in this figure is that an initial profile, which is not an exact solitary-wave solution, will evolve into two (or perhaps more) waves which move at different speeds and tend to individual solitary waves as time increases. Another observation, itself an extension of what Russell reported, is shown in Figure 3.2. This time we have an initial profile comprising two peaks, the taller to the left of the shorter, but both propagating to the right. The taller is moving faster (since it is locally similar to a solitary wave), and so catches up and then interacts with the shorter, and thereafter moves ahead of it. At first sight, the interaction appears to involve no interaction at all, as would be the case if the two waves satisfied the linear superposition principle; cf. equation (1.75) *et seq*. However, a nonlinear event does occur here, and that it is clearly *not* linear is confirmed by the fact that the waves are phase-shifted (by the interaction) from the

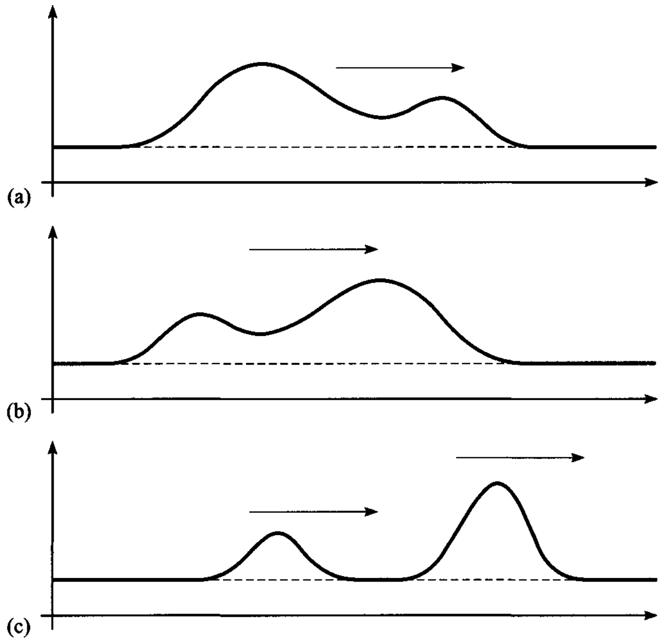


Figure 3.2. An extension of the situation depicted in Figure 3.1, where the larger wave is first to the left of the smaller; it catches up the smaller, interacts with it and then moves off to the right.

positions they would have taken had both waves travelled at constant speed througout. These, and many associated properties, will be briefly described in Section 3.3; our primary objective here is to show how this important class of completely integrable equations arise in water-wave theory. We shall then extend the ideas to more general problems, which usually do not give rise to completely integrable equations, but which do provide models for more realistic applications.

The various problems that we shall describe are based on the equation for an inviscid fluid, and for the propagation of gravity waves only (so $W_e = 0$, but see Q3.3). Although much of our early work will be for irrotational flow, some of the important applications presented later will allow an underlying rotational state; we therefore choose to develop all the work here from the Euler equations. (In contrast, a derivation of the KdV equation directly from Laplace's equation was given in Section 2.9.1.) The relevant governing equations will be found in Section 1.3.2, and they are reproduced here.

In rectangular Cartesian coordinates we have

$$\left.\begin{aligned} &\frac{\mathrm{D}u}{\mathrm{D}t}=-\frac{\partial p}{\partial x},\quad \frac{\mathrm{D}v}{\mathrm{D}t}=-\frac{\partial p}{\partial y},\quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{\partial p}{\partial z}\\ &\text{where}\\ &\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+\varepsilon\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\end{aligned}\right\}\tag{3.1}$$

with

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3.2}$$

and

$$p=\eta \quad \text{and} \quad w=\frac{\partial \eta}{\partial t}+\varepsilon\left(u\frac{\partial \eta}{\partial x}+v\frac{\partial \eta}{\partial y}\right) \quad \text{on} \quad z=1+\varepsilon\eta \tag{3.3}$$

$$w=u\frac{\partial b}{\partial x}+v\frac{\partial b}{\partial y} \quad \text{on} \quad z=b(x,y). \tag{3.4}$$

Correspondingly, in cylindrical coordinates, we have

$$\left.\begin{aligned} &\frac{\mathrm{D}u}{\mathrm{D}t}-\frac{\varepsilon v^2}{r}=-\frac{\partial p}{\partial r},\quad \frac{\mathrm{D}v}{\mathrm{D}t}+\frac{\varepsilon uv}{r}=-\frac{1}{r}\frac{\partial p}{\partial \theta},\quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{\partial p}{\partial z}\\ &\text{where}\\ &\frac{\mathrm{D}}{\mathrm{D}t}\equiv\frac{\partial}{\partial t}+\varepsilon\left(u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial \theta}+w\frac{\partial}{\partial z}\right)\end{aligned}\right\}\tag{3.5}$$

with

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0 \tag{3.6}$$

and

$$p=\eta \quad \text{and} \quad w=\frac{\partial \eta}{\partial t}+\varepsilon\left(u\frac{\partial \eta}{\partial r}+\frac{v}{r}\frac{\partial \eta}{\partial \theta}\right) \quad \text{on} \quad z=1+\varepsilon\eta \tag{3.7}$$

$$w=u\frac{\partial b}{\partial r}+\frac{v}{r}\frac{\partial b}{\partial \theta} \quad \text{on} \quad z=b(r,\theta). \tag{3.8}$$

In these equations, and for the following calculations, we consider only bottom topographies ($z=b$) which are independent of time.

3.2 The Korteweg–de Vries family of equations

We first present a derivation of the classical Korteweg–de Vries equation, from the Euler equations, being careful to describe the necessary (and minimal) assumptions that are required. We then show how this approach can be generalised to obtain corresponding equations valid in both different and higher-dimensional coordinate systems.

3.2.1 Korteweg–de Vries (KdV) equation

We consider surface gravity waves propagating in the positive x-direction over stationary water of constant depth (so $b = 0$). Thus, from equations (3.1)–(3.4), we have

$$\left.\begin{aligned} u_t + \varepsilon(uu_x + wu_z) = -p_x; \quad \delta^2\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z; \\ u_x + w_z = 0, \\ \text{with} \\ p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta \\ \text{and} \\ w = 0 \quad \text{on} \quad z = 0. \end{aligned}\right\} \tag{3.9}$$

This problem, when previously discussed via Laplace's equation in Section 2.9.1, led us to invoke a special choice of the parameters, namely $\delta^2 = O(\varepsilon)$ as $\varepsilon \to 0$. If this were to be a necessary condition in order to obtain the appropriate balance between nonlinearity and dispersion (and so to produce the KdV equation and hence to model solitary waves, for example), we might expect these waves to occur rather rarely in nature: solitary waves would be infrequently observed, and this is not the case. It is therefore no surprise that we can readily demonstrate that, for any δ as $\varepsilon \to 0$, there always exists a region of (x, t)-space where this balance comes about. Thus, for as long as no other physical effects intervene, we can expect to be able to generate KdV solitary waves (and solitons etc.) somewhere, provided only that the amplitude is small (in the sense of $\varepsilon \to 0$).

The region of interest is defined by a scaling of the independent variables. First we transform

$$x \to \frac{\delta}{\varepsilon^{1/2}}x, \quad t \to \frac{\delta}{\varepsilon^{1/2}}t, \tag{3.10}$$

for any ε and δ; this transformation then implies, for consistency from the equation of mass conservation, that we also transform

$$w \to \frac{\varepsilon^{1/2}}{\delta} w; \tag{3.11}$$

cf. Q1.34. The governing equations (3.9) then become

$$u_t + \varepsilon(uu_x + wu_z) = -p_x; \quad \varepsilon\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z; \tag{3.12}$$

$$u_x + w_z = 0, \tag{3.13}$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u \eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta \tag{3.14}$$

and

$$w = 0 \quad \text{on} \quad z = 0, \tag{3.15}$$

so the net outcome of the transformation is to replace δ^2 by ε in equations (3.9). (The presence of δ in transformations (3.10) and (3.11) is merely equivalent to using h_0 alone as the relevant length scale; see Section 1.3.1.)

Now, for $\varepsilon \to 0$, we see that a first approximation to equations (3.12) and (3.14) is

$$p(x, t, z) = \eta(x, t), \quad 0 \le z \le 1$$

with

$$u_t + \eta_x = 0. \tag{3.16}$$

Then, from equation (3.13), we obtain

$$w = -zu_x$$

which satisfies (3.15), and from condition (3.14) we require

$$\eta_t = -u_x;$$

this combined with equation (3.16) yields

$$\eta_{tt} - \eta_{xx} = 0,$$

as we should expect (cf. equation (2.10)). We choose to follow right-going waves (but see Q3.2), and so we introduce

$$\xi = x - t. \tag{3.17}$$

However, an asymptotic expansion which is based on the classical wave equation (with higher-order nonlinear and dispersive terms) necessarily leads to a non-uniformity as t (or x) $\to \infty$; this is discussed in equation (1.95) *et seq.* Thus we define a suitable long-time variable

$$\tau = \varepsilon t; \tag{3.18}$$

cf. equation (1.99). Consequently $\xi = \mathrm{O}(1)$, $\tau = \mathrm{O}(1)$, together describe the *far-field* region for this problem, and therefore the region where we expect a KdV-type of balance to occur. (We observe that these scaling arguments have been generated by the existence of the surface wave propagating in the x-direction, and no scalings are required to describe different regions of the z-structure of the problem.)

With the choice of far-field variables, (3.17) and (3.18), the equations (3.12)–(3.15) become

$$-u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi; \quad \varepsilon\{-w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\} = -p_z; \tag{3.19}$$

$$u_\xi + w_z = 0, \tag{3.20}$$

with

$$p = \eta \quad \text{and} \quad w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0. \tag{3.21}$$

We seek an asymptotic solution of this system of equations and boundary conditions in the form

$$q(\xi, \tau, z; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi, \tau, z), \quad \eta(\xi, \tau; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \tau), \tag{3.22}$$

where q (and correspondingly q_n) represents each of u, w and p. The leading-order then becomes

$$u_{0\xi} = p_{0\xi}; \quad p_{0z} = 0; \quad u_{0\xi} + w_{0z} = 0$$

with

$$p_0 = \eta_0 \quad \text{and} \quad w_0 = -\eta_{0\xi} \quad \text{on} \quad z = 1$$

and

$$w_0 = 0 \quad \text{on} \quad z = 0.$$

These equations directly lead to

$$p_0 = \eta_0, \quad u_0 = \eta_0, \quad w_0 = -z\eta_{0\xi}, \quad 0 \le z \le 1,$$

where we have imposed the condition that the perturbation in u is caused only by the passage of the wave; that is, $u_0 = 0$ whenever $\eta_0 = 0$. We see that the surface ($z = 1$) boundary condition on w_0 is automatically satisfied; thus, at this order, $\eta_0(\xi, \tau)$ is an arbitrary function. To proceed, and hence to determine η_0, we must first treat the surface boundary conditions with more care.

The two boundary conditions on $z = 1 + \varepsilon\eta$ are rewritten as evaluations on $z = 1$, by developing Taylor expansions of u, w and p about $z = 1$. (The usual convergence requirements need not be investigated since, strictly, these expansions are to exist only in the asymptotic sense as $\varepsilon \to 0$; see also Section 2.5.) These boundary conditions are therefore expressed in the form

$$p_0 + \varepsilon\eta_0 p_{0z} + \varepsilon p_1 = \eta_0 + \varepsilon\eta_1 + \mathrm{O}(\varepsilon^2) \tag{3.23}$$

and

$$w_0 + \varepsilon\eta_0 w_{0z} + \varepsilon w_1 = -\eta_{0\xi} - \varepsilon\eta_{1\xi} + \varepsilon(\eta_{0\tau} + u_0\eta_{0\xi}) + \mathrm{O}(\varepsilon^2) \tag{3.24}$$

$$\left.\right\} \text{ on } z = 1$$

which are to be used in conjunction with equations (3.19), (3.20) and (3.21).

The leading order has already been found, and the equations that define the next order are

$$-u_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi}; \quad p_{1z} = w_{0\xi}; \quad u_{1\xi} + w_{1z} = 0,$$

with

$$p_1 + \eta_0 p_{0z} = \eta_1 \quad \text{and} \quad w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + \eta_{0\tau} + u_0\eta_{0\xi} \quad \text{on} \quad z = 1$$

and

$$w_1 = 0 \quad \text{on} \quad z = 0.$$

When we note that

$$u_{0z} = 0, \quad p_{oz} = 0 \quad \text{and} \quad w_{0z} = -\eta_{0\xi}; \tag{3.25}$$

then

$$p_1 = \frac{1}{2}(1 - z^2)\eta_{0\xi\xi} + \eta_1, \tag{3.26}$$

and hence

$$\begin{aligned} w_{1z} = -u_{1\xi} &= -\, p_{1\xi} - u_{0\tau} - u_0 u_{0\xi} \\ &= -\, \eta_{1\xi} - \frac{1}{2}(1 - z^2)\eta_{0\xi\xi\xi} - \eta_{0\tau} - \eta_0\eta_{0\xi}. \end{aligned}$$

Thus

$$w_1 = -(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi})z + \frac{1}{6}z^3\eta_{0\xi\xi\xi} \tag{3.27}$$

which satisfies the bottom boundary condition; finally the surface boundary condition (now on $z = 1$) yields

$$\begin{aligned} w_1|_{z=1} &= -(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi}) + \frac{1}{6}\eta_{0\xi\xi\xi} \\ &= -\eta_{1\xi} + \eta_{0\tau} + 2\eta_0\eta_{0\xi} \end{aligned}$$

resulting in

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0; \tag{3.28}$$

at this order, η_1 is unknown. Equation (3.28) is the Korteweg–de Vries equation which describes the leading-order contribution to the surface wave; see also equation (1.102). This is the equation first derived (but not in our form) by Korteweg and de Vries (1895), which they did by seeking a solution of Laplace's equation as a power series in z. Furthermore, these authors also included the effects of surface tension, which here is left as an exercise (Q3.3). The significance of the KdV equation, together with some of its properties, will be discussed later.

Provided that bounded solutions of equation (3.28) exist, at least for $\xi = O(1)$, $\tau = O(1)$, and for all the higher-order terms, η_n, in the same region of space, then the function $\eta_0(\xi, \tau)$ gives the dominant behaviour that we seek. Clearly we may ask, in addition, if the asymptotic expansion for η (and hence for the other dependent variables) is uniformly valid as $|\xi| \to \infty$ and as $\tau \to \infty$. In the case of $\tau \to \infty$, this question is far from straightforward to answer completely, mainly because the equations for η_n, $n \geq 1$, are not readily solved. (The problem for η_1 is included as Q3.4; see also equation (1.100) *et seq.*) All the available evidence, some of which is numerical, suggests that our asymptotic representation of η is indeed uniformly valid as $\tau \to \infty$ (at least for solutions that satisfy $\eta \to 0$ as $|\xi| \to \infty$). The validity as $|\xi| \to \infty$ for $\tau < \infty$ does not normally raise any particular difficulties. These aspects are not pursued here since, although we believe that the theory just presented describes some important properties of real water waves, the relevance of $\tau \to \infty$ is questionable. Clearly, if the waves are allowed to propagate indefinitely, then other physical effects cannot be ignored; the most prominent of these is likely to be viscous damping (which we shall briefly discuss in Chapter 5). Usually, in practice, the damping is sufficiently weak to allow the nonlinear and dispersive effects to dominate before the waves eventually decay completely.

3.2.2 Two-dimensional Korteweg–de Vries (2D KdV) equation

The Korteweg–de Vries equation, (3.28), describes nonlinear plane waves that propagate in the x-direction. An obvious question to pose is: how is the wave propagation modified when the waves move on a two-dimensional surface (which, of course, is the physical situation)? Although a plane wave can propagate in any direction (at least on stationary water), and we may label this to be the x-direction, the waves that we wish to describe may not be plane. An important example arises when two (or more) waves, that are plane waves at infinity, cross; for the nonlinear interaction of these crossing waves, the y-dependence will not be trivial. We investigate the situation in which the wave configuration is propagating predominantly in the x-direction, with the appropriate balance of nonlinear and dispersive effects (also in the x-direction). However, in addition, we include the relevant dependence on the y-variable, this contribution appearing at the same order as the nonlinearity and dispersion.

The simplest way to see what this implies is to consider, first, the linear propagation of long waves on the surface; the leading-order problem is described by the classical wave equation

$$\eta_{tt} - (\eta_{xx} + \eta_{yy}) = 0,$$

written here in nondimensional variables (cf. equation (2.14)). This equation has a solution

$$\eta \propto \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)} \quad \text{where} \quad \omega^2 = k^2 + l^2;$$

see Q2.7. Now, for waves that propagate predominantly in the x-direction, we require l to be small (since the wave propagates in the direction of the wave number vector $\mathbf{k} \equiv (k, l)$) and then the dispersion relation gives

$$\omega \sim k\left(1 + \frac{1}{2}\frac{l^2}{k^2}\right) \quad \text{as} \quad l \to 0.$$

This expression represents propagation at the (nondimensional) speed of unity (cf. equation (3.17)), together with a correction provided by the wave-number component in the y-direction. In order that this correction be the same size as the nonlinearity and dispersion, we require $l^2 = O(\varepsilon)$ or $l = O(\varepsilon^{1/2})$; equivalently, we may accommodate this by transforming $y \to \varepsilon^{1/2} y$ (and then we require $v \to \varepsilon^{1/2} v$ so that we have consistency with, for example, the representation in terms of a velocity potential, $\mathbf{u} = \nabla\phi$). Thus we introduce the variables

$$\xi = x - t, \quad \tau = \varepsilon t, \quad Y = \varepsilon^{1/2} y, \quad v = \varepsilon^{1/2} V \tag{3.29}$$

and then equations (3.1)–(3.4), with δ^2 replaced by ε (see Section 3.2.1) and with $b(x, y) = 0$, become

$$\begin{aligned} -u_\xi + \varepsilon(u_\tau + uu_\xi + \varepsilon V u_Y + wu_z) &= -p_\xi; \\ -V_\xi + \varepsilon(V_\tau + uV_\xi + \varepsilon VV_Y + wV_z) &= -p_Y; \\ \varepsilon\{-w_\xi + \varepsilon(w_\tau + uw_\xi + \varepsilon V w_Y + ww_z)\} &= -p_z; \\ u_\xi + \varepsilon V_Y + w_z &= 0, \end{aligned}$$

with

$$p = \eta \quad \text{and} \quad w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi + \varepsilon V \eta_Y) \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0;$$

cf. equations (3.19)–(3.21). We seek an asymptotic solution, valid as $\varepsilon \to 0$, in the same form as before (see (3.22)); the leading order problem is then unchanged (except that the variables may now depend on Y), so that

$$p_0 = \eta_0, \quad u_0 = \eta_0, \quad w_0 = -z\eta_{0\xi}, \quad 0 \le z \le 1,$$

and with $V_{0\xi} = \eta_{0Y}$. At the next order, the only difference from the derivation of the KdV equation (Section 3.2.1) arises in the equation of mass conservation, which here reads

$$w_{1z} = -u_{1\xi} - V_{0Y}.$$

The result of this change is to give

$$w_1 = -(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi} + V_{0Y})z + \frac{1}{6}z^3\eta_{0\xi\xi\xi};$$

cf. equation (3.27). Thus we obtain the equation for the leading-order representation of the surface wave in the form

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} + V_{0Y} = 0$$

where $V_{0\xi} = \eta_{0Y}$; upon the elimination of V_0 this yields

$$\left(2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi}\right)_\xi + \eta_{0YY} = 0, \tag{3.30}$$

the *two-dimensional Korteweg–de Vries* (*2D KdV*) equation. (The small amount of detail omitted from this derivation, which follows that leading to equations (3.23)–(3.28), is left as an exercise.) We note that this result recovers the KdV equation, (3.28), when there is no dependence on Y; that is, $V_{0Y} = 0$.

Equation (3.30), which in the literature is often called the *Kadomtsev–Petviashvili* (*KP*) equation (Kadomtsev & Petviashvili, 1970), turns out to be another of those very special completely integrable equations. This equation admits, as an exact analytical solution, any number of waves that cross obliquely and interact nonlinearly; this in turn, for the case of three such waves, leads to special solutions which correspond to a *resonance* condition. We shall comment in more detail about the solution of this equation later, and some of the exercises allow further exploration.

Before we leave this equation for the present, it is instructive to interpret the scalings, (3.29), that have led to the 2D KdV equation. This nonlinear dispersive wave appears at times $\tau = \mathrm{O}(1)$, so t is large, and where $\xi = \mathrm{O}(1)$; that is, the measure of the 'width' of the wave remains finite and non-zero as $\varepsilon \to 0$. However, the wave also depends on $Y = \varepsilon^{1/2}y$, and this is most conveniently thought of as weak dependence on the y-coordinate. That is, along any wavefront, $\mathrm{d}y/\mathrm{d}\xi = \mathrm{O}(\varepsilon^{1/2})$ and so, in physical coordinates, the waves deviate only a little ($\mathrm{O}(\varepsilon^{1/2})$) away from plane waves ($\xi =$ constant). Thus, for example, in the case of two obliquely crossing waves – an exact solution of the 2D KdV equation – the angle between them (in physical coordinates) is $\mathrm{O}(\varepsilon^{1/2})$: they are nearly parallel waves in this approximation. We shall discuss more general aspects of obliquely crossing waves later (Section 3.4.5), as an example of a non-uniform environment.

3.2.3 Concentric Korteweg–de Vries (cKdV) equation

The two equations derived above, the KdV and 2D KdV equations, are the relevant weakly nonlinear dispersive wave equations that arise in Cartesian geometry. It is now reasonable to ask if a corresponding pair of equations exists in cylindrical geometry. In this section and the next we shall demonstrate that this is, indeed, the case, although the change of coordinates is not an altogether trivial exercise, since important differences arise. To see what the essential changes are we first consider, for purely concentric waves, the linearised problem for large radius.

The equation for linear concentric waves (in the long-wave approximation) is

$$\eta_{tt} - \left(\eta_{rr} + \frac{1}{r}\eta_r\right) = 0;$$

see equation (2.14), with the dependence on the angular coordinate, θ, absent. It is convenient to introduce the characteristic coordinate $\xi = r - t$ (for outward propagation) and $R = \alpha r$ (so that $\alpha \to 0$ will correspond to large radius; that is, $R = O(1)$, $\alpha \to 0$, yields $r \to \infty$). The equation therefore becomes

$$2\eta_{\xi R} + \frac{1}{R}\eta_\xi + \alpha\left(\eta_{RR} + \frac{1}{R}\eta_R\right) = 0,$$

and as $\alpha \to 0$ we see that

$$\sqrt{R}\eta_\xi \sim g(\xi),$$

where $g(\xi)$ is an arbitrary function. Thus, for outwardly propagating waves, the relevant solution takes the form

$$\eta \sim \frac{1}{\sqrt{R}}f(\xi) \quad \text{as} \quad \alpha \to 0, \tag{3.31}$$

where $f = \int g\,d\xi$ and we have chosen $\eta = 0$ when $f = 0$. This dominant behaviour, (3.31), for large radius, describes the expected geometrical decay of the wave: as the radius increases, so the length of the wavefront increases and the amplitude must correspondingly decrease. This presents a very different picture from that encountered in the derivation of the KdV equation. In that case the amplitude remained uniformly $O(\varepsilon)$; here the amplitude decreases as the radius increases – and we expect the relevant region of balance to occur for some suitably large radius, yet this could imply that the amplitude is so small that the nonlinear terms play no rôle at leading order. We shall now establish that a scaling does exist which ensures that all the relevant conditions are met.

The equations that describe concentric gravity waves are (from (3.5)–(3.8))

$$u_t + \varepsilon(uu_r + wu_z) = -p_r; \quad \delta^2\{w_t + \varepsilon(uw_r + ww_z)\} = -p_z;$$

$$u_r + \frac{1}{r}u + w_z = 0,$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u \eta_r \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0,$$

where, as before, we have chosen $b = 0$. To proceed, we introduce

$$\xi = \frac{\varepsilon^2}{\delta^2}(r - t), \quad R = \frac{\varepsilon^6}{\delta^4} r, \tag{3.32}$$

where a large radius variable is used here in preference to large time (but see below), and we write

$$\eta = \frac{\varepsilon^3}{\delta^2} H, \quad p = \frac{\varepsilon^3}{\delta^2} P, \quad u = \frac{\varepsilon^3}{\delta^2} U, \quad w = \frac{\varepsilon^5}{\delta^4} W; \tag{3.33}$$

in this transformation, large distance/time is measured by the scale δ^4/ε^6, so $1/\sqrt{\delta^4/\varepsilon^6} = \varepsilon^3/\delta^2$, which is the scale of the wave amplitude, consistent with the decay at large radius. The original amplitude parameter, ε, is now to be interpreted as based on the amplitude of the wave for $r = \mathrm{O}(1)$ and $t = \mathrm{O}(1)$. The governing equations thus become

$$-U_\xi + \Delta(UU_\xi + WU_z + \Delta UU_R) = -(P_\xi + \Delta P_R)$$

$$\Delta\{-W_\xi + \Delta(UW_\xi + WW_z + \Delta UW_R)\} = -P_z;$$

$$U_\xi + W_z + \Delta\left(U_R + \frac{1}{R}U\right) = 0,$$

with

$$P = H \quad \text{and} \quad W = -H_\xi + \Delta(UH_\xi + \Delta UH_R) \quad \text{on} \quad z = 1 + \Delta H$$

and

$$W = 0 \quad \text{on} \quad z = 0,$$

where $\Delta = \varepsilon^4/\delta^2$. These equations are identical, in terms of their general structure, to those discussed in Sections 3.2.1 and 3.2.2, with ε replaced by Δ; here we therefore require only that $\Delta \to 0$. This condition is satisfied, for example, with δ fixed and $\varepsilon \to 0$, and the scalings (3.32) then describe the region where the required balance occurs; the amplitude of the wave in this region is $\mathrm{O}(\Delta)$.

We seek an asymptotic solution in the usual form

$$Q \sim \sum_{n=0}^{\infty} \Delta^n Q_n, \quad \Delta \to 0,$$

where Q represents each of H, P, U and W. Directly, we see that the leading order yields the familiar result

$$P_0 = H_0, \quad U_0 = H_0, \quad W_0 = -zH_{0\xi}, \quad 0 \leq z \leq 1,$$

and then (similar to the derivation of the 2D KdV equation) we observe that the new important ingredient comes from the equation of mass conservation:

$$W_{1z} = -U_{1\xi} - \left(U_{0R} + \frac{1}{R}U_0\right).$$

We leave the few small details in this calculation to the reader; it should be clear, however, that the equation for $H_0(\xi, \tau)$ will be

$$2H_{0R} + \frac{1}{R}H_0 + 3H_0H_{0\xi} + \frac{1}{3}H_{0\xi\xi\xi} = 0, \tag{3.34}$$

the *concentric Korteweg–de Vries* (*cKdV*) equation. (Equivalently, we could work throughout using a large time variable, $\tau = \varepsilon^6 t/\delta^4$, and write $R = \tau + \Delta\xi \sim \tau$; this option is left as an exercise in Q3.7.)

This equation is, in a significant way, different from the first two KdV-type equations that we have derived: equation (3.34) includes a term (H_0/R) which involves a variable coefficient. Nevertheless, the cKdV equation is also one of the set of completely integrable equations, as we shall describe later.

3.2.4 *Nearly concentric Korteweg–de Vries (ncKdV) equation*

In Section 3.2.2 we described how the classical KdV equation can be extended to include (weak) dependence on the y-coordinate; this addition leads to the 2D KdV equation. We now explore how the concentric KdV equation might have a counterpart which represents an appropriate (weak) dependence on the angular variable, θ. Indeed, if we follow the philosophy adopted for the 2D KdV equation (where the relevant scaling in the y-direction was $\varepsilon^{1/2}$), then we might expect that the scaling on θ is $\Delta^{1/2}$. (Remember that the small parameter for the cKdV equation, after rescaling all the variables, turned out to be Δ – which plays the rôle of ε

in that derivation). Thus we use the variables introduced for the cKdV, namely (3.32) and (3.33), and also define

$$\theta = \Delta^{1/2}\Theta = \frac{\varepsilon^2}{\delta}\Theta \tag{3.35}$$

so that the angular distortion away from purely concentric is small. This is precisely equivalent to the case of nearly plane waves which satisfy $\mathrm{d}y/\mathrm{d}\xi = \mathrm{O}(\varepsilon^{1/2})$, so that $\theta = \mathrm{O}(\varepsilon^{1/2})$ with $\mathrm{d}y/\mathrm{d}\xi = \tan\theta$. Further, as we found before (see (3.29)), a scaling is then implied for the θ-component of the velocity vector; here we write

$$v = \frac{\varepsilon^5}{\delta^3}V \tag{3.36}$$

in order to maintain consistency between the scalings on $u = \phi_r$ and on $v = \phi_\theta/r$.

The governing equations, which follow from equations (3.5)–(3.8) with (3.32), (3.33), (3.35) and (3.36), become

$$-U_\xi + \Delta\left\{UU_\xi + WU_z + \Delta\left(UU_R + \frac{V}{R}U_\Theta\right)\right\} - \frac{\Delta^3}{R}V^2 = -(P_\xi + \Delta P_R);$$

$$-V_\xi + \Delta\left\{UV_\xi + WV_z + \Delta\left(UV_R + \frac{1}{R}VV_\Theta\right)\right\} + \Delta^2\frac{UV}{R} = -\frac{1}{R}P_\Theta;$$

$$\Delta\left\{-W_\xi + \Delta\left(UW_\xi + WW_z + \Delta UW_R + \frac{\Delta}{R}VW_\Theta\right)\right\} = -P_z;$$

$$U_\xi + W_z + \Delta\left\{U_R + \frac{1}{R}(U + V_\Theta)\right\} = 0,$$

with

$$P = H \quad \text{and} \quad W = -H_\xi + \Delta\left\{UH_\xi + \Delta\left(UH_R + \frac{1}{R}VH_\Theta\right)\right\} \quad \text{on } z = 1 + \Delta H$$

and

$$W = 0 \quad \text{on} \quad z = 0.$$

We have written $\Delta = \varepsilon^4/\delta^2$ and used the large radius variable, $R = \varepsilon^6 r/\delta^4$, as we did in Section 3.2.3. The asymptotic solution follows the familiar pattern, with

$$P_0 = H_0, \quad U_0 = H_0, \quad W_0 = -zH_{0\xi}, \quad 0 \le z \le 1,$$

and $V_{0\xi} = H_{0\Theta}/R$. At the next order the important difference again arises from the equation of mass conservation, where

$$W_{1z} = -U_{1\xi} - U_{0R} - \frac{1}{R}(U_0 + V_{0\Theta})$$

which leads to a fourth KdV-type equation

$$2H_{0R} + \frac{1}{R}H_0 + 3H_0H_{0\xi} + \frac{1}{3}H_{0\xi\xi\xi} + \frac{1}{R}V_{0\Theta} = 0$$

where $V_{0\xi} = H_{0\Theta}/R$. When we eliminate $V_0(\xi, R, \Theta)$, we obtain the single equation

$$\left(2H_{0R} + \frac{1}{R}H_0 + 3H_0H_{0\xi} + \frac{1}{3}H_{0\xi\xi\xi}\right)_\xi + \frac{1}{R^2}H_{0\Theta\Theta} = 0, \tag{3.37}$$

the *nearly concentric Korteweg–de Vries* (*ncKdV*) equation (although a few authors have named this *Johnson's* equation since it appeared first in Johnson (1980)). When there is no dependence on the angular coordinate, Θ, we have $V_{0\Theta} = 0$ and the equation becomes the cKdV equation.

3.2.5 Boussinesq equation

The four equations derived in the previous sections all relate to propagation in one direction. We now consider the problem of describing waves that propagate in both the positive and negative x-directions and which are also weakly nonlinear and weakly dispersive. To start, we recall the governing equations for one-dimensional propagation, incorporating the scaling that replaces δ^2 by ε; these are equation (3.12)–(3.15):

$$u_t + \varepsilon(uu_x + wu_z) = -p_x; \quad \varepsilon\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z;$$
$$u_x + w_z = 0,$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

We seek an asymptotic solution, as $\varepsilon \to 0$, in the usual form (that is, in integer powers of ε) and so obtain, at leading order,

$$p_0 = \eta_0, \quad u_{0t} = -\eta_{0x}, \quad w_0 = -zu_{0x}, \quad u_{0x} = -\eta_{0t}, \quad 0 \le z \le 1, \tag{3.38}$$

and hence

$$\eta_{0tt} - \eta_{0xx} = 0 \tag{3.39}$$

exactly as in Section 3.2.1. Note that, in this development, we are seeking a solution which is valid for $x = \mathrm{O}(1)$ and $t = \mathrm{O}(1)$; cf. the far-field scalings adopted in Sections 3.2.1–3.2.4.

At $\mathrm{O}(\varepsilon)$ we see that

$$p_1 = -\frac{1}{2}(1 - z^2)u_{0xt} + \eta_1,$$

so

$$u_{1t} + u_0 u_{0x} = \frac{1}{2}(1 - z^2)u_{0xxt} - \eta_{1x};$$

then

$$w_{1z} = -u_{1x}$$

gives

$$w_{1zt} = -u_{1xt} = \eta_{1xx} - \frac{1}{2}(1 - z^2)u_{0xxxt} + (u_0 u_{0x})_x.$$

Thus

$$w_{1t} = \left\{(u_0 u_{0x})_x + \eta_{1xx} - \frac{1}{2}u_{0xxxt}\right\}z + \frac{1}{6}z^3 u_{0xxxt},$$

which satisfies $w_{1t} = 0$ (equivalently $w_1 = 0$) on $z = 0$; the boundary condition on $z = 1$ then yields, after differentiating with respect to t,

$$(w_1 + \eta_0 w_{0z})_t = (\eta_{1t} + u_0 \eta_{0x})_t,$$

and so

$$(u_0 u_{0x})_x + \eta_{1xx} - \frac{1}{3}u_{0xxxt} - (\eta_0 u_{0x})_t = \eta_{1tt} + (u_0 \eta_{0x})_t.$$

This equation can be rewritten as

$$\eta_{1tt} - \eta_{1xx} - \left(\frac{1}{2}\eta_0^2 + u_0^2\right)_{xx} - \frac{1}{3}\eta_{oxxxx} = 0, \tag{3.40}$$

where we have used the equations (3.38), as necessary. Finally, we combine equations (3.39) and (3.40) to obtain a single equation for

$$\eta = \eta_0 + \varepsilon\eta_1 + \mathrm{O}(\varepsilon^2)$$

which is correct at $O(\varepsilon)$; this is

$$\eta_{tt} - \eta_{xx} - \varepsilon\left\{\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t \mathrm{d}x\right)^2\right\}_{xx} - \frac{\varepsilon}{3}\eta_{xxxx} = O(\varepsilon^2), \tag{3.41}$$

where we have written

$$u_0 = -\int_{-\infty}^{x} \eta_{0t} \mathrm{d}x,$$

with the assumption that $u_0 \to 0$ as $x \to -\infty$. (We could equally have chosen

$$u_0 = \int_{x}^{\infty} \eta_{0t} \mathrm{d}x,$$

so that $u_0 \to 0$ as $x \to +\infty$, if that was appropriate.)

Equation (3.41) (or, more precisely, equation (3.41) with zero on the right-hand side) is one version of the *Boussinesq equation* (Boussinesq, 1871). This equation possesses solutions that describe propagation both to the left and to the right; furthermore, the waves also interact *weakly* and are *weakly dispersive*. Nevertheless, these $O(\varepsilon)$ terms are exactly the ones associated with the KdV equation and, indeed, equation (3.41) recovers precisely the KdV equation of our earlier work; see Q3.9. So, although these terms are $O(\varepsilon)$ here, they are the *relevant* and *dominant* contributions in the characterisation of our nonlinear dispersive waves.

We have mentioned that the equations which describe unidirectional propagation belong to the class of completely integrable equations. The Boussinesq equation, suitably approximated (Q3.9), gives rise to the KdV equation which is one of these remarkable equations. At first sight the Boussinesq equation, (3.41), appears rather more complicated (for example, second derivative in time) than our previous equations and is therefore, perhaps, not a member of this special class. However, if we set

$$H = \eta - \varepsilon\eta^2$$

and define

$$X = x + \varepsilon \int_{-\infty}^{x} \eta(x, t; \varepsilon)\, \mathrm{d}x$$

then the equation for $H(X, t; \varepsilon)$ becomes

$$H_{tt} - H_{XX} - \frac{3\varepsilon}{2}(H^2)_{XX} - \frac{\varepsilon}{3}H_{XXXX} = \mathrm{O}(\varepsilon^2); \tag{3.42}$$

see Q3.10. Equation (3.42) is the more conventional version of the Boussinesq equation; this equation, with zero on the right-hand side, turns out to be completely integrable for any $\varepsilon > 0$. That is, the equation

$$H_{tt} - H_{XX} + 3(H^2)_{XX} - H_{XXXX} = 0, \tag{3.43}$$

written now in its most usual form, is completely integrable. (The transformation from (3.42) to (3.43) is simply

$$H \to -\frac{2}{\varepsilon}H, \quad X \to \sqrt{\frac{\varepsilon}{3}}X, \quad t \to \sqrt{\frac{\varepsilon}{3}}\,t,$$

the confirmation of which is left to the reader.)

3.2.6 Transformations between these equations

We have already commented that, under suitable conditions, the Boussinesq equation recovers the KdV equation, a demonstration that has been left as an exercise (Q3.9). Here, we examine the nature of transformations between the KdV, cKdV, 2D KdV and ncKdV equations; that transformations should exist is easily established. These four equations are written in either Cartesian or cylindrical coordinates, so

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x,$$

for the variables used in equations (3.1)–(3.8). Thus for a nearly plane wavefront, for which $y/x \to 0$, we may write

$$r - t \sim x\left(1 + \frac{1}{2}\frac{y^2}{x^2}\right) - t,$$

and because we are in the neighbourhood of the wavefront (that is, $\xi = \mathrm{O}(1)$, $t = \mathrm{O}(\varepsilon^{-1})$; see (3.29)) we obtain

$$\begin{aligned} r - t &\sim x - t + \frac{1}{2}y^2/t \\ &= \xi + \frac{1}{2}Y^2/\tau. \end{aligned}$$

Of course, this is only a rough-and-ready argument, but the suggestion is, for example, that we should seek a solution of the 2D KdV equation, (3.30), namely

$$(2\eta_\tau + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi})_\xi + \eta_{YY} = 0,$$

in the form

$$\eta = H(\zeta, \tau), \quad \zeta = \xi + \frac{1}{2}Y^2/\tau. \tag{3.44}$$

This yields

$$\frac{\partial}{\partial\zeta}(2H_\tau - \frac{Y^2}{\tau^2}H_\zeta + 3HH_\zeta + \frac{1}{3}H_{\zeta\zeta\zeta}) + \frac{1}{\tau}H_\zeta + \frac{Y^2}{\tau^2}H_{\zeta\zeta} = 0$$

which gives, after one integration in ζ (and upon assuming decay conditions for $\zeta \to \infty$, for example), the cKdV equation

$$2H_\tau + \frac{1}{\tau}H + 3HH_\zeta + \frac{1}{3}H_{\zeta\zeta\zeta} = 0. \tag{3.45}$$

This is the form of equation (3.34), when we read R for τ and ξ for ζ, and is even closer to the equation derived in Q3.7. In confirmation of our earlier derivation, Section 3.2.3, we also note that equation (3.45) is invariant under the scaling transformation

$$\zeta \to \frac{\delta}{\varepsilon^{3/2}}\zeta, \quad \tau \to \frac{\delta^3}{\varepsilon^{9/2}}\tau, \quad H \to \frac{\varepsilon^2}{\delta^2}H$$

which describes the choice of variables consistent with (3.10), (3.18) and (3.32) (with τ replacing R).

To take this idea further, we might now expect that a corresponding transformation exists which involves the angular variable (Θ) in the ncKdV equation, and which takes this equation into the KdV equation. Following the same philosophy as above, we write (with $x = r\cos\theta$)

$$\begin{aligned} x - t &\sim r - t - \frac{1}{2}r\theta^2 \quad \text{as} \quad \theta \to 0 \\ &= \frac{\delta^2}{\varepsilon^2}(\xi - \frac{1}{2}R\Theta^2); \end{aligned}$$

see (3.32) and (3.35). This suggests that we seek a solution of

$$(2H_R + \frac{1}{R}H + 3HH_\xi + \frac{1}{3}H_{\xi\xi\xi})_\xi + \frac{1}{R^2}H_{\Theta\Theta} = 0 \tag{3.46}$$

in the form

$$H = \eta(\zeta, R), \quad \zeta = \xi - \frac{1}{2}R\Theta^2 \tag{3.47}$$

which yields

$$2\eta_R + 3\eta\eta_\zeta + \frac{1}{3}\eta_{\zeta\zeta\zeta} = 0 \tag{3.48}$$

after one integration (as described above). This is the KdV equation, with R replacing τ; since $R = \tau + \Delta\xi \sim \tau$ as $\Delta \to 0$, we may interpret the R derivative as a τ derivative, to leading order, and hence recover equation (3.28).

These two results show, for example, that for suitable initial data our four KdV-type equations can be reduced to the solution of just two of them (the KdV and cKdV equations). Of course, in general, the initial profiles will not necessarily conform with the transformations (3.44) or (3.47), and then we must seek solutions of the original 2D KdV and ncKdV equations. We shall briefly describe the near-field problems, and their rôle in providing initial data for our various KdV-type equations, in the next section.

Finally, we remark that the transformations we have presented here are capable of a small extension which then enables the 2D KdV and ncKdV equations to be directly related; this is explored in Q3.12. (This idea turns out to be useful in obtaining certain classes of solution of the ncKdV equation; see Q3.13.)

3.2.7 Matching to the near-field

The equations that we have derived in this chapter, with the exception of the Boussinesq equation, describe waves that are characterised by the balance of nonlinear and dispersive effects in an appropriate far-field. The complete prescription for the solution of these equations requires boundary conditions (such as decay behaviour ahead and behind the wavefront) and initial data provided by the near-field problem; cf. equation (1.94) *et seq*. (The Boussinesq equation is written in near-field variables, and its far-field is represented by a KdV equation, as described in Q3.9.) We now briefly explore the relation between the near-field and far-field problems.

First, for the KdV equation for $\eta_0(\xi, \tau)$,

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0,$$

(3.28), we require the initial profile $\eta_0(\xi, 0)$. From the derivation given in Section 3.2.1, and using the near-field variables (x, t) (defined in (3.10) with (3.11)), we showed that

$$\eta_{tt} - \eta_{xx} = 0,$$

to leading order. Thus, for right running waves, we have

$$\eta = f(x - t) = f(\xi),$$

were $f(\cdot)$ is determined by the initial conditions provided (on $t = 0$) for the wave equation. The matching of the near-field and far-field solutions is then stated as: the two functions

$$\begin{array}{lll} f(\xi) & \text{as } t \to \infty & \text{for } \xi = O(1) \\ \eta_0(\xi, \tau) & \text{as } \tau \to 0 & \text{for } \xi = O(1) \end{array}$$

are to be identical. Hence, to leading order, we must have the initial condition

$$\eta_0(\xi, 0) = f(\xi);$$

this shows that the solution of the KdV equation provides a uniformly valid solution for $\tau \in [0, T]$, certainly for any $T = O(1)$, to leading order in ε.

The corresponding development for the 2D KdV equation is essentially the same, after the additional variable Y (see (3.29)) is introduced. Then the near-field yields

$$\eta = f(\xi, Y),$$

to leading order, which provides the matching condition for the 2D KdV, for the function $\eta_0(\xi, Y, \tau)$, as the initial condition

$$\eta_0(\xi, Y, 0) = f(\xi, Y).$$

Finally, we turn to the related problem for the concentric KdV equation, (3.34). In terms of the appropriate near-field variables, defined by the transformation

$$r \to \frac{\delta^2}{\varepsilon^2} r, \quad t =\to \frac{\delta^2}{\varepsilon^2} t, \quad \eta \to \frac{\varepsilon}{\delta}\eta,$$

we obtain, to leading order, the concentric wave equation

$$\eta_{tt} - \left(\eta_{rr} + \frac{1}{r}\eta_r\right) = 0,$$

already mentioned in Section 3.2.3. For large r this yields

$$\eta \sim \frac{1}{\sqrt{r}} f(r - t) \quad \text{as} \quad r \to \infty, \quad r - t = \xi = \mathrm{O}(1);$$

see equation (3.31). On the other hand, the cKdV equation has a solution of the form

$$H_0(\xi, R) \sim \frac{1}{\sqrt{R}} F(\xi) \quad \text{as} \quad R \to 0, \quad \xi = \mathrm{O}(1)$$

(obtained from the dominant balance $2H_{0R} \sim -H_0/R$); the matching condition therefore provides the initial condition (that is, as $R \to 0$) for the cKdV equation:

$$F(\xi) = f(\xi).$$

The function $f(\cdot)$ is available from the solution of the concentric wave equation which is valid in the near-field.

This leaves the nearly concentric KdV equation for consideration. Unhappily, this equation is not so easily analysed; either the dependence on Θ is absent from the leading-order near-field problem (in which case the calculation reduces, essentially, to that for the cKdV equation) or the terms involving Θ in the solution of the ncKdV equation are exponentially small as $R \to 0$. The structure of the near-field in this latter case is then quite involved. This description is beyond the scope of our text, but the ideas are touched on in Johnson (1980), where the problem of matching to *similarity solutions* of the various KdV equations is also discussed in some detail.

3.3 Completely integrable equations: some results from soliton theory

Wave after wave, each mightier than the last.

The Coming of Arthur

In the introduction to this chapter we mentioned the existence of special equations together with solitary waves, solitons and complete integrability. We have now met a number of these special equations, and our purpose here is to write a little about these equations, their properties and methods of solution. It is not the rôle of this text to provide a comprehensive discussion of these equations, nor to present a careful development of *inverse scattering transform theory* (to use the more accurate title for these studies). Certainly these ideas, usually grouped together under the simpler title of *soliton theory*, are relevant to our further exploration of water-wave theory, but only to the extent of having available solutions and, perhaps, some methods of solution. There are many good texts now available which provide the basis for further study; some of these offer excellent introductions to the theory, whereas others describe advanced and deep ideas. An extensive list of Further Reading is provided at the end of this chapter.

The last 25 years or so have seen the rise of this exciting and powerful approach to our understanding of wave propagation. In particular, the existence of families of solutions of nonlinear wave equations that describe nonlinear interactions without the expected destruction (and, perhaps, resulting chaos), was a considerable surprise. Apart from the diverse observations in nature of many of these phenomena, from our wave interactions on water to the red spot on Jupiter, this work has also led, for example, to the important and very practical application to signal propagation along fibre-optic cables of great length. Furthermore, it turns out that many fundamental concepts in various branches of physics, applied mathematics and pure mathematics also have an important place in this work. Thus both Hamiltonian mechanics and the geometry of surfaces – to mention but two – play a fundamental rôle in soliton theory. In addition, quite new mathematical techniques have been developed and, even more, some longstanding mathematical problems have been solved using soliton theory (for example, the solution of Painlevé equations).

Briefly, the story begins with the KdV equation and its numerical solution, first in a related problem by Fermi, Pasta and Ulam in 1955, and then by Zabusky and Kruskal in 1965. (It was Zabusky and Kruskal who coined the word 'soliton' to describe these new nonlinear waves, because they possess the properties of both solitary waves and particles such as the electron and the photon.) The results were so surprising – primarily the nonlinear interaction of waves that retain their identities – that a group at Princeton University (Gardner, Greene, Kruskal and Miura) set out to understand the processes involved. This led them to develop (in 1967) a method of solution which treats a function that satisfies the KdV equation – the required solution – as the (time-dependent) potential of a one-dimensional linear scattering problem.

The linear scattering problem, and the associated inverse scattering problem, coupled with the time evolution deduced by invoking the KdV equation, produce a solution method which ultimately transforms the *nonlinear* partial differential equation into a *linear* integral equation. Although this integral equation cannot be solved in closed form for arbitrary initial data of the KdV equation, it does possess simple exact solutions which correspond to the soliton solutions (and which enable these solutions of the KdV equation to be written down in a fairly simple and compact form).

From this small beginning – one equation and apparently a tailor-made method of solution – has sprung a whole range of methods which are applicable to many different equations; it has also led to many alternative approaches to the construction of some of the special solutions. We shall present the method of solution for the KdV equation (but without its rather lengthy derivation), and likewise for a few other equations that are important and relevant to water-wave theory. We shall also describe one of the simple direct methods of solution (Hirota's bilinear transformation) and the rôle of conservation laws both in the theory of these equations and, of course, in their application to water waves. In the space available, and in the context of water waves, we cannot explore the many other equally important aspects of soliton theory, such as the Bäcklund transform, Hamiltonian systems and prolongation structure.

3.3.1 Solution of the Korteweg–de Vries equation

The solution, $u(x, t)$, of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{3.49}$$

(which is obtained by a simple scaling transforming from equation (3.28); see Q3.1) is related to a function $K(x, z; t)$ by the transformation

$$u(x, t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x, x; t) \tag{3.50}$$

where K satisfies the integral equation

$$K(x, z; t) + F(x, z, t) + \int_x^\infty K(x, y; t)F(y, z, t)\mathrm{d}y = 0, \tag{3.51}$$

usually called the *Marchenko* (or sometimes *Gel'fand–Levitan*) *equation.* In this equation, $F(x, z, t)$ satisfies both the equations

$$F_{xx} - F_{zz} = 0; \quad F_t + 4(F_{xxx} + F_{zzz}) = 0, \tag{3.52}$$

but since the final evaluation which leads to u is on $z = x$, the relevant solution for F is a function only of $(x + z)$; thus it is convenient to write $F = F(x + z, t)$ so that we now have

$$F_t + 8F_{\xi\xi\xi} = 0 \quad (\xi = x + z) \tag{3.53}$$

and

$$K(x, z; t) + F(x + z, t) + \int_x^\infty K(x, y; t)F(y + z, t)\mathrm{d}y = 0. \tag{3.54}$$

The formulation of this method of solution, via the scattering and inverse scattering problems, enables the initial-value (Cauchy) problem for the KdV equation to be solved, at least provided that certain existence conditions are satisfied, for example

$$\int_{-\infty}^{\infty} |u(x, t)|\mathrm{d}x < \infty, \qquad \int_{-\infty}^{\infty} (1 + |x|)|u(x, t)|\mathrm{d}x \leq \infty, \quad \forall\, t.$$

In particular the initial profile, $u(x, 0)$, must satisfy these conditions. (The first of these says that u must be absolutely integrable and the second – the *Faddeev condition* – says that u must actually decay quite rapidly at infinity.) The solitary wave and soliton solutions certainly do satisfy these conditions (because they decay exponentially as $|x| \to \infty$ for all t), although periodic solutions clearly do not.

In order to gain some familiarity with these equations, and with the method of solution, we shall show how the solitary-wave solution can be recovered. We then extend the technique to obtain the two-soliton solution, and thereafter the generalisation to N–solitons is easily explained (although the details of the calculation are rather lengthy and are not reproduced here).

Example 1: solitary-wave solution

We start from the simplest exponential solution of equation (3.53), which we choose to write as

$$F = \mathrm{e}^{-k\xi+\omega t+\alpha}, \quad \xi = x + z, \tag{3.55}$$

where k (> 0) is a constant, α is an arbitrary constant (equivalent to writing $F = Ae^{-k\xi+\omega t}$) and $\omega(k)$ is to be determined; this solution ensures that $F \to 0$ as $x \to +\infty$. Substitution of (3.55) into (3.53) yields directly the dispersion relation for ω in terms of the wave number (k)

$$\omega = 8k^3,$$

and then the integral equation, (3.51), becomes

$$K(x, z; t) + \exp\{-k(x+z) + 8k^3 t + \alpha\} + \int_x^\infty K(x, y; t)\exp\{-k(y+z) + 8k^3 t + \alpha\}\,\mathrm{d}y = 0.$$

It is immediately clear that the solution takes the (separable) form

$$K(x, z; t) = \mathrm{e}^{-kz} L(x; t) \tag{3.56}$$

so that

$$L + \exp(-kx + 8k^3 t + \alpha) + L\exp(8k^3 t + \alpha)\int_x^\infty \mathrm{e}^{-2ky}\mathrm{d}y = 0$$

and hence

$$L\left\{1 + \frac{1}{2k}\exp(-2kx + 8k^3 t + \alpha)\right\} + \exp(-kx + 8k^3 t + \alpha) = 0,$$

which gives $L(x; t)$. Thus

$$K(x, x; t) = \mathrm{e}^{-kx} L(x; t) = \frac{-1}{\dfrac{1}{2k} + \exp(2kx - 8k^3 t - \alpha)}$$

and then

$$u(x, t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x, x; t) = \frac{-4k\exp(2kx - 8k^3 t - \alpha)}{\left(\dfrac{1}{2k} + \exp(2kx - 8k^3 t - a)\right)^2}$$

$$= \frac{-8k^2}{(\sqrt{2k}\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}/\sqrt{2k})^2}, \quad \theta = kx - 4k^3 t - \alpha/2,$$

or

$$u(x, t) = -2k^2 \operatorname{sech}^2\{k(x - x_0) - 4k^3 t\} \tag{3.57}$$

where we have written

$$\sqrt{2k}\mathrm{e}^{-\alpha/2} = \mathrm{e}^{-kx_0}$$

so that x_0 now describes an arbitrary shift in x. Solution (3.57) is the solitary-wave solution, of amplitude $-2k^2$, of the KdV equation; cf. equation (2.174) and Q1.55(a).

Example 2: two-soliton solution

The extension to two (and eventually N) solitons is surprisingly simple, even though the solution that we obtain describes the *nonlinear* interaction of two (or more) waves. Such a description cannot apply to the solitary wave, since it propagates at constant speed with unchanging form: it is merely an example of a travelling-wave solution. The method hinges on the property that both the equations for F and K are linear, and therefore we may choose to develop a more general solution by taking a linear combination of functions. Thus we write, in place of (3.55),

$$F = \exp(\theta_1) + \exp(\theta_2), \quad \theta_i = -k_i(x+z) + 8k_i^3 t + \alpha_i,$$

where the dispersion relation ($\omega_i = 8k_i^3$) has been incorporated; we are interested in solutions for which $k_1 \neq k_2$, α_1 and α_2 are arbitrary constants. The integral equation for $K(x, z; t)$ now becomes

$$\begin{aligned} K(x,z;t) &+ \exp\{-k_1(x+z) + 8k_1^3 t + \alpha_1\} + \exp\{-k_2(x+z) + 8k_2^3 t + \alpha_2\} \\ &+ \int_x^\infty K(x,y;t)\{\exp[-k_1(y+z) + 8k_1^3 t + \alpha_1] \\ &\qquad + \exp[-k_2(y+z) + 8k_2^3 t + \alpha_2]\}\,\mathrm{d}y = 0 \end{aligned}$$

so that the solution must take the form

$$K(x,z;t) = \exp(-k_1 z)L_1(x;t) + \exp(-k_2 z)L_2(x;t);$$

our problem is an example of a *separable* integral equation, leading to this simple method of solution, which extends what we did in Example 1.

Since $k_1 \neq k_2$, the integral equation separates into two (algebraic) equations for L_1 and L_2; these are

$$L_1 + \exp(-k_1 x + 8k_1^3 t + \alpha_1)$$
$$+ \exp(8k_1^3 + \alpha_1)\left\{ L_1 \int_x^\infty \exp(-2k_1 y)\mathrm{d}y + L_2 \int_x^\infty \exp[-(k_1 + k_2)y]\mathrm{d}y \right\} = 0;$$
$$L_2 + \exp(-k_2 x + 8k_2^3 t + \alpha_2)$$
$$+ \exp(8k_2^3 + \alpha_2)\left\{ L_1 \int_x^\infty \exp[-(k_1 + k_2)y]\mathrm{d}y + L_2 \int_x^\infty \exp(-2k_2 y)\mathrm{d}y \right\} = 0.$$

The integrations, like that in Example 1, are very easily accomplished, yielding the pair of equations

$$L_1 + \exp(-k_1 x + \phi_1)$$
$$+ \frac{L_1}{2k_1}\exp(-2k_1 x + \phi_1) + \frac{L_2}{k_1 + k_2}\exp\{-(k_1 + k_2)x + \phi_1\} = 0;$$
$$L_2 + \exp(-k_2 x + \phi_2)$$
$$+ \frac{L_1}{k_1 + k_2}\exp\{-(k_1 + k_2)x + \phi_2\} + \frac{L_2}{2k_2}\exp(-2k_2 + \phi_2) = 0$$

where

$$\phi_i = 8k_i^3 t + \alpha_i.$$

These equations are solved for L_1 and L_2, and then we form

$$K(x, x; t) = \exp(-k_1 x)L_1(x; t) + \exp(-k_2 x)L_2(x; t)$$

from which we then calculate

$$u(x, t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x, x; t).$$

The manipulative details, which are altogether straightforward, are left as an exercise; the resulting solution can be expressed in a number of ways, one of which is

$$u(x, t) = -8\frac{k_1^2 E_1 + k_2^2 E_2 + 2(k_1 - k_2)^2 E_1 E_2 + A(k_2^2 E_1 + k_1^2 E_2)E_1 E_2}{(1 + E_1 + E_2 + AE_1 E_2)^2} \tag{3.58}$$

where

$$E_i = \exp\{2k_i(x - x_{0i}) - 8k_i^3 t\}, \quad i = 1, 2,$$

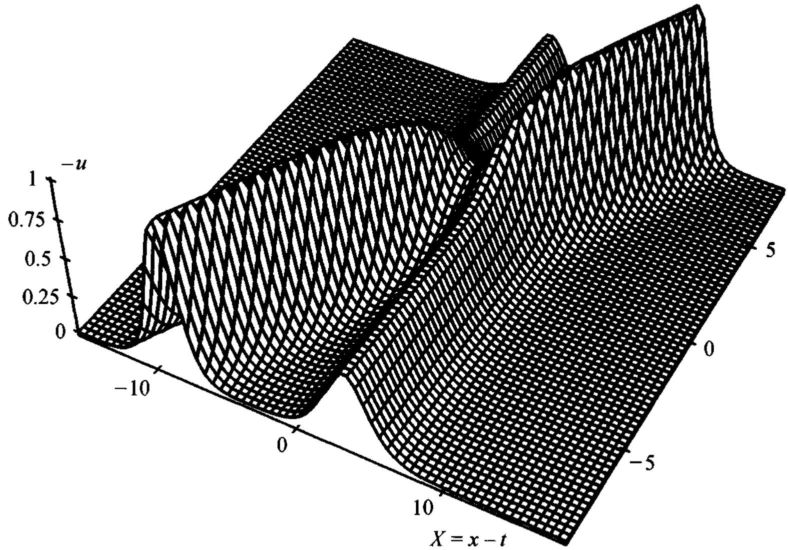


Figure 3.3. A perspective view of a two-soliton solution of the Korteweg–de Vries equation (for $k_1 = 1$ and $k_2 = \sqrt{2}$), drawn in the frame $X = x - t$. Note that $-u$ is plotted here.

and

$$A = (k_1 - k_2)^2/(k_1 + k_2)^2.$$

The two arbitrary phase shifts are x_{0i}, $i = 1, 2$. Solution (3.58) is the most general two-soliton solution of the KdV equation, an example of which is shown in Figure 3.3. A special case of this solution, in which $k_1 = 1$, $k_2 = 2$, $x_{01} = x_{02} = 0$, is

$$u(x, t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\{3\cosh(x - 28t) + \cosh(3x - 36t)\}^2} \tag{3.59}$$

(after some further manipulation); this particular solution is the first that was obtained (Gardner *et al.*, 1967) and corresponds to the initial profile

$$u(x, 0) = -6\,\mathrm{sech}^2 x.$$

The observed water wave (of positive amplitude) is recovered by transforming $u \to -u$ (cf. equations (3.28), (3.49)), and this solution (3.59) is depicted in Figure 3.4.

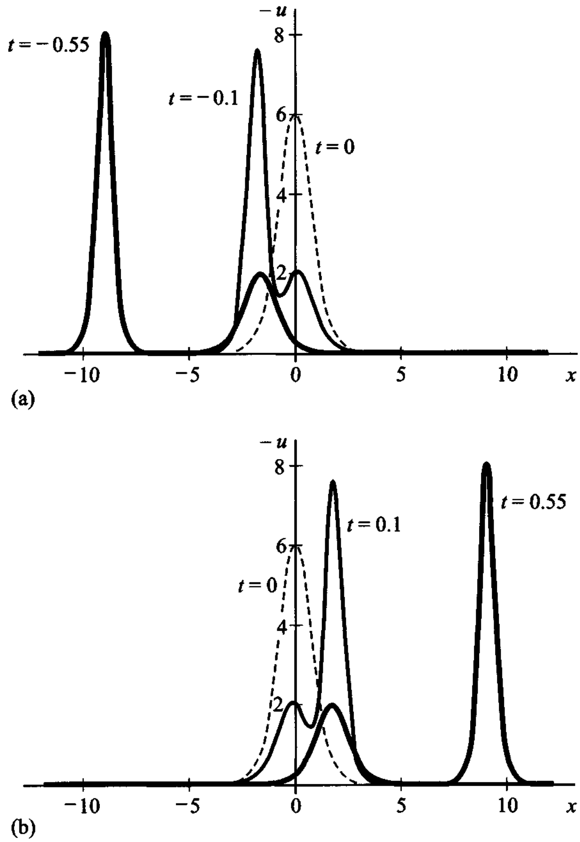


Figure 3.4. The two-soliton solution of the Korteweg–de Vries equation with $u(x, 0) = -6\ \mathrm{sech}^2 x$, shown at times (a) $t = -0.55$, -0.1, 0 and (b) $t = 0$, 0.1, 0.55. Note that $-u$ is plotted against x.

The generalisation to N solitons is obtained in the obvious way by writing

$$F = \sum_{i=1}^{N} \exp(\theta_i), \qquad \theta_i = -k_i(x + z) + 8k_i^3 t + \alpha_i; \tag{3.60}$$

the 3-soliton solution is explored in Q3.17 and Q3.25. For N solitons, the initial profile, which is simply a sech^2 function (which arises when the k_i are suitable integers), takes the form

$$u(x, 0) = -N(N + 1)\,\mathrm{sech}^2 x.$$

Both specific and general forms of the N-soliton solution are discussed in the literature, and many interesting and useful properties have been described. A particularly significant property of these nonlinear wave interactions is evident in Figure 3.4, which represents solution (3.59). The two waves, which are locally almost solitary waves for $t < 0$, combine to form a single wave (the $-6\,\mathrm{sech}^2 x$ profile) at the instant $t = 0$. Thereafter, the taller wave, which had caught up the shorter, moves ahead and away from the shorter as t increases. The two waves that move into $x > 0$ are (asymptotically) identical to the pair that moved in $x < 0$. It might seem, at first sight, that this process is purely linear: the faster (taller) wave catches up and then overtakes the slower (shorter) one, the full solution at any time being the sum of the two. However, a more careful examination of the sequence shown in Figure 3.4 makes it clear that the taller wave has moved *forward*, and the shorter one *backward*, relative to the positions that they would have reached if the two waves had moved at constant speeds throughout. The net result of the nonlinear interaction, therefore, is to produce a *phase shift* of the waves; this property is generally regarded as the hallmark of this type of nonlinear interaction; that is, of soliton solutions. The relevant calculation for solution (3.59) is left as an exercise (Q3.14), and the phase shifts are represented in Figure 3.5.

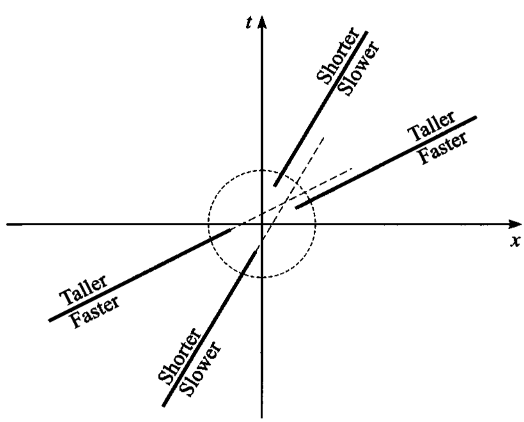


Figure 3.5. A representation of the paths of the two wave crests in a two-soliton solution of the KdV equation. The circle indicates the region inside which the dominant interaction occurs, and the dotted lines show the paths that would have been taken by the waves if no interaction had occurred.

3.3.2 Soliton theory for other equations

The development for the KdV equation is now extended to other equations that are relevant to water waves. We shall present the details for the methods of solution for the 2D KdV and cKdV equations. (Another important example, the Nonlinear Schrödinger (NLS) equation, will be described in the next chapter.)

The solution of the two-dimensional Korteweg–de Vries equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \tag{3.61}$$

follows that for the KdV equation very closely (Dryuma, 1974). The transformation between u and $K(x, x; t, y)$ is the same, namely

$$u(x, t, y) = -\frac{\mathrm{d}K}{\mathrm{d}x}K(x, x; t, y), \tag{3.62}$$

where K satisfies

$$K(x, z; t, y) + F(x, z, t, y) + \int_x^\infty K(x, Y; t, y)F(Y, z, t, y)\mathrm{d}Y = 0$$

(and we have written the integration variable here as Y, to avoid the obvious confusion). The function F satisfies the pair of equations

$$F_{xx} - F_{zz} - F_y = 0, \quad F_t + 4(F_{xxx} + F_{zzz}) = 0; \tag{3.63}$$

cf. equation (3.52). Then, for example, the solitary-wave solution is obtained by choosing

$$F = \exp\{-(kx + lz) + (k^2 - l^2)y + 4(k^3 + l^3)t + \alpha\};$$

see Q3.18 and Q3.19.

The concentric Korteweg–de Vries equation,

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0, \tag{3.64}$$

is solved by a similar method, although the details are not so straightforward. As before, $K(x, z; t)$ is a solution of the Marchenko equation

$$K(x, z; t) + F(x, z, t) + \int_x^\infty K(x, y; t)F(y, z, t)\mathrm{d}y = 0$$

where F is now a solution of the pair of equations

$$\left.\begin{aligned} F_{xx} - F_{zz} &= (x-z)F; \\ 3tF_t - F + F_{xxx} + F_{zzz} &= xF_x + zF_z. \end{aligned}\right\} \tag{3.65}$$

The solution of the cKdV equation is then obtained in the form

$$u(x,t) = -2(12t)^{-2/3}\frac{\mathrm{d}K}{\mathrm{d}\xi},\ K = K(\xi,\xi;t), \tag{3.66}$$

where $\xi = x/(12t)^{1/3}$; it is the requirement to use the similarity variable that particularly complicates the procedure in this case. Another mild irritant is that equations (3.65) do not admit exponential solutions, the simplest solutions being based on the *Airy* functions. Further exploration of this method is provided in Q3.22.

3.3.3 Hirota's bilinear method

The solitary-wave solution of the KdV equation,

$$u(x,t) = -2k^2\operatorname{sech}^2\{k(x-x_0) - 4k^3t\},$$

as given in (3.57), can be written as

$$\begin{aligned} u(x,t) &= -2k\frac{\partial}{\partial x}\tanh\{k(x-x_0) - 4k^3t\} \\ &= -2\frac{\partial^2}{\partial x^2}\log(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}), \quad \theta = k(x-x_0) - 4k^3t, \\ &= -2\frac{\partial^2}{\partial x^2}\left\{-k(x-x_0) + 4k^3t + \log(1+\mathrm{e}^{2\theta})\right\} \\ &= -2\frac{\partial^2}{\partial x^2}\log f, \quad f = 1 + \exp\{2k(x-x_0) - 8k^3t\}. \end{aligned} \tag{3.67}$$

Indeed, the N-soliton solution can be written in precisely the same form:

$$u = -2\frac{\partial^2}{\partial x^2}\log f, \tag{3.68}$$

where $f(x,t)$ turns out to be the determinant of an $N \times N$ matrix of coefficients that arise in the solution of the Marchenko equation, when F is a sum of N exponential terms (see (3.60)). Hirota's idea, first published in 1971, was to explore the possibility of solving the KdV equation (at least for the soliton solutions) by constructing $f(x,t)$ directly. At first sight it might appear that the problem for f is more difficult than that for u; however, Hirota showed that it eventually leads to a very neat method of solution. And the idea is not restricted to the KdV equation: all the soliton-type equations can be tackled in a similar way (although the transformation (3.68) is not always the relevant one). In the context of solitary wave and soliton solutions, which we have already suggested are of some interest in water-wave theory, this method often provides a convenient method for their construction. We regard this technique as a powerful addition to the more general method of solution that is based on the Marchenko integral equation. There are yet other approaches available, but we believe that Hirota's method is sufficiently simple and useful to warrant a place in this text.

We describe the elements of the method by developing the details for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

with

$$u = -2\frac{\partial^2}{\partial x^2}\log f,$$

where $f_x, f_t, f_{xx}, f_{tt}, \ldots \to 0$ as $x \to +\infty$ or $x \to -\infty$ (see (3.67)). The process is made a little simpler if, first, we write $u = \phi_x$ and then integrate once in x to yield

$$\phi_t - 3\phi_x^2 + \phi_{xxx} = 0, \tag{3.69}$$

where the decay conditions on f imply corresponding conditions on ϕ $(= -2f_x/f)$ and these have been used to give equation (3.69). (This version of the KdV equation is often called the *potential KdV equation*.) Now we introduce f so that

$$\phi_t = -2(ff_{xt} - f_xf_t)/f^2, \quad \phi_x = -2(ff_{xx} - f_x^2)/f^2$$

and

$$\phi_{xxx} = -2(ff_{xxxx} - 4f_xf_{xxx} - 3f_{xx}^2)/f^2 - 24f_{xx}f_x^2/f^3 + 12f_x^4/f^4;$$

it is clear that when these are substituted into equation (3.69) we obtain (after multiplication by f^2)

$$ff_{xt} - f_xf_t + ff_{xxxx} - 4f_xf_{xxx} + 3f_{xx}^2 = 0. \tag{3.70}$$

This equation certainly appears more difficult to solve than the original KdV equation, although we do note one significant improvement: every term is now quadratic in f. So how do we tackle the solution of equation (3.70)?

The crucial step was provided by Hirota when he introduced the *bilinear operator*, $\mathrm{D}_t^m \mathrm{D}_x^n(a \cdot b)$, defined as

$$\mathrm{D}_t^m \mathrm{D}_x^n(a \cdot b) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x, t)b(x', t')\bigg|_{\substack{x'=x \\ t'=t}} \tag{3.71}$$

for non-negative integers m and n. As an example, let us consider the case of $m = n = 1$ for which

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)a(x, t)b(x', t') &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)(a_x b - a b_{x'}) \\ &= a_{xt} b - a_t b_{x'} - a_x b_{t'} + a b_{x't'}.\end{aligned}$$

This is evaluated on $x' = x$, $t' = t$, to give

$$\mathrm{D}_t \mathrm{D}_x(a \cdot b) = a_{xt} b + a b_{xt} - a_t b_x - a_x b_t$$

and if we choose the special case of $a = b$, for all x, t, then

$$\mathrm{D}_t \mathrm{D}_x(a \cdot b) = 2(a a_{xt} - a_x a_t). \tag{3.72}$$

Another useful example is to find $\mathrm{D}_x^4(a \cdot b)$, so that now $m = 0$ and $n = 4$; this yields

$$\begin{aligned}&\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^4 a(x, t)b(x', t') \\ &\qquad = a_{xxxx} b - 4a_{xxx} b_{x'} + 6a_{xx} b_{x'x'} - 4a_x b_{x'x'x'} + a b_{x'x'x'x'}.\end{aligned}$$

We evaluate on $x' = x$, $t' = t$, and again make the special choice $a = b$, to give

$$\mathrm{D}_x^4(a \cdot a) = 2(a a_{xxxx} - 4a_x a_{xxx} + 3a_{xx}^2). \tag{3.73}$$

It is immediately clear, if we compare equations (3.72) and (3.73) with (3.70), that our equation for f can be expressed as

$$(\mathrm{D}_x \mathrm{D}_t + \mathrm{D}_x^4)(f \cdot f) = 0, \tag{3.74}$$

the *bilinear form* of the KdV equation.

Before we describe how the bilinear equation, (3.74), is solved, we offer two comments. First, examination of the differential operator

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)^n a(x, t)b(x', t')\bigg|_{\substack{x'=x \\ t'=t}},$$

which is left as an exercise, shows that this is precisely the familiar derivative of a product:

$$\frac{\partial^{m+n}}{\partial t^m \partial x^n}(ab).$$

In other words, Hirota's novel differential operator simply uses the difference – rather than the sum – of the derivatives in t and t', and in x and x'. Second, and often used as a guide in the quick construction of a bilinear form, is the *interpretation* of D_x and D_t as the conventional derivatives $\partial/\partial x$ and $\partial/\partial t$, respectively. If we use this interpretation of $\mathrm{D}_t + \mathrm{D}_x^3$ then this operator becomes the linearised operator in the KdV equation, obtained by letting $u \to 0$. Thus the underlying structure of the bilinear form is that of the corresponding *linear* differential equation, at least here for KdV equation; we shall meet later some other equations which possess this same property.

In order to solve the bilinear equation we require some properties of the bilinear operator, and in particular the two results

$$\mathrm{D}_t^m \mathrm{D}_x^n (a \cdot 1) = \mathrm{D}_t^m \mathrm{D}_x^n (1 \cdot a) = \frac{\partial^{m+n} a}{\partial t^m \partial x^n}, \quad \text{for } m+n \text{ even}, \tag{3.75}$$

and

$$\mathrm{D}_t^m \mathrm{D}_x^n \{\exp(\theta_1) \cdot \exp(\theta_2)\} = (\omega_2 - \omega_1)^m (k_1 - k_2)^n \exp(\theta_1 + \theta_2), \tag{3.76}$$

where $\theta_i = k_i x - \omega_i t + \alpha_i$; these and other properties are explored in Q3.24. Now for B any bilinear operator and

$$f = 1 + \mathrm{e}^{\theta}, \quad \theta = 2k(x - x_0) - 8k^3 t, \tag{3.77}$$

(see (3.74)), then

$$\mathrm{B}(f \cdot f) = \mathrm{B}(1 \cdot 1) + \mathrm{B}(1 \cdot \mathrm{e}^{\theta}) + \mathrm{B}(\mathrm{e}^{\theta} \cdot 1) + \mathrm{B}(\mathrm{e}^{\theta} \cdot \mathrm{e}^{\theta}).$$

Consequently, with (3.75) and (3.76), the bilinear form of our KdV equation gives

$$(\mathrm{D}_x \mathrm{D}_t + \mathrm{D}_x^4)(f \cdot f) = 2(2k)(-8k^3) + (2k)^4\}\mathrm{e}^{\theta} = 0$$

which confirms that (3.77) is an exact solution (which, of course, generates the solitary-wave solution). The extension of this approach to the construction of the N-soliton solution is now addressed.

The neatest way to set up this problem is to introduce an arbitrary parameter ε, with the assumption that f can be expanded in integral powers of ε. The aim is to show that the series that we obtain terminates after a finite number of terms; in this situation we may then arbitrarily assign ε: for example, $\varepsilon = 1$. Thus we write

$$f = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x, t)$$

and then the general bilinear form becomes

$$\mathrm{B}(f \cdot f) = \mathrm{B}(1 \cdot 1) + \varepsilon \mathrm{B}(f_1 \cdot 1 + 1 \cdot f_1) + \varepsilon^2 \mathrm{B}(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2)$$
$$+ \cdots + \varepsilon^r \mathrm{B}\left(\sum_{m=0}^{\infty} f_{r-m} \cdot f_m \right) + \cdots = 0$$

where $f_0 = 1$. We know that $\mathrm{B}(1 \cdot 1) = 0$, and we ask that each coefficient of ε^r $(r = 1, 2, \ldots)$ be zero, so

$$\mathrm{B}(f_1 \cdot 1 + 1 \cdot f_1) = 0, \tag{3.78}$$

$$\mathrm{B}(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0, \text{ etc.} \tag{3.79}$$

Because B is a linear differential operator, we have

$$\mathrm{B}(a \cdot b + c \cdot d) = \mathrm{B}(a \cdot b) + \mathrm{B}(c \cdot d)$$

and then equation (3.78), for our KdV equation, becomes

$$2 \frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} \right) f_1 = 0$$

or, after one integration,

$$\mathrm{D} f_1 = 0, \quad \mathrm{D} \equiv \frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}, \tag{3.80}$$

where we have again used $f_{1t}, f_{1x}, \ldots \to 0$ as $x \to +\infty$ or $x \to -\infty$. The next two equations in this sequence are written as

$$2 \frac{\partial}{\partial x}(\mathrm{D} f_2) = -\mathrm{B}(f_1 \cdot f_1); \quad 2 \frac{\partial}{\partial x}(\mathrm{D} f_3) = -\mathrm{B}(f_1 \cdot f_2 + f_2 \cdot f_1), \tag{3.81}$$

where $\mathrm{B} = \mathrm{D}_x \mathrm{D}_t + \mathrm{D}_x^4$. It is immediately clear that a solution of this set is

$$f_1 = \mathrm{e}^{\theta}, \quad \theta = kx - k^3 t + \alpha; \quad f_n = 0, \ \forall n \geq 2,$$

where we have written k here for $2k$ and α for $-2kx_0$; see (3.67). Thus we have a solution which terminates after $n = 1$, and so we may set $\varepsilon = 1$; this recovers (3.67) for the solitary wave. (We note that the presence of ε is equivalent to a phase shift, but the term α already provides an arbitrary phase shift in the solution.)

The equation for f_1, (3.78), is linear and so we may construct more general solutions by taking a linear combination of exponential terms; let us choose

$$f_1 = \exp(\theta_1) + \exp(\theta_2), \quad \theta_i = k_i x - k_i^3 t + \alpha_i.$$

The equation for f_2, from (3.81), now becomes

$$\begin{aligned} 2\frac{\partial}{\partial x}(\mathrm{D}f_2) = &- \mathrm{B}\{\exp(\theta_1)\cdot\exp(\theta_1)\} - \mathrm{B}\{\exp(\theta_1)\cdot\exp(\theta_2)\} \\ &- \mathrm{B}\{\exp(\theta_2)\cdot\exp(\theta_1)\} - \mathrm{B}\{\exp(\theta_2)\cdot\exp(\theta_2)\} \end{aligned}$$

and the terms involving either only θ_1 or only θ_2 are zero. Otherwise, we see that

$$2\frac{\partial}{\partial x}(\mathrm{D}f_2) = -2\{(k_1 - k_2)(k_2^3 - k_1^3) + (k_1 - k_2)^4\}\exp(\theta_1 + \theta_2)$$

and this equation clearly has a particular integral of the form

$$f_2 = A\exp(\theta_1 + \theta_2). \tag{3.82}$$

This yields the equation

$$\begin{aligned} &A\{-(k_1 + k_2)(k_1^3 + k_2^3) + (k_1 + k_2)^4\} \\ &\qquad = (k_1 - k_2)^2\{k_1^2 + k_1 k_2 + k_2^2 - (k_1 - k_2)^2\} \end{aligned}$$

for the constant A, which simplifies to give

$$A = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{3.83}$$

We use only the particular integral for f_2; any additional contributions (as part of a complementary function) could be moved from f_2 to f_1 – at least when $\varepsilon = 1$ – and we have already made a choice for f_1.

The equation for f_3 then becomes

$$\begin{aligned} 2\frac{\partial}{\partial x}(\mathrm{D}f_3) &= -A\mathrm{B}\{\exp(\theta_1)\cdot\exp(\theta_1 + \theta_2)\} - A\mathrm{B}\{\exp(\theta_1 + \theta_2)\cdot\exp(\theta_1)\} \\ &\quad - A\mathrm{B}\{\exp(\theta_2)\cdot\exp(\theta_1 + \theta_2)\} - A\mathrm{B}\{\exp(\theta_1 + \theta_2)\cdot\exp(\theta_2)\} \\ &= -2A\{-k_2(k_2^3) + k_2^4\}\exp(2\theta_1 + \theta_2) \\ &\qquad - 2A\{-k_1(k_1^3) + k_1^4\}\exp(2\theta_2 + \theta_1) \\ &= 0, \end{aligned}$$

and so a solution for f_3 is $f_3 = 0$. The equation for f_4 is

$$2\frac{\partial}{\partial x}(\mathrm{D}f_4) = -\mathrm{B}(f_1 \cdot f_3 + f_2 \cdot f_2 + f_3 \cdot f_1)$$
$$= 0$$

since $f_3 = 0$ and f_2 is a single exponential (from (3.82)); it is clear, therefore, that we may choose $f_n = 0$, $\forall n \geq 3$. Thus we have another exact solution which, for $\varepsilon = 1$, is

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \exp(\theta_1 + \theta_2); \tag{3.84}$$

this generates the most general two-soliton solution of the KdV equation, previously written down in (3.58).

This method of solution can be extended to produce an exact solution which represents the N-soliton solution of the KdV equation. This is accomplished simply by writing

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i)$$

and then it can be shown that the series for f terminates after the term f_N. The construction of this solution is routine but rather tedious and therefore will not be pursued here, although the case $N = 3$ is set as an exercise in Q3.25, and a 3-soliton solution is depicted in Figure 3.6. The form that f takes, for example as given in (3.84) for $N = 2$, represents a *nonlinear superposition principle* for the soliton solutions from which their explicit construction follows directly.

Finally, the other nonlinear equations that we have introduced in Section 3.2 can also be written in bilinear form. (The details are left for the reader to explore in the exercises.) Thus we find that the 2D KdV equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

has the bilinear form

$$(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + 3\mathrm{D}_y^2)(f \cdot f) = 0,$$

and the cKdV equation

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0$$

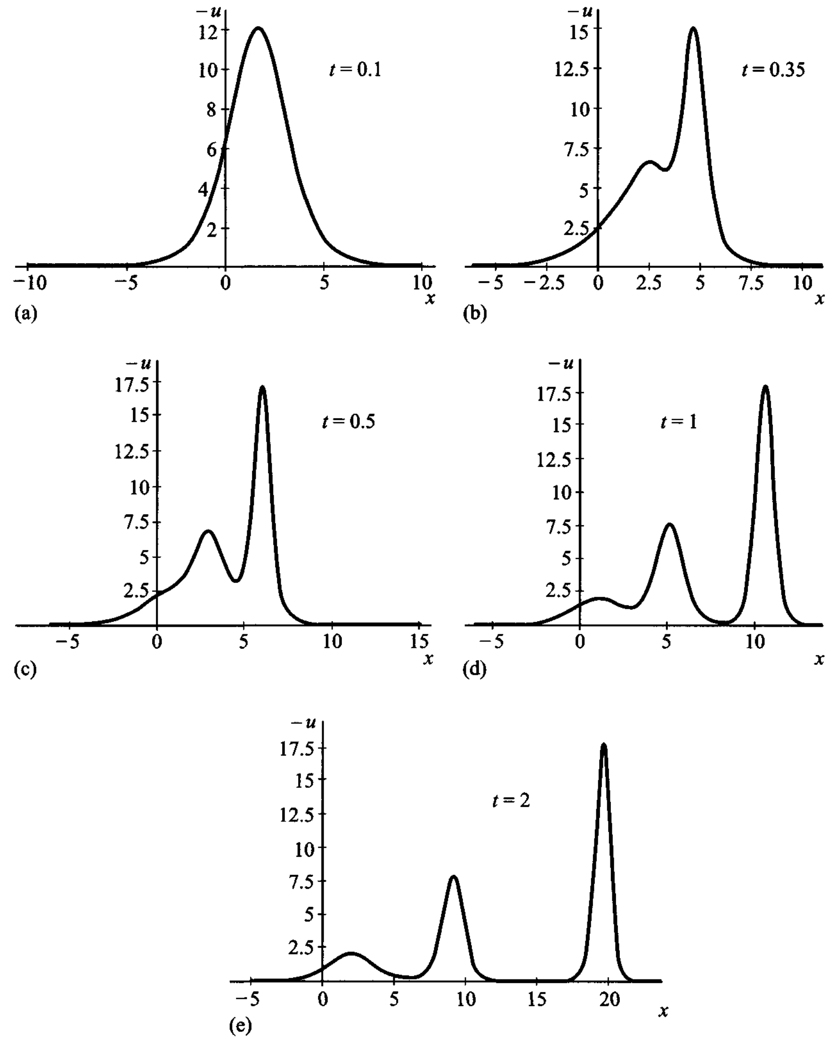


Figure 3.6. A 3-soliton solution of the Korteweg–de Vries equation, for $k_1 = 1$, $k_2 = 2$, and $k_3 = 3$, at times $t = 0.1$(a), 0.35(b), 0.5(c), 1(d) and 2(e). Note that $-u$ is plotted here.

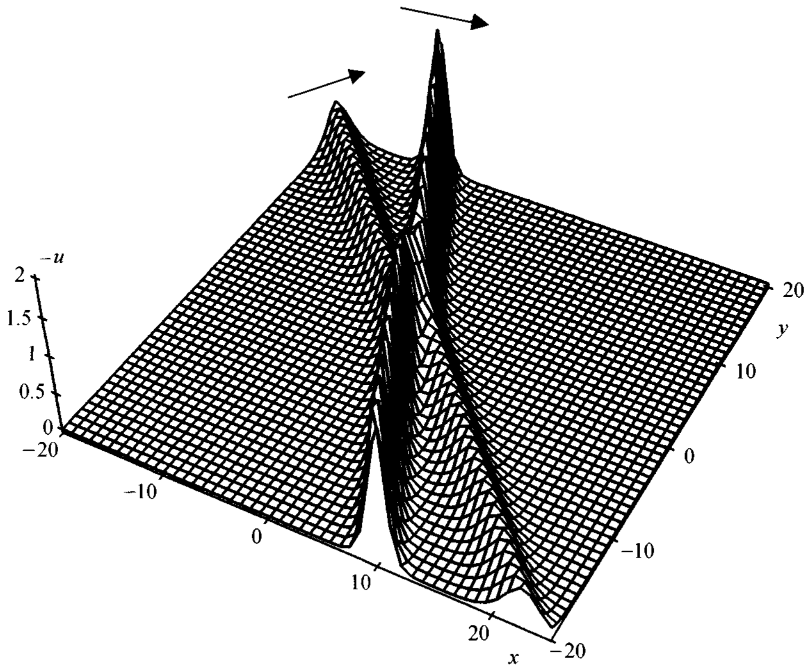


Figure 3.7. A 2-soliton solution of the 2D Korteweg–de Vries equation, for $l_1 = l_2 = 1$ and $k_1 = 1$, $k_2 = 2$. Note that $-u$ is plotted here.

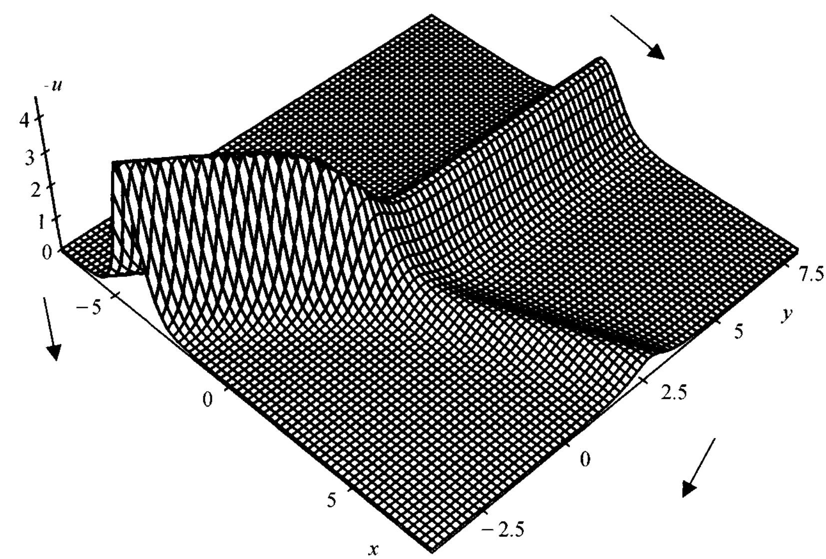


Figure 3.8. A *resonant* 2-soliton solution of the 2D KdV equation, for $l_1 = 0$, $l_2 = -3$, $k_1 = 2$, and $k_2 = 3$. Note that $-u$ is plotted here.

becomes

$$\left(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + \frac{1}{2t}\frac{\partial}{\partial x}\right)(f\cdot f) = 0$$

where

$$\frac{\partial}{\partial x}(f\cdot f) = f\frac{\partial f}{\partial x};$$

in both these equations the transformation is

$$u = -2\frac{\partial^2}{\partial x^2}\log f.$$

With this same transformation, the Boussinesq equation

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$$

has the bilinear form

$$(\mathrm{D}_t^2 - \mathrm{D}_x^2 - \mathrm{D}_x^4)(f\cdot f) = 0.$$

As the exercises should demonstrate, the construction of solitary wave and soliton solutions from the bilinear form is, in most cases, a fairly straightforward and routine operation. A 2-soliton solution of the 2D KdV equation is shown in Figure 3.7 (see Q3.30), and a *resonant* solution is shown in Figure 3.8 (see Q3.32). A solution of the Boussinesq equation, which describes both head-on and overtaking soliton collisions, is given in Figure 3.9.

3.3.4 Conservation laws

We are already familiar with the equation of mass conservation (Section 1.1.1) and how this equation can be integrated in z (Section 1.2.4) to produce the form

$$d_t + \nabla_\perp \cdot \bar{\mathbf{u}}_\perp = 0$$

(equation (1.38)). This is a general equation for water waves, where $d = h - b$ is the local depth and

$$\bar{\mathbf{u}}_\perp = \int_b^h \mathbf{u}_\perp \mathrm{d}z.$$

Furthermore, in the case of one-dimensional propagation with decay conditions at infinity, we found (equation (1.40)) that

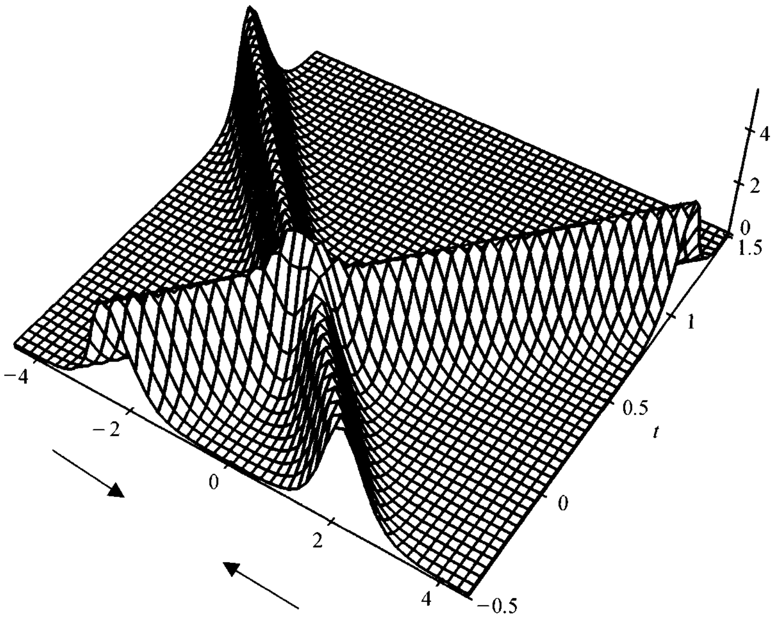


Figure 3.9. A solution of the Boussinesq equation depicting the head-on collision of two solitons, each of amplitude 2.

$$\int_{-\infty}^{\infty} H(x, t)\mathrm{d}x = \text{constant},$$

where $h(x, t) = h_0 + H(x, t)$ and $H \to 0$ as $|x| \to \infty$; this is a very convenient and transparent version of the statement of mass conservation in water waves. Of course, this result can be obtained – very simply – directly from the equations for one-dimensional gravity-wave propagation:

$$\left.\begin{aligned}
&u_t + \varepsilon(uu_x + wu_z) = -p_x; \quad \delta^2\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z; \\
&\qquad\qquad u_x + w_z = 0, \\
&\text{with} \\
&\qquad p = \eta \quad \text{and} \quad w = \eta_t + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta \\
&\text{and} \\
&\qquad\qquad w = 0 \quad \text{on} \quad z = 0;
\end{aligned}\right\} \tag{3.85}$$

see equations (3.9). Thus, employing the technique of differentiation under the integral sign, we obtain

$$\frac{\partial}{\partial x}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) - \varepsilon u\eta_x\bigg|_{1+\varepsilon\eta} + [w]_0^{1+\varepsilon\eta} = 0,$$

so

$$\eta_t + \frac{\partial}{\partial x}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) = 0 \tag{3.86}$$

from which we get

$$\int_{-\infty}^{\infty} \eta(x, t)\mathrm{d}x = \text{constant}. \tag{3.87}$$

Similarly, for energy (see Section 1.2.5), we obtain directly

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\eta^2 + \frac{1}{2}\int_0^{1+\varepsilon\eta} (u^2 + \delta^2 w^2)\mathrm{d}z\right\} + \frac{\partial}{\partial x}\left\{\int_0^{1+\varepsilon\eta}\left(\frac{\varepsilon}{2}u^3 + \frac{\varepsilon\delta^2}{2}uw^2 + up\right)\mathrm{d}z\right\} = 0 \tag{3.88}$$

from equations (3.85); cf. equation (1.47). (The derivation of this result is left as an exercise (Q3.33), although all the essential details are described in Section 1.2.5.) The resulting conserved energy in the motion is therefore

$$\int_{-\infty}^{\infty}\left\{\eta^2 + \int_0^{1+\varepsilon\eta} (u^2 + \delta^2 w^2)\mathrm{d}z\right\}\mathrm{d}x = \text{constant}, \tag{3.89}$$

where decay conditions at infinity have again been invoked. Generally, expressions of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{3.90}$$

where $T(x, t)$ (the *density*) and $X(x, t)$ (the *flux*) do not normally contain derivatives with respect to t, are called *conservation laws*. If both T and X_x are integrable over all x, so that

$$X \to X_0 \quad \text{as} \quad |x| \to \infty,$$

where X_0 is a constant, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{\infty} T\mathrm{d}x\right) = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} T(x, t)\,\mathrm{d}x = \text{constant}: \tag{3.91}$$

the integral of $T(x, t)$ over all x is a *conserved quantity* (often called a *constant of the motion*, especially when t is interpreted as a time-like variable). We have so far written down the conservation laws for mass (3.86) and for energy (3.88), as they apply to our one-dimensional water-wave problem. It is no surprise that there is a third conservation law, namely the one that describes the conservation of momentum. There has been no need to express this in the form (3.90) in our earlier work, but we will now show how it can be obtained very simply from equations (3.85). (Ideas that are closely related to all three conservation laws have been developed in the discussion of the jump conditions in Section 2.7.)

Referring to equations (3.85), we see that the first added to εu times the third yields

$$u_t + 2\varepsilon u u_x + \varepsilon(uw)_z + p_x = 0,$$

and so

$$\int_0^{1+\varepsilon\eta} (u_t + 2\varepsilon u u_x + p_x)\,\mathrm{d}z + \varepsilon[uw]_0^{1+\varepsilon\eta} = 0.$$

The boundary conditions that describe w then give

$$\int_0^{1+\varepsilon\eta} (u_t + 2\varepsilon u u_x + p_x)\,\mathrm{d}z + \varepsilon u_s(\eta_t + \varepsilon u_s \eta_x) = 0$$

where u_s is $u(x, t, z)$ evaluated on the surface, $z = 1 + \varepsilon\eta$. Again, application the method of differentiating under the integral sign produces

$$\frac{\partial}{\partial t}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) + \frac{\partial}{\partial x}\left\{\int_0^{1+\varepsilon\eta} (\varepsilon u^2 + p)\,\mathrm{d}z - \frac{1}{2}\varepsilon\eta^2\right\} = 0 \tag{3.92}$$

where we have used the surface boundary condition for p (where $p = \eta$). Equation (3.92) is the conservation law for momentum which, with an undisturbed background state in place, gives

$$\int_{-\infty}^{\infty}\left(\int_{0}^{1+\varepsilon\eta} u\,\mathrm{d}z\right)\mathrm{d}x = \text{constant}, \tag{3.93}$$

the conserved momentum in the motion. Thus, as we must expect, the passage of the gravity wave (within the confines of our model), conserves the three fundamental properties of motion: mass, momentum and energy. The question that we now address is how these conservation laws – and associated conserved quantities – manifest themselves in our various KdV-type equations.

We begin this discussion with the KdV equation itself; the leading-order contribution (η_0) to the representation of the surface wave satisfies equation (3.28):

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0$$

or

$$\frac{\partial}{\partial\tau}(2\eta_0) + \frac{\partial}{\partial\xi}\left(\frac{3}{2}\eta_0^2 + \frac{1}{3}\eta_{0\xi\xi}\right) = 0,$$

when written in the form of a conservation law. With the assumption that the wave decays at infinity, so that $\eta_0 \to 0$ as $|\xi| \to \infty$ (which is equivalent to $|x| \to \infty$ for any t), we have

$$\int_{-\infty}^{\infty} \eta_0\,\mathrm{d}\xi = \text{constant}; \tag{3.94}$$

this is the conservation of mass, equation (3.87). Another conserved quantity is obtained by multiplying the KdV equation by η_0, to give

$$\frac{\partial}{\partial\tau}(\eta_0^2) + \frac{\partial}{\partial\xi}\left\{\eta_0^3 + \frac{1}{3}\left(\eta_0\eta_{0\xi\xi} - \frac{1}{2}\eta_{0\xi}^2\right)\right\} = 0,$$

so

$$\int_{-\infty}^{\infty} \eta_0^2\,\mathrm{d}\xi = \text{constant}. \tag{3.95}$$

It is clear that the two constants of the motion, (3.94) and (3.95), are general properties of all solutions of the KdV equations which decay rapidly enough at infinity. (This also means that, for example, periodic solutions do not satisfy these particular integral constraints, although an analogous set of results can be obtained if the integral is taken over just one period.) The second result, (3.95), should – we must surmise – correspond to the conservation of momentum, equation (3.93). Now from Section 3.2.1 we find that

$$u \sim u_0 + \varepsilon\left\{\eta_1 + \left(\frac{1}{3} - \frac{1}{2}z^2\right)\eta_{0\xi\xi} - \frac{1}{4}\eta_0^2\right\}; \quad u_0 = \eta_0$$

(where the KdV equation has been used to eliminate the term $\eta_{0\tau}$ in u_1), so

$$\int_{-\infty}^{\infty}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right)\mathrm{d}x \sim \int_{-\infty}^{\infty}\left\{u_0 + \varepsilon(u_0\eta_0 + \eta_1 + \frac{1}{3}\eta_{0\xi\xi} - \frac{1}{4}\eta_0^2)\right\}\mathrm{d}\xi$$
$$-\frac{\varepsilon}{2}\int_{-\infty}^{\infty}\eta_{0\xi\xi}\left(\int_0^1 z^2\mathrm{d}z\right)\mathrm{d}\xi$$
$$= \int_{-\infty}^{\infty}\left(\eta_0 + \varepsilon\eta_1 + \frac{3\varepsilon}{4}\eta_0^2\right)\mathrm{d}\xi,$$

which is correct at $\mathrm{O}(\varepsilon)$. The first two terms in this integral appear in the conservation of mass, and consequently we must have

$$\int_{-\infty}^{\infty}\eta_0^2\,\mathrm{d}\xi = \text{constant},$$

which recovers (3.95). (The confirmation that the integral of η_1 alone is itself a constant follows directly from the equation for η_1 obtained in Q3.4.)

We have obtained two conserved densities for our KdV equation, η_0 and η_0^2, which correspond to the conservation of mass and momentum, respectively. We can anticipate that the equation possesses a third conserved density, which is associated with the energy of the motion. To see that this is indeed the case, we construct $3\eta_0^2 \times$ (KdV) minus $(2\eta_{0\xi}/3) \times (\partial/\partial\xi)$(KdV) to give

$$3\eta_0^2\left(2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi}\right)$$
$$-\frac{2}{3}\eta_{0\xi}(2\eta_{0\xi\tau} + 3\eta_0\eta_{0\xi\xi} + 3\eta_{0\xi}^2 + \frac{1}{3}\eta_{0\xi\xi\xi\xi}) = 0$$

which can be written as

$$2\frac{\partial}{\partial \tau}\left(\eta_0^3 - \frac{1}{3}\eta_{0\xi}^2\right) + \frac{\partial}{\partial \xi}\left(\frac{9}{4}\eta_0^4 + \eta_0^2\eta_{0\xi\xi} - 2\eta_0\eta_{0\xi}^2 - \frac{2}{9}\eta_{0\xi}\eta_{0\xi\xi\xi} + \frac{1}{9}\eta_{0\xi\xi}^2\right) = 0.$$

This is in the form of a conservation law, so we obtain a third conserved quantity

$$\int_{-\infty}^{\infty}\left(\eta_0^3 - \frac{1}{3}\eta_{0\xi}^2\right) d\xi = \text{constant}, \tag{3.96}$$

which is indeed directly related to the total energy given in (3.89) (see Q3.34). (In the context of the KdV equation treated in isolation, it would seem reasonable to regard (3.95) as a statement of energy conservation, since the integrand is a square (that is, proportional to (amplitude)^2). However, as we have seen, when the appropriate physical interpretation is adopted, it is (3.96) which corresponds to the conservation of energy.)

The existence of these three conservation laws is to be expected since our underlying water-wave equations exhibit this same property (where only conservative forces are involved). However, there is now a real surprise: the KdV equation possesses an *infinite* number of conservation laws. In the early stages of the study of the KdV equation (Miura, Gardner & Kruskal, 1968), eight further conservation laws were written down explicitly (having been obtained by extraordinary perseverance); for example, the next two conserved densities are

$$\frac{45}{4}\eta_0^4 - 15\eta_0\eta_{0\xi}^2 + \eta_{0\xi\xi}^2$$

and

$$63\eta_0^5 - 210\eta_0^2\eta_{0\xi}^2 + 28\eta_0\eta_{0\xi\xi}^2 - \frac{8}{9}\eta_{0\xi\xi\xi}^2;$$

see Q3.35. The existence of an infinite set of conservation laws (which will not be proved here) relates directly to the important idea that the KdV equation, and other 'soliton' equations, each constitute a *completely integrable Hamiltonian system*; equivalently, this is to say that the KdV equation can be written as a *Hamiltonian flow*. This aspect of soliton theory is quite beyond the scope of a text that is centred on water-wave theory, but much has been written on these matters; see the section on Further Reading at the end of this chapter.

Finally, we briefly indicate the form of some of the conservation laws that are associated with the standard KdV-type equations. We shall use here the simplest – we might say normalised – versions of these equations that were introduced in Section 3.3. First we consider the concentric KdV equation, (3.64):

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0.$$

It is clear that, for the first two terms, $t^{1/2}$ is an integrating factor; thus we multiply by $t^{1/2}$ to give

$$\frac{\partial}{\partial t}(t^{1/2}u) + \frac{\partial}{\partial x}\left\{t^{1/2}(u_{xx} - 3u^2)\right\} = 0,$$

so $t^{1/2}u$ is a conserved density. This describes the geometrical decay that is required to maintain the conservation of mass (cf. equation (3.31)). Similarly, if we multiply by $2tu$, then we obtain

$$\frac{\partial}{\partial t}(tu^2) + \frac{\partial}{\partial x}\left\{t(2uu_{xx} - u_x^2 - 4u^3)\right\} = 0$$

so that another conserved density is tu^2; further conserved densities are discussed in Q3.39.

The Boussinesq equation, (3.43), is

$$H_{tt} - H_{XX} + 3(H^2)_{XX} - H_{XXXX} = 0$$

which, for our current purpose, is most conveniently written as the pair of equations

$$H_t = -U_X, \quad U_t + H_X - 3(H^2)_X + H_{XXX} = 0;$$

cf. equation (3.38). The second equation here is obtained after one integration in X, coupled with the assumption of decay conditions as $|X| \to \infty$. We now obtain directly

$$\int_{-\infty}^{\infty} H_t \,\mathrm{d}X = -[U]_{-\infty}^{\infty}; \qquad \int_{-\infty}^{\infty} U_t \,\mathrm{d}X = -[H - 3H^2 + H_{XX}]_{-\infty}^{\infty}$$

and so

$$\int_{-\infty}^{\infty} H \,\mathrm{d}X = \text{constant} \quad \text{and} \quad \int_{-\infty}^{\infty} U_t \,\mathrm{d}X = \text{constant}. \tag{3.97}$$

The first of these is the conservation of mass and the second is the conservation of momentum, an identification which becomes clearer if we revert to the original x, where

$$X = x + \varepsilon \int_{-\infty}^{x} \eta \, \mathrm{d}x$$

(see Section 3.2.5), so that

$$\int_{-\infty}^{\infty} U(1 + \varepsilon\eta) \, \mathrm{d}x = \text{constant};$$

cf. equation (3.93). A few other conserved densities are given in Q3.40.

Our third and final example is the two-dimensional KdV equation which is written here as

$$u_t - 6uu_x + u_{xxx} + 3v_y = 0, \quad u_y = v_x;$$

see equation (3.61) and Section 3.2.2. No longer do we have the classical form of a (two-dimensional) conservation law: u is a function of three variables here. This complication produces a development that is less straightforward. When we integrate the second equation with respect to x, and impose decay conditions at infinity, we obtain

$$\frac{\partial}{\partial y}\left(\int_{-\infty}^{\infty} u \, \mathrm{d}x\right) = 0 \quad \text{so} \quad \int_{-\infty}^{\infty} u \, \mathrm{d}x = f(t).$$

However, this is true for all y; let us evaluate the integral for any y that is far-removed from any wave interaction in, say, the N-soliton solution. (The N-soliton solution of the 2D KdV equation describes the interaction of waves that asymptote to plane oblique solitary waves at infinity; see Section 3.3.2 and Q3.19.) In this situation, the function $f(t)$ is a constant; consequently we obtain

$$\int_{-\infty}^{\infty} u \, \mathrm{d}x = \text{constant}, \tag{3.98}$$

at least for this class of solutions. A similar argument yields the result

$$\int_{-\infty}^{\infty} v \, \mathrm{d}y = \text{constant}; \tag{3.99}$$

these two conserved quantities are analogous to the pair (3.97) that we derived for the Boussinesq equation. To proceed, the integral in x of the first equation of this pair yields

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} u\,\mathrm{d}x\right)+[-3u^2+u_{xx}]_{-\infty}^{\infty}+3\frac{\partial}{\partial y}\left(\int_{-\infty}^{\infty} v\,\mathrm{d}x\right)=0$$

and so, making use of (3.98) and the same argument as above, we also have

$$\int_{-\infty}^{\infty} v\,\mathrm{d}x = \text{constant}. \tag{3.100}$$

The obvious interpretation of equations (3.99) and (3.100) is that momentum is conserved in both the y- and x- directions; other conservation laws are even less straightforward to obtain and to interpret.

As an intriguing postscript, we mention the equations for shallow water (obtained in Section 2.6). Following the choices we made there (of setting $\varepsilon = 1$ and writing $1+\varepsilon\eta(x, t) = h(x, t)$), these equations are

$$\left.\begin{aligned} &u_t + uu_x + wu_z + h_x = 0; \quad u_x + w_z = 0,\\ \text{with}\quad &\\ &w = h_t + uh_x \text{ on } z = h \text{ and } w = 0 \text{ on } z = 0. \end{aligned}\right\} \tag{3.101}$$

We have seen that our water-wave equations, (3.85), admit just the three physical conservation laws (of mass, momentum and energy). On the other hand, all our KdV-type equations – that is, completely integrable equations – possess an infinity of conservation laws. The question we pose is: how many conservation laws does the set (3.101) possess? The obvious answer, surely, is just three; let us investigate further.

First, the now very familiar procedure of forming

$$\int_0^h u_x\,\mathrm{d}z + [w]_0^h = \frac{\partial}{\partial x}\left(\int_0^h u\,\mathrm{d}z\right)+\frac{\partial h}{\partial t} = 0$$

yields the conservation of mass

$$\int_{-\infty}^{\infty} h(x, t)\,\mathrm{d}x = \text{constant}, \tag{3.102}$$

provided decay conditions obtain. Next, the second equation in (3.101) is multiplied by u and added to the first to give

$$u_t + 2uu_x + (uw)_z + h_x = 0,$$

so (cf. equation (3.92)) we obtain

$$\frac{\partial}{\partial t}\left(\int_0^h u\,\mathrm{d}z\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}h^2 + \int_0^h u^2\,\mathrm{d}z\right) = 0,$$

from which we obtain the conservation of momentum

$$\int_{-\infty}^{\infty}\left(\int_0^h u\,\mathrm{d}z\right)\mathrm{d}x = \text{constant}. \qquad (3.103)$$

Finally, multiply the first equation by u to produce

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{3}u^3\right) + \frac{\partial}{\partial z}\left(\frac{1}{2}u^2 w\right) - \frac{1}{2}u^2 w_z + \frac{\partial}{\partial x}(uh) - hu_x = 0$$

and then substitute from the second equation for w_z and for u_x:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}u^3 + uh\right) + \frac{\partial}{\partial z}\left(\frac{1}{2}u^2 w + hw\right) = 0$$

since $h = h(x, t)$ only. Consequently the integration in z, coupled with differentiation under the integral sign, yields

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}h^2 + \frac{1}{2}\int_0^h u^2\,\mathrm{d}z\right\} + \frac{\partial}{\partial x}\left\{\int_0^h\left(\frac{1}{2}u^3 + uh\right)\mathrm{d}z\right\} = 0,$$

so

$$\int_{-\infty}^{\infty}\left(h^2 + \int_0^h u^2\,\mathrm{d}z\right)\mathrm{d}x = \text{constant}, \qquad (3.104)$$

the conserved energy. These three conserved quantities, (3.102), (3.103) and (3.104), are to be compared with those derived earlier ((3.87), (3.93) and (3.89)). No surprises here: we have derived the expected conservation laws for mass, momentum and energy.

We now explore an extension of this process by multiplying the first equation of (3.101) by u^2 (and follow the development described by Benney (1974) and Miura (1974)), to give

$$\frac{\partial}{\partial t}\left(\frac{1}{3}u^3\right) + \frac{\partial}{\partial x}\left(\frac{1}{4}u^4\right) + wu^2 u_z + u^2 h_x = 0;$$

this is rewritten as

$$\frac{\partial}{\partial t}\left(\frac{1}{3}u^3\right)+\frac{\partial}{\partial x}\left(\frac{1}{4}u^4+u^2h\right)+\frac{\partial}{\partial z}\left(\frac{1}{3}u^3w\right)-\frac{1}{3}u^3w_z-2huu_x=0, \quad (3.105)$$

The same equation is multiplied by h to produce

$$(hu)_t-uh_t+huu_x+whu_z+\left(\frac{1}{2}h^2\right)_x=0$$

which is added to equation (3.105) to yield

$$\frac{\partial}{\partial t}\left(\frac{1}{3}u^3+hu\right)+\frac{\partial}{\partial x}\left(\frac{1}{3}u^4+u^2h+\frac{1}{2}h^2\right)$$
$$+\frac{\partial}{\partial z}\left(\frac{1}{3}u^3w\right)-huu_x-uh_t+hwu_z=0. \qquad (3.106)$$

Here we write

$$hwu_z=(huw)_z-huw_z$$

and then introduce

$$[w]_0^h+\int_0^h u_x\,\mathrm{d}z=0; \quad \text{that is,} \quad h_t+m_x=0, \quad m=\int_0^h u\,\mathrm{d}z,$$

to give

$$\begin{aligned}hwu_z-uh_t-huu_x&=(huw)_z-huw_z+um_x-huu_x\\&=(huw)_z+(um)_x-u_xm=(huw)_z+(um)_x+(mw)_z.\end{aligned}$$

Thus equation (3.106) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{3}u^3+hu\right)+\frac{\partial}{\partial x}\left(\frac{1}{3}u^4+hu^2+\frac{1}{2}h^2+um\right)$$
$$+\frac{\partial}{\partial z}\left\{w\left(\frac{1}{3}u^3+hu+m\right)\right\}=0,$$

which provides a *fourth* conservation law which, after an integration in z over $(0, h)$, yields

$$\left(\frac{1}{3}m_3+hm_1\right)_t+\left(\frac{1}{3}m_4+hm_2+\frac{1}{2}m_1^2+\frac{1}{3}h^3\right)_x=0,$$

where we have written

$$m_n = \int_0^h u^n \,\mathrm{d}z.$$

Indeed, as Benney and Miura demonstrate, the set of equations (3.101) – like our special evolution equations – possesses an *infinite* set of conservation laws; see Q3.42 and Q3.43 for more about these laws.

We have now introduced some of the important equations of soliton theory that arise in the study of water waves, together with a description of some of their properties. We now extend our studies to show how other physically relevant properties can be introduced into our nonlinear evolution equations (although the resulting equations that govern the wave propagation are unlikely to be completely integrable).

3.4 Waves in a non-uniform environment

The equations that we have derived so far – the KdV family of equations – appear to arise in very special circumstances. In particular, we have assumed that the water is stationary and that the bottom is both flat and horizontal. It is clear that any application of these methods to situations that model physical reality more closely must encompass variable depth and an underlying (non-uniform) flow, at the very least. Certainly it would be a disappointment to find that all the interesting phenomena of nonlinear wave propagation (that we have described earlier) occur only under ideal conditions that hardly ever obtain in the physical world. One of our objectives in this section will be to demonstrate that the derivation and existence of KdV equations (in water-wave theory) are fairly robust to changes in the underlying physical properties. Specifically, we shall see how the derivation of some of the family of KdV equations is affected by the inclusion of (a) an underlying shear flow and (b) variable depth. In addition we shall briefly look at some properties of obliquely interacting waves.

3.4.1 Waves over a shear flow

The purpose here is to derive the classical Korteweg–de Vries equation, for long gravity waves, propagating in the x-direction, over water which is moving only in the x-direction, with a velocity profile which depends only on z: $u = U(z)$. This is the prescribed underlying shear flow, although the terminology that we adopt is not to imply that the profile is generated by viscous stresses. This description is used in order to indicate what type of profile could be chosen; in the undisturbed state – no waves – the governing equations (for inviscid flow) admit a solution for arbitrary $U(z)$, provided that the depth is constant; see Q1.13. Thus we set $b = 0$ for all x in the equations for one-dimensional flow; we use equations (3.12)–(3.15), where the shear flow is introduced by writing

$$U(z) + \varepsilon u \quad \text{for} \quad \varepsilon u \tag{3.107}$$

since we want $U = \mathrm{O}(1)$ as $\varepsilon \to 0$. Hence the equations that we shall now examine are

$$\left.\begin{aligned}
&u_t + Uu_x + U'w + \varepsilon(uu_x + wu_z) = -p_x\\
&\varepsilon\{w_t + Uw_x + \varepsilon(uw_x + ww_z)\} = -p_z;\\
&u_x + w_z = 0,\\
&\text{with}\\
&p = \eta \quad \text{and} \quad w = \eta_t + U\eta_x + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta\\
&\text{and}\\
&w = 0 \quad \text{on} \quad z = 0,
\end{aligned}\right\} \tag{3.108}$$

where $U' \equiv \mathrm{d}U/\mathrm{d}z$. (We note in passing that, indeed, these equations are satisfied with $u = w = p = \eta = 0$ – no disturbances – for arbitrary $U(z)$.)

The first task here is to determine the nature of the linear problem, that is, the leading order problem in the asymptotic expansion for $\varepsilon \to 0$. This is described by the equations

$$\left.\begin{aligned}
&u_t + Uu_x + U'w = -p_x; \quad p_z = 0; \quad u_x + w_z = 0,\\
&\text{with}\\
&p = \eta \quad \text{and} \quad w = \eta_t + U\eta_x \quad \text{on} \quad z = 1\\
&\text{and}\\
&w = 0 \quad \text{on} \quad z = 0.
\end{aligned}\right\} \tag{3.109}$$

We are interested (at this order) in waves that propagate at constant speed with unchanging form. Do any such solutions of equations (3.109) exist? Let us suppose that they do, and so introduce a coordinate that is moving with the waves at a constant speed c; we therefore transform from (x, t, z) to $(x - ct, z)$. Our equations (3.109) become

$$(U - c)u_\xi + U'w = -p_\xi; \quad u_\xi + w_z = 0; \quad p = \eta \quad (0 \le z \le 1), \tag{3.110}$$

with

$$w = (U - c)\eta_\xi \quad \text{on} \quad z = 1; \quad w = 0 \quad \text{on} \quad z = 0. \tag{3.111}$$

Here we have obtained $p(= \eta)$ in the familiar way, and have written $\xi = x - ct$.

To proceed, we eliminate u_ξ between the equations in (3.110) to give

$$U'w - (U - c)w_z = \eta_\xi \quad \text{or} \quad (U - c)^2 \frac{\partial}{\partial z}\left(\frac{w}{U - c}\right) = \eta_\xi,$$

so

$$w = (U - c)\eta_\xi \int_0^z \frac{\mathrm{d}z}{(U - c)^2}$$

which satisfies the bottom boundary condition (in (3.111)). The surface boundary condition (on $z = 1$; see (3.111) again) requires that

$$\int_0^1 \frac{\mathrm{d}z}{(U - c)^2} = 1, \tag{3.112}$$

and then $\eta(\xi)$ is arbitrary: the waves propagate at a constant speed (c) determined by equation (3.112), given $U(z)$, and – at this order – they move with an unchanging shape which is arbitrary. The equation for c, (3.112), is very different from that which has appeared in any of our other work that has led to an expression for the speed of propagation for gravity waves. This is an important equation in water-wave theory (and its counterparts appear in other problems which incorporate an underlying flow); it is known as the *Burns condition* (Burns (1953), although it seems to have appeared first in Thompson (1949)). But it turns out that its real interest is evident in the cases where solutions of (3.112) do not exist!

Solutions of (3.112) for c exist only provided $U(z) \neq c$ for $0 \leq z \leq 1$. If $U(z_c) = c$ for some $z_c \in (0, 1)$ – and $U(0) = c$ or $U(1) = c$ can never happen – then the left-hand side of (3.112) is not defined; $z = z_c$ is called a *critical level* or *layer*. We shall make a few comments about the nature of the Burns condition later (Section 3.4.2), but it is sufficient for our present purposes to assume that solutions of (3.112) *do* exist. For example, the simple choice

$$U(z) = U_0 + (U_1 - U_0)z, \tag{3.113}$$

where U_0 and U_1 are constants, yields

$$-\frac{1}{(U_1-U_0)}\left[\frac{1}{U_1-c}-\frac{1}{U_0-c}\right]=1$$

and so

$$c=\frac{1}{2}\left[U_0+U_1\pm\sqrt{4+(U_1-U_0)^2}\right]. \tag{3.114}$$

This solution describes two possible speeds of propagation, one of which satisfies $c > U_1$ and the other $c < U_0$. That is, for the linear shear (3.113), the two speeds of propagation are: greater than the surface speed of the flow and less than the bottom speed. We note that, for $U(z)=0$, $0 \le z \le 1$, (that is, $U_0 = U_1 = 0$) we recover $c = \pm 1$ (see equation (2.10)). (The case of a uniform stream corresponds to the choice $U_0 = U_1$; cf. Q2.11.)

We now proceed to the derivation of the KdV equation as relevant to this problem, and to accomplish this we follow the method described in Section 3.2.1. Thus we introduce a local characteristic variable (ξ) and a far-field variable (τ) defined by

$$\xi = x - ct, \qquad \tau = \varepsilon t, \tag{3.115}$$

where c is a solution of the Burns condition. Equations (3.108) become

$$\left.\begin{gathered}(U-c)u_\xi + U'w + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi;\\ \varepsilon\{(U-c)w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\} = -p_z;\\ u_\xi + w_z = 0,\\ \text{with}\\ p=\eta \quad\text{and}\quad w=(U-c)\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi)\\ \text{on}\quad z = 1+\varepsilon\eta\\ \text{and}\\ w=0 \quad\text{on}\quad z=0.\end{gathered}\right\} \tag{3.116}$$

We seek a solution of this set, as usual, in the form of an asymptotic expansion

$$q \sim \sum_{n=0}^{\infty}\varepsilon^n q_n, \qquad \varepsilon \to 0,$$

where q represents each of u, w, p and η. Thus, at leading order, we have

$$(U-c)u_{0\xi}+U'w_0=-p_{0\xi};\quad p_{0z}=0;\quad u_{0\xi}+w_{0z}=0$$

with

$$p_0=\eta_0 \quad \text{and} \quad w_0=(U-c)\eta_{0\xi} \quad \text{on} \quad z=1$$

and

$$w_0=0 \quad \text{on} \quad z=0.$$

This is, as expected, the linear problem that we have just described: see equations (3.109)–(3.111). Thus c is a solution of

$$\int_0^1 \frac{\mathrm{d}z}{(U-c)^2}=1, \tag{3.117}$$

the Burns condition, and

$$\left.\begin{aligned} w_0&=(U-c)\eta_{0\xi}\int_0^z \frac{\mathrm{d}z}{(U-c)^2};\\ u_0&=-\eta_0\left\{\frac{1}{U-c}+U'\int_0^z \frac{\mathrm{d}z}{(U-c)^2}\right\};\quad p_0=\eta_0, \end{aligned}\right. \tag{3.118}$$

where we have assumed that $u_0=0$ wherever $\eta_0=0$. The solution (3.118), with (3.117), is valid for arbitrary $\eta_0(\xi,\tau)$, at this order; to find η_0 we must construct the problem at $\mathrm{O}(\varepsilon)$.

The $\mathrm{O}(\varepsilon)$ terms from equations (3.116) give rise to the set of equations

$$(U-c)u_{1\xi}+U'w_1+u_{0\tau}+u_0u_{0\xi}+w_0u_{0z}=-p_{1\xi}; \tag{3.119}$$

$$(U-c)w_{0\xi}=-p_{1z};\quad u_{1\xi}+w_{1z}=0 \tag{3.120}$$

with

$$p_1+\eta_0p_{0z}=\eta_1 \tag{3.121}$$

$$w_1+\eta_0w_{0z}=(U-c)\eta_{1\xi}+U'\eta_0\eta_{0\xi}+\eta_{0\tau}+u_0\eta_{0\xi} \tag{3.122}$$

(3.121) and (3.122) on $z=1$

and

$$w_1=0 \quad \text{on} \quad z=0. \tag{3.123}$$

At this stage it is convenient to introduce a compact notation to cope with the integrals that arise, namely

$$I_n(z) = \int_0^z \frac{dz}{(U-c)^n}; \tag{3.125}$$

then, for example, the Burns condition (3.117) becomes simply

$$I_2(1) \; (= I_{21}) = 1. \tag{3.126}$$

Similarly, equations (3.118) are written as

$$w_0 = (U-c)I_2\eta_{0\xi}; \quad u_0 = -\{(U-c)^{-1} + U'I_2\}\eta_0; \quad p_0 = \eta_0 \tag{3.127}$$

and then from equations (3.120) and (3.121) we obtain

$$p_1 = \eta_1 + \eta_{0\xi\xi} \int_z^1 (U-c)^2 I_2 dz.$$

Now we eliminate $u_{1\xi}$ between equations (3.119) and (3.120) to give

$$\begin{aligned}(U-c)^2\left\{\frac{w_1}{U-c}\right\}_z &+ \{(U-c)^{-1} + U'I_2\}\eta_{0\tau} - \{(U-c)^{-1} + U'I_2\}^2\eta_0\eta_{0\xi} \\ &+ (U-c)U''I_2^2\eta_0\eta_{0\xi} = \eta_{1\xi} + \eta_{0\xi\xi\xi}\int_z^1 (U-c)^2 I_2 dz,\end{aligned}$$

and then the solution which satisfies the bottom boundary condition, (3.123), can be written as

$$\begin{aligned}w_1 = (U-c)\Bigg\{&\left(\frac{I_2}{U-c} - 2I_3\right)\eta_{0\tau} + \left(I_4 + 4\int_0^z \frac{U'I_2}{(U-c)^3}dz - \frac{U'I_2}{U-c}\right)\eta_0\eta_{0\xi} \\ &+ I_2\eta_{1\xi} + \eta_{0\xi\xi\xi}\int_0^z (U-c)^{-2}\left[\int_z^1 (U-c)^2 I_2 dz\right]dz\Bigg\}.\end{aligned}$$

Finally, the surface boundary condition for w_1, (3.121), requires that

$$(U_1 - c)\eta_{1\xi} + U_1'\eta_0\eta_{0\xi} + \eta_{0\tau} - 2\left\{\frac{1}{U_1 - c} + U_1'I_{21}\right\}\eta_0\eta_{0\xi}$$
$$= (U_1 - c)\left\{\left(\frac{I_{21}}{U_1 - c} - 2I_{31}\right)\eta_{0\tau}\right.$$
$$\left. + \left(I_{41} + 4\int_0^1 \frac{U'I_2}{(U-c)^3}\,\mathrm{d}z - \frac{U_1'I_{21}}{U_1 - c}\right)\eta_0\eta_{0\xi} + I_{21}\eta_{1\xi} + J_1\eta_{0\xi\xi\xi}\right\}$$

where the additional subscript '1' denotes evaluation on $z = 1$, and

$$J_1 = \int_0^1\int_z^1\int_0^{z_1} \frac{[U(z_1) - c]^2}{[U(z) - c]^2[U(z_2) - c]^2}\,\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}z.$$

After we use $I_{21} = 1$ (see (3.126)) and simplify, the equation for η_0 becomes

$$-2I_{31}\eta_{0\tau} + 3I_{41}\eta_0\eta_{0\xi} + J_1\eta_{0\xi\xi\xi} = 0, \tag{3.128}$$

since η_1 cancels identically from the problem at this order.

Equation (3.128) is an altogether satisfying result: it is a (classical) Korteweg–de Vries equation, since it has constant coefficients (and so may be transformed into any suitable variant of the KdV equation; see Q3.1). The presence of an ambient arbitrary velocity profile (and hence an arbitrary vorticity distribution in the flow) is evident only through the three constants I_{31}, I_{41} and J_1. Thus a problem that, we might have supposed, is significantly more involved than for the case of propagation on a stationary flow, reduces essentially to the same result. Hence nonlinear dispersive waves (for example the solitary wave) can exist on arbitrary flows. It is now a simple exercise, first, to check that we recover our previous KdV equation for stationary flow and, second, to obtain the form of the KdV equation for any given flow (at least in the absence of a critical level); see Q3.44 & Q3.45. (More details of this derivation will be found in Freeman & Johnson (1970); the corresponding calculation for the sech^2 solitary wave is described by Benjamin (1962).)

3.4.2 The Burns condition

The derivation of the Korteweg–de Vries equation for flow over an arbitrary shear, as we have presented it, is valid only if a critical level does not arise. If $U(z)$ and c are such that $U(z_c) = c$ for some z_c $(0 < z_c < 1)$, it is clear that we must examine the nature of the problem in the neighbourhood of $z = z_c$ (since, for example, the integrals over z no longer exist). It turns out that, in the context of inviscid fluid dynamics, a region exists in the neighbourhood of $z = z_c$ (in fact, where $z - z_c = O(\varepsilon^{1/2})$) where nonlinear effects are important. The inclusion of the appropriate contribution from the nonlinearity enables the singularity at $z = z_c$ to be removed. This calculation is, however, altogether beyond the scope of this text; those interested in this aspect of the problem should consult some of the references given in the Further Reading at the end of this chapter. Suffice it to record here that the Burns condition and, indeed, the KdV equation, are both recovered even when a critical level is present. The only change from the results that we have described is that all the integrals are now defined by their *finite parts*, that is, their *Cauchy principal values*.

One way to define the finite part of our integrals is as the

$$\text{finite part as } \varepsilon \to 0^+ \left\{ \int_0^{z_c-\varepsilon} f(z)\,dz + \int_{z_c+\varepsilon}^{1} f(z)\,dz \right\}, \tag{3.129}$$

from which it is clear that the finite part recovers the classical value of any integral which is defined for all $z \in [0, 1]$. The usual shorthand for the finite part is to write $⨍$ for $\int$, or $\mathrm{I\!\!\!-}$ for I; in this notation, the Burns condition (3.117) becomes

$$\mathrm{I\!\!\!-}_{21} = ⨍_0^1 \frac{dz}{[U(z) - c]^2} = 1. \tag{3.130}$$

It can be shown (Burns, 1953) that, for a monotonic profile which satisfies $U(0) \leq U(z) \leq U(1)$, there are always at least two solutions of equation (3.130):

$$c > U(1) \quad \text{and} \quad c < U(0),$$

exactly as we mentioned earlier. Depending on the form of the function $U(z)$ there may, or may not, be one or more *critical-layer solutions* for which $c = U(z_c)$, $0 < z_c < 1$. We conclude by noting that both the linear-shear profile given in (3.113) and the parabolic (Poiseuille) profile

$$U(z) = U_1(2z - z^2), \quad U_1 = \text{constant},$$

do not admit any critical-layer solutions; see Q3.46. Two 'model' profiles that do give rise to critical-layer solutions – one each – are discussed in Q3.47, Q3.48.

3.4.3 Ring waves over a shear flow

In the two preceding sections we have presented some linear and nonlinear aspects of the problem of *unidirectional* propagation over an arbitrary shear flow. We now address the corresponding problem of a *ring wave* moving over a (unidirectional) shear flow. We refer to this wave as a ring wave, rather than a concentric wave (cf. Sections 2.1.3 and 3.2.3) because it turns out that the wave is concentric only in the case of uniform flow ($U =$ constant everywhere). The presentation here will include some details of the linear problem, and we will mention only briefly the related nonlinear problem.

The underlying shear flow, exactly as in Section 3.4.1, is written as $u = U(z)$, and this is given; the wave, however, propagates outwards from some initial central disturbance. This coordinate mix – rectangular Cartesian for the shear flow and plane polar for the wave – leads to a rather involved formulation of this problem. First we recall the governing equations expressed in rectangular Cartesian coordinates, suitably written with our choice of parameters. Thus we use equations (3.1)–(3.4), but with δ^2 replaced by ε; see equations (3.12)–(3.15) *et seq*. In addition we introduce our standard representation for a horizontal flat bottom ($b = 0$), and then replace εu by $U(z) + \varepsilon u$; see equation (3.107). These manoeuvres yield the set of equations

$$\left.\begin{aligned} u_t + (U + \varepsilon u)u_x + \varepsilon v u_y + U'w + \varepsilon w u_z &= -p_x; \\ v_t + (U + \varepsilon u)v_x + \varepsilon v v_y + \varepsilon w v_z &= -p_y; \\ \varepsilon\{w_t + (U + \varepsilon u)w_x + \varepsilon v w_y + \varepsilon w w_z\} &= -p_z; \\ u_x + v_y + w_z &= 0, \\ \text{with} \qquad & \\ p = \eta \quad \text{and} \quad w = \eta_t + (U + \varepsilon u)\eta_x + \varepsilon v \eta_y & \\ \text{on} \quad z = 1 + \varepsilon\eta & \\ \text{and} \qquad & \\ w = 0 \quad \text{on} \quad z = 0. & \end{aligned}\right\} \qquad (3.31)$$

Now we introduce a plane polar coordinate system which is moving at a constant speed c in the x-direction; we shall make a suitable choice of c later. Thus we transform $(x, y, t) \to (r, \theta, t)$, where

$$x = ct + r\cos\theta, \quad y = r\sin\theta \tag{3.132}$$

and, correspondingly, we define the velocity perturbation in the (x, y)-plane by the transformation

$$u \to u\cos\theta - v\sin\theta, \quad v \to u\sin\theta + v\cos\theta, \tag{3.133}$$

so that (u, v) now represents the perturbation velocity vector for the horizontal components of the motion, written in the polar coordinate frame. Further, we choose to describe a wave whose wavefront has reached an appropriate far-field (so that we may, eventually, construct the relevant KdV equation); thus we define

$$\xi = rk(\theta) - t, \quad R = \varepsilon rk(\theta), \tag{3.134}$$

where the wavefront is represented by $\xi =$ constant and $k(\theta)$ is to be determined. (A concentric wave corresponds to the case $k(\theta) =$ constant, for all θ.) This choice of far-field variables, (3.134), is to be compared with those used in Sections 3.2.1 and 3.2.3; in particular, in this latter case, we see that (3.134) is equivalent to setting $\delta = \varepsilon$ there and then writing ε for ε^2.

The set of equations (3.131), under the transformations (3.132)–(3.134), becomes

$$\left.\begin{aligned}
&(D_1 + \varepsilon D_2 + \varepsilon D_3 + \varepsilon^2 D_4)u + \varepsilon(U - c)\frac{k}{R}v\sin\theta + U'w\cos\theta \\
&\qquad\qquad\qquad\qquad - \varepsilon^2\frac{kv^2}{R} = -k(p_\xi + \varepsilon p_R); \\
&(D_1 + \varepsilon D_2 + \varepsilon D_3 + \varepsilon^2 D_4)v - \varepsilon(U - c)\frac{k}{R}u\sin\theta - U'u\sin\theta \\
&\qquad\qquad + \varepsilon^2\frac{kuv}{R} = -k'(p_\xi + \varepsilon p_R) - \varepsilon\frac{k}{R}p_\theta; \\
&\varepsilon\{(D_1 + \varepsilon D_2 + \varepsilon D_3 + \varepsilon^2 D_4)w\} = -p_z \\
&ku_\xi + k'v_\xi + w_z + \varepsilon(ku_R + \frac{ku}{R} + k'v_R + \frac{k}{R}v_\theta) = 0, \\
&\text{with} \\
&\left.\begin{aligned} p = \eta \quad \text{and} \quad w = (D_1 + \varepsilon D_2 + \varepsilon^2 D_4)\eta + \varepsilon(ku + k'v)\eta_\xi \\ \text{on } z = 1 + \varepsilon\eta \end{aligned}\right\}
\end{aligned}\right\} \tag{3.135}$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

The differential operators (D_n) are defined by

$$D_1 \equiv \{-1 + [U(z) - c](k\cos\theta - k'\sin\theta)\}\frac{\partial}{\partial\xi};$$

$$D_2 \equiv [U(z) - c]\left\{(k\cos\theta - k'\sin\theta)\frac{\partial}{\partial R} - \frac{k}{R}\sin\theta\frac{\partial}{\partial\theta}\right\};$$

$$D_3 \equiv (ku + k'v)\frac{\partial}{\partial\xi} + w\frac{\partial}{\partial z}; \qquad D_4 \equiv (ku + k'v)\frac{\partial}{\partial R} + \frac{kv}{R}\frac{\partial}{\partial\theta},$$

where $k' = \mathrm{d}k/\mathrm{d}\theta$ and, as before, $U' = \mathrm{d}U/\mathrm{d}z$. (The routine but rather tedious calculation that leads to equations (3.135) is left as an exercise.)

We seek an asymptotic solution of the set (3.135) in the usual fashion:

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n, \qquad \varepsilon \to 0,$$

where q represents each of u, v, w, p and η. The leading-order, linear problem is therefore

$$\begin{aligned}
-u_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)u_{0\xi} + U'w_0\cos\theta &= -k'p_{0\xi};\\
-v_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)v_{0\xi} - U'w_0\sin\theta &= -k'p_{0\xi};\\
p_{0z} = 0; \quad ku_{0\xi} + k'v_{0\xi} + w_{0z} &= 0,
\end{aligned}$$

with

$$p_0 = \eta_0 \text{ and } w_0 = -\eta_{0\xi} + (U - c)(k\cos\theta - k'\sin\theta)\eta_{0\xi} \text{ on } z = 1$$

and

$$w_0 = 0 \quad \text{on} \quad z = 0.$$

In common with our previous calculation of this type, we see that $p_0 = \eta_0$ for $z \in [0, 1]$, and then adding the first equation $\times k$ to $k' \times$ the second and eliminating $(ku_{0\xi} + k'v_{0\xi})$ with the equation of mass conservation yields

$$-Fw_{0z} + F_z w_0 = -(k^2 + k'^2)\eta_{0\xi}$$

where we have written

$$F(z, \theta) = -1 + \{U(z) - c\}(k\cos\theta - k'\sin\theta).$$

Thus

$$w_0 = (k^2 + k'^2)F\eta_{0\xi} \int_0^z \frac{\mathrm{d}z}{F^2}$$

satisfies the bottom boundary condition, and then the surface boundary condition for w_0 requires that

$$(k^2 + k'^2) \int_0^1 \frac{\mathrm{d}z}{[1 - \{U(z) - c\}(k\cos\theta - k'\sin\theta)]^2} = 1, \tag{3.136}$$

for arbitrary η_0.

Equation (3.136) is a *generalised Burns condition*, which reduces to our previous Burns condition, (3.117), when we introduce the choice for one-dimensional plane waves: $k(\theta) = 1$, $\theta = 0$ and write c for $1 + c$ (since the characteristic, ξ, contributes a wave speed of 1). In this case, equation (3.136) is used to determine c for a given $U(z)$. However, in the context of a ring wave, this equation is used to define $k(\theta)$ given both $U(z)$ and the speed (c) of the frame of reference. (We note that, in this frame, the speed of the outward propagating wave is $1/k(\theta)$ at any θ, provided $k(\theta) > 0$.) The derivation that has been described assumes that a critical level, $z = z_c$ ($z_c \in (0, 1)$), is not present; if a critical level does occur, so that $F(z_c, \theta) = 0$, then the generalised Burns condition is still (3.136) but now interpreted as the finite part of the integral.

A simple example of the use of the generalised Burns condition is afforded by the choice (see Q3.47)

$$U(z) = \begin{cases} U_1, & d \le z \le 1 \\ U_1 z/d, & 0 \le z < d, \end{cases} \tag{3.137}$$

where U_1 and $d \in [0, 1]$ are constants; this model shear flow was used in Johnson (1990), where more properties of the ring wave are described. The generalised Burns condition (3.136), with (3.137), becomes

$$(k^2 + k'^2)\left\{ \frac{1 - d}{[1 - (U_1 - c)(k\cos\theta - k'\sin\theta)]^2} + \left[\frac{d/\{U_1(k\cos\theta - k'\sin\theta)\}}{\{1 - (U_1 z/d - c)(k\cos\theta - k'\sin\theta)\}} \right]_0^d \right\} = 1,$$

and we now make a choice for c (the speed of the polar coordinate frame). The form of this expression for $k(\theta)$ suggests that we set $c = U_1$, an obvious selection on physical grounds since this ensures that the frame is moving at the surface speed of the shear flow. Our equation for $k(\theta)$ then reduces to

$$(k^2 + k'^2)[1 - d + d/\{1 + U_1(k\cos\theta - k'\sin\theta)\}] = 1, \qquad (3.138)$$

a nonlinear first-order ordinary differential equation for $k(\theta)$. This equation possesses, quite clearly, the general solution

$$\left.\begin{aligned} &k(\theta) = a\cos\theta + b(a)\sin\theta \\ &\text{where} \\ &(a^2 + b^2)\{1 - d + d/(1 + aU_1)\} = 1, \end{aligned}\right\} \qquad (3.139)$$

an approach that can also be adopted for general $U(z)$; see Q3.49. Unfortunately, solutions of the form (3.139) for any $a > 0$ (provided that b is real) do not admit $k(\theta) > 0$ for all θ: at some $\theta \in (0, \pi)$ (and also again for $\theta \in (0, -\pi)$), $k(\theta) = 0$ and thereafter $k(\theta) < 0$. Thus at two (symmetric) points the wavefront has moved to infinity (that is, $r = \{t + \text{constant}\}/k(\theta) \to \infty$) and, where $k(\theta) < 0$, it is moving *inwards*. But we are seeking an outward propagating wave and this, it turns out, is represented by the *singular solution* of equation (3.138). This solution (see Q3.52) can be written in the form

$$\left.\begin{aligned} &k(\theta) = a\cos\theta + b(a)\sin\theta \\ &\text{with} \\ &\tan\theta = -1/b'(a) \\ &\text{where} \\ &(a^2 + b^2)\{1 - d + d/(1 + aU_1)\} = 1. \end{aligned}\right\} \qquad (3.140)$$

Three examples ($d = 0.5$; $U_1 = 0.5, 1, 2$) are presented in Figure 3.10, which shows clearly how the shear flow distorts the wavefront from the circular; these results have been obtained directly from equations (3.140) that define the singular solution.

Finally, we briefly state what happens when we construct the problem that arises at $O(\varepsilon)$. As we know, at this order, we shall find that η_1 is arbitrary, but we expect to obtain a KdV-type equation that describes the evolution of η_0. The calculation follows the lines of that already presented in Section 3.4.1, although it involves more complicated integrals that define the coefficients of the equation for η_0. This equation takes the form

$$A\eta_{0R} + \frac{B}{R}\eta_0 + \frac{C}{R}\eta_{0\theta} + D\eta_0\eta_{0\xi} + E\eta_{0\xi\xi\xi} = 0, \qquad (3.141)$$

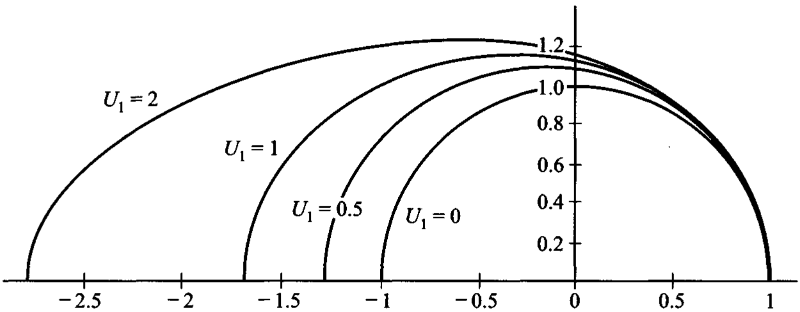


Figure 3.10. The shape of the wavefronts for the ring wave over a shear flow (equation (3.137)) for $0 \le \theta \le \pi$, with $d = 0.5$ and $U_1 = 0.5$, 1, 2. The corresponding circular ring wave ($U_1 = 0$) is included for comparison.

where A–E are the coefficients (which here depend on θ); more details can be found in Johnson (1990). This equation is clearly of KdV-type but, for arbitrary $U(z)$ (and therefore general coefficients), it is not one of the family of completely integrable equations. It does, however, recover the concentric KdV (cKdV) equation when $U(z) =$ constant (and we set $k(\theta) = 1$) for then $F(z, \theta) = -1$ and $A = 2$, $B = 1$, $C = 0$, $D = 3$, $E = 1/3$. Equation (3.141) can be discussed, in the general case, only via a numerical approach (which we do not pursue here).

3.4.4 The Korteweg–de Vries equation for variable depth

Another problem of some practical interest is the propagation of nonlinear dispersive waves (such as a solitary wave) over variable depth. We now address this situation in the case of one-dimensional propagation. Here, as we shall see, the important decision that we must make concerns the scale on which the depth variation occurs. In order to explain what is involved, we consider the classical situation that gives rise to the KdV equation (described in Section 3.2.1). We have shown that the relevant scales are

$$\xi = x - t, \qquad \tau = \varepsilon t,$$

for right-going waves; thus $x - t = \mathrm{O}(1)$ and $t = \mathrm{O}(\varepsilon^{-1})$. This is equivalent to the choice $x - t = \mathrm{O}(1)$ and $x = \mathrm{O}(\varepsilon^{-1})$ (cf. Figure 1.7), which is the convenient interpretation to adopt here, for, if the depth varies on a scale which is either faster or slower than $\mathrm{O}(\varepsilon^{-1})$, we shall obtain appropriately simplified KdV problems. In the former case, we have a situation where, to leading order as $\varepsilon \to 0$, the depth changes rapidly relative to any changes due to the natural evolution of the nonlinear wave. On the other hand, in the latter case, the wave will evolve at essentially the (local) constant depth. Of course, both these problems are of considerable interest in their own right and they have received some attention – particularly the latter choice; more details can found from the Further Reading at the end of this chapter. However, for the development that we present here, the most interesting case arises when the scale of the depth variation is the same as the scale on which the wave will naturally evolve even over constant depth. In the context of the KdV derivations that we have presented so far, this can be thought of as the 'worst case' scenario. Of course, we may then use this case to gain some insight into the problems for faster and for slower depth variations; we shall touch on these two extremes later.

The governing equations for one-dimensional propagation (cf. equations (3.12)–(3.15) with (3.4)) are

$$\left.\begin{aligned} u_t + \varepsilon(uu_x + wu_z) &= -p_x; \\ \varepsilon\{w_t + \varepsilon(uw_x + ww_z)\} &= -p_z; \\ u_x + w_z &= 0, \\ \text{with} \qquad & \\ p = \eta \text{ and } w = \eta_t + \varepsilon u\eta_x &\text{ on } z = 1 + \varepsilon\eta \\ \text{and} \qquad & \\ w = 0 \quad \text{on} \quad z &= 0. \end{aligned}\right\} \tag{3.142}$$

The important choice, described above, is to set

$$b(x) = B(\varepsilon x),$$

and we shall usually define $B(\varepsilon x) = 0$ in $x < 0$, so that the wave propagates (rightwards) from a region of constant depth. The appropriate variables to use for the far-field (cf. $\xi = x - t$, $\tau = \varepsilon t$) must accommodate the variation of wave speed with depth (see equation (2.47) and Q2.33), and the slow spatial scale (εx). Thus we introduce

$$\xi = \frac{1}{\varepsilon}\chi(X) - t, \qquad X = \varepsilon x,$$

where $\chi(X)$ is to be determined, so we transform according to

$$\frac{\partial}{\partial x} \equiv \chi' \frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} \equiv -\frac{\partial}{\partial \xi}.$$

This makes clear why the factor ε^{-1} is required in the definition of ξ: $\chi' = \mathrm{O}(1)$ plays the rôle of the speed of propagation, $c = \mathrm{O}(1)$ (actually χ' is equivalent to $1/c$). Further, we anticipate that, for constant depth ($B = 0$), we shall have $\chi(X) = X = \varepsilon x$ and then ξ recovers our former expression $(x - t)$. The set of governing equations, (3.142), therefore becomes

$$\left.\begin{aligned}
-u_\xi + \varepsilon\{u(\chi' u_\xi + \varepsilon u_X) + w u_z\} &= -(\chi' p_\xi + \varepsilon p_X);\\
\varepsilon[-w_\xi + \varepsilon\{u(\chi' w_\xi + \varepsilon w_X) + w w_z\}] &= -p_z;\\
\chi' u_\xi + \varepsilon u_X + w_z &= 0,\\
&\text{with}\\
p = \eta \quad \text{and} \quad w = -\eta_\xi + \varepsilon u(\chi'\eta_\xi + \varepsilon \eta_X)&\\
\text{on} \quad z = 1 + \varepsilon\eta&\\
&\text{and}\\
w = \varepsilon u B'(X) \quad \text{on} \quad z = B(X).&
\end{aligned}\right\} \tag{3.143}$$

We adopt the standard form of solution:

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n, \quad \varepsilon \to 0,$$

where q represents each of u, w, p and η; the leading-order problem from (3.143) is then described by the equations

$$u_{0\xi} = \chi' p_{0\xi}; \quad p_{0z} = 0; \quad \chi' u_{0\xi} + w_{0z} = 0,$$

with

$$p_0 = \eta_0 \quad \text{and} \quad w_0 = -\eta_{0\xi} \quad \text{on} \quad z = 1$$

and

$$w_0 = 0 \quad \text{on} \quad z = B(X).$$

These equations are in a form that we recognise as typical of these problems; the solution is immediately

$$p_0 = \eta_0, \quad 0 \le z \le 1; \quad u_0 = \chi'\eta_0; \quad w_0 = (B - z)\chi'^2 \eta_{0\xi}, \tag{3.144}$$

where we have chosen $u_0 = 0$ wherever $\eta_0 = 0$. The function w_0 satisfies the bottom boundary condition (on $z = B$), and in order to satisfy the corresponding surface condition we require

$$\chi'^2 = \frac{1}{D(X)}, \tag{3.145}$$

where $D(X) = 1 - B(X)$ (> 0) is the local depth. Thus we write

$$\chi(X) = \int_0^X \frac{\mathrm{d}X'}{\sqrt{D(X')}} \tag{3.146}$$

for right-going waves, which agrees precisely with the form given in equation (2.47). Here, for convenience, we have chosen $\chi(0) = 0$. At this order $\eta_0(\xi, X)$ is arbitrary, so we move to the $O(\varepsilon)$ terms, which will provide the equation for η_0.

From equations (3.143) we see that

$$\left.\begin{aligned}
&-u_{1\xi} + \chi' u_0 u_{0\xi} + w_0 u_{0z} = -(\chi' p_{1\xi} + p_{0X}); \\
&w_{0\xi} = p_{1z}; \quad \chi' u_{1\xi} + u_{0X} + w_{1z} = 0, \\
&\text{with} \\
&p_1 + \eta_0 p_{0z} = \eta_1 \text{ and } w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + \chi' u_0 \eta_{0\xi} \\
&\qquad \text{on } z = 1 \\
&\text{and} \\
&w_1 = u_0 B'(X) \quad \text{on} \quad z = B(X).
\end{aligned}\right\} \tag{3.147}$$

(We note in passing that, at least in this problem, there is no need to expand χ as

$$\chi \sim \sum_{n=0}^{\infty} \varepsilon^n \chi_n(X),$$

although in other problems this might be necessary in order to obtain a uniformly valid representation.) Thus we obtain (with $\chi' = 1\sqrt{D}$)

$$p_1 = \frac{1}{D}\left\{B(z-1) + \frac{1}{2}(1-z^2)\right\}\eta_{0\xi\xi} + \eta_1,$$

since $p_{0z} = 0$ and we have used (3.144). From the first and third equations in (3.147), upon the elimination of $u_{1\xi}$, we obtain

$$u_{0X} + w_{1z} + \frac{1}{D}u_0 u_{0\xi} + \frac{1}{\sqrt{D}} w_0 u_{0z}$$
$$= -\frac{1}{D}\left[\frac{1}{D}\left\{B(z-1) + \frac{1}{2}(1-z^2)\right\}\eta_{0\xi\xi\xi} + \eta_{1\xi}\right] - \frac{1}{\sqrt{D}}\eta_{0X}$$

which yields

$$w_1 = (B - z)\left\{\left(\frac{1}{\sqrt{D}}\eta_0\right)_X + \frac{1}{D^2}\eta_0\eta_{0\xi} + \frac{1}{\sqrt{D}}\eta_{0X} + \frac{1}{D}\eta_{1\xi}\right\}$$
$$-\frac{1}{D^2}\left\{B\left(\frac{1}{2}z^2 - z\right) + \frac{1}{2}\left(z - \frac{z^3}{3}\right) - \frac{1}{3}B^3 + B^2 - \frac{1}{2}B\right\}\eta_{0\xi\xi\xi} + \frac{B'}{\sqrt{D}}\eta_0,$$

where the bottom boundary condition (on $z = B$) is satisfied. Finally, the surface boundary condition requires

$$-\eta_{1\xi} + \frac{2}{D}\eta_0\eta_{0\xi} = -D\left\{\left(\frac{1}{\sqrt{D}}\eta_0\right)_X + \frac{1}{D^2}\eta_0\eta_{0\xi} + \frac{1}{\sqrt{D}}\eta_{0X} + \frac{1}{D}\eta_{1\xi}\right\}$$
$$-\frac{1}{3}D\eta_{0\xi\xi\xi} - \frac{D'}{\sqrt{D}}\eta_0$$

in which, as expected, $\eta_{1\xi}$ cancels identically, leaving

$$2\sqrt{D}\eta_{0X} + \frac{1}{2}\frac{D'}{\sqrt{D}}\eta_0 + \frac{3}{D}\eta_0\eta_{0\xi} + \frac{1}{3}D\eta_{0\xi\xi\xi} = 0, \qquad (3.148)$$

where $D = D(X)$. This is a variable-coefficient KdV equation, which clearly reduces to our classical KdV equation, (3.28), when we introduce the constant depth, $D = 1$:

$$2\eta_{0X} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0,$$

although we must now interpret X as τ (which is legitimate at this order).

Our new KdV-type equation, (3.148), is not one of the special completely integrable equations (for arbitrary $D(X)$), but special reductions are possible (as for $D = 1$; see also Q3.53). However, the general equation can be usefully written by first multiplying by $D^{-1/4}$, to give

$$2(D^{1/4}\eta_0)_X + \frac{3}{D^{5/4}}\eta_0\eta_{0\xi} + \frac{1}{3}D^{3/4}\eta_{0\xi\xi\xi} = 0,$$

where the first term embodies Green's law (as described in equation (2.47) *et seq.* and in Q2.34). Indeed, it is convenient to introduce

$$H_0(\xi, X) = D^{1/4}\eta_0$$

so that we obtain

$$2H_{0X} + \frac{3}{D^{7/4}}H_0H_{0\xi} + \frac{1}{3}D^{1/2}H_{0\xi\xi\xi} = 0. \qquad (3.149)$$

Although this equation can be solved in any complete sense only numerically, we can make some important observations about the nature of its solutions.

For a wave profile that tends to zero both ahead of and behind the wavefront, we see that the integral in ξ of equation (3.149) yields

$$2\frac{\mathrm{d}}{\mathrm{d}X}\int_{-\infty}^{\infty} H_0 \mathrm{d}\xi + \left[\frac{3}{2}D^{-7/4}H_0^2 + \frac{1}{3}D^{1/2}H_{0\xi\xi}\right]_{-\infty}^{\infty} = 0,$$

so

$$\int_{-\infty}^{\infty} H_0(\xi, X)\mathrm{d}\xi = \text{constant},$$

which is equivalent to the conservation of mass. However, this does not describe the correct mass conservation for the water-wave problem. To see this, consider

$$\int_{-\infty}^{\infty} H_0 \mathrm{d}\xi = D^{1/4}(X)\int_{-\infty}^{\infty} \eta_0(\xi, X)\,\mathrm{d}\xi = \text{constant};$$

let us suppose that a wave is moving in a region of constant depth ($D = 1$) and is carrying a total mass of m_0; then

$$D^{1/4}\int_{-\infty}^{\infty} \eta_0(\xi, X)\,\mathrm{d}\xi = m_0.$$

But the mass carried by the wave is always

$$\int_{-\infty}^{\infty} \eta_0(\xi, X)\,\mathrm{d}\xi$$

and this is clearly *not* conserved as D varies, since

$$\int_{-\infty}^{\infty} \eta_0(\xi, X)\,\mathrm{d}\xi = m_0 D^{-1/4}(X).$$

The difficulty has arisen because the mass conservation applies to the complete water-wave problem, and not necessarily to a single element of the solution taken in isolation – here the solution of our KdV equation. Indeed, it is this inconsistency which has led to much detailed study of this problem (particularly in the cases of faster and slower depth variations, where considerable headway can be made). The important observation (see Miles, 1979; Knickerbocker and Newell 1980, 1985) is that other wave components of smaller amplitude, but which carry O(1) mass, are required to complete the description. In particular it has been found that a *left-going* wave (that is, a *reflected* wave) is necessary, and that this supplies the major correction to the overall mass conservation. (Other conservation laws for equation (3.149) are discussed in Q3.54.)

In conclusion, we briefly describe some properties of the wave component that is represented by the solution of the KdV equation, in the two extreme cases where

$$D(X) = \hat{D}(\sigma X), \quad \sigma \to 0 \quad \text{or} \quad \sigma \to \infty.$$

Of course, a more complete discussion of these problems – and indeed for the case of $\sigma = O(1)$ – requires a study of the other wave components, as we have just outlined, but this is beyond the scope of the presentation here. Nevertheless, the resulting wave evolution does give the correct picture to leading order in amplitude (even though the mass carried by the waves is incorrect to leading order).

First, in the case of $\sigma \to 0$, where the depth variation occurs on a scale that is slower than the evolution scale (X) of the wave, the variable coefficients in equation (3.149) are treated as independent of X. (This approach can be formalised by introducing an appropriate multiple-scale representation:

$$H_0 = H_0(\xi, X, \hat{X}), \quad \hat{X} = \sigma X, \quad \sigma \to 0.)$$

For example, the solitary-wave solution of this equation can be expressed, for η_0, as

$$\eta_0 = D^{-1/4} H_0 = \frac{A_0}{D} \operatorname{sech}^2 \left\{ \sqrt{\frac{3A_0}{4D^3}} \left(\xi - \frac{1}{2} D^{-5/2} A_0 X \right) \right\}, \tag{3.150}$$

where A_0 is the amplitude of the wave on the constant depth $D = 1$. We have chosen to write the solution in this form in order to ensure that the conservation law in H_0^2, for equation (3.149), is satisfied; see Q3.54 and Q3.55. An example of the evolution of the solitary wave, according to (3.150), is shown in Figure 3.11.

The second case that we describe is where the depth variation is fast ($\sigma \to \infty$) compared with the evolution of the wave. In this situation, the depth varies rapidly – instantaneously in the limit $\sigma \to \infty$ – so a wave moving on one depth must instantaneously begin to evolve as it adjusts to a new depth. As before, let us consider the example of a solitary wave, of

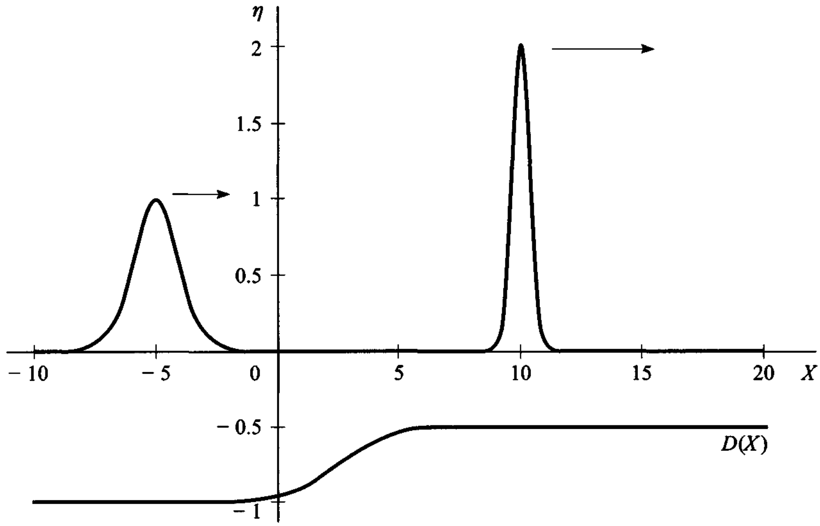


Figure 3.11. A representation of the distortion of a solitary wave (of amplitude 1) as it moves over a *slow* depth variation, from depth 1 to depth 0.5.

amplitude A_0, which is propagating in a region of constant depth, $D = 1$. Then, directly from equation (3.150), we have that

$$H_0 = A_0 \operatorname{sech}^2\left\{\frac{1}{2}\sqrt{3A_0}(\xi - \frac{1}{2}A_0 X)\right\}. \tag{3.151}$$

Now suppose that the depth changes suddenly from $D = 1$ (in $X < 0$, say) to $D = D_0$ (in $X > 0$); the profile (3.151) will move into $X > 0$ but cannot immediately adjust to the new depth. Thus this profile becomes an initial condition for the KdV equation, (3.149), evaluated for $D = D_0$:

$$2H_{0X} + 3D_0^{-7/4} H_0 H_{0\xi} + \frac{1}{3} D_0^{1/2} H_{0\xi\xi\xi} = 0.$$

We compare this version of the (constant coefficient) KdV equation with the standard form (see (3.49)):

$$u_t - 6uu_x + u_{xxx} = 0,$$

which possesses an N-soliton solution if

$$u(x, 0) = -N(N+1)\operatorname{sech}^2 x;$$

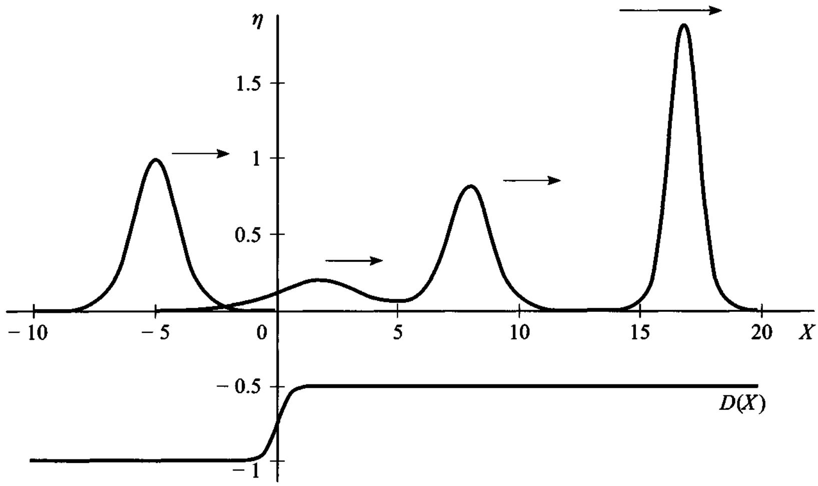


Figure 3.12. A representation of the distortion of a solitary wave (of amplitude 1) as it moves over a *fast* depth variation, from depth 1 to 0.451 (which corresponds to $N = 3$, the 3-soliton solution).

see equation (3.60) *et seq*. Thus we transform according to

$$\hat{\xi} = \frac{1}{2}\sqrt{3A_0}\xi, \quad \hat{X} = \frac{1}{6}D_0^{1/2}\left(\frac{1}{2}\sqrt{3A_0}\right)^3 X, \quad \hat{H}_0 = -\frac{2}{A_0}D_0^{-9/4}H_0$$

which gives

$$\hat{H}_{0\hat{X}} - 6\hat{H}_0\hat{H}_{0\hat{\xi}} + \hat{H}_{0\hat{\xi}\hat{\xi}\hat{\xi}} = 0;$$

an N-soliton solution is possible if

$$\hat{H}_0(\hat{\xi}, 0) = -N(N+1)\text{sech}^2\hat{\xi}.$$

But

$$\hat{H}_0(\hat{\xi}, 0) = -\frac{2}{A_0}D_0^{-9/4}\hat{H}_0(\hat{\xi}, 0) = -2D_0^{-9/4}\text{sech}^2\hat{\xi},$$

and so the solitary wave on $D = 1$ will evolve into N solitons on $D = D_0$ if

$$D_0 = \left\{\frac{1}{2}N(N+1)\right\}^{-4/9},$$

a result obtained and described in Tappert & Zabusky (1971) and Johnson (1973). We see immediately that solitons can appear only if the depth *decreases*, because $N = 2, 3, \ldots$ for two or more solitons. (If the depth increases, then the wave collapses into a nonlinear oscillatory wave; see Johnson (1973).) An example of 3-soliton production ($N = 3$, $D_0 \approx 0.451$) is shown in Figure 3.12, where $\eta_0 = D^{-1/4} H_0$ is reproduced.

3.4.5 Oblique interaction of waves

We have already met the two-dimensional KdV equation (Section 3.2.2), which admits solutions that represent obliquely crossing waves. In that analysis we were guided by the requirement to find the scaling that led to a KdV-type equation. Here, we address the problem of obliquely crossing waves (of small amplitude) directly from the governing equations, without the restriction to producing a KdV-type of balance. This approach will provide deeper insight into how such waves interact and, indeed, also provide a different interpretation of the rôle of the 2D KdV equation.

We shall consider the propagation of a plane wave – perhaps a solitary wave – moving in an arbitrary direction across the surface of stationary water of constant depth. However, the surface contains another plane wave which is also propagating in an arbitrary direction. Thus, as far as the first wave is concerned, the environment is no longer uniform. (This section will therefore complete a discussion of various non-uniform environments, namely: (a) an underlying shear flow; (b) variable depth; (c) a disturbed surface.)

The discussion of this problem that we shall present will follow closely the seminal work of Miles (1977a), although we shall cast it in a form that is consistent with much of our earlier work. This, it turns out, is an occasion when the most convenient approach is to take full advantage of the irrotationality of the flow, and so we shall formulate the problem in terms of Laplace's equation and the pressure equation; see Q1.38 and equations (2.132). We replace δ^2 by ε (as before) and set $b = 0$, so we have

$$\phi_{zz} + \varepsilon(\phi_{xx} + \phi_{yy}) = 0 \tag{3.152}$$

$$\left.\begin{aligned} &\phi_z = \varepsilon\{\eta_t + \varepsilon(\phi_x\eta_x + \phi_y\eta_y)\} \\ &\eta + \phi_t + \frac{1}{2}\phi_z^2 + \frac{1}{2}\varepsilon(\phi_x^2 + \phi_y^2) = 0 \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta \tag{3.153}$$

and

$$\phi_z = 0 \quad \text{on} \quad z = 0. \tag{3.154}$$

(The way in which ϕ_z appears in these equations, and particularly the term ϕ_z^2, need cause no alarm since we shall find that, although $\phi = \mathrm{O}(1)$, $\phi_z = \mathrm{O}(\varepsilon)$.)

The Laplace equation, (3.152), can be solved in the form of an asymptotic expansion in ε which satisfies the bottom boundary condition, (3.154). To see how this proceeds, let us first write

$$\phi \sim \sum_{n=0}^{\infty} \varepsilon^n \phi_n$$

and then

$$\phi_{0zz} = 0; \quad \phi_{1zz} = -(\phi_{0xx} + \phi_{0yy}), \quad \text{etc.},$$

and so

$$\phi_0 = f_0(x, y, t); \quad \phi_1 = -\frac{1}{2} z^2 (f_{0xx} + f_{0yy}) + f_1(x, y, t), \quad \text{etc.},$$

each satisfying $\phi_{nz} = 0$ on $z = 0$; the structure of this expansion is also evident from our work in Section 2.9.1. However, in this analysis we do not wish to be specific, at this early stage, about how f (that is, $f_0, f_1, \ldots$) is related to ε (and so how η relates to ε). It is clear that we may write the solution of Laplace's equation, and satisfy the bottom boundary condition, by writing

$$\phi \sim \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} (-\varepsilon \nabla_\perp^2)^n f, \quad \nabla_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

where $f = f(x, y, t; \varepsilon)$ is arbitrary. Of course, the complete asymptotic structure of ϕ, for $\varepsilon \to 0$, will be determined once we have settled on the form of f. The two equations that are required in order to define f and η are obtained by substituting for ϕ into the two surface boundary conditions, (3.153), with

$$\phi_z \sim \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!} (-\varepsilon \nabla_\perp^2)^n f \quad (= \mathrm{O}(\varepsilon))$$

since the first term in ϕ is absent in ϕ_z. We must evaluate on $z = 1 + \varepsilon\eta$ and so, for example, ϕ_z becomes

$$\phi_z \sim \sum_{n=1}^{\infty} \frac{(1+\varepsilon\eta)^{2n-1}}{(2n-1)!}(-\varepsilon\nabla_\perp^2)^n f;$$

we shall retain terms in both boundary conditions that will allow us to find both the leading order and O(ε) contributions. Thus equations (3.153) yield

$$-(1+\varepsilon\eta)\nabla_\perp^2 f + \frac{\varepsilon}{3!}\nabla_\perp^4 f \sim \eta_t + \varepsilon(f_x\eta_x + f_y\eta_y)$$

and

$$\eta + f_t - \frac{\varepsilon}{2!}\nabla_\perp^2 f_t + \frac{\varepsilon}{2}(f_x^2 + f_y^2) = \mathrm{O}(\varepsilon^2) \tag{3.155}$$

from which a single equation for f may be obtained:

$$-(1-\varepsilon f_t)\nabla_\perp^2 f + \frac{\varepsilon}{6}\nabla_\perp^4 f + f_{tt} - \frac{\varepsilon}{2}\nabla_\perp^2 f_{tt} + 2\varepsilon(f_x f_{xt} + f_y f_{yt}) = \mathrm{O}(\varepsilon^2). \tag{3.156}$$

Equation (3.156) enables $f(x, y, t; \varepsilon)$ to be determined, and then equation (3.155) gives $\eta(x, y, t; \varepsilon)$ directly. It is left as an exercise (Q3.56) to show that a single plane wave of the form

$$f = F(\xi, \tau; \varepsilon), \quad \eta = H(\xi, \tau; \varepsilon), \quad \xi = kx + ly - t, \quad \tau = \varepsilon t, \tag{3.157}$$

where $k^2 + l^2 = 1$ is the dispersion relation, recovers the KdV equation (cf. (3.28))

$$2H_\tau + 3HH_\xi + \frac{1}{3}H_{\xi\xi\xi} = 0, \tag{3.158}$$

to leading order. This wave propagates in the direction of the wave-number vector (k, l); indeed, it is convenient to write

$$k = \cos\theta, \ l = \sin\theta, \tag{3.159}$$

and then the wavefront moves in the direction that makes an angle θ with the positive x-axis.

The problem that we wish to address is the situation where a (nonlinear) plane wave, a solution of the KdV equation (3.158), moves on a surface (in an arbitrary direction) which contains another plane wave. Our first approach is to seek a solution which comprises two waves, each satisfying a KdV equation, together with an interaction between them that is *weak* (that is, O(ε)); see Q1.49. To this end we introduce

$$\zeta = px + qy - t; \quad p = \cos\psi, \quad q = \sin\psi, \tag{3.160}$$

and then write

$$f = F(\xi, \tau) + G(\zeta, \tau) + \varepsilon I(\xi, \zeta, \tau) + O(\varepsilon^2), \tag{3.161}$$

where I represents the interaction of the two waves. Direct substitution into equation (3.156) then yields

$$\begin{aligned}
&-\{1+\varepsilon(F_\xi + G_\zeta)\}\{F_{\xi\xi} + G_{\zeta\zeta} + \varepsilon(I_{\xi\xi} + I_{\zeta\zeta}) + 2\varepsilon(kp+lq)I_{\xi\zeta}\} \\
&\quad + \frac{\varepsilon}{6}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) + F_{\xi\xi} + G_{\zeta\zeta} - 2\varepsilon(F_{\xi\tau} + G_{\zeta\tau}) \\
&\quad + \varepsilon(I_{\xi\xi} + 2I_{\xi\zeta} + I_{\zeta\zeta}) - \frac{\varepsilon}{2}(F_{\xi\xi\xi\xi} + G_{\zeta\zeta\zeta\zeta}) \\
&\quad - 2\varepsilon\{(kF_\xi + pG_\zeta)(kF_{\xi\xi} + pG_{\zeta\zeta}) + (lF_\xi + qG_\zeta)(lF_{\xi\xi} + qG_{\zeta\zeta})\} = O(\varepsilon^2),
\end{aligned}$$

where we have used $k^2 + l^2 = 1$ and $p^2 + q^2 = 1$. But F_ξ and G_ζ satisfy appropriate KdV equations (see Q3.56); that is

$$2F_{\xi\tau} + 3F_\xi F_{\xi\xi} + \frac{1}{3}F_{\xi\xi\xi\xi} = 0; \quad 2G_{\zeta\tau} + 3G_\zeta G_{\zeta\zeta} + \frac{1}{3}G_{\zeta\zeta\zeta\zeta} = 0,$$

so we obtain

$$2\{1-(kp+lq)\}I_{\xi\zeta} - \{1+2(kp+lq)\}(F_\xi G_{\zeta\zeta} + F_{\xi\xi}G_\zeta) = O(\varepsilon) \tag{3.162}$$

which is the equation for I, at this order of approximation.

The coefficients of equation (3.162) are conveniently written in terms of

$$\begin{aligned}
kp + lq &= \cos\theta\cos\psi + \sin\theta\sin\psi \\
&= \cos(\theta - \psi) = 1 - 2\sin^2\{(\theta-\psi)/2\}
\end{aligned}$$

and we set $\lambda = \sin^2\{(\theta-\psi)/2\}$ so that

$$kp + lq = 1 - 2\lambda.$$

Further, on noting that $F_\zeta = G_\xi = 0$, we see that equation (3.162) reduces to

$$4\lambda I_{\xi\zeta} - (3-4\lambda)\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\zeta}\right)F_\xi G_\zeta = O(\varepsilon) \tag{3.163}$$

which may be integrated directly and so, to leading order, we obtain

$$I = \left(\frac{3}{4}\lambda^{-1} - 1\right)(F_\xi G + FG_\zeta) \tag{3.164}$$

where we assume that $I = 0$ if either $F = 0$ or $G = 0$. The solution for f is therefore

$$f = F(\xi, \tau) + G(\zeta, \tau) + \varepsilon\left(\frac{3}{4}\lambda^{-1} - 1\right)(F_\xi G + FG_\zeta) + O(\varepsilon^2)$$

which can be written in the more compact form

$$f = F(\xi + \varepsilon\mu G, \tau) + G(\zeta + \varepsilon\mu F, \tau) + O(\varepsilon^2), \tag{3.165}$$

where $\mu = (3/4\lambda - 1)$; cf. Q1.53.

The surface wave, η, is obtained from equation (3.155) as

$$\eta = F_\xi + \varepsilon\left(\frac{1}{4}F_\xi^2 - \frac{1}{3}F_{\xi\xi\xi}\right) + G_\zeta + \varepsilon\left(\frac{1}{4}G_\zeta^2 - \frac{1}{3}G_{\zeta\zeta\zeta}\right)$$
$$+ \varepsilon\left(\frac{3}{2\lambda} - 3 + 2\lambda\right)F_\xi G_\zeta + O(\varepsilon^2), \tag{3.166}$$

where F_τ and G_τ have been eliminated by using the KdV equations for F_ξ and G_ζ (and by invoking decay conditions at $+\infty$); the derivation of (3.166) is left as an exercise (Q3.57). An example of the surface profile described by (3.166), for two solitary waves, is shown in Figure 3.13 (where we have taken $\varepsilon = 0.2$ to make clear the nature of the interaction). This figure also includes, for comparison, the solution for the same pair of plane waves when the interaction is absent; that is, $\varepsilon = 0$.

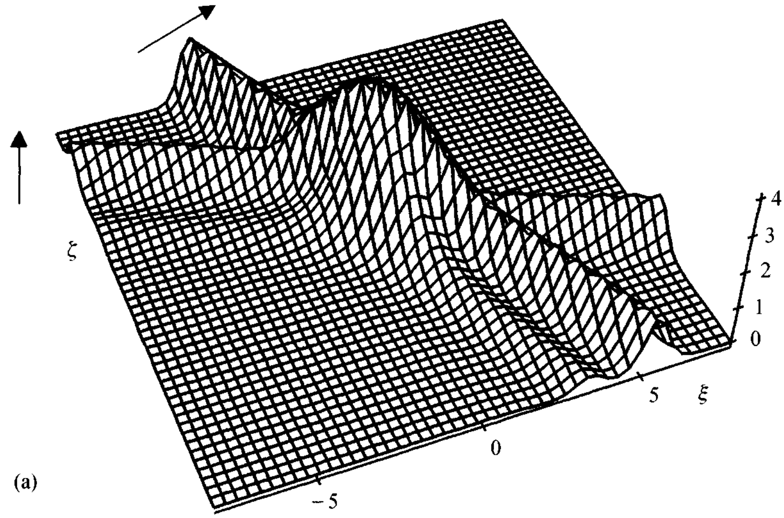
The solution that we have described so far assumes that the interaction between the waves is weak as $\varepsilon \to 0$ or, equivalently, $I = O(1)$. However, it is clear from equation (3.163), or from the solution (3.164), that I grows without bound as $\lambda \to 0$ (that is, as $\theta \to \psi$), so that the two waves approach the parallel orientation. This important observation was made by Miles (1977a), who proceeded to examine what happens as λ decreases; we shall follow a similar path here. Now, since the interaction term is εI, and $I = O(\lambda^{-1})$ as $\lambda \to O$, our weak interaction theory is not uniformly valid when $\lambda = O(\varepsilon)$; this has become a *strong* interaction. Let us write $\psi = \theta + \alpha$, with $\alpha \to 0$; then

$$\lambda = \sin^2\{(\theta - \psi)/2\} = \sin^2(\alpha/2) = O(\alpha^2),$$

so the wave number (p, q) becomes

$$(p, q) = (\cos\psi, \sin\psi) = (\cos\theta, \sin\theta) + \alpha(-\sin\theta, \cos\theta) + O(\alpha^2)$$
$$= (k, l) + \alpha(-l, k) + O(\alpha^2).$$

Thus, as we must expect, the wave number (p, q) is nearly parallel to the wave number (k, l); consequently we must use coordinates based on the



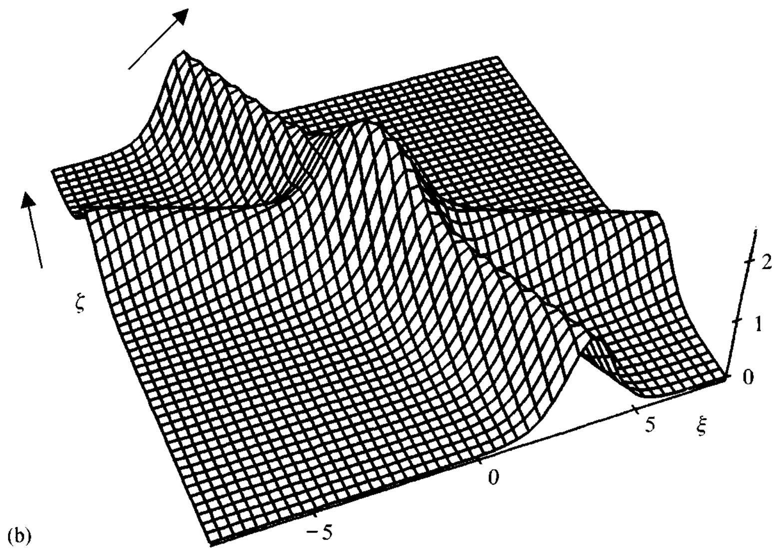


Figure 3.13. The oblique interaction of two solitary waves, for the case of a *weak* interaction, for amplitudes 1 and 1.5, with $\theta = 3\pi/8$ and $\psi = \pi/8$; (a) $\varepsilon = 0.2$, (b) $\varepsilon = 0$, no interaction.

wave number (k, l), and on $(-l, k)$ – this latter suitably scaled. Indeed, since $\lambda = \mathrm{O}(\alpha^2)$ and the non-uniformity arises for $\lambda = \mathrm{O}(\varepsilon)$, the relevant scaling is $\sqrt{\varepsilon}$; further, since $(-l, k)$ is perpendicular to (k, l), the configuration that we are led to is equivalent to that employed in the derivation of the 2D KdV equation (Section 3.2.2).

We introduce

$$\xi = kx + ly - t, \quad \zeta = \sqrt{\varepsilon}(-lk + ky), \quad \tau = \varepsilon t;$$

cf. equations (3.157) and (3.160), where ζ is now a (scaled) coordinate perpendicular to the characteristic coordinate, ξ. The equation for $f = f(\xi, \zeta, \tau; \varepsilon)$, obtained from (3.156), is therefore

$$-(1 + \varepsilon f_\xi)(f_{\xi\xi} + \varepsilon f_{\zeta\zeta}) + \frac{\varepsilon}{6} f_{\xi\xi\xi\xi} + f_{\xi\xi} - 2\varepsilon f_{\xi\tau} - \frac{\varepsilon}{2} f_{\xi\xi\xi\xi} - 2\varepsilon f_\xi f_{\xi\xi} = \mathrm{O}(\varepsilon^2),$$

which simplifies to give

$$2f_{\xi\tau} + 3f_\xi f_{\xi\xi} + \frac{1}{3} f_{\xi\xi\xi\xi} + f_{\zeta\zeta} = \mathrm{O}(\varepsilon).$$

And so, finally, with $\eta \sim -f_t \sim f_\xi$ (see equation (3.155)), we obtain for the surface wave

$$\left(2\eta_\tau + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi}\right)_\xi + \eta_{\zeta\zeta} = 0,$$

to leading order: the 2D Korteweg–de Vries equation (Section 3.2.2, equation (3.30)). All that we have written about this equation is now applicable here.

In our earlier discussion of the 2D KdV equation (Section 3.2.2), we were guided by the requirement to obtain a KdV-type equation which incorporated some (weak) dependence on a transverse coordinate. This was a purely 'theoretical' exercise, whose success rested on a very special and precise choice for the way in which y (here, ζ) appears in the equation. What we have now demonstrated is that the 2D KdV equation arises quite naturally as the appropriate equation for the strong interaction of obliquely crossing waves. The interaction becomes more pronounced (eventually leading to a strongly nonlinear interaction) as the wave configuration is more nearly that of parallel waves. We can interpret this situation as one in which the waves interact over a much larger distance, thereby producing a greater effect – distortion – one upon the other.

This concludes all that we shall present in relation to the Korteweg–de Vries equation and other members of the family. We have demonstrated how the equations can arise in many different situations and, in particular, how they are relevant in configurations that model physical phenomena more closely (such as variable depth and shear flows). There is much that we have not included, not least the extensive breadth and depth of ideas that now constitute soliton theory. The exercises to some extent, and the Further reading more so, enable the interested reader to take many of these ideas much further.

Further reading

There are many texts now available that describe either general or very specific aspects of soliton theory; some of these texts are listed below. The derivation and properties of the various members of the KdV family of equations that arise in water-wave theory are described mainly in research papers; we provide a small selection for the interested reader.

3.2 Two texts that cover some of the ground that we have described here are Infeld & Rowlands (1990) and Debnath (1994). Various derivations and discussions of these equations will be found in Korteweg & de Vries (1895), Kadomtsev & Petviashvili (1970), Miles (1978, 1981) and Johnson (1980).

3.3 A few of the texts that provide most of the essential features of soliton theory are Lamb (1980), Ablowitz & Segur (1981), Dodd *et al.* (1982), Drazin & Johnson (1994) and Ablowitz & Clarkson (1991). The texts by Lamb and Drazin & Johnson, in particular, present an elementary introduction to many of the ideas. More advanced texts, generally touching on deeper issues, are Calogero & Degasperis (1982) and Newell (1985). An excellent introduction to the ideas, coupled with a description of some simple experiments, is provided by Remoissenet (1994). In addition there are publications that describe specific topics in soliton theory: Rogers & Shadwick (1982) for Bäcklund transformations; Matsuno (1984) for the bilinear transform; and Schuur (1986) for the asymptotic structure of soliton solutions. Finally, the broader and deeper concepts that relate soliton theory to Hamiltonian methods are described by Faddeev & Takhtajan (1987) and Dickey (1991).

3.4 Nonlinear waves propagating over shear flows are described by Benjamin (1962) and Freeman & Johnson (1970); the Burns condition is discussed by Thompson (1949), Burns (1953), Velthuizen & van Wijngaarden (1969), Yih (1972), Brotherton-Ratcliffe & Smith (1989) and Johnson (1991). The nature of the critical layer, and particularly the rôle of nonlinearity, is described in Benney & Bergeron (1969), Davis (1969), Haberman (1972) and Varley & Blythe (1983); and the connection between nonlinear wave propagation and the critical layer is examined by Redekopp (1977), Maslowe & Redekopp (1980) and Johnson (1986). A theory for linear and nonlinear ring waves over a shear flow is presented in Johnson (1990).

The problems of waves moving over a variable depth have a long history, starting with Green (1837) and Boussinesq (1871). Some of the more recent papers, with the emphasis on nonlinear wave propagation, are Peregrine (1967), Grimshaw (1970, 1971), Kakutani (1971), Tappert & Zabusky (1971), Johnson (1973, 1994), Leibovich & Randall (1973), Miles (1979) and, most importantly, Knickerbocker & Newell (1980, 1985). The oblique interaction of nonlinear plane waves is described in Miles (1977a,b); the case of a large solitary wave interacting obliquely with a sech^2 wave is discussed in Johnson (1982); see also Tanaka (1993).

Exercises

Q3.1 *Standard KdV equation.* Show that a scaling transformation, $u \to \alpha u$, $x \to \beta x$, $t \to \gamma t$, for non-zero real constants α, β, γ, enables the general KdV equation

$$Au_t + Buu_x + Cu_{xxx} = 0$$

(for real, non-zero, constants A, B, C) to be transformed into

$$u_t - 6uu_x + u_{xxx} = 0.$$

Q3.2 *KdV for left-going waves.* Repeat the calculation given in Section 3.2.1, leading to the KdV equation (3.28), for waves that propagate to the left (cf. Q1.47 and Q1.48).

Q3.3 *KdV with surface tension.* Repeat the calculation given in Section 3.2.1, but retain the surface tension contribution (characterised by the parameter W (or W_e); see equation (1.64)) and derive the corresponding KdV equation. Show that the inclusion of surface tension alters only the coefficient of the third derivative term, that is, the dispersive contribution.

Q3.4 *Higher-order correction to the KdV equation.* Continue the calculation described in Section 3.2.1 to find the equation that defines $\eta_1(\xi, \tau)$. In the case of the travelling-wave solution, where both η_0

and η_1 are functions only of $\xi - c\tau$ (cf. Q1.55), obtain an expression for η_1 in terms of η_0 by seeking a solution $\eta_1 = \eta_0' F(\xi - c\tau)$ (where the prime denotes the derivative with respect to $\xi - c\tau$).

Q3.5 *KdV similarity solution.* Show that the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

possesses a similarity solution of the form $u(x, t) = -(3t)^m F(\eta)$, $\eta = x(3t)^n$, for suitable values of the constants m and n. (The inclusion of the factor 3, and the use of the negative sign, are merely for convenience.) Hence obtain the equation for F:

$$F''' + (6F - \eta)F' - 2F = 0$$

and, by writing $F = \lambda V' - V^2$ with $V = V(\eta)$ and where λ is a constant to be determined, show that

$$V'' - \eta V - 2V^3 = 0$$

after two integrations, provided that V decays sufficiently rapidly as either $\eta \to +\infty$ or $\eta \to -\infty$.

[The equation for $V(\eta)$ is a *Painlevé equation* of the second kind; see Ince (1927). The use of soliton methods enables the Painlevé equations to be solved; see Ablowitz & Clarkson (1991), Drazin & Johnson (1993) and Airault (1979) as an introduction to these ideas.]

Q3.6 *KdV rational solution.* Show that

$$u(x, t) = 6x(x^3 - 24t)/(x^3 + 12t)^2$$

is a solution of the KdV equation given in Q3.5.

[This solution is not particularly useful since it is *singular* on $x^3 + 12t = 0$, although some other soliton equations do have rational solutions that exist everywhere.]

Q3.7 *A cKdV equation.* Follow the calculation described in Section 3.2.3 for the concentric KdV equation, but now use a large time variable $\tau = \varepsilon^6 t/\delta^4$; cf. equation (3.32). Hence obtain the appropriate cKdV equation.

Q3.8 *Solitary-wave solution of the Boussinesq equation.* Obtain the solitary-wave solution of the Boussinesq equation

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$$

in the form $u(x, t) = a\,\mathrm{sech}^2\{b(x - ct) + \alpha\}$ for suitable relations between the constants a, b, c and α. Show that the wave may propagate in either direction.

Q3.9 *Boussinesq → KdV.* By means of a suitable choice of far-field variables, recover the KdV equation for right-going waves from the Boussinesq equation, (3.41). Repeat this calculation for left-going waves; cf. equation (3.28) and Q3.2.

Q3.10 *Boussinesq → standard Boussinesq.* Use the transformation

$$H = \eta - \varepsilon\eta^2, \quad X = x + \varepsilon \int_{-\infty}^{x} \eta(x', t; \varepsilon)\mathrm{d}x'$$

and thereby obtain equation (3.42) from equation (3.41).

[This transformation is equivalent to writing the equation in a Lagrangian rather than an Eulerian frame.]

Q3.11 *Solitary-wave solution of the 2D KdV equation I.* Show that the 2D KdV equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

has the solitary-wave solution $u(x, t) = a\,\mathrm{sech}^2(kx + ly - \omega t + \alpha)$ for suitable relations between the constants a, k, l, ω and α.

Q3.12 *Transformations between ncKdV and 2D KdV equations.* Show that the nearly concentric KdV equation, (3.46), transforms into the 2D KdV equation if we write

$$H = \eta(\zeta, R, Y), \quad \zeta = \xi - \frac{1}{2}R\Theta^2, \quad Y = R\Theta.$$

Conversely, show that the choice

$$\eta = H(\zeta, \tau, \Theta), \quad \zeta = \xi + \frac{1}{2}Y^2/\tau, \quad \Theta = Y/\tau$$

transforms the 2D KdV equation, (3.20), into the ncKdV equation.

Q3.13 *Solution of the ncKdV equation.* Given that $v(x, t, y)$ is a solution of the 2D KdV equation

$$(v_t - 6vv_x + v_{xxx})_x + 3v_{yy} = 0,$$

show that $u(x, t, y) = v(x - y^2t/12, t, yt)$ is a soluton of the ncKdV equation

$$\left(u_t + \frac{u}{2t} - 6uu_x + u_{xxx}\right)_x + \frac{3}{t^2}u_{yy} = 0.$$

[This enables solutions for u, which decay sufficiently rapidly as $(x^2+y^2)^{-1} \to 0$, to be obtained from the solutions for v which satisfy this same condition. However, solutions for u which have a different behaviour at infinity cannot be obtained via this transformation; see Dryuma (1983), Matveev & Salle (1991).]

Q3.14 *Phase shifts for a 2-soliton KdV solution.* The phase shifts exhibited by the soliton behaviour in solution (3.59) can be examined in this fashion: we consider the asymptotic form of the solitary waves that appear as $t \to \pm\infty$. First, for $\xi = x - 16t = \mathrm{O}(1)$ as $t \to \pm\infty$, show that

$$u \sim -8\operatorname{sech}^2(2\xi \mp \frac{1}{2}\log 3),$$

and then for $\eta = x - 4t = \mathrm{O}(1)$ as $t \to \pm\infty$, show that

$$u \sim -2\operatorname{sech}^2(\eta \pm \frac{1}{2}\log 3);$$

all signs are vertically ordered. Hence deduce that the taller wave moves forward by an amount $x = \frac{1}{2}\log 3$, and the shorter back by $x = \log 3$, relative to where they would have been if moving throughout at constant speed.

Q3.15 *Phase-shifts: general.* Recast the calculation of Q3.14 in order to find the phase shifts for the general 2-soliton solution, (3.58).

Q3.16 *Character of the 2-soliton KdV solution.* Show that a special case of the 2-soliton solution, (3.58), takes the form of a sech^2 pulse at $t = 0$. Further, show that the pulse at $t = 0$ may have either one or two local maxima.

[In this calculation you should first define x so that a symmetric profile occurs at $t = 0$; for further details see Lax (1968).]

Q3.17 *Three-soliton solution of the KdV equation I.* Extend the calculation in Section 3.3.1 (Example 2) to obtain the general 3-soliton solution of the KdV equation. Show that this can be written in the form $u(x, t) = -2(\partial^2/\partial x^2)\log A$, where

$$A = 1 + \sum_{i=1}^{3} E_i + \sum_{<i=1>}^{3} A_{ij}E_iE_j + \prod_{<i=1>}^{3} A_{ij}E_i$$

with $E_i = \exp\{2k_i(x - x_{0i}) - 8k_i^3 t\}$, $A_{ij} = (k_i - k_j)^2/(k_i + k_j)^2$, and $< >$ denotes that j is to be chosen cyclically with respect to i.

Q3.18 *Solitary-wave solution of the 2D KdV equation II.* Use the choice

$$F = \exp\{-(kx + lz) + (k^2 - l^2)y + 4(k^3 + l^3)t + \alpha\}$$

in the Marchenko equation, and hence recover the solitary-wave solution obtained in Q3.11; see Section 3.3.2. (Direct correspondence with Q3.11 requires here that $k + l \to -2k$, $k^2 - l^2 \to 2l$, $k^3 + l^3 = -\omega/2$.)

Q3.19 *Two-soliton solution of the 2D KdV equation I.* See Q3.18; now write F as the sum of two appropriate exponentials and hence derive the two-soliton solution in the form

$$u(x, t) = -2\frac{\partial^2}{\partial x^2}\log(1 + E_1 + E_2 + AE_1E_2)$$

where $E_i = \exp\{-(k_i + l_i)x + (k_i^2 - l_i^2)y + 4(k_i^3 + l_i^3)t + \alpha_i\}$ and

$$A = \frac{(k_1 - k_2)(l_1 - l_2)}{(k_1 + k_2)(l_1 + l_2)}.$$

[This solution describes various configurations of two plane waves that intersect obliquely and suffer a nonlinear interaction; an excellent discussion of these solutions is to be found in Freeman (1980).]

Q3.20 *A 2D Boussinesq equation.* Follow the derivation of the Boussinesq equation (Section 3.2.5), but include the weak y-dependence as required for the two-dimensional KdV equation (Section 3.2.2). Hence show that, correct at $O(\varepsilon)$, the surface wave satisfies the equation

$$\eta_{tt} - \eta_{xx} - \varepsilon\left\{\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t \mathrm{d}x\right)^2\right\}_{xx} - \frac{\varepsilon}{3}\eta_{xxxx} + \varepsilon V_{Yt} = O(\varepsilon^2),$$

where $V_t = -\eta_Y$. Finally, transform and rescale (exactly as in Section 3.2.5) to obtain the 2D Boussinesq equation

$$H_{tt} - H_{XX} + 3(H^2)_{XX} - H_{XXXX} - H_{YY} = 0.$$

Q3.21 *Solitary-wave solution of the 2D Boussinesq equation.* Seek a solution of the equation for $H(X, t, Y)$ given in Q3.20 in the form $H = a\,\mathrm{sech}^2\{kX + lY - \omega t + \alpha\}$ for suitable relations between the constants a, k, l, ω and α. Confirm that your solution is an oblique wave that may propagate in one of two directions.

[This 2D Boussinesq equation is not a completely integrable equation, although it still provides a description of the *head-on* collision of *oblique* waves (cf. Q3.19) and it does possess some interesting properties; see Johnson (1996).]

Q3.22 *Solitary-wave solution of the cKdV equation.* Show that a solution for F of the pair of equations (3.65) is

$$F(x, z; t) = \int_{-\infty}^{\infty} f(st^{1/3})Ai(x+s)Ai(s+z)\mathrm{d}s,$$

where f is an arbitrary function and Ai is the Airy function. The solitary-wave solution is usually regarded as that solution obtained from the choice $f(\cdot) = k\delta(\cdot)$, where δ is the Dirac delta function and k is a positive constant; construct the solitary-wave solution of the cKdV equation, (3.64).

Q3.23 *A similarity solution of the cKdV equation.* Show that

$$u(x, t) = -\frac{x}{12t} - \frac{2\lambda^2}{t}\operatorname{sech}^2\{\lambda(x+8\lambda^2)/t^{1/2}\}, \quad t > 0,$$

is a solution of the concentric KdV equation, (3.64), for any real constant λ.

[This solution is undefined on $t = 0$, is not real for $t < 0$ and grows without bound as $|x| \to \infty$ at any fixed t.]

Q3.24 *Bilinear operator.* Prove these identities, where $\mathrm{D}_t^m\mathrm{D}_x^n(a \cdot b)$ is the bilinear operator defined in equation (3.71):

(a) $\mathrm{D}_t^m\mathrm{D}_x^n(a \cdot b) = \mathrm{D}_x^n\mathrm{D}_t^m(a \cdot b)$;

(b) $\mathrm{D}_x^n(a \cdot b) = (-1)^n\mathrm{D}_x^n(b \cdot a)$ and hence that $\mathrm{D}_x^n(a \cdot a) = 0$ for n odd;

(c) $\mathrm{D}_t^m\mathrm{D}_x^n(a \cdot 1) = \mathrm{D}_t^m\mathrm{D}_x^n(1 \cdot a) = \partial^{m+n}a/\partial x^n\partial t^m$ for $m+n$ even;

(d) $\mathrm{D}_t^m\mathrm{D}_x^n\{\exp(\theta_1) \cdot \exp(\theta_2)\} = (\omega_2 - \omega_1)^m(k_1 - k_2)^n \exp(\theta_1 + \theta_2)$ where $\theta_i = k_i x - \omega_i t + \alpha_i$, $i = 1, 2$.

Q3.25 *Three-soliton solution of the KdV equation II.* Use Hirota's bilinear method to find the expression for $f(x, t)$ which generates the 3-soliton solution of the KdV equation.

Q3.26 *Two-dimensional KdV equation.* Show that the bilinear form of the equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

is

$$(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + 3\mathrm{D}_y^2)(f \cdot f) = 0,$$

where $u(x, t) = -2(\partial^2/\partial x^2)\log f$.

Q3.27 *Boussinesq equation.* Show that the bilinear form of the equation

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0$$

is

$$(\mathrm{D}_t^2 - \mathrm{D}_x^2 - \mathrm{D}_x^4)(f \cdot f) = 0,$$

where $u(x, t) = -2(\partial^2/\partial x^2)\log f$.

Q3.28 *Concentric KdV equation.* Show that the bilinear form of the equation

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0$$

is

$$\left(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + \frac{1}{2t}\frac{\partial}{\partial x}\right)(f \cdot f) = 0,$$

where $u(x, t) = -2(\partial^2/\partial x^2)\log f$ and $(\partial/\partial x)(f \cdot f) = ff_x$.

Q3.29 *Solitary-wave solutions.* Obtain the solitary-wave solutions of the equations given in Q3.26–Q3.28 by seeking appropriate simple solutions of the corresponding bilinear forms.

[Check your answers with those obtained in Q3.11, Q3.18, Q3.8 and Q3.22, and compare the various methods employed.]

Q3.30 *Two-soliton solution of the 2D KdV equation II.* See Q3.26 and Q3.29; obtain the expression for $f(x, t, y)$ from which the two-soliton solution of the 2D KdV can be constructed (cf. Q3.19).

Q3.31 *Two-soliton solution of the Boussinesq equation.* See Q3.27 and Q3.29; obtain the expression for $f(x, t)$ from which the two-soliton solution of the Boussinesq equation can be constructed. Show that your solution admits solitons which travel in either the same or opposite directions.

Q3.32 *A resonant solution of the 2D KdV equation.* The solutions obtained in Q3.30 can be written as

$$f = 1 + E_1 + E_2 + AE_1E_2$$

where $E_i = \exp(\theta_i)$, $\theta_i = k_ix + l_iy - \omega_it + \alpha_i$ with $\omega_i = k_i^3 + 3l_i^2/k_i$; A is a function of the k_i and $l_i\,(\mathrm{i} = 1, 2)$. Show that this f is a solution even if $A = 0$, and describe this solution by examining $\theta_1 \to -\infty$ with θ_2 fixed; $\theta_2 \to -\infty$ with θ_1 fixed; $\theta_1 \to +\infty$ with $\theta_3 = \theta_1 - \theta_2$ fixed.

Introduce a parameterisation of the dispersion relation $\omega_i = k_i^3 + 3l_i^2/k_i$ in the form

$$k_i = m_i + n_i, \quad l_i = m_i^2 - n_i^2, \quad \omega_i = 4(m_i^3 + n_i^3), \quad i = 1, 2,$$

(cf. Q3.19). Hence show that $A = 0$ if, for example, $m_1 = m_2$. Write $\theta_3 = k_3 x + l_3 y - \omega_3 t + \alpha_3$ and show that, if $m_1 = m_2$, $n_3 = n_2$ and $m_3 = -n_1$, then $\omega_3 = k_3^2 + 3l_3^2/k_3$.

[These definitions of ω_3, k_3 and l_3 (that is, $\omega_3 = \omega_1 - \omega_2$, etc., and $\omega_i, k_i, l_i, i = 1, 2, 3$, satisfying the dispersion relation) are the conditions for a *resonant wave interaction* or *phase-locked waves*; see Miles (1977b), Freeman (1980).]

Q3.33 *Energy conservation law for water waves.* See equations (3.85); multiply the first by u, use the third twice (once for w_z and once for u_x) and then the second (for p_z), and hence derive equation (3.88). Also confirm that $\mathscr{E}$ (given in Section 2.1.2) can be used to obtain (3.89).

Q3.34 *Energy conservation for the KdV equation.* Show that the third conserved quantity for the KdV equation, (3.96), can be deduced from the statement of energy conservation for water waves, (3.89).

Q3.35 *KdV conserved density.* Show that

$$\frac{45}{4}u^4 - 15uu_x^2 + u_{xx}^2$$

is a conserved density of the KdV equation

$$2u_t + 3uu_x + \frac{1}{3}u_{xxx} = 0.$$

Q3.36 *KdV equation: another conserved density.* Show that $xu + 3tu^2$ is a conserved density for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

Q3.37 *KdV equation: a 'centre of mass' property.* Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{\infty} xu \,\mathrm{d}x\right) = \text{constant},$$

where u satisfies the KdV equation in Q3.36 (provided $u \to 0$ sufficiently rapidly as $|x| \to \infty$). Interpret this result as the conservation of linear momentum of a linear mass distribution with density $u(x, t)$. Further, confirm that this result is consistent with the phase shifts associated with the two-soliton solution (discussed in Q3.14 and Q3.15).

Q3.38 *N-soliton solution and the conserved quantities.* Given that $u(x, t)$ evolves, according to the KdV equation (Q3.36), into an N-soliton solution from a given initial profile $u(x, 0)$, consider the profile at $t = 0$ and the solution as $t \to \infty$; describe how the conserved quantities can be used to determine the amplitudes of the resulting solitons. Use the first two conservation laws, and then the first three, to verify your method for the 2-soliton and 3-soliton solutions, respectively.

[This idea is developed in Berezin & Karpman (1967).]

Q3.39 *cKdV: conserved densities.* Show that

$$xt^{1/2}u + 6t^{3/2}u^2$$

and

$$x^2t^{1/2}u + 12xt^{3/2}u^2 + 48t^{5/2}u^3 + 24t^{5/2}u_x^2$$

are conserved densities of the concentric KdV equation

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0.$$

[An interesting observation is that this cKdV equation is, approximately, the KdV equation (Q3.36) for large t with u_t dominating u/t. You may wish to confirm that the coefficients of the dominant terms in the first three conserved densities for the cKdV equation are the conserved densities of the KdV equation.]

Q3.40 *Boussinesq equation: conserved quantities.* Show that

$$\int_{-\infty}^{\infty} HU\mathrm{d}X, \quad \int_{-\infty}^{\infty} U\mathrm{d}t$$

and

$$\int_{-\infty}^{\infty} (H^2 + U^2 - 4H^3 + 2HH_{XX} - H_X^2)\mathrm{d}t$$

are conserved quantities for the Boussinesq equation written in the form

$$U_t + H_X - 3(H^2)_X + H_{XXX} = 0; \quad H_t = -U_X.$$

[See Hirota (1973).]

Q3.41 *Conserved quantities and the N-soliton solution.* Use the results obtained in Q3.40, and described in Section 3.3.4, to show in principle how the amplitudes of the solitons of the Boussinesq equation can be determined from given initial data; see Q3.38. Give an example of the method for the 2-soliton solution. (The solitary-wave solution of the Boussinesq equation is discussed in Q3.8.)

Q3.42 *Shallow water equations: conservation laws I.* Show that

$$\left(\frac{1}{4}u^4 + hu^2 + um_1 + \frac{1}{2}h^2\right)_t + \left(\frac{1}{4}u^5 + hu^3 + u^2m_1 + um_2 + \frac{3}{2}h^2u + hm_1\right)_x + \left\{\left(\frac{1}{4}u^4 + hu^2 + um_1 + m_2 + \frac{3}{2}h^2\right)w\right\}_z = 0$$

is a conservation law for the shallow water equations, (3.101). Hence obtain the corresponding conserved quantity.

Q3.43 *Shallow water equations: conservation laws II.* See Q3.42; show that

$$(tu)_t + \{t(u^2 + h) - xu\}_x + \{(tu - x)w\}_z = 0$$

and

$$\{t(u^2 + h) - xu\}_t + \{t(u^3 + 2h + m_{1x}) - x(u^2 + h)\}_x + \{[t(u^2 + 2h) - xu]w\}_z = 0$$

are also conservation laws.

Q3.44 *Reduction to the classical KdV equations.* Show that the KdV equation for shear flow, (3.128), together with the Burns condition, (3.117), lead to the classical KdV equations ((3.28), Q3.2) for right- and left-going waves, respectively, when $U(z) = 0$.

Q3.45 *KdV equation for linear shear.* Obtain the form of the KdV equation, (3.128), when the shear flow is

$$U(z) = U_0 + (U_1 - U_0)z, \quad 0 \le z \le 1;$$

see (3.113).

Q3.46 *Burns condition.* For the two shear profiles

(a) $U(z) = U_0 + (U_1 - U_0)z$;

(b) $U(z) = U_1(2z - z^2), \quad 0 \le z \le 1,$
show that no critical level exists.

[Hint: assume that a critical level does exist, use the definition (3.129) and then show that the *only* solutions are *not* critical.]

Q3.47 *Burns condition with critical level I.* Show that, for $0 < d < 1$, the Burns condition for the model profile

$$U(z) = \begin{cases} U_1, & d \le z \le 1 \\ U_1 z/d, & 0 \le z < d \end{cases}$$

where U_1 is a constant, gives rise to three solutions for c, one of which corresponds to a critical level.

Q3.48 *Burns condition with critical level II.* Show that the conditions described in Q3.47 obtain also for the model profile

$$U(z) = \begin{cases} U_1, & d \le z \le 1 \\ U_1(2dz - z^2)/d^2, & 0 \le z < d. \end{cases}$$

Q3.49 *Generalised Burns condition.* Show that the generalised Burns condition, (3.136), has a solution

$$k(\theta) = a\cos\theta + b(a)\sin\theta,$$

where a is a parameter, and $b(a)$ is to be determined.

Q3.50 *Generalised Burns condition for oblique waves.* Determine the generalised Burns condition, (3.136), for plane oblique waves; that is, $k(\theta) = 1$ and $\theta = \theta_0 =$ constant. In the case $U = U_0$ $=$ constant, find the speed of the wave.

Q3.51 *Generalised Burns condition for linear shear.* Determine $k(\theta)$, using the method of Q3.49, for the case of a linear shear

$$U(z) = U_0 + (U_1 - U_0)z, \quad 0 \le z \le 1.$$

[Note: You are advised to make a convenient choice for c; see how we obtained (3.138).]

Q3.52 *Singular solution.* Derive the solution (3.140) from the general solution (3.139), using standard methods.

[Note: These ideas are described in any good text on (ordinary) differential equations, for example Forsyth (1921) or Piaggio (1933).]

Q3.53 *Variable coefficients → cKdV.* Show that the variable coefficient KdV equation, (3.148), transforms to the concentric KdV equation, (3.34), for H, where

$$\eta_0 = D^2 H\left(\int \sqrt{D}\,\mathrm{d}X, \xi\right),$$

provided that a special choice of $D(X)$ is made. What is this $D(X)$?

Q3.54 *Conservation laws.* Show that the variable coefficient KdV equation, with $D = D(X)$,

$$2H_{0X} + 3D^{-7/4}H_0H_{0\xi} + \frac{1}{3}D^{1/2}H_{0\xi\xi\xi} = 0$$

has a conserved density H_0^2. Also investigate the form of the next equation in this sequence (which involves $(H_0^3)_X$; cf. equation (3.96)).

Q3.55 *Variable depth solitary wave.* Obtain the most general solitary-wave solution of the equation given in Q3.54, where D is treated as a variable parameter. Now impose the conservation law associated with H_0^2 (also in Q3.54) and hence obtain the form of $\eta_0 = D^{-1/4}H_0$; see equation (3.150).

Q3.56 *Oblique plane wave.* Obtain, at leading order as $\varepsilon \to 0$, the KdV equation, (3.158), from equations (3.155) and (3.156), by seeking a solution which is a function of $\xi = kx + ly - t,\ \tau = \varepsilon t$.

Q3.57 *Oblique waves: weak interaction.* Obtain the expression for the surface wave, (3.166), correct at $O(\varepsilon)$, from the solution for f, (3.165).

4

Slow modulation of dispersive waves

'But let me tell thee now another tale'
The Coming of Arthur

In ever climbing up the climbing wave
The Lotos-Eaters: Choric song IV

The Korteweg–de Vries equation, members of its family and the applications to more realistic situations, cover only one general area of interest in the modern theory of nonlinear water waves. In particular, all our discussions in Chapter 3 have been based on the requirement that the waves are long; this was accomplished by the condition $\delta \to 0$ or, rather, by the rescaling

$$x \to \frac{\delta}{\varepsilon^{\frac{1}{2}}} x, \quad t \to \frac{\delta}{\varepsilon^{\frac{1}{2}}} t,$$

with $\varepsilon \to 0$; see equation (3.10). In this discussion we shall now allow the wave to be of any wavelength, so that the wave number (k) plays the rôle of a parameter in our calculations. The amplitude parameter, ε, is then used to describe the slow evolution of an harmonic wave of wave number k; the wave is thus slowly modulated as described by $\varepsilon \to 0$. The approach that we adopt is to be found in Section 1.4.2 (equation (1.103) *et seq.*) where the appropriate multiple-scale technique is used there to obtain the asymptotic solution of a partial differential equation.

We shall follow a similar route to that developed in Chapter 3, namely, a presentation of the derivation of the basic evolution equation together with the application of these ideas to more realistic situations. It turns out that the fundamental equation (the *Nonlinear Schrödinger equation*) – and some of its relations – are again special equations of the completely integrable (soliton) type. We shall describe a few properties of these equations, and how solutions can be readily obtained. Not surprisingly, the long-wave limit of these various problems that we present here recovers the essential features of the KdV description; we shall show how this comes about.

4.1 The evolution of wave packets

We shall present two derivations that lead to a description of the evolution of wave packets (for gravity waves) on the surface of water of finite depth. First we examine the problem of the propagation of a plane wave and then, just as in Chapter 3, we construct a two-dimensional version of this problem that incorporates a suitable (weak) dependence on the coordinate that is transverse to the predominant direction of propagation; cf. the 2D KdV equation. This two-dimensional surface wave is described by a pair of equations: the *Davey–Stewartson* (DS) equations.

4.1.1 Nonlinear Schrödinger (NLS) equation

In keeping with much that has gone before, we shall start with an examination of gravity waves (moving in one direction) on stationary water of constant depth ($b = 0$). The most direct approach is to formulate the problem in terms of the equations for irrotational flow (although we shall not always be able to follow this route). Thus, from equations (2.132) with $\partial/\partial y \equiv 0$, we have

$$\left.\begin{aligned} &\phi_{zz} + \delta^2\phi_{xx} = 0; \\ &\left.\begin{aligned} &\phi_z = \delta^2(\eta_t + \varepsilon\phi_x\eta_x) \\ &\phi_t + \eta + \frac{1}{2}\varepsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2\right) = 0 \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta \end{aligned}\right\} \tag{4.1}$$

and

$$\phi_z = 0 \quad \text{on} \quad z = 0.$$

In these equations we have retained the shallowness parameter, δ, and we shall consider $\varepsilon \to 0$ for δ fixed. The solution that we seek is a harmonic wave with wave number k – a solution of the linear equations ($\varepsilon = 0$) – which is allowed to evolve slowly on scales determined by ε. We have already seen (equation (1.103) *et seq.*) that the relevant scales would seem to be associated with both ε and ε^2. Here, therefore, we introduce

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t, \tag{4.2}$$

where $c_p(k)$ and $c_g(k)$ are to be determined (but the notation should be suggestive!). The justification for this choice is, ultimately, that it produces a consistent solution of the equations; a simple argument based on the Fourier integral representation of a general plane wave also confirms this choice (Q4.1).

The governing equations, (4.1), under the transformation (4.2), become

$$\left.\begin{aligned}
&\phi_{zz} + \delta^2\left(\phi_{\xi\xi} + 2\varepsilon\phi_{\xi\zeta} + \varepsilon^2\phi_{\zeta\zeta}\right) = 0;\\
&\left.\begin{aligned}
&\phi_z = \delta^2\{\varepsilon^2\eta_\tau - \varepsilon c_g\eta_\zeta - c_p\eta_\xi\\
&\qquad + \varepsilon(\phi_\xi + \varepsilon\phi_\zeta)(\eta_\xi + \varepsilon\eta_\zeta)\}\\
&\varepsilon^2\phi_\tau - \varepsilon c_g\phi_\zeta - c_p\phi_\xi + \eta\\
&\qquad + \frac{1}{2}\varepsilon\left\{\frac{1}{\delta^2}\phi_z^2 + \left(\phi_\xi + \varepsilon\phi_\zeta\right)^2\right\} = 0
\end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta\\
&\text{and}\\
&\qquad\qquad \phi_z = 0 \quad \text{on} \quad z = 0.
\end{aligned}\right\} \tag{4.3}$$

We seek an asymptotic solution of these equations in the form

$$\phi \sim \sum_{n=0}^{\infty} \varepsilon^n \phi_n(\xi, \zeta, \tau, z); \quad \eta \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \zeta, \tau) \quad \text{as} \quad \varepsilon \to 0,$$

which is to be periodic in ξ. The leading-order problem is clearly

$$\phi_{0zz} + \delta^2\phi_{0\xi\xi} = 0 \tag{4.4}$$

with

$$\phi_{0z} = -\delta^2 c_p \eta_{0\xi} \quad \text{and} \quad -c_p\phi_{0\xi} + \eta_0 = 0 \quad \text{on} \quad z = 1 \tag{4.5}$$

and

$$\phi_{0z} = 0 \quad \text{on} \quad z = 0. \tag{4.6}$$

The solution of interest to us takes the form

$$\eta_0 = A_0 E + \text{c.c.}; \quad \phi_0 = f_0 + F_0 E + \text{c.c.}, \tag{4.7}$$

where $E = \exp(\mathrm{i}k\xi)$, $A_0 = A_0(\zeta, \tau)$, $F_0 = F_0(z, \zeta, \tau)$, $f_0 = f_0(\zeta, \tau)$ and c.c. denotes the complex conjugate of the terms in E. The real term $f_0(\zeta, \tau)$ is required in order to accommodate the mean drift component; see Section 2.5. This solution describes a single harmonic wave, of wave number k, which is propagating at speed c_p. We see that Laplace's equation, (4.4), with (4.7), becomes

$$F_{0zz} - \delta^2 k^2 F_0 = 0,$$

so the solution which satisfies the bottom boundary condition, (4.6), is

$$F_0 = G_0(\zeta, \tau)\cosh(\delta k z),$$

where $G_0(\zeta, \tau)$ is yet to be determined. The two surface boundary conditions, (4.5), yield

$$\delta k G_0 \sinh \delta k = -\mathrm{i}\delta^2 k c_p A_0 \quad \text{and} \quad \mathrm{i}k c_p G_0 \cosh \delta k = A_0,$$

from which we obtain

$$c_p^2 = \frac{\tanh \delta k}{\delta k} \quad \text{and} \quad G_0 = -\frac{\mathrm{i}A_0}{k c_p}\operatorname{sech}\delta k; \tag{4.8}$$

thus we may write

$$F_0 = -\mathrm{i}\delta c_p A_0 \frac{\cosh \delta k z}{\sinh \delta k}, \tag{4.9}$$

all of which is familiar; see equation (2.4) *et seq.*, equation (2.13) and Q2.5. At this order, the amplitude function $A_0(\zeta, \tau)$ is unknown; we now proceed to the problem given by the $\mathrm{O}(\varepsilon)$ terms.

Equations (4.3), upon collecting the terms of $\mathrm{O}(\varepsilon)$ and expanding about $z = 1$ in the surface boundary conditions, yield

$$\phi_{1zz} + \delta^2 \phi_{1\xi\xi} + 2\delta^2 \phi_{0\xi\zeta} = 0; \tag{4.10}$$

with

$$\left.\begin{aligned} &\phi_{1z} + \eta_0 \phi_{0zz} = \delta^2(-c_g \eta_{0\zeta} - c_p \eta_{1\xi} + \phi_{0\xi}\eta_{0\xi}) && (4.11)\\ &-c_g\phi_{0\zeta} - c_p(\phi_{1\xi} + \eta_0\phi_{0\xi z}) + \eta_1 + \frac{1}{2}\left(\frac{1}{\delta^2}\phi_{0z}^2 + \phi_{0\xi}^2\right) = 0 && (4.12)\end{aligned}\right\} \text{ on } z = 1$$

and

$$\phi_{1z} = 0 \quad \text{and} \quad z = 0. \tag{4.13}$$

It is clear that these equations will produce terms E^2, E^{-2} and E^0 (from $E^1 E^{-1}$) by virtue of the nonlinearity of the surface boundary conditions. The contributions E^2 (with E^{-2}, the complex conjugate) are the first of the higher harmonics that are generated by the nonlinear interaction; the fundamental is E^1 (with E^{-1}), introduced in (4.7). Since we are seeking a solution which is periodic in ξ, we choose to build in this requirement at this stage. We do this by imposing a periodic structure on the form of solution that we seek hereafter. An alternative approach is to solve at each $\mathrm{O}(\varepsilon^n)$ and determine the various functions that are available, in such a fashion as to remove all those terms that contribute to the non-periodic (or *secular*) terms; this is how we tackled the problem in Section 1.4.2 (equations (1.103) *et seq.*). The two methods produce, eventually, exactly the same result, but the former presents us with a more straightforward calculation, as we shall now demonstrate.

In order to implement this idea, we write

$$\phi_n = \sum_{m=0}^{n+1} F_{nm}E^m + \text{c.c.}; \quad \eta_n = \sum_{m=0}^{n+1} A_{nm}E^m + \text{c.c.} \tag{4.14}$$

for $n = 1, 2, \ldots$, where $F_{nm}(z, \zeta, \tau)$ and $A_{nm}(\zeta, \tau)$ are to be determined; the complex conjugate relates only to terms in E^m, $m = 1, 2, \ldots$. We note that terms $m = 0$, although not harmonic (oscillatory) functions, do not destroy the periodicity in ξ. The solution described by (4.14) incorporates the phenomenon that, at each higher order in ε, higher harmonics progressively appear, so E^2 appears first at $\mathrm{O}(\varepsilon)$, E^3 first at $\mathrm{O}(\varepsilon^2)$, and so on.

Laplace's equation for ϕ_1, (4.10), therefore gives

$$F_{10zz} = 0; \quad F_{12zz} - 4\delta^2 k^2 F_{12} = 0$$

and

$$F_{11zz} - \delta^2 k^2 F_{11} + 2\mathrm{i}\delta^2 k F_{0\zeta} = 0,$$

which have solutions (see Q4.2) satisfying the bottom boundary condition, (4.13),

$$\left.\begin{aligned} &F_{10} = G_{10}(\zeta, \tau); \quad F_{12} = G_{12}(\zeta, \tau)\cosh(2\delta k z) \\ \text{and} \\ &F_{11} = G_{11}(\zeta, \tau)\cosh(\delta k z) - \mathrm{i}\delta G_{0\zeta} z \sinh(\delta k z), \end{aligned}\right\} \tag{4.15}$$

where the $G_{1m}(\zeta, \tau)$ are arbitrary functions at this stage. These results are then used in the two surface boundary conditions, (4.11) and (4.12), to give six equations (arising from the coefficients of E^0, E^1 and E^2 in each). Equation (4.11) yields (with the asterisk denoting the complex conjugate)

$$E^0\colon\ (A_0 G_0^* + A_0^* G_0)\delta^2 k^2 \cosh\delta k = \delta^2 k^2 (A_0 G_0^* + A_0^* G_0)\cosh\delta k; \tag{4.16}$$

$$\begin{aligned} E^1\colon\ &\delta k G_{11}\sinh\delta k - \mathrm{i}\delta G_{0\zeta}(\sinh\delta k + \delta k\cosh\delta k) \\ &= -\delta^2(c_g A_{0\zeta} + \mathrm{i}k c_p A_{11}) \end{aligned} \tag{4.17}$$

$$\begin{aligned} E^2\colon\ &2\delta k G_{12}\sinh 2\delta k + \delta^2 k^2 A_0 G_0 \cosh\delta k \\ &= -\delta^2(2\mathrm{i}k c_p A_{12} + k^2 A_0 G_0\cosh\delta k), \end{aligned} \tag{4.18}$$

and, correspondingly, equation (4.12) gives

$$E^0: \ -c_g f_{0\zeta} + \mathrm{i}\delta k^2 c_p(A_0 G_0^* + A_0^* G_0)\sinh\delta k + A_{10} + k^2 G_0 G_0^*(\sinh^2\delta k + \cosh^2\delta k) = 0; \quad (4.19)$$

$$E^1: \ -c_g G_{0\zeta}\cosh\delta k - \mathrm{i}kc_p(G_{11}\cosh\delta k - \mathrm{i}\delta G_{0\zeta}\sinh\delta k) + A_{11} = 0; \quad (4.20)$$

$$E^2: \ -\mathrm{i}kc_p(2G_{12}\cosh 2\delta k + \delta k A_0 G_0 \sinh\delta k) + A_{12} - \frac{1}{2}k^2 G_0^2 = 0. \quad (4.21)$$

We see that equation (4.16) is identically satisfied, and that with equation (4.8) (for G_0) used in (4.19) we obtain

$$A_{10} = -\frac{2\delta k}{\sinh 2\delta k}A_0 A_0^* + c_g f_{0\zeta}. \quad (4.22)$$

Equation (4.20) gives us directly that

$$A_{11} = c_g G_{0\zeta}\cosh\delta k + \mathrm{i}kc_p(G_{11}\cosh\delta k - \mathrm{i}\delta G_{0\zeta}\sinh\delta k),$$

and when this is used in equation (4.17) we find, first, that G_{11} cancels identically when we invoke the expression for c_p^2 (in (4.8)); then, with G_0 from (4.8), we see that $A_{0\zeta}$ ($\neq 0$) also cancels, leaving

$$c_g = \frac{1}{2}c_p(1 + 2\delta k\,\mathrm{cosech}\,2\delta k), \quad (4.23)$$

which is the group speed for gravity waves; see equation (2.29) *et seq.*, and Q2.26. Finally, equations (4.18) and (4.21) are solved for G_{12} and A_{12} (using (4.8) as necessary) to give

$$G_{12} = -\frac{3\mathrm{i}}{4}\frac{\delta^2 kc_p A_0^2}{\sinh^4\delta k}; \quad A_{12} = \frac{\delta k\cosh\delta k}{2\sinh^3\delta k}(2\cosh^2\delta k + 1)A_0^2, \quad (4.24)$$

and $A_0(\zeta, \tau)$ is still undetermined. (It is clear that the solution of this problem requires some fairly extensive manipulation – and it is considerably worse at the next order – the details of which are left to the sufficiently enthusiastic reader.) We now examine the next order, $\mathrm{O}(\varepsilon^2)$, where we expect the equation for A_0 to emerge.

From equations (4.3), with the usual expansion of the surface boundary conditions about $z = 1$, we obtain the problem at $\mathrm{O}(\varepsilon^2)$ as

$$\phi_{2zz} + \delta^2\phi_{2\xi\xi} + 2\delta^2\phi_{1\xi\zeta} + \delta^2\phi_{0\zeta\zeta} = 0; \quad (4.25)$$

with

$$\phi_{2z} + \eta_0\phi_{1zz} + \frac{1}{2}\eta_0^2\phi_{0zzz} + \eta_1\phi_{0zz}$$
$$= \delta^2\{\eta_{0\tau} - c_g\eta_{1\zeta} - c_p\eta_{2\xi} + \phi_{0\xi}(\eta_{1\xi} + \eta_{0\zeta}) + \eta_{0\xi}(\eta_0\phi_{0\xi z} + \phi_{1\xi} + \phi_{0\zeta})\}; \quad (4.26)$$
$$\phi_{0\tau} - c_g\eta_0\phi_{0\zeta z} - c_p\phi_{1\zeta} - c_p(\phi_{2\xi} + \eta_0\phi_{1\xi z} + \eta_1\phi_{0\xi z} + \frac{1}{2}\eta_0^2\phi_{0\xi zz})$$
$$+ \eta_2 + \frac{1}{\delta^2}\phi_{0z}(\eta_0\phi_{0zz} + \phi_{1z}) + \phi_{0\xi}(\eta_0\phi_{0\xi z} + \phi_{1\xi} + \phi_{0\zeta}) = 0, \quad (4.27)$$

both on $z = 1$, and

$$\phi_{2z} = 0 \quad \text{on} \quad z = 0. \quad (4.28)$$

The periodic solution described in equations (4.14) is now used for $n = 2$ (so the higher harmonic E^3 now appears for the first time); equation (4.25) then gives

$$F_{20zz} + \delta^2 f_{0\zeta\zeta} = 0; \quad F_{21zz} - \delta^2k^2F_{21} + 2\mathrm{i}k\delta^2F_{11\zeta} + \delta^2F_{0\zeta\zeta} = 0,$$

and so on. It soon becomes evident that the equation for A_0 appears from the terms that arise at E^1 (because this is equivalent to the removal of secular terms at E^1; cf. equation (1.108) *et seq.*), so we examine only this problem in any detail. The solution for $F_{21}(z, \zeta, \tau)$ which satisfies the bottom boundary condition, (4.28), becomes

$$F_{21} = G_{21}\cosh\delta kz - (\mathrm{i}\delta G_{11\zeta} + \frac{\delta}{2k}G_{0\zeta\zeta})z\sinh\delta kz$$
$$+ \frac{1}{2}\delta^2G_{0\zeta\zeta}\left(\frac{1}{\delta k}z\sinh\delta kz - z^2\cosh\delta kz\right); \quad (4.29)$$

see Q4.2. The boundary condition (4.26) gives, for terms E^1,

$$F_{21z} + A_0^*F_{12zz} + \frac{1}{2}(A_0^2F_{0zzz}^* + 2A_0A_0^*F_{0zzz}) + A_{10}F_{0zz} + A_{12}F_{0zz}^*$$
$$= \delta^2\{A_{0\tau} - c_gA_{11\zeta} - \mathrm{i}kc_pA_{21} + 2k^2A_{12}F_0^*$$
$$-k^2A_0(A_0^*F_{0z} - A_0F_{0z}^*) + k^2A_0^*(A_0F_{0z} + 2F_{12})\} \quad (4.30)$$

on $z = 1$. The second boundary condition on $z = 1$, (4.27), similarly gives

$$\begin{aligned} F_{0\tau} - c_g F_{11\zeta} - \mathrm{i}kc_p F_{21} - 2\mathrm{i}kc_p A_0^* F_{12z} - \mathrm{i}kc_p(A_{10}F_{0z} - A_{12}F_{0z}^*) \\ - \frac{1}{2}\mathrm{i}kc_p(2A_0A_0^*F_{0zz} - A_0^2F_{0zz}^*) + A_{21} \\ + \frac{1}{\delta^2}\{(A_0F_{0zz}^* + A_0^*F_{0zz})F_{0z} + (A_0F_{0zz} + F_{12z})F_{0z}^*\} \\ - k^2(A_0^*F_{0z} - A_0F_{0z}^*)F_0 + k^2(A_0F_{0z} + 2F_{12})F_0^* = 0. \quad (4.31) \end{aligned}$$

The procedure that we follow is easy to describe, but rather tiresome to perform: eliminate A_{21} between equations (4.30) and (4.31), introduce the functions obtained at earlier stages (including F_{21} from (4.29)) and simplify. We find that G_{21} cancels identically by virtue of the definition of c_p^2, and that A_{11} (or G_{11}) also cancels when the expression for c_g, (4.23), is used. This leaves an equation for $A_0(\zeta, \tau)$, incorporating the terms $A_{0\tau}$, $A_{0\zeta\zeta}$ and $A_0|A_0|^2$:

$$-2\mathrm{i}kc_pA_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0|A_0|^2 = 0, \quad (4.32)$$

which is one form of the *Nonlinear Schrödinger* (*NLS*) *equation*, where here

$$\alpha = c_g^2 - (1 - \delta k \tanh \delta k)\mathrm{sech}^2\delta k \quad (4.33)$$

and

$$\beta = \frac{k^2}{c_p^2}\left\{\frac{1}{2}(1 + 9\coth^2 \delta k - 13\mathrm{sech}^2\delta k - 2\tanh^4 \delta k) - (2c_p + c_g\mathrm{sech}^2\delta k)^2(1 - c_g^2)^{-1}\right\}, \quad (4.34)$$

although a more instructive form for α is

$$\alpha = -kc_p\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}, \quad \omega(k) = kc_p;$$

cf. Q4.1 and see Q4.3. (The NLS equation is sometimes called the *cubic Schrödinger equation.*) It is a straightforward calculation – which is left as an exercise – to show that $\alpha > 0$ for all δk, but that β changes in sign from positive to negative as δk decreases, across $\delta k \approx 1.363$; see Section 4.3.1 and Figure 4.6 (on p. 336). We comment here that the condition $\alpha\beta > 0$ turns out to be significant for the existence of certain important solutions of the NLS equation; see Section 4.2. The consideration of terms that arise at ε^2E^0 is left as an exercise; see Q4.4.

Some relevant properties of the NLS equation, and the interpretation of its solutions in the understanding of water-wave phenomena, will be presented later.

4.1.2 *Davey–Stewartson (DS) equations*

The classical NLS equation applies to the situation where the wave properties only in one direction, and for which the profile evolves only in this same direction. Such a wave would be generated by an initial profile which takes the form

$$A(\varepsilon x)\mathrm{e}^{\mathrm{i}kx} + \text{c.c.};$$

we now consider (following Davey & Stewartson, 1974) the development of a wave which, at $t = 0$, is described by

$$A(\varepsilon x, \varepsilon y)\mathrm{e}^{\mathrm{i}kx} + \text{c.c.}$$

We see that the slow (or weak) dependence occurs equally in both the x- and y-directions, but that the fast oscillation is only in the x-direction: the wave packet will propagate in the x-direction with a slowly evolving structure in both x- and y-directions. The group speed is, of course, still associated with the propagation in the x-direction. The appropriate form of solution will be sought from the governing equations (see equations (2.132)); these are

$$\left.\begin{aligned}
&\phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0; \\
&\text{with} \\
&\left.\begin{aligned}
\phi_z &= \delta^2\{\eta_t + \varepsilon(\phi_x\eta_x + \phi_y\eta_y)\} \\
\phi_t + \eta &+ \frac{1}{2}\varepsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2 + \phi_y^2\right) = 0
\end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta \\
&\text{and} \\
&\phi_z = 0 \quad \text{on} \quad z = 0.
\end{aligned}\right\} \tag{4.35}$$

We introduce the variables (cf. equation (4.2))

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad Y = \varepsilon y, \quad \tau = \varepsilon^2 t$$

and so equations (4.35) become (cf. equations (4.3))

$$\phi_{zz} + \delta^2(\phi_{\xi\xi} + 2\varepsilon\phi_{\xi\zeta} + \varepsilon^2\phi_{\zeta\zeta} + \varepsilon^2\phi_{YY}) = 0;$$

$$\left.\begin{aligned} \phi_z = \delta^2\{\varepsilon^2\eta_\tau - \varepsilon c_g\eta_\zeta - c_p\eta_\xi \\ +\varepsilon(\phi_\xi + \varepsilon\phi_\zeta)(\eta_\xi + \varepsilon\eta_\zeta) + \varepsilon^3\phi_Y\eta_Y\} \\ \varepsilon^2\phi_\tau - \varepsilon c_g\phi_\zeta - c_p\phi_\xi + \eta \\ +\frac{1}{2}\varepsilon\left\{\frac{1}{\delta^2}\phi_z^2 + (\phi_\xi + \varepsilon\phi_\zeta)^2 + \varepsilon\phi_Y^2\right\} = 0 \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta$$

and

$$\phi_z = 0 \quad \text{on} \quad z = 0.$$

It is immediately clear that, if we proceed no further than $O(\varepsilon^2)$, the only contribution from the dependence in Y will arise from the term ϕ_{YY} in Laplace's equation. The other terms involving derivatives in Y produce new nonlinear interactions that appear first at $O(\varepsilon^3)$. The calculation therefore follows very closely that already presented for the NLS equation, so we shall not give the details here.

We seek a solution in the form

$$\left.\begin{aligned} \phi \sim f_0(\zeta, Y, \tau) + \sum_{n=0}^{\infty}\varepsilon^n\left\{\sum_{m=0}^{n+1}F_{nm}(z, \zeta, Y, \tau)E^m + \text{c.c.}\right\}; \\ \eta \sim \sum_{n=0}^{\infty}\varepsilon^n\left\{\sum_{m=0}^{n+1}A_{nm}(\zeta, Y, \tau)E^m + \text{c.c.}\right\}; \end{aligned}\right\} \tag{4.36}$$

where $E = \exp(\mathrm{i}k\xi)$ and $A_{00} = 0$ (so that the first approximation to the surface wave is purely harmonic). The results mirror those already obtained, for all the terms at $O(1)$ and $O(\varepsilon)$; the differences first appear at $O(\varepsilon^2)$. It turns out that the problem at ε^2E^0 gives

$$(1 - c_g^2)f_{0\zeta\zeta} + f_{0YY} = -\frac{1}{c_p^2}(2c_p + c_g\operatorname{sech}^2\delta k)(|A_0|^2)_\zeta, \tag{4.37}$$

the equation for f_0, given A_0 ($\equiv A_{01}$); the surface boundary conditions for the terms ε^2E^1 produce

$$\left.\begin{aligned} -2\mathrm{i}kc_pA_{0\tau} + \alpha A_{0\zeta\zeta} - c_pc_gA_{0YY} \\ +\frac{k^2}{2c_p^2}(1 + 9\coth^2\delta k - 13\operatorname{sech}^2\delta k - 2\tanh^4\delta k)A_0|A_0|^2 \\ + k^2(2c_p + c_g\operatorname{sech}^2\delta k)A_0f_{0\zeta} = 0. \end{aligned}\right\} \tag{4.38}$$

These two equations, (4.37) and (4.38), are the *Davey–Stewartson* (*DS*) *equations* for the modulation of harmonic waves. It is clear that for no dependence on Y, so that (4.37) gives

$$(1 - c_g^2)f_{0\zeta} = -\frac{1}{c_p^2}(2c_p + c_g\mathrm{sech}^2\delta k)|A_0|^2 \tag{4.39}$$

(on the assumption that $f_{0\zeta} = 0$ where $A_0 = 0$), equation (4.38) then recovers the NLS equation

$$-2\mathrm{i}kc_pA_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0|A_0|^2 = 0$$

as given in (4.32); also see Q4.24. Equation (4.39) provides the leading contribution to the mean drift generated by the nonlinear interaction of the wave; see Q4.4 and Q2.32.

The DS equations are more compactly written in the form

$$(1 - c_g^2)f_{0\zeta\zeta} + f_{0YY} = -\frac{\gamma}{c_p^2}(|A_0|^2)_\zeta; \tag{4.40}$$

$$-2\mathrm{i}kc_pA_{0\tau} + \alpha A_{0\zeta\zeta} - c_pc_gA_{0YY} + \left\{\beta + \frac{\gamma^2k^2}{c_p^2(1 - c_g^2)}\right\}A_0|A_0|^2 + k^2\gamma A_0f_{0\zeta} = 0, \tag{4.41}$$

where α and β were given earlier ((4.33), (4.34)) and

$$\gamma = 2c_p + c_g\mathrm{sech}^2\delta k; \tag{4.42}$$

we observe that $\gamma > 0$ and note that $c_pc_g > 0$. These equations (and, of course, the NLS equation) may be further approximated for long or short waves ($\delta \to 0$, $\delta \to \infty$, respectively), although their validity must remain in doubt: that is, for sufficiently small/large δ, other terms will presumably become important. However, as model equations for the evolution of wave packets in these two limits, they do provide useful insights; these limiting cases are considered in Q4.6. Furthermore, as a mathematical exercise to confirm the overall consistency of our equations, the result of letting $\delta \to 0$ (so that we have long waves) is important. We know, for the one-dimensional propagation of long waves, that the relevant equation is the Korteweg–de Vries equation. The existence of a close relationship between the NLS and KdV equations is now explored.

4.1.3 Matching between the NLS and KdV equations

The two fundamental equations for weakly nonlinear waves that we have introduced are the KdV and NLS equations. The former equation describes long waves, which can be obtained by letting $\delta \to 0$ and $\varepsilon \to 0$ with $\delta^2 = \mathrm{O}(\varepsilon)$; see Section 2.9.1. Alternatively, and more generally, we use a suitable rescaling of the variables which allows us to obtain the KdV equation for arbitrary δ. However, this transformation results in the replacement of δ^2 by ε in the governing equations (see equations (3.10)–(3.15)) with $\varepsilon \to 0$; thus the transformation, coupled with $\varepsilon \to 0$, is equivalent to $\delta \to 0$: long waves. On the other hand, the NLS equation uses scaled variables which are defined with respect to ε only, with δ $(= \mathrm{O}(1))$ retained as a parameter throughout. Thus, at least for a class of waves, we have two representations:

$$\eta(x, t; \varepsilon, \delta) \text{ with } \varepsilon \to 0,\ \ \delta \to 0 - \text{KdV};$$
$$\eta(x, t; \varepsilon, \delta) \text{ with } \varepsilon \to 0,\ \ \delta \text{ fixed } \ - \text{NLS}.$$

We might, therefore, suppose that the two descriptions satisfy some appropriate matching condition in δ. That is, the KdV representation with $\delta \to \infty$ might match with the NLS representation with $\delta \to 0$. So we take the short-wave limit of the KdV equation (but, as we shall see, written in an appropriate form) and the long-wave limit of the NLS equation.

Let us first construct the limiting form of the NLS equation as $\delta \to 0$; this requires that we determine the dominant behaviours of the coefficients of the equation

$$-2\mathrm{i}kc_p A_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0|A_0|^2 = 0,$$

where α and β are given in equations (4.33) and (4.34). (The details of this calculation, for the DS equations and then for the NLS equation, are rehearsed in Q4.6 but we shall record the salient features here.) From Q4.5 we have that

$$c_p \sim 1 - \frac{1}{6}\delta^2 k^2; \quad c_g \sim 1 - \frac{1}{2}\delta^2 k^2 \quad \text{as} \quad \delta \to 0 \tag{4.43}$$

(cf. equation (2.137) *et seq.*; the behaviours of c_p and c_g, as functions of δk, are also shown in Figure 4.1), and so

$$-2\mathrm{i}kc_p \sim -2\mathrm{i}k$$

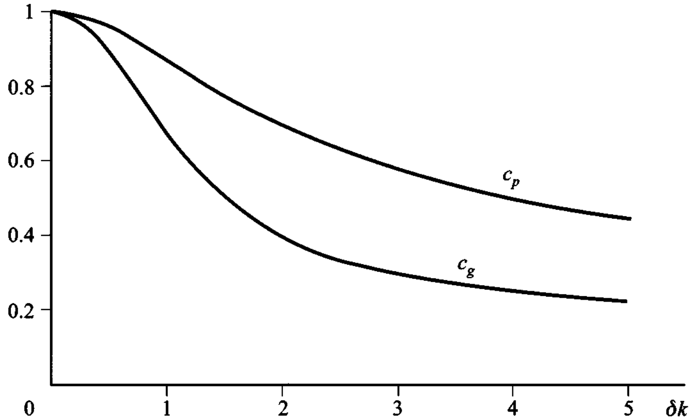


Figure 4.1. Plots of c_p and c_g as functions of $\delta k\,(\geq 0)$.

and

$$\alpha \sim (1 - \frac{1}{2}\delta^2 k^2)^2 - (1 - \delta^2 k^2)(1 - \frac{1}{2}\delta^2 k^2)^2$$
$$\sim \delta^2 k^2,$$

both as $\delta \to 0$. Similarly we obtain

$$\beta \sim k^2 \left\{ \frac{1}{2}\left[1 + 9\left(\frac{1}{\delta k} + \frac{\delta k}{3}\right)^2 - 13(1 - \frac{1}{2}\delta^2 k^2)^2 - 2(\delta k)^4 \right] - \left[2(1 - \frac{1}{6}\delta^2 k^2) + (1 - \frac{1}{2}\delta^2 k^2)(1 - \frac{1}{2}\delta^2 k^2)^2 \right]^2 \times \left[1 - \left(1 - \frac{1}{2}\delta^2 k^2\right)^2 \right]^{-1} \right\}$$

$$\sim k^2\left(\frac{9}{2\delta^2 k^2} - \frac{9}{\delta^2 k^2}\right) = -\frac{9}{2\delta^2},$$

and hence our NLS equation, approximated for long waves, becomes

$$-2\mathrm{i}kA_{0\tau} + \delta^2 k^2 A_{0\zeta\zeta} - \frac{9}{2\delta^2} A_0 |A_0|^2 = 0;$$

in the light of what we describe below, it is convenient to multiply by δ^2 to give

$$-2\mathrm{i}k\delta^2 A_{0\tau} + \delta^4 k^2 A_{0\zeta\zeta} - \frac{9}{2} A_0 |A_0|^2 = 0. \qquad (4.44)$$

Now we turn to the examination of the KdV equation, for $\delta \to \infty$, which proves to be rather less straightforward.

Our KdV equation is

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0 \tag{4.45}$$

(equation (3.28)), where

$$\xi = \frac{\varepsilon^{1/2}}{\delta}(x - t), \quad \tau = \frac{\varepsilon^{3/2}}{\delta}t,$$

and x, t are the original nondimensional variables. In order to produce the explicit dependence on δ in the equation, we write

$$\xi = \frac{\varepsilon^{1/2}}{\delta}\hat{\xi}, \quad \tau = \frac{\varepsilon^{1/2}}{\delta}\hat{\tau} \quad (\text{so } \hat{\xi} = x - t, \hat{\tau} = \varepsilon t)$$

to give

$$2\eta_{0\hat{\tau}} + 3\eta_0\eta_{0\hat{\xi}} + \frac{\lambda}{3}\eta_{0\hat{\xi}\hat{\xi}\hat{\xi}} = 0, \tag{4.46}$$

where $\lambda = \delta^2/\varepsilon$ (and we see here the relevance of the special choice $\delta^2 = O(\varepsilon)$ alluded to above, and used in Section 2.9.1). In this form, the limiting process that allows us to describe short waves is $\lambda \to \infty$. However, we also require a solution which produces a direct correspondence with the form of solution used in the derivation of the NLS equation. Thus we seek a modulated harmonic wave solution of equation (4.45) with $\lambda \to \infty$; to this end we introduce

$$\left.\begin{aligned} X &= \hat{\xi} + c_{p_1}\lambda\hat{\tau} & &[= x - (1 - \delta^2 c_{p_1})t]; \\ Z &= \lambda^{-1}(\hat{\xi} + c_{g_1}\lambda\hat{\tau}) & &\left[= \frac{\varepsilon}{\delta^2}\{x - (1 - \delta^2 c_{g_1})t\}\right]; \\ T &= \lambda^{-1}\hat{\tau} & &\left[= \frac{\varepsilon^2}{\delta^2}t\right], \end{aligned}\right\} \tag{4.47}$$

where the notation c_{p_1}, c_{g_1} indicates the correction – to be found – to the phase and group speeds, respectively. The KdV equation, (4.46), therefore becomes

$$\begin{aligned} &2(\lambda c_{p_1}\eta_{0X} + c_{g_1}\eta_{0Z} + \lambda^{-1}\eta_{0T}) + 3\eta_0(\eta_{0X} + \lambda^{-1}\eta_{0Z}) \\ &\qquad + \frac{\lambda}{3}(\eta_{0XXX} + 3\lambda^{-1}\eta_{0ZXX} + 3\lambda^{-2}\eta_{0ZZX} + \lambda^{-3}\eta_{0ZZZ}) = 0 \end{aligned}$$

and we write the solution (cf. equation (4.36)) as

$$\eta_0 \sim \sum_{n=0}^{\infty} \lambda^{-n} \left\{ \sum_{m=0}^{n+1} A_{nm}(Z, T) E^m + \text{c.c.} \right\}, \tag{4.48}$$

where $E = \exp(\mathrm{i}kX)$, c.c. denotes the complex conjugate of the terms $m \geq 1$ and $A_{00} = 0$. The wave number of the fundamental is k, and this is a fixed parameter in the solution. We collect the various terms from the equation, and these are listed to the left; we obtain

$$\lambda E^1: \quad c_{p_1} = k^2/6; \tag{4.49}$$

$$E^1: \quad c_{g_1} = k^2/2 \text{ (provided } A_{01Z} \neq 0); \tag{4.50}$$

$$E^2: \quad A_{12} = \tfrac{3}{2k^2} A_{01}^2;$$

$$\lambda^{-1} E^0: \quad A_{10} = -\tfrac{3}{k^2} |A_{01}|^2;$$

$$\lambda^{-1} E^1: \quad -2\mathrm{i}kA_{01T} + k^2 A_{01ZZ} - \tfrac{9}{2} A_{01} |A_{01}|^2 = 0 \tag{4.51}$$

where each earlier result is used, as necessary, to produce later results. We note that the corrections (to c_p and c_g) given in (4.49) and (4.50) agree precisely with the approximations used in the NLS equation; see equations (4.43). Equation (4.51) is the required NLS equation which describes the evolution of the amplitude of the fundamental; this is to be compared with the NLS equation valid for long waves, (4.44):

$$-2\mathrm{i}k\delta^2 A_{0\tau} + \delta^4 k^2 A_{0\zeta\zeta} - \frac{9}{2} A_0 |A_0|^2 = 0. \tag{4.52}$$

When we introduce the variables (4.47) into (4.52), that is

$$\tau = \varepsilon^2 t = \delta^2 T, \quad \zeta = \varepsilon\{x - (1 - \frac{1}{2}\delta^2 k^2)t\} = \delta^2 Z,$$

we obtain

$$-2\mathrm{i}kA_{0T} + k^2 A_{0ZZ} - \frac{9}{2} A_0 |A_0|^2 = 0,$$

which is precisely equation (4.51) (since $A_0 \equiv A_{01}$): the short-wave limit of the KdV equation (for harmonic waves) matches the long-wave limit of the NLS equation. The same match also occurs between the Davey–Stewartson equations, (4.37) and (4.38), and the 2D KdV equation, (3.30); this is discussed in Q4.7 and by Freeman & Davey (1975).

4.2 NLS and DS equations: some results from soliton theory

The Nonlinear Schrödinger (NLS) equation, which in one version is often written in the simple (normalised) form as

$$\mathrm{i}u_t + u_{xx} + u|u|^2 = 0 \tag{4.53}$$

(see Q4.8 and below), is one of the *completely integrable* equations; we will call (4.53) the NLS+ equation (see below). The method of solution involves an important extension of that used for the solution of the KdV equation (described in Section 3.3.1). The central idea is to replace the scalar functions F and K (as used, for example, in equations (3.51) and (3.52)) by 2×2 matrix functions, in an approach developed first by Zakharov & Shabat (1972); see also Shabat (1973), Zakharov & Shabat (1974). On this basis we shall present the general method of solution for the NLS equation, written both in the form (4.53) and also in the (normalised) form (called NLS–)

$$\mathrm{i}u_t + u_{xx} - u|u|^2 = 0. \tag{4.54}$$

It turns out that the relative sign of the terms u_{xx} and $u|u|^2$ is important in determining the essential character of the solution of the NLS equation (hence: NLS+, NLS–); for some simple solutions see Q4.9–Q4.12. To change the sign of the term $\mathrm{i}u_t$ is simply equivalent to taking the complex conjugate of the equation. We shall later mention the Davey–Stewartson equations, and how the bilinear method (see Section 3.3.3) and conservation laws (Section 3.3.4) are relevant to this NLS family of equations.

4.2.1 Solution of the Nonlinear Schrödinger equation

We follow the notation used in Section 3.3.1; but here, $F(x, z, t)$ is a 2×2 matrix function which satisfies the pair of (matrix) equations

$$\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} F_x + F_z \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} = 0; \quad F_{xx} - F_{zz} - \mathrm{i}\alpha F_t = 0, \tag{4.55}$$

where l, m and α are arbitrary real constants. The 2×2 function $K(x, z; t)$ is then a solution of the *matrix* Marchenko equation

$$K(x, z; t) + F(x, z, t) + \int_x^\infty K(x, y; t) F(y, z, t)\,\mathrm{d}y = 0. \tag{4.56}$$

We write this solution as

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \text{then} \quad u(x, t) = b(x, x; t) \tag{4.57}$$

is a solution of the general NLS± equation

$$\mathrm{i}\alpha(l - m)u_t + (l + m)u_{xx} \pm \frac{2}{lm}(l - m)(l^2 - m^2)u|u|^2 = 0; \tag{4.58}$$

see Q4.17. The choice of signs in equation (4.58) is governed by the two possibilities for $c(x, x; t)$, namely $\pm u^*$ (where the asterisk denotes the complex conjugate). Armed with this information, we will describe how the equations are solved in order to generate the solitary-wave solution of the NLS+ equation; we shall then indicate how this approach is extended to embrace the N-soliton solution.

Example: solitary-wave solution

The required solutions are obtained when we set

$$F = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix};$$

then the first of equations (4.55) yields

$$f = f(mx - lz, t) \quad g = g(lx - mz, t),$$

but f and g are otherwise arbitrary functions. The second of equations (4.55) is now satisfied by the choice of the (single) exponential solutions

$$\left.\begin{aligned} f &= f_0 \exp\{\lambda(mx - lz) + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}; \\ g &= g_0 \exp\{\mu(lx - mz) + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\}, \end{aligned}\right\} \tag{4.59}$$

where f_0, g_0, λ and μ are arbitrary constants. This introduction of an exponential solution is to be compared with equation (3.55), for the KdV equation.

The matrix Marchenko equation, (4.56), with K given by the expression in (4.57), becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} + \int_x^\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} \mathrm{d}y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.60}$$

where we have suppressed the arguments of the functions; the functions f and g are given by (4.59). Thus, from equation (4.60), we obtain four scalar equations:

$$a(x, z; t) + \int_x^\infty b(x, y; t) g_0 \exp\{\mu(ly - mz) + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\}\mathrm{d}y = 0; \quad (4.61)$$

$$b(x, z; t) + f_0 \exp\{\lambda(mx - lz) + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}$$
$$+ \int_x^\infty a(x, y; t) f_0 \exp\{\lambda(my - lz) + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}\mathrm{d}y = 0, \quad (4.62)$$

and two similar equations for $c(x, z; t)$ and $d(x, z; t)$ (whose identification is left as an exercise in Q4.18). The integral equations, (4.61) and (4.62), clearly possess solutions

$$a(x, z; t) = \mathrm{e}^{-\mu mz} M(x; t), \quad b(x, z; t) = \mathrm{e}^{-\lambda lz} L(x; t),$$

respectively, and so (4.61) and (4.62) yield

$$M + g_0 L \int_x^\infty \exp\{l(\mu - \lambda)y + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\}\mathrm{d}y = 0;$$

$$L + f_0 \exp\{\lambda mx + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}$$
$$+ f_0 M \int_x^\infty \exp\{m(\lambda - \mu)y + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}\mathrm{d}y = 0.$$

These two equations exist only if

$$\mathscr{R}\{l(\mu - \lambda)\} < 0 \quad \text{and} \quad \mathscr{R}\{m(\lambda - \mu)\} < 0, \quad (4.63)$$

for otherwise the integrals would not be finite. These two conditions imply that l and m must be of opposite sign; let us, for ease of further calculation, choose

$$l = 2, \quad m = -1 \quad \text{and} \quad \alpha = \frac{1}{3}. \quad (4.64)$$

Equation (4.58), with these choices, then becomes

$$\mathrm{i}u_t + u_{xx} \mp 9u|u|^2 = 0 \quad (4.65)$$

(and consequently equations (4.53) and (4.54) are recovered if we transform $u \to u/3$). The solution for $L(x; t)$, with $\mathscr{R}(\mu - \lambda) < 0$, is now obtained directly as

$$L(x; t) = \frac{f_0 \exp(-\lambda x + 9\mathrm{i}\lambda^2 t)}{\frac{1}{2} f_0 g_0 (\lambda - \mu)^{-2} \exp\{3(\mu - \lambda)[x - 3\mathrm{i}(\lambda + \mu)t]\} - 1}$$

and a convenient choice affording further simplification is

$$\frac{1}{2}f_0 g_0(\lambda - \mu)^{-2} = -1; \tag{4.66}$$

the solution that we seek is therefore

$$\begin{aligned} u(x, t) = b(x, x; t) &= \mathrm{e}^{-\lambda l x} L(x; t) \\ &= \frac{-f_0 \exp\{-3\lambda(x - 3\mathrm{i}\lambda t)\}}{1 + \exp\{3(\mu - \lambda)[x - 3\mathrm{i}(\lambda + \mu)t]\}}. \end{aligned} \tag{4.67}$$

Finally, we introduce real parameters k and p such that

$$\lambda = k + \mathrm{i}p, \quad \mu = -k + \mathrm{i}p \quad (k > 0)$$

and then $f_0 g_0 = -8k^2$.

The corresponding calculation for $d(x, z; t)$ and $c(x, z; t)$ can be followed through (see Q4.18, Q4.19); when we impose the condition $c(x, x; t) = -u^*$, we find that $g_0 = -f_0^*$ and hence $f_0 = \pm 2\sqrt{2}k$ (choosing a real f_0). The solution (4.67) now becomes

$$u(x, t) = \pm\sqrt{2}k \exp\{-3\mathrm{i}px + 9\mathrm{i}(k^2 - p^2)t\}\operatorname{sech}(3kx + 18kpt)$$

and if we identify

$$a = 3\sqrt{2}k, \quad c = -6p \quad \text{and} \quad n = \frac{1}{2}(a^2 + \frac{1}{2}c^2)$$

we obtain the solitary-wave solution

$$u(x, t) = \pm a \exp\{\mathrm{i}[\tfrac{1}{2}c(x - ct) + nt]\}\operatorname{sech}\{a(x - ct)/\sqrt{2}\} \tag{4.68}$$

of the NLS+ equation

$$\mathrm{i}u_t + u_{xx} + u|u|^2 = 0,$$

which in the form (4.68) is discussed in Q4.9. The NLS solitary wave is an oscillatory wave packet which propagates at a speed c, the underlying oscillation being governed by the frequency n (which is a function of the wave amplitude and speed). An example of this wave is given in Figure 4.2 (for the choice $a = 1$, $c = 10$ and at two different times) where we have elected to show only the real part of u. The imaginary part is, of course, very similar, and the modulus of u is simply

$$|u| = a \operatorname{sech}\{a(x - ct)/\sqrt{2}\},$$

a sech profile. Other simple solutions of the NLS+ equation are possible; see Q4.11 and Q4.12. Examples of these two solutions (the *Ma* and

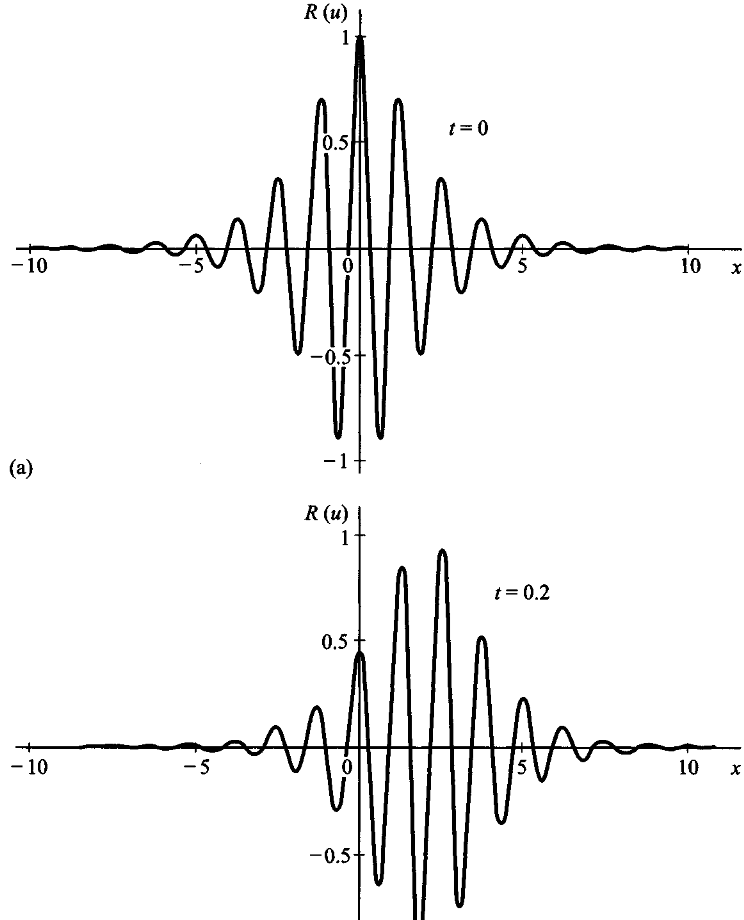
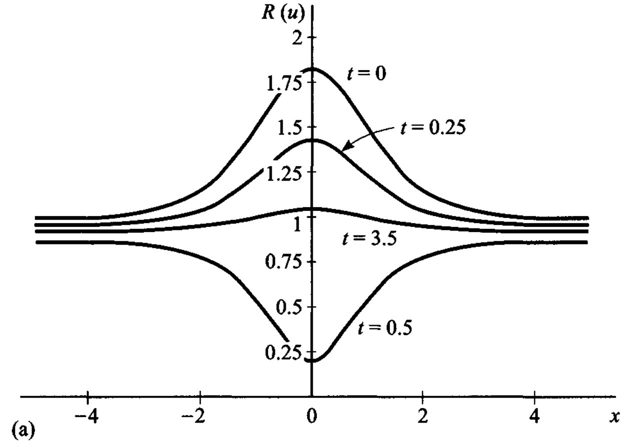


Figure 4.2. Real part of the solitary-wave solution, (4.68), of the NLS+ equation, for $a = 1$ and $c = 10$, at times $t = 0$(a), 0.2(b).

rational-cum-oscillatory waves) are depicted in Figure 4.3. As the figures clearly demonstrate, these solutions take the form of standing waves; not surprisingly, the solutions of most interest to use are those that represent the propagation – and interaction – of the nonlinear waves. Finally, we comment that the N-soliton solution of the NLS equation is obtained in the obvious way. That is, for the function f, for example (see equation (4.59)), we write



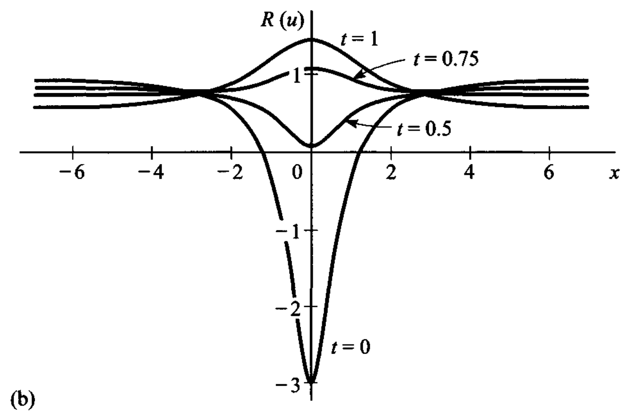


Figure 4.3. (a) Real part of the Ma solitary wave, given in Q4.11, for $a = m = 1$, at times $t = 0$, 0.25, 0.35, 0.5. (b) Real part of the rational-cum-oscillatory solution, given in Q4.12, at times $t = 0$, 0.5, 0.75, 1.

$$f(x, z; t) = \sum_{n=1}^{N} f_n \exp\{\lambda_n(mx - lz) + i\lambda_n^2(l^2 - m^2)t/\alpha\}, \tag{4.69}$$

where f_n and λ_n are arbitrary constants. We shall write more about these solutions, and their interpretation in the context of water-wave theory, later. Here, we comment that the N-soliton solution is more readily obtained by direct methods, such as Hirota's bilinear method, which we now present.

4.2.2 Bilinear method for the NLS equation

In Section 3.3.3 we introduced Hirota's bilinear method for the KdV equation, which led to the bilinear form of that equation:

$$(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4)(f \cdot f) = 0.$$

This equation and its solution will be found in (3.74) *et seq*. Of some importance for us was that this approach provided a rather direct route to the construction of N-soliton solutions. Furthermore, we also gave the bilinear form of a number of other equations that belong to the KdV family of completely integrable equations.

Now, the NLS equation is a completely integrable equation (as we mentioned in Section 4.2.1), so it is no surprise to learn that this equation can be expressed in a bilinear form. It can be shown (see Q4.20) that the NLS equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon = \pm 1) \tag{4.70}$$

with $u = g/f$, where f is a real function, can be written as a *pair* of bilinear equations:

$$(\mathrm{iD}_t + \mathrm{D}_x^2)(g \cdot f) = 0; \quad \mathrm{D}_x^2(f \cdot f) = \varepsilon|g|^2. \tag{4.71}$$

We observe, just as we found with the KdV family, that the linearised operator which appears in the NLS equation (that is, $\mathrm{i}\partial/\partial t + \partial^2/\partial x^2$) has a direct counterpart in equations (4.71), namely $(\mathrm{iD}_t + \mathrm{D}_x^2)$. As an example of the method of solution here, we seek the solitary-wave solution of (4.70) via (4.71).

From equations (4.67) or (4.68), we see that an obvious way to proceed is, first, to introduce

$$\theta = kx + \omega t + \alpha, \tag{4.72}$$

where k, ω and α are complex constants, and then to write

$$g = \mathrm{e}^{\theta}, \quad f = 1 + A\exp(\theta + \theta^*) \tag{4.73}$$

where A is a real constant (and the asterisk denotes the complex conjugate). On recalling the properties of the bilinear operator (described in Section 3.3.3 and in Q3.24), we find that the second equation in (4.71) gives (the non-zero contributions)

$$\mathrm{D}_x^2\{1 \cdot A\exp(\theta + \theta^*) + A\exp(\theta + \theta^*) \cdot 1\} = \varepsilon\exp(\theta + \theta^*),$$

so

$$A = \frac{1}{2}\varepsilon(k + k^*)^{-2}. \tag{4.74}$$

The first equation of (4.71) becomes

$$(\mathrm{iD}_t + \mathrm{D}_x)\{\mathrm{e}^\theta \cdot 1 + \mathrm{e}^\theta \cdot A\exp(\theta + \theta^*)\} = 0$$

which yields

$$(\mathrm{i}\omega + k^2)\mathrm{e}^\theta + A\left[\mathrm{i}(\omega - \omega - \omega^*) + (k - k - k^*)^2\right]\exp(2\theta + \theta^*) = 0,$$

which requires

$$\mathrm{i}\omega + k^2 = 0 \quad \text{and} \quad -\mathrm{i}\omega^* + k^{*2} = 0.$$

These are clearly consistent with

$$\omega = \mathrm{i}k^2; \tag{4.75}$$

thus we have a solution

$$u = g/f = \mathrm{e}^\theta \Big/ \left\{1 + \frac{1}{2}\varepsilon(k + k^*)^{-2}\exp(\theta + \theta^*)\right\} \tag{4.76}$$

with

$$\theta = kx + \mathrm{i}k^2 t + \alpha,$$

where k and α are arbitrary (complex) parameters. It is clear that (4.76) provides a bounded solution only in the case $\varepsilon = +1$; that is, for the NLS+ equation; cf. Q4.9 and Q4.10. Then, for this case, solution (4.76) can be recast precisely in the form of (4.68) if we make the identification

$$k = \frac{a}{\sqrt{2}} + \mathrm{i}\frac{c}{2} \tag{4.77}$$

and choose α to be real, and such that

$$\sqrt{A}\,\mathrm{e}^\alpha = 1. \tag{4.78}$$

(The rôle of α is simply to provide a constant phase-shift in the solution.) The use of (4.77) and (4.78) in (4.76) gives, after a little manipulation,

$$u = \pm a\exp\left\{\mathrm{i}\left[\frac{1}{2}c(x - ct) + nt\right]\right\}\operatorname{sech}\left\{a(x - ct)/\sqrt{2}\right\},$$

with $n = \frac{1}{2}(a^2 + \frac{1}{2}c^2)$, all exactly as in (4.68). In summary, therefore, the solitary-wave solution of the NLS+ equation can be expressed as

$$g = \mathrm{e}^{\theta}, \quad f = 1 + A\exp(\theta + \theta^*), \quad \theta = kx + \mathrm{i}k^2 t + \alpha.$$

The method that we have described can be extended to obtain the N-soliton solution, although the calculation – even for the case $N = 2$ – is considerably more involved than for the KdV family of equations. We shall present the results that produce the 2-soliton solution, but the details are left as an exercise (Q4.30). First, we write

$$g = E_1(1 + b_2 E_2 E_2^*) + E_2(1 + b_1 E_1 E_1^*) \tag{4.79}$$

with

$$E_m = \exp(k_m x + \mathrm{i}k_m^2 t + \alpha_m), \quad m = 1, 2,$$

where b_m are constants. Correspondingly, we have

$$f = 1 + f_1 E_1 E_1^* + f_2 E_2 E_2^* + cE_1 E_2^* + c^* E_1^* E_2 + dE_1 E_1^* E_2 E_2^*, \tag{4.80}$$

where f_m, c and d are constants. These two expressions are substituted into

$$(\mathrm{i}\mathrm{D}_t + \mathrm{D}_x^2)(g \cdot f) = 0, \quad \mathrm{D}_x^2(f \cdot f) = |g|^2;$$

we find that the given f and g satisfy both equations provided

$$f_m = \frac{1}{2}(k_m + k_m^*)^{-2}; \quad c = \frac{1}{2}(k_1 + k_2^*)^{-2}; \quad d = \left(\frac{k_1 + k_2^*}{k_1^* + k_2}\right)^2 b_1^* b_2$$

with

$$b_m = \frac{(k_1 - k_2)^2}{2(k_m + k_m^*)^2(k_m^* + k_n)^2} \quad (n = 1, 2; n \neq m).$$

The solution, (4.79) with (4.80), represents the interaction of two solitons which asymptotically take the form

$$u_m = a_m \exp\left\{\mathrm{i}\left[\frac{1}{2}c_m(x - c_m t) + n_m t\right]\right\} \operatorname{sech}\left\{a_m(x - c_m t)/\sqrt{2}\right\} \tag{4.81}$$

where

$$k_m = \frac{a_m}{\sqrt{2}} + \mathrm{i}\frac{c_m}{2} \quad \text{with} \quad n_m = \frac{1}{2}(a_m^2 + \frac{1}{2}c_m^2), \tag{4.82}$$

for $m = 1, 2$. An example of this solution is depicted in Figure 4.4 (where we have chosen $a_1 = \sqrt{2}$, $a_2 = 2\sqrt{2}$, $c_1 = -2$ and $c_2 = 2$ and we have

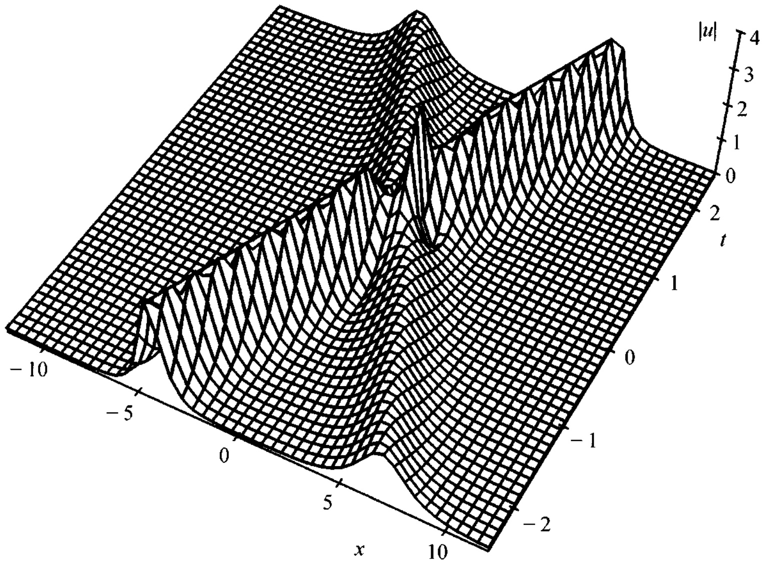


Figure 4.4. Two-soliton solution of the NLS+ equation (based on equations (4.79) and (4.80)); $|u|$ is plotted here for the case $a_1 = \sqrt{2}$, $a_2 = 2\sqrt{2}$, $c_1 = -2$, $c_2 = 2$.

plotted $|u|$); the interaction, together with one of the resulting phase shifts, are clearly shown in this figure.

Finally, before we leave this 2-soliton example altogether, we make an important observation that distinguishes this type of interaction from the KdV-type. In the case of KdV solitons (and others of this family), each separate soliton must have its own distinct speed at infinity. (The general 2-soliton solution of the KdV equation, (3.58), with $k_1 = k_2$ merely recovers the solitary-wave solution with parameter k_1.) However, the 2-soliton solution here, (4.79) and (4.80), contains essentially the four real parameters a_m, c_m $(m = 1, 2)$ given by (4.82) (since the α_m simply provide arbitrary phase shifts at a prescribed instant in time). The speed of the NLS+ soliton – the envelope function is the relevant part – is given by c_m; see (4.81). If we set $c_1 = c_2$, and retain $a_1 \neq a_2$, the two solitons remain distinct but *do not move apart*: they stay bound together and forever interact. The object so produced is itself a new type of solitary wave (with three parameters: a_1, a_2 and $c_1 = c_2$); it is called a *bound soliton* or *bi-soliton*, and it can interact with other similar or different solitons. These different solitons might be the classical ones for the NSL+

equation (such as given in (4.81)) or higher-order bound solitons formed by producing the N-soliton solution (following (4.79) and (4.80)) and then choosing $c_m = c$, for $m = 1, 2, \ldots, N$. A description of the bi-soliton solution that is obtained from (4.79) and (4.80) is left as an exercise (Q4.31); an example of a bi-soliton solution is given in Figure 4.5 (where we have used $a_1 = \sqrt{2}$, $a_2 = 2\sqrt{2}$, $c_1 = c_2 = -2$). The bound, but varying, nature of this solution is quite evident from the figure.

It is clear from our examination of some of the solutions of the NLS+ equation that it possesses a very rich set of solutions – far more than for the KdV equation and other members of that family. Of course, which solution is the relevant one in a given situation is controlled by the precise details of the initial profile. As a significant addition to this brief description of the solutions of the NLS+ equation, we shall later present an important application of the equation (to study the stability of the Stokes wave; Section 4.3.1).

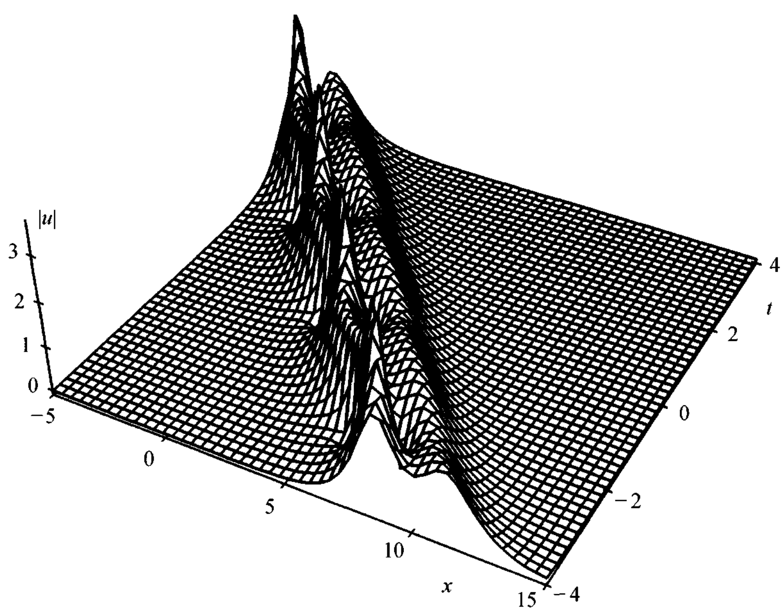


Figure 4.5. Bi-soliton (or *bound* soliton) of the NLS+ equation (given in Q4.31); $|u|$ is plotted here for the case $a_1 = \sqrt{2}$, $a_2 = 2\sqrt{2}$, $c_1 = c_2 = -2$.

4.2.3 Bilinear form of the DS equations for long waves

The long wave ($\delta \to 0$) approximation of the Davey–Stewartson equations (see Section 4.1.2), which is discussed in Q4.6, can be written as

$$-2ikA_{0\tau} + \delta^2 k^2 A_{0\zeta\zeta} - A_{0YY} + \frac{9}{2\delta^2} A_0 |A_0|^2 + 3k^2 A_0 f_{0\zeta} = 0 \tag{4.83}$$

with

$$\delta^2 k^2 f_{0\zeta\zeta} + f_{0YY} = -3(|A_0|^2)_\zeta, \tag{4.84}$$

and these equations possess a compact bilinear representation. However, it is necessary first to change the variables (both dependent and independent) by introducing a more convenient set; we follow the ideas described by Anker & Freeman (1978) and Freeman (1984).

First we define

$$f_{0\zeta} = \phi_{YY} + \lambda |A_0|^2$$

where $\phi = \phi(\zeta, Y, \tau)$ and λ is a (complex) constant to be determined; then we see that equation (4.84) can be written, after one differentiation with respect to ζ, as

$$\delta^2 k^2 f_{0\zeta\zeta\zeta} + f_{0YY\zeta} = -3(|A_0|^2)_{\zeta\zeta},$$

so

$$\delta^2 k^2 (\phi_{YY} + \lambda |A_0|^2)_{\zeta\zeta} + (\phi_{YY} + \lambda |A_0|^2)_{YY} = -3(|A_0^2|)_{\zeta\zeta}.$$

We choose $\delta^2 k^2 \lambda = -3$ (so λ turns out to be real), to leave

$$\delta^2 k^2 \phi_{YY\zeta\zeta} + \phi_{YYYY} - \frac{3}{\delta^2 k^2} |A_0|^2_{YY} = 0$$

or

$$\delta^2 k^2 \phi_{\zeta\zeta} + \phi_{YY} = \frac{3}{\delta^2 k^2} |A_0|^2 \tag{4.85}$$

when we integrate and then invoke decay conditions at infinity. In equation (4.83) we substitute for $f_{0\zeta}$ to give

$$-2ikA_{0\tau} + \delta^2 k^2 A_{0\zeta\zeta} - A_{0YY} + \frac{3}{2} k^2 A_0 \left(2\phi_{YY} - \frac{3}{\delta^2 k^2} |A_0|^2\right) = 0$$

and then upon substituting for $|A_0|^2$ from (4.85) this yields

$$-2ikA_{0\tau} + \delta^2 k^2 A_{0\zeta\zeta} - A_{0YY} + \frac{3}{2} k^2 A_0 (\phi_{YY} - \delta^2 k^2 \phi_{\zeta\zeta}) = 0. \tag{4.86}$$

At this stage we define

$$x = \frac{\zeta}{\delta k} - Y, \quad y = \frac{\zeta}{\delta k} + Y, \tag{4.87}$$

and hence equations (4.85) and (4.86) become

$$2(\phi_{xx} + \phi_{yy}) = \frac{3}{\delta^2 k^2} |A_0|^2;$$

$$-\mathrm{i}kA_{0\tau} + 2A_{0xy} + 3k^2 A_0 \phi_{xy} = 0.$$

Let us write

$$\phi_x = u, \quad \phi_y = -v$$

(where u and v are real which, with the definition of λ from above, implies then f_0 is real), then we obtain the pair of equations

$$2\delta^2 k^2 (u_x - v_y) = 3|A_0|^2;$$

$$-\mathrm{i}kA_{0\tau} + 2A_{0xy} + 3k^2 A_0 u_y = 0.$$

Finally, we define the complex function

$$Z = u + \mathrm{i}v$$

so that

$$u = \frac{1}{2}(Z + Z^*) \quad \text{and} \quad u_x - v_y = Z_x + \mathrm{i}Z_y$$

(since $u_y = -v_x$); our pair of equations therefore becomes

$$Z_x + \mathrm{i}Z_y = \frac{3}{2\delta^2 k^2} |A_0|^2;$$

$$-\mathrm{i}\frac{k}{2} A_{0\tau} + A_{0xy} + \frac{3k^2}{4} A_0 (Z + Z^*)_y = 0$$

which, with the scaling transformations

$$\tau \to -\frac{2\tau}{k}, \quad Z \to \frac{8}{3k^2} Z, \quad A_0 \to \frac{4}{3\delta} A_0$$

yields

$$\mathrm{i}A_{0\tau} + A_{0xy} + 2A_0(Z + Z^*)_y = 0; \quad Z_x + \mathrm{i}Z_y = |A_0|^2; \tag{4.88}$$

this is essentially the form of the equations discussed by Anker & Freeman (1978) and Freeman (1984). (Our scaling transformation also involves a change in sign of τ; this is avoided if the second equation is expressed in terms of the conjugate, A_0^*.) It can be shown (Q4.27) that equations (4.88) possess a simple bilinear representation:

$$(\mathrm{iD}_\tau + \mathrm{D}_x\mathrm{D}_y)(g \cdot f) = 0; \quad (\mathrm{D}_x^2 + \mathrm{D}_y^2)(f \cdot f) = 2|g|^2 \tag{4.89}$$

where

$$A_0 = \frac{g}{f}, \quad Z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) \ln f \quad (f \text{ real});$$

cf. equations (4.71) for the NLS equation.

A simple solution of equations (4.89) is obtained by following the development that has been described for the NLS equation; see equations (4.73) *et seq*. Thus, if we set

$$g = \mathrm{e}^\theta, \quad f = 1 + \mu \exp(\theta + \theta^*),$$

where $\theta = kx + ly + \omega\tau + \alpha$ and μ is a real constant, we find (Q4.28) that, for example,

$$A_0 = e^\theta / \{1 + [(k + k^*)^2 + (l + l^*)^2]^{-1} \exp(\theta + \theta^*)\} \tag{4.90}$$

with

$$\theta = kx + ly + \mathrm{i}kl\tau + \alpha,$$

where k, l and α are arbitrary (complex) constants. Solution (4.90) is the solitary wave solution of the long-wave Davey-Stewartson equations (although we should remember that x and y are not the physical coordinates used to describe the horizontal plane in which the wave propagates; see (4.87)). This solution, (4.90), should be compared (see Q4.29) with that discussed in Q4.26; the generalisation to N solitons follows the method adopted for the NLS equation, and presented in equations (4.79) and (4.80).

4.2.4 Conservation laws for the NLS and DS equations

All completely integrable equations possess an infinite number of conservation laws, the first few of which – certainly the first three – have simple and direct physical interpretations. These ideas were introduced and explored in the context of the KdV equation (and its associated family of equations) in Section 3.3.4. We now describe how the corresponding picture is developed for the NLS equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon = \pm 1). \tag{4.91}$$

The construction of the conservation laws for this equation is fairly straightforward, although the procedure becomes progressively more tedious for the 'higher' laws.

First, we write down the equation which is satisfied by u^*, the conjugate of u, namely

$$-\mathrm{i}u_t^* + u_{xx}^* + \varepsilon u^*|u|^2 = 0, \tag{4.92}$$

and then we form $u^* \times (4.91) - u \times (4.92)$ to give

$$\mathrm{i}(u^*u_t + uu_t^*) + u^*u_{xx} - uu_{xx}^* = 0.$$

This equation is immediately

$$\mathrm{i}\frac{\partial}{\partial t}(uu^*) + \frac{\partial}{\partial x}(u^*u_x - uu_x^*) = 0,$$

which we now integrate over all x; provided that ambient conditions exist at infinity (so that conditions at $\pm\infty$ are identical), we obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}|u|^2\mathrm{d}x = 0,$$

so

$$\int_{-\infty}^{\infty}|u|^2\mathrm{d}x = \text{constant}. \tag{4.93}$$

Equation (4.93) is the first conservation law (for both versions of the NLS equation). Based on our experience with the KdV equation (see Section 3.3.4), we would expect this law to be associated with the conservation of mass; we shall confirm this interpretation shortly.

A second conservation law is derived by forming $u_x^* \times (4.91) + u_x \times (4.92)$, to produce

$$\mathrm{i}(u_x^*u_t - u_xu_t^*) + u_x^*u_{xx} + u_xu_{xx}^* + \varepsilon(u_x^*u + u_xu^*)|u|^2 = 0. \tag{4.94}$$

Further, we also construct $u^* \times (4.91)_x + u \times (4.92)_x$ to give

$$\mathrm{i}(u^*u_{xt} - uu_{xt}^*) + u^*u_{xxx} + uu_{xxx}^* + \varepsilon\{u^*(u|u|^2)_x + u(u^*|u|^2)_x\} = 0 \tag{4.95}$$

and then we form (4.94) – (4.95):

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial t}(uu_x^* - u^*u_x) + \frac{\partial}{\partial x}(u_xu_x^*) - (u^*u_{xxx} + uu_{xxx}^*)&\\ + \varepsilon|u|^2(uu^*)_x - \varepsilon\{(uu^*)_x|u|^2 + 2uu^*(|u|^2)_x\} &= 0.\end{aligned}$$

This equation is re-expressed as

$$\mathrm{i}\frac{\partial}{\partial t}(uu_x^* - u^*u_x) + \frac{\partial}{\partial x}(u_x u_x^*) - \frac{\partial}{\partial x}(u^*u_{xx} + uu_{xx}^*) \\ + \frac{\partial}{\partial x}(u_x u_x^*) - 2\varepsilon|u|^2(|u|^2)_x = 0$$

which is

$$\mathrm{i}\frac{\partial}{\partial t}(uu_x^* - u^*u_x) + \frac{\partial}{\partial x}\{2u_x u_x^* - (u^*u_{xx} + uu_{xx}^*) - \varepsilon|u|^4\} = 0.$$

Hence, with ambient conditions existing at infinity (as invoked above), this yields

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(uu_x^* - u^*u_x)\,\mathrm{d}x = 0,$$

so

$$\int_{-\infty}^{\infty}(uu_x^* - u^*u_x)\,\mathrm{d}x = \text{constant} \tag{4.96}$$

is the second conservation law.

A third conservation law, whose derivation is left as an exercise (see Q4.32), is

$$\int_{-\infty}^{\infty}\left\{|u_x|^2 - \frac{1}{2}\varepsilon|u|^4\right\}\mathrm{d}x = \text{constant}; \tag{4.97}$$

this is the simplest law that includes $\varepsilon\,(=\pm1)$ and therefore takes different forms for the two NLS equations (NLS+, NLS–). We have produced the first three conservation laws; that an infinity exists is proved by Zakharov & Shabat (1972), and a little further exploration is provided in Q4.32 and Q4.33. The relation between these conservation laws, and the conserved quantities that arise directly from the governing equations, will now be briefly investigated.

The conservation of mass, equation (3.86), is

$$\eta_t + \frac{\partial}{\partial x}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) = 0; \tag{4.98}$$

this must be written in terms of the variables used in the derivation of the NLS equation (see (4.2)). Equation (4.98) therefore becomes

$$\varepsilon^2\eta_\tau - \varepsilon c_g \eta_\zeta - c_p \eta_\xi + \left(\frac{\partial}{\partial\xi} + \varepsilon\frac{\partial}{\partial\zeta}\right)\left\{\int_0^{1+\varepsilon\eta} (\phi_\xi + \varepsilon\phi_\zeta)\mathrm{d}z\right\} = 0 \qquad (4.99)$$

where we have introduced $u = \phi_x = \phi_\xi + \varepsilon\phi_\zeta$. The conservation law we require is expressed in terms of (τ, ζ); see our NLS equation for water waves, (4.32). First, therefore, equation (4.99) is integrated in ξ over one period (for example, from 0 to $2\pi/k$) to give

$$\varepsilon\bar{\eta}_\tau - c_g\bar{\eta}_\zeta + \frac{\partial}{\partial\zeta}\overline{\left\{\int_0^{1+\varepsilon\eta} (\phi_\xi + \varepsilon\phi_\zeta)\,\mathrm{d}z\right\}} = 0 \qquad (4.100)$$

where the overbar denotes the integral in ξ, and we have used the property that our solution is strictly periodic in ξ, at fixed τ, ζ. Now we integrate equation (4.100) over all ζ, and again use ambient conditions as $\zeta \to \pm\infty$, to obtain

$$\int_{-\infty}^{\infty} \bar{\eta}\,\mathrm{d}\zeta = \text{constant}, \qquad (4.101)$$

which is the appropriate form of the conservation of mass that we need here. However, from equation (4.14) (see also equations (4.36)), we have

$$\eta \sim \sum_{n=0}^{\infty} \varepsilon^n \left\{\sum_{m=0}^{n+1} A_{nm}(\zeta, \tau)E^m + \text{c.c.}\right\}, \qquad \varepsilon \to 0,$$

with $A_{00} = 0$; thus

$$\bar{\eta} = \int_0^{2\pi/k} \eta\,\mathrm{d}\xi \sim \frac{2\pi}{k}\sum_{n=1}^{\infty} \varepsilon^n A_{n0}. \qquad (4.102)$$

The dominant behaviour (as $\varepsilon \to 0$) with (4.102) used in (4.101) then yields

$$\int_{-\infty}^{\infty} A_{10}\,\mathrm{d}\zeta = \text{constant},$$

and from equation (4.22) we have

$$A_{10} = -\frac{2\delta k}{\sinh \delta k}|A_0|^2 + c_g f_{0\zeta}$$

(where A_0 is written for A_{01}), and so

$$\int_{-\infty}^{\infty} |A_0|^2 \mathrm{d}\zeta = \text{constant}$$

which is therefore the conservation of mass for our NLS equation, (4.32), for $A_0(\zeta, \tau)$; this recovers our first conservation law (4.93), where for u read A_0.

The equivalent calculation for the conservation of momentum, starting from equation (3.92):

$$\frac{\partial}{\partial t}\left(\int_0^{1+\varepsilon\eta} u\,\mathrm{d}z\right) + \frac{\partial}{\partial x}\left\{\int_0^{1+\varepsilon\eta} (\varepsilon u^2 + p)\mathrm{d}z - \frac{1}{2}\varepsilon\eta^2\right\} = 0,$$

and leading to a conservation law of the type given in (4.96), is left as an exercise (Q4.34). The correspondence between the conservation of energy for the wave motion, and the third conservation law (4.97), is obtained in a similar way (although in this case the connection is less easily confirmed).

Before we leave the discussion of the conservation laws altogether, we briefly mention the situation with regard to the Davey–Stewartson equations. These are by no means straightforward to analyse, and this is because the waves depend on two variables (ζ and Y) in the horizontal plane. (Similar difficulties were encountered with the 2D KdV equation; see Section 3.3.4, equation (3.98) *et seq.*) However, we provide the first stage in the discussion of these equations; let us write them in the form (cf. equations (4.37), (4.38))

$$f_{0YY} + \lambda f_{0\zeta\zeta} = \mu(|A_0|^2)_\zeta$$

$$-\mathrm{i}\alpha A_{0\tau} + \beta A_{0\zeta\zeta} - \gamma A_{0YY} + \delta A_0|A_0|^2 + A_0 f_{0\zeta} = 0$$

where λ, μ, α, β, γ and δ are real constants. We take f_0 to be a real function (as we found for the solutions described in Section 4.2.3), so the conjugate of the second equation becomes

$$\mathrm{i}\alpha A^*_{0\tau} + \beta A^*_{0\zeta\zeta} - \gamma A^*_{0YY} + \delta A^*_0|A_0|^2 + A^*_0 f_{0\zeta} = 0.$$

The procedure adopted for the NLS equation (see equation (4.92) *et seq.*) then gives

$$\mathrm{i}\alpha\frac{\partial}{\partial\tau}(A_0A_0^*)+\beta\frac{\partial}{\partial\zeta}(A_0A_{0\zeta}^*-A_0^*A_{0\zeta})+\gamma\frac{\partial}{\partial Y}(A_0^*A_{0Y}-A_0A_{0Y}^*)=0,$$

which is in conservation form. Thus we obtain

$$\mathrm{i}\alpha\frac{\partial}{\partial\tau}\left(\int_{-\infty}^{\infty}|A_0|^2\,\mathrm{d}\zeta\right)+\gamma\frac{\partial}{\partial Y}\left(\int_{-\infty}^{\infty}(A_0^*A_{0Y}-A_0A_{0Y}^*)\,\mathrm{d}\zeta\right)=0$$

and

$$\mathrm{i}\alpha\frac{\partial}{\partial\tau}\left(\int_{-\infty}^{\infty}|A_0|^2\,\mathrm{d}Y\right)+\beta\frac{\partial}{\partial\zeta}\left(\int_{-\infty}^{\infty}(A_0A_{0\zeta}^*-A_0^*A_{0\zeta})\mathrm{d}Y\right)=0,$$

provided decay conditions exist as $|\zeta|\to\infty$, at fixed Y, and as $|Y|\to\infty$, at fixed ζ. This requires that the waves at infinity are not parallel to either the ζ or the Y coordinates; it is this type of additional assumption or restriction that complicates the issue. Furthermore, if decay conditions exist as $Y\to\pm\infty$, and as $\zeta\to\pm\infty$, that is, the solution vanishes (sufficiently rapidly) as $Y^2+\zeta^2\to\infty$, we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|A_0|^2\,\mathrm{d}\zeta\,\mathrm{d}Y=\text{constant},$$

a conserved quantity that applies only for a limited class of solutions. Nevertheless, albeit with some important restrictions, we have derived a rather conventional type of conservation law – clearly the conservation of mass.

Finally, the other equation in the DS pair is already in conservation form, namely

$$\frac{\partial}{\partial Y}(f_{0Y})+\frac{\partial}{\partial\zeta}(\lambda f_{0\zeta}-\mu|A_0|^2)=0$$

and so, for example, we obtain

$$\frac{\partial}{\partial\zeta}\left(\int_{-\infty}^{\infty}(\lambda f_{0\zeta}-\mu|A_0|^2)\,\mathrm{d}Y\right)+[f_{0Y}]_{-\infty}^{\infty}=0;$$

if conditions are the same as $\zeta \to \pm\infty$, then

$$\lambda \frac{\partial}{\partial \zeta}\left(\int_{-\infty}^{\infty} f_0 \, \mathrm{d}Y\right) - \mu \int_{-\infty}^{\infty} |A_0|^2 \, \mathrm{d}Y = g(\tau),$$

where $g(\tau)$ is an arbitrary function. If, for some ζ, the left-hand side of this equation is zero (because, for example, the solution decays in this region), we must have $g(\tau) = 0$ for all τ. Hence

$$\lambda \int_{-\infty}^{\infty} f_0 \, \mathrm{d}Y = \mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A_0|^2 \mathrm{d}Y \, \mathrm{d}\zeta = \text{constant}$$

if we follow our previous discussion; thus

$$\int_{-\infty}^{\infty} f_0 \, \mathrm{d}Y = \text{constant}$$

is another conservation law. A further small exploration of some special conservation laws of the DS equations will be found in Q4.35; other conservation laws are to be found in Q4.36.

4.3 Applications of the NLS and DS equations

We have presented a theory which describes how modulated harmonic waves arise in the study of water waves. So far, for both the one-dimensional and two-dimensional problems, we have restricted the scenario to the simplest possible: constant depth and stationary water, in the undisturbed state. As we explained for the various KdV problems (Section 3.4), an important question to pose is whether the simple ideas and constructions carry over to more realistic situations. Thus we shall now – without spelling-out all the details, because of the complexity of much of the work – show how the effects of an underlying shear, and of variable depth, manifest themselves in the modulation problems. In addition, and as our first application, we shall use the NLS and DS equations (precisely as already derived) to examine the stability of wave trains; these procedures can also be employed, with appropriate adjustments, when a shear or variable depth is included. Other ingredients, such as the inclusion of surface tension, will not be entertained here (since our interest, in this introductory text, still remains principally the study of gravity waves). However, these and other aspects are left to the interested reader, who may follow the various avenues through the references that appear later.

4.3.1 Stability of the Stokes wave

Our first and most direct application of the results obtained in Sections 4.1 and 4.2 is to the Stokes wave, which we introduced in Section 2.5. In order to see the relevance of the Nonlinear Schrödinger equation (and, indeed, the DS equations), we make use of a very special and simple solution of the NLS equation; for another, see Q4.40. From equation (4.32), our NLS equation is written as

$$-2\mathrm{i}kc_pA_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0|A_0|^2 = 0, \tag{4.103}$$

where α and β are given in (4.33) and (4.34). The nonlinear plane wave solution, with constant amplitude, of this equation is

$$A_0 = A\exp\{\mathrm{i}(K\zeta - \Omega\tau)\}, \tag{4.104}$$

where A is a complex constant and K is a real constant; (4.104) is then a solution of (4.103) provided that Ω satisfies the dispersion relation

$$2kc_p\Omega = \beta|A|^2 - \alpha K^2. \tag{4.105}$$

Here, α, β and c_p are all functions of k (> 0), the wave number of the carrier wave (as described in equations (4.7)); K (> 0) is the wave number of the modulation. The primary wave (from (4.7)) therefore becomes

$$\eta_0 = A\exp\{\mathrm{i}(kx + K\zeta - \omega t - \Omega\tau)\} + \text{c.c.},$$

and if we choose the wave number of this solution to be precisely k, then we set $K = 0$ to yield

$$\eta_0 = A\exp\{\mathrm{i}[kx - (\omega + \varepsilon^2\Omega)t]\} + \text{c.c.}, \tag{4.106}$$

with $\tau = \varepsilon^2 t$ and Ω given by (4.105) with $K = 0$.

Solution (4.106) is the Stokes wave (cf. equation (2.133)) of amplitude A and wave number k, with the frequency (dispersion function) taken as far as terms of $O(\varepsilon^2)$ (cf. equation (2.137)). However, we note that our description via the NLS equation also incorporates an additional component to the *set-down*, which is associated with the *mean drift* (see Section 2.5 and Q4.4), although this is not needed here. Further details of this connection will be found in Q4.42. Now, with this identification as the Stokes-wave solution, we can use the NLS equation to provide an estimate for the stability of the Stokes wave.

The NLS equation describes the modulation of the amplitude of the harmonic wave, represented by $E(+\text{c.c.})$, for initial data which depends on x (and the parameter ε) in a way consistent with the NLS equation. The Stokes wave is recovered by introducing the plane wave solution of constant amplitude; thus we seek a solution which takes, as its initial form, a small perturbation about the nonlinear plane wave solution, (4.106). Thus we set

$$A_0 = A(1+\Delta a)\exp\{\mathrm{i}(-\Omega\tau + \Delta\theta)\} \tag{4.107}$$

where $a = a(\zeta, \tau)$, $\theta = \theta(\zeta, \tau)$ (both taken to be real functions) and we have chosen $K = 0$ (as used above), so

$$2kc_p\Omega = \beta|A|^2; \tag{4.108}$$

Δ is a parameter that we shall regard as small in what follows. Direct substitution of (4.107) into (4.103) yields

$$\begin{aligned} &-2\mathrm{i}kc_p\{\Delta a_\tau + \mathrm{i}(1+\Delta a)(\Delta\theta_\tau - \Omega)\}A\mathscr{E} \\ &\qquad + \alpha\{\Delta a_{\zeta\zeta} + 2\mathrm{i}\Delta^2 a_\zeta\theta_\zeta + \Delta(1+\Delta a)(\mathrm{i}\theta_{\zeta\zeta} - \Delta\theta_\zeta^2)\}A\mathscr{E} \\ &\qquad\qquad + \beta(1+\Delta a)^3 A|A|^2\mathscr{E} = 0 \end{aligned}$$

where $\mathscr{E}$ is the exponential term in (4.107). The leading terms (that is, the O(1) terms as $\Delta \to 0$) cancel by virtue of (4.108), and then the leading perturbation terms (of O(Δ)) give

$$-2\mathrm{i}kc_p(a_\tau + \mathrm{i}\theta_\tau - \mathrm{i}\Omega) + \alpha(a_{\zeta\zeta} + \mathrm{i}\theta_{\zeta\zeta}) + 3\beta|A|^2 a = 0.$$

Since a and θ are real functions, and again invoking (4.108) to eliminate Ω, we obtain

$$\begin{aligned} 2kc_p\theta_\tau + \alpha a_{\zeta\zeta} + 2\beta|A|^2 a &= 0; \\ -2kc_p a_\tau + \alpha\theta_{\zeta\zeta} &= 0. \end{aligned}$$

This pair of equations is linear, with constant coefficients, so we have a solution

$$\begin{pmatrix} a \\ \theta \end{pmatrix} = \begin{pmatrix} a_0 \\ \theta_0 \end{pmatrix} \exp\{\mathrm{i}(\kappa\zeta - \lambda\tau)\} + \text{c.c.},$$

where a_0, θ_0, κ (> 0) and λ are constants; this solution exists provided (see Q4.43)

$$(2kc_p\lambda)^2 = \alpha^2\kappa^2(\kappa^2 - 2\beta|A|^2/\alpha), \tag{4.109}$$

the dispersion relation for λ.

Thus, from equation (4.109), it is immediately evident that for $\beta/\alpha < 0$ (or we could write $\alpha\beta < 0$), λ is real for all values of κ; the Stokes wave is stable or, more precisely, it is *neutrally* stable, since the wave perturbation persists but does not grow. However, if $\beta/\alpha > 0$, then λ will be imaginary for some κ, namely for

$$0 < \kappa < 2|A|\sqrt{\beta/\alpha}; \tag{4.110}$$

in this case a solution exists which grows exponentially as $\tau \to +\infty$: the Stokes wave is now unstable. We have already seen, in Section 4.2, that the sign of β/α is critical to the existence of certain types of solution of the NLS equations. In particular, for $\beta/\alpha > 0$ which corresponds to the NLS+ equation, we have a modulation which approaches zero at infinity (see equation (4.68)); no such solution exists for $\beta/\alpha < 0$. We might surmise that, for the unstable wave, there is a growth which continues until a balance is reached between the nonlinear and dispersive effects represented in the NLS+ equation. Once this has occurred, the amplitude modulation will evolve in line with the structure of a soliton solution: the solution will therefore not grow indefinitely. There is some observational and numerical evidence to support this sequence of events.

It is clear that, in order to see the relevance of the condition that heralds instability (that is, $\beta/\alpha > 0$ with equation (4.110)) – which we shall interpret more fully shortly – we need to know more about the coefficients of the NLS equation, (4.103). The expressions for the coefficients α and β are given in equations (4.33) and (4.34); these are rather complicated functions of δk. So that we can readily see their character, they are presented in Figure 4.6 where α and $\delta^2\beta$ are plotted against $\delta k > 0$. (It is usual practice to treat the given wave number, k, as positive.) The behaviours of α and β should be compared with the results obtained for $\delta k \to 0$ and $\delta k \to \infty$ in Q4.6. We see that the coefficient α is positive for all $\delta k\,(> 0)$, but that β changes sign from positive to negative as δk decreases across $\delta k = \delta k_0 \approx 1.363$. Thus the Stokes wave, based on the analysis above, is stable to small disturbances if $\delta k < \delta k_0$ (that is, for sufficiently long Stokes waves); on the other hand, if $\delta k > \delta k_0$ so that $\beta/\alpha > 0$, there exists a range of wave numbers κ for which the Stokes wave is unstable. (The problem in which δk is close to δk_0 must be treated separately; see Johnson (1977).) How should we interpret these κ?

The most straightforward approach is to construct the leading order term (the *fundamental*), η_0; further, its initial ($t = 0$) form is quite sufficient for our purposes, so we have

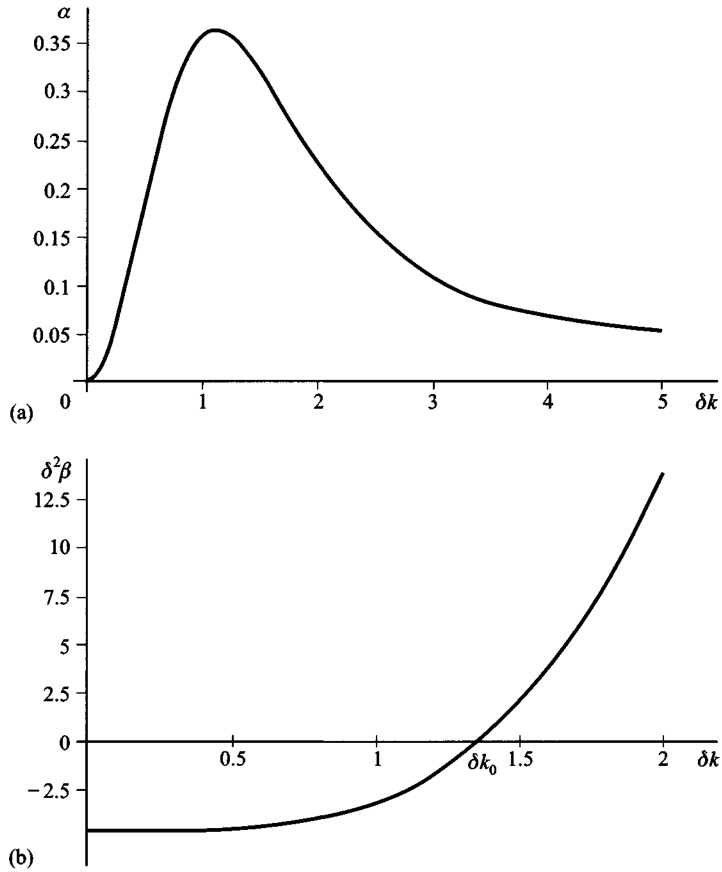


Figure 4.6. Plots of α (in (a)) and $\delta^2\beta$ (in (b)) as functions of δk, as required for the analysis of the stability of the Stokes wave.

$$\begin{aligned}\eta_0 &= A_0E + \text{c.c.} = A(1+\Delta a)\mathscr{E}E + \text{c.c.}\\ &\sim A\exp\big\{\mathrm{i}(kx+\Delta\theta)\big\} + \Delta A a_0 \exp\big\{\mathrm{i}(k+\varepsilon\kappa)x + \mathrm{i}\Delta\theta\big\} + \text{c.c.}\end{aligned}$$

for $\Delta \to 0$ (at fixed x and ε). Here we have used $\zeta = \varepsilon(x - c_g t)$, with $t = 0$, and retained the term $\Delta\theta$ in the exponent (although it plays no rôle in this interpretation). The perturbation to the fundamental, the term in Δa_0, has a wave number $k + \varepsilon\kappa$; that is, close to k. Hence a perturbation which has a wave number close to that of the fundamental, will generate an instability whenever $\beta/\alpha > 0$. Since, both in nature and in the laboratory, it is impossible to produce waves with a precisely fixed wave number, waves with a small deviation in k will occur and give rise to this phenomenon. This is often observed: what starts out as a set of plane waves gradually breaks down (*along* the wavefronts) into a number of wave groups. This type of stability, because it is associated with a small change in the fundamental wave number, is called a *side-band* instability; it was first described in a seminal paper, in 1967, by Benjamin and Feir (and it is therefore often referred to as *Benjamin–Feir* instability).

We conclude this discussion of the rôle of the NLS equation, in the study of the Stokes wave, by extending the analysis to encompass the DS equations. In Q4.24, solutions of the DS equations which depend only on τ and $X = l\zeta + mY$ were obtained; with this choice of variables, coupled with appropriate decay conditions, this pair of equations then recovers the NLS equation in the form

$$-2\mathrm{i}kc_pA_{0\tau} + (\alpha l^2 - c_pc_gm^2)A_{0XX} + \left\{\beta + \frac{\gamma^2k^2m^2}{c_p^2(1-c_g^2)[m^2+(1-c_g^2)l^2]}\right\}A_0|A_0|^2 = 0. \quad (4.111)$$

Here, α and β are exactly as used above (and given in (4.33) and (4.34)), and γ is given in (4.42). The solution of this NLS equation, (4.111), describes a modulation that is oblique to the carrier wave, which itself propagates with its wavefronts normal to the x-direction. If we now use equation (4.111) as the basis for investigating the stability of the Stokes wave, then we are considering the perturbation to be at any angle relative to the carrier wave; this is clearly a more general perturbation. What effect does this have on the stability of the Stokes wave?

Following the analysis that we gave for the NLS equation, (4.103), which produced the stability condition $\beta/\alpha > 0$, we see that the corresponding condition for equation (4.111) is

$$(\alpha l^2 - c_pc_gm^2)\left\{\beta + \frac{\gamma^2k^2m^2}{c_p^2(1-c_g^2)[m^2+(1-c_g^2)l^2]}\right\} < 0. \quad (4.112)$$

(Here we have chosen, for convenience, to express the condition as the product rather than the ratio of the coefficients; for the case $m = 0$ this yields $\alpha\beta < 0$, which is equivalent to $\beta/\alpha < 0$.) A slightly more transparent version of (4.112) is

$$(\alpha l^2 - c_pc_gm^2)\{(\beta + k^2\hat{\gamma}^2)m^2 + \beta(1-c_g^2)l^2\} < 0 \quad (4.113)$$

where

$$\hat{\gamma}^2 = \gamma^2/\{c_p^2(1 - c_g^2)\}$$

and we note that $\alpha > 0$, $c_p c_g > 0$, $c_g^2 < 1$ (cf. Figure 4.6), but that we can have $\beta < 0$. It is clear that it is always possible to find a pair (l, m) which leads to a violation of (4.113), except in one special case (and we consider only $\delta k > 0$). This case arises when the value of δk is such that

$$\frac{\beta + k^2\hat{\gamma}^2}{\beta(1 - c_g^2)} = -\frac{c_p c_g}{\alpha}$$

for then (4.113) becomes

$$-(\alpha l^2 - c_p c_g m^2)^2 < 0$$

which is always true. (The very special case for which $l^2/m^2 = c_p c_g/\alpha$ is of no practical interest.) The value of δk where this occurs is $\delta k \approx 0.38$; thus for all other values of δk, the Stokes wave is always unstable to some oblique perturbation. Our conclusion, therefore, is that we cannot expect the Stokes wave to propagate without, eventually, suffering significant distortion.

4.3.2 Modulation of waves over a shear flow

The problem of nonlinear wave propagation, described by some type of KdV equation, in the presence of an underlying arbitrary shear has been described (Section 3.4.1). Where we might expect a quite dramatic disruption of the propagation process, we found that the effect of the shear was only to change the (constant) coefficients of the classical KdV equation. We now investigate how a shear flow manifests itself in the problem of the modulation of a wave. (We remember that there is no suggestion that the term 'shear flow' is to imply that our model accommodates any viscous contribution.) Again, if the presence of an arbitrary shear flow merely adjusts the constant coefficients of the NLS equation, then we should have much greater confidence in the predictions offered by that equation.

The starting point for this description is, in all essentials, the same as that adopted for the derivation of the KdV equation with shear. However, here we retain the parameter δ^2 because the waves are of arbitrary wavelength (and, therefore, we do not use the transformation which takes $\delta^2 \to \varepsilon$, with $\varepsilon \to 0$, in the equations). Thus from equations (3.108) (but see also (3.9)) we have

$$u_t + Uu_x + U'w + \varepsilon(uu_x + wu_z) = -p_x;$$
$$\delta^2\{w_t + Uw_x + \varepsilon(uw_x + ww_z)\} = -p_z;$$
$$u_x + w_z = 0,$$

with

$$p = \eta \quad \text{and} \quad w = \eta_t + U\eta_x + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

The underlying shear flow is represented by $U(z)$ and $U' = dU/dz$. The solution that describes a modulated harmonic wave requires the choice of variables (see (4.2))

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t$$

and then we write, for $\varepsilon \to 0$

$$\eta \sim \sum_{n=0}^{\infty} \varepsilon^n \left\{ \sum_{m=0}^{n+1} A_{nm}(\zeta, \tau) E^m + \text{c.c.} \right\}$$

with $A_{00} = 0$, and

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n \left\{ \sum_{m=0}^{n+1} Q_{nm}(\zeta, \tau, z) E^m + \text{c.c.} \right\}$$

where q (and Q_{nm}) stands for each of u, w and p; cf. equations (4.14) and (4.36).

The construction of the solution follows closely that described for both the NLS and DS equations (Section 4.1), but here the details are even more intricate. Thus we choose to present only the main features and results of the calculation, which, with the inclusion of a little more detail, can be found in Johnson (1976).

The terms $\varepsilon^0 E$, with $P_{01} = P(z)A_{01}$, yield the equation for $P(z)$:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{1}{(U - c_p)^2}\frac{\mathrm{d}P}{\mathrm{d}z}\right\} - \frac{\delta^2 k^2}{(U - c_p)^2} P = 0, \qquad (4.114)$$

which corresponds to the earlier equation for F_0; see equation (4.7) *et seq.* The boundary conditions for $P(z)$ are

$$P(1) = 1; \quad P'(0) = 0, \tag{4.115}$$

together with a third condition which leads to the determination of c_p, namely

$$P'(1) = (\delta k)^2 W_1$$

where we have written

$$W(z) = U(z) - c_p, \quad W_1 = W(1);$$

this gives

$$\int_0^1 \frac{P(z)}{\{W(z)\}^2}\,\mathrm{d}z = 1. \tag{4.116}$$

It is evident that equation (4.116) is a generalisation of the Burns condition given previously in (3.112); here it defines the phase speed, $c_p(k)$, for the given shear. A simple check on this result is afforded by the choice $U = 0$ (or, indeed, $U =$ constant), leading to the solution of (4.114) and then the determination of c_p from (4.116); see Q4.44. We now proceed on the assumption that there is no critical layer for the given $U(z)$, so that $W(z) \neq 0$, $z \in [0, 1]$.

The terms that arise at εE generate an expression for the group speed, c_g, in the form

$$c_g = c_p - \left\{ \frac{\int_0^1 (WI')^2 \mathrm{d}z - 1}{W_1^{-1} + \int_0^1 II'W'\mathrm{d}z} \right\} \tag{4.117}$$

where we have written

$$I(z) = \int_0^z \frac{P(z)}{\{W(z)\}^2}\,\mathrm{d}z$$

(so the Burns condition, (4.116), now becomes $I_1 = I(1) = 1$). It is far from clear that the group speed given by (4.117) satisfies the classical relation (see, for example, Q2.26)

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}, \quad \text{where} \quad c_p = \frac{\omega}{k};$$

that this is indeed the case is left as an exercise (Q4.45).

Finally, terms $\varepsilon^2 E$ produce the Nonlinear Schrödinger equation for A_{01}:

$$2\mathrm{i}kW_1\left(1+W_1\int_0^1 \Pi'W'\,\mathrm{d}z\right)A_{01\tau}+\hat{\alpha}A_{01\zeta\zeta}+\hat{\beta}A_{01}|A_{01}|^2=0 \qquad (4.118)$$

where the coefficients $\hat{\alpha}$ and $\hat{\beta}$ are extremely complicated functions of k and $U(z)$. The expressions for $\hat{\alpha}$ and $\hat{\beta}$ are given in Johnson (1976). The important observation is that, for arbitrary $U(z)$ and given wave number k, all the coefficients of the NLS equation, (4.118), are constant. Thus the description of a modulated wave, its various properties via solutions of the NLS equation and, for example, its relevance to the stability of the Stokes wave, all follow the various analyses already given. The only requirement is, for a given $U(z)$, to compute the coefficients (as functions of k) and then to use this information in the desired solutions. This computation, however, is very lengthy except for the very simplest choices of $U(z)$.

We complete this section by applying our new NLS equation, (4.118), to the problem of the stability of Stokes waves that are moving over an arbitrary shear; stability is governed by the condition

$$\hat{\alpha}\hat{\beta}<0;$$

cf. equation (4.112). The details of where this condition is violated, for a given $U(z)$, require (as just mentioned) a lengthy computation that is quite beyond the scope of this text. Suffice it here to describe the situation that obtains for long waves; that is, $\delta k \to 0$. (We already know that the Stokes wave on stationary water is stable for $\delta k < \delta k_0 \approx 1.363$.) For $\delta k \to 0$, but allowing $U(z)$ to be arbitrary, the NLS equation reduces (after much tiresome calculation) to

$$2\mathrm{i}k\delta^2 I_{31}A_{01\tau}+3k^2\delta^4 J_1 A_{01\zeta\zeta}-\frac{3}{2}\frac{(I_{41})^2}{J_1}A_{01}|A_{01}|^2=0; \qquad (4.119)$$

cf. equation (4.44). Here we have used the notation that was employed for the problem of the KdV equation associated with arbitrary shear (given in Section 3.4.1), namely

$$I_{n1}=\int_0^1 \frac{\mathrm{d}z}{(U-c_p)^n}; \quad J_1=\int_0^1\int_z^1\int_0^{z_1}\frac{[U(z_1)-c_p]^2}{[U(z)-c_p]^2[U(z_2)-c_p]^2}\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}z.$$

The condition for stability of the Stokes wave, from equation (4.119) (and cf. (4.118)) is

$$\hat{\alpha}\hat{\beta} = -\frac{9}{2}k^2\delta^2(I_{41})^2 < 0,$$

which is clearly satisfied for *all* shear flows. Thus, for sufficiently long waves, the Stokes wave is stable *no matter* the form of the underlying shear (at least, in the absence of a critical layer). This result has important implications for Stokes waves that are observed in nature (or in the laboratory): for long waves, the underlying flow is essentially irrelevant. Of course, the value of δk at which the Stokes wave becomes *unstable* for a given shear – a far more significant result – cannot be obtained in any direct manner. Indeed, to be practically useful, an observed shear profile would have to be the basis for the choice of $U(z)$, followed by a computation of the coefficients for each δk.

A final mathematical comment: the NLS equation for long waves, (4.119), matches directly with the KdV equation for arbitrary shear, (3.128). This calculation is easily reproduced by following the method described in Section 4.1.3; indeed, merely noting the appropriate changes to the coefficients in the two pairs of equations confirms the matching. (A small additional calculation relevant to the derivation of equation (4.119) is discussed in Q4.46.)

4.3.3 Modulation of waves over variable depth

We have seen (Section 3.4.4) that the propagation of long waves, as they move over variable depth, produces a distortion of the waves; it is therefore no surprise to find that the same occurs for modulated harmonic waves. Since the derivation of the standard NLS equation (Section 4.1.1) is itself rather lengthy, we shall present here only briefest outline of the corresponding calculation for variable depth. Far more details, with a much fuller discussion, can be found in Djordjevic & Redekopp (1978), and in Turpin, Benmoussa & Mei (1983). This latter paper describes the result of combining both a slowly varying depth and a slowly varying current.

The problem is formulated in the same vein as we approached the derivation of the variable coefficient KdV equation (Section 3.4.4); that is, we first seek the appropriate scale on which the depth should vary. (Of course, other scales are possible – faster or slower – but these will generate simpler fundamental equations, in some sense.) The original

Nonlinear Schrödinger equation was obtained by introducing the variables

$$\xi = x - c_p t, \quad \zeta = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t;$$

see equations (4.2). On the basis of this, we anticipate that the most general NLS equation will arise when the depth varies on the scale ε^2; cf. the argument used for the KdV equation with variable depth, given in Section 3.4.4. This assumption then requires some adjustments to our choice of variables here.

Let us write $X = \varepsilon^2 x$, so that the bottom is now defined by

$$z = b(x; \varepsilon) = B(X),$$

and we shall use X rather than $\tau = \varepsilon^2 t$ to represent the longest scale in the problem. The variable that is associated with the propagation of the group is written as

$$\zeta = \varepsilon\left(\frac{1}{\varepsilon^2}\int_0^X \gamma_g(X')\mathrm{d}X' - t\right),$$

where it is consistent to write $\gamma_g(X) = c_g^{-1}(X)$ with $c_g(X)$ the (local) group speed. The most convenient representation of the variable that provides the harmonic component is obtained by writing

$$E = \mathrm{e}^{\mathrm{i}\phi} \quad \text{with} \quad \frac{\partial \phi}{\partial x} = k \quad \text{and} \quad \frac{\partial \phi}{\partial t} = -kc_p(X).$$

The derivation follows precisely the route described in Section 4.1.1, and results in the Nonlinear Schrödinger equation with variable coefficients:

$$-2\mathrm{i}kc_p c_g A_X - \mathrm{i}k^2 c_p^2 \left\{\frac{\partial}{\partial X}\left(\frac{c_g}{\omega}\right)\right\} A + \frac{\hat{\alpha}}{c_g^2} A_{\zeta\zeta} + \hat{\beta} A|A|^2 = 0; \qquad (4.120)$$

cf. equation (4.32). The coefficients depend on X, through the local depth $D = 1 - B(X)$, with

$$c_p^2 = \frac{\tanh \delta kD}{\delta k}, \quad c_g = \frac{1}{2}c_p(1 + 2\delta kD \operatorname{cosech} 2\delta kD), \quad \omega = kc_p,$$

and $\hat{\alpha}$, $\hat{\beta}$ are precisely α, β (see equations (4.33) and (4.34)) with δ replaced by δD. In equation (4.120) we have the new term that arises by virtue of the dependence on X: a term proportional to A (which corresponds to the term in η_0 that appeared in the variable-depth KdV equation, (3.148)). It is clear that equation (4.120) recovers the standard NLS equation when we have constant coefficients, for then we set $D = 1$ and transform

$c_g\zeta \to \zeta$, $X \to c_g T$ (which is the appropriate leading-order equivalence for the propagation of the group); obviously $\hat{\alpha} \to \alpha$ and $\hat{\beta} \to \beta$.

The first two terms in equation (4.120) can be written as

$$-2ik^2c_p^2\left\{\frac{c_g}{\omega}A_X + \frac{1}{2}A\left(\frac{c_g}{\omega}\right)_X\right\} = -2ik^2c_p^2\sqrt{\frac{c_g}{\omega}}\left(\sqrt{\frac{c_g}{\omega}}A\right)_X$$

from which we see that we can write the equation as

$$-2ikc_pc_gB_X + \frac{\hat{\alpha}}{c_g^2}B_{\zeta\zeta} + \frac{\omega\hat{\beta}}{c_g}B|B|^2 = 0, \tag{4.121}$$

where $B = A\sqrt{c_g/\omega}$; see Q4.47. This equation, (4.121), can now be discussed in much the same way that we adopted for the variable coefficient KdV equation (in Section 3.4.4). That is, we may use the equation to give some insight into the development of, for example, a solitary wave as it enters a region of very rapid or very slow depth change; that is, on a scale shorter than ε^{-2}, or longer than ε^{-2}, respectively. Of course, as we mentioned in the case of the KdV equation, a complete study of these problems requires an analysis of the full equations, with the inclusion of the appropriate depth scales. The particular case of very slow depth change, which results in a distortion of the solitary wave only (in this representation), is left as an exercise (Q4.48); we shall, however, briefly describe the case of a rapid depth change.

Equation (4.121), with constant coefficients, has a solitary-wave solution (of amplitude b) if

$$|B| = b\,\mathrm{sech}\left(b\zeta\sqrt{\frac{\omega c_g\hat{\beta}}{2\hat{\alpha}}}\right)$$

on $X = 0$ (cf. equation (4.68) and Q4.9); we choose, in order to make the results more transparent, to work with the envelope $|B|$ rather than B itself. Equivalently, when we write $b = a\sqrt{c_g/\omega}$, we have

$$|A| = a\,\mathrm{sech}\left(ac_g\zeta\sqrt{\frac{\hat{\beta}}{2\hat{\alpha}}}\right)$$

as the corresponding initial ($X = 0$) profile for equation (4.120). Similarly, it turns out that if the initial profile is

$$|A| = a\,\mathrm{sech}\left(\frac{ac_g\zeta}{N}\sqrt{\frac{\hat\beta}{2\hat\alpha}}\right),$$

then N solitons will eventually appear as the solution evolves in X; see, for example, Satsuma & Yajima (1974). Thus, if there is a rapid change in depth so that

$$c_g\sqrt{\frac{\hat\beta}{2\hat\alpha}} = \mu_1 \text{ on } D = 1$$

changes to

$$\mu_0 = c_g\sqrt{\frac{\hat\beta}{2\hat\alpha}} = \mu_1/N \quad \text{on} \quad D = D_0,$$

an initial profile which is a solitary wave on $D = 1$ will evolve into N solitons on $D = D_0$; cf. the result for the KdV equation, given in equation (3.151) *et seq.* Figure 4.7 shows the result of plotting μ_1/μ_0 for various δk; we see that two solitons appear for the cases $\delta k = 20$, 30, but not for $\delta k = 10$. (Note that we are interested only in the solution for which D decreases monotonically to its final value of D_0, and therefore at the point on these curves where this is first attained.) When μ_1/μ_0 is not precisely

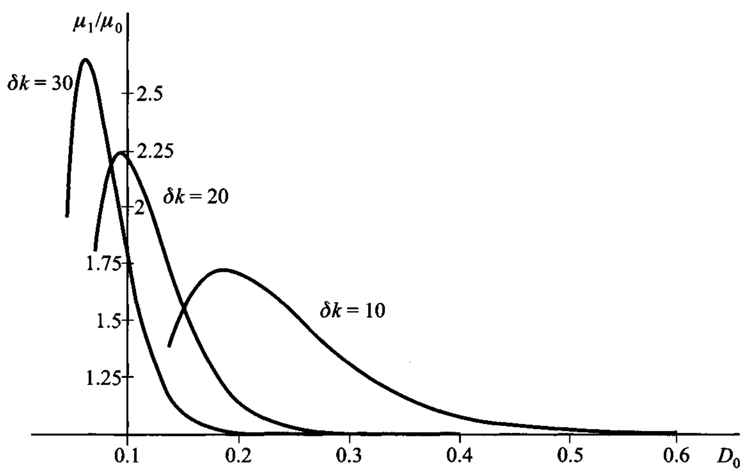


Figure 4.7. Plots of μ_1/μ_0 against D_0, for $\delta_k = 10$, 20, and 30, as used in the discussion of the solutions of the NLS equation with a rapid depth change.

an integer (that is, $\mu_1/\mu_0 = N + \Delta, 0 < \Delta < 1$), then the solution evolves into N solitons plus an oscillatory (dispersive) tail. These results mirror precisely those for the KdV equation, although here the relation between depth change and the number of solitons is considerably more involved.

Further reading

All the references to various aspects of soliton theory that were given at the end of the preceding chapter are relevant here (and will not be repeated). These texts describe the applications to both the NLS equations and the KdV family of equations. Below, we add a few further references that may prove of some interest to the reader who wishes to explore more deeply.

4.1 The initial work was done by Hasimoto & Ono (1972), Davey & Stewartson (1974) and Freeman & Davey (1975). Many other aspects of this work, which includes some mention of applications in other flow problems, can be found in Mei (1989), Infeld & Rowlands (1990) and Debnath (1994). An excellent text which touches on many more ideas in wave propagation, and which goes well beyond surface waves, is Craik (1988). All these texts and papers contain numerous references for still further reading.

4.3 A discussion of how these results apply to the stability of the Stokes wave is expanded in some of the references given above, and also in Whitham (1974). The particular applications that incorporate a shear or variable depth are mentioned in the texts by Mei and by Debnath. More information can be obtained from the papers by Johnson (1976), Djordjevic & Redekopp (1978) and Turpin *et al.* (1983).

Exercises

Q4.1 *Modulated wave from a Fourier representation.* Suppose that a wave is described by

$$\phi(x, t) = \int_{-\infty}^{\infty} F(k)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\mathrm{d}k$$

for some given $F(k)$, and a given dispersion function $\omega = \omega(k)$. Consider the situation where the profile obtains its main contribution near the wave number $k = k_0$; define $k = k_0 + \varepsilon\kappa$, and assume that $\omega(k)$ may be expanded in a Taylor series about $k = k_0$ (as far as the term in ε^2). Write $F(k_0 + \varepsilon\kappa) = f(\kappa; \varepsilon)/\varepsilon$ and hence show that

$$\phi(x, t) \sim A(\zeta, \tau)\exp\{\mathrm{i}(k_0 x - \omega_0 t)\} \quad \text{as } \varepsilon \to 0,$$

where $\omega_0 = \omega(k_0)$, $\zeta = \varepsilon\{x - \omega'(k_0)t\}$ and $\tau = \varepsilon^2 t$, for some function $A(\zeta, \tau)$ which should be determined.

Q4.2 *Inhomogeneous differential equations.* Obtain the general solutions of the ordinary differential equations

$$\text{(a)}\ \frac{\mathrm{d}^2F}{\mathrm{d}z^2} - \omega^2 F = \cosh(\omega z); \qquad \text{(b)}\ \frac{\mathrm{d}^2F}{\mathrm{d}z^2} - \omega^2 F = z\sinh(\omega z),$$

where $\omega\,(> 0)$ is a constant.

Q4.3 *Second derivative of $\omega(k)$.* Given that

$$c_p^2 = \frac{\tanh\delta k}{\delta k} \quad \text{and} \quad c_g = \frac{\mathrm{d}}{\mathrm{d}k}(kc_p) = \frac{1}{2}c_p(1 + 2\delta k\,\mathrm{cosech}\,2\delta k)$$

show that

$$kc_p\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} = -\{c_g^2 - (1 - \delta k\tanh\delta k)\mathrm{sech}^2\delta k\},$$

where $\omega = kc_p$.

[Observe how $\omega''(k_0)$ appears in the solution to Q4.1.]

Q4.4 *Modulated wave: mean drift component.* Use the terms that arise at $\varepsilon^2 E^0$ in the derivation of the NLS equation (and see equation (4.7)) to show that

$$f_{0\zeta} = -c_p^{-2}(1 - c_g^2)^{-1}(2c_p + c_g\mathrm{sech}^2\delta k)|A_0|^2.$$

Hence show how this term is relevant to the particle velocity in the direction of propagation.

[This, you will find, provides the leading term to the non-periodic part of the velocity; it is a mean drift generated by the nonlinear interaction of the wave motion, usually called the *Stokes drift*.]

Q4.5 *Phase and group speeds for long waves.* Find the first two terms in the asymptotic expansions of c_p and c_g, as $\delta \to 0$; see equations (4.8) and (4.23).

Q4.6 *NLS and DS equations: long and short wave limits.* Obtain the long ($\delta \to 0$) and short ($\delta \to \infty$) wave limits of the Davey–Stewartson equations, retaining only the dominant contributions to each coefficient of the equations. Write down the corresponding Nonlinear Schrödinger equations that arise when there is no dependence on Y.

[The coefficients α and β are shown in Figure 4.6.]

Q4.7 *Matching of the DS and 2D KdV equations.* Follow the technique used in Section 4.1.3 to show that the DS equations in the long-wave limit ($\delta \to 0$; see Q4.6) match with the 2D KdV equation, (3.30),

$$(2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi})_\xi + \eta_{0YY} = 0$$

in the short-wave limit ($\delta \to \infty$). Construct the solution to this 2D KdV equation exactly as before, but now seek a solution which also depends on Y (the variable used in the 2D KdV equation). You will find that the correspondence requires that $A_{10} = f_{0z}$ (and you will need terms $\lambda^{-2}E^0$).

Q4.8 *Transformation of NLS equations.* Use scale transformations of u, x and t (as necessary) to transform

$$\mathrm{i}\alpha u_t + \beta u_{xx} \pm \gamma u|u|^2 = 0,$$

where α, β and γ are positive real constants, into

$$\mathrm{i}u_t + u_{xx} \pm u|u|^2 = 0.$$

Q4.9 *NLS+ equation: solitary wave.* Consider the NLS+ equation

$$\mathrm{i}u_t + u_{xx} + u|u|^2 = 0,$$

and seek a travelling-wave solution in the form

$$u = r\mathrm{e}^{\mathrm{i}(\theta + nt)}, \quad r = r(x - ct), \quad \theta = \theta(x - ct),$$

where r, θ, c and n are real (c, n being constants). Show that there is a solution for which

$$\theta' = \frac{1}{2}c; \quad 2(r')^2 = 2(n - \frac{1}{4}c^2)r^2 - r^4$$

and hence obtain the solitary-wave solution

$$u(x, t) = a\exp\left\{\mathrm{i}\left[\frac{1}{2}c(x - ct) + nt\right]\right\}\operatorname{sech}\{a(x - ct)/\sqrt{2}\}$$

for all $a^2 = 2(n - \frac{1}{4}c^2) > 0$.

[This solution represents an oscillatory wavepacket for which the amplitude approaches zero as $|x - ct| \to \infty$].

Q4.10 *NLS– equation: solitary wave.* Follow the procedure described in Q4.9, but now for the NLS– equation

$$\mathrm{i}u_t + u_{xx} - u|u|^2 = 0.$$

Show that there exists a solution for which

$$r^2 = -n - 2a^2\operatorname{sech}^2(a\xi), \quad \theta = -\arctan\left\{\frac{2a}{c}\tanh(a\xi)\right\},$$

where $\xi = x - ct$, for all c and $a = \frac{1}{2}\sqrt{-2n - c^2}$, provided $n < -\frac{1}{2}c^2$. What is the behaviour of this solution as $|\xi| \to \infty$?

[This solution is sometimes called a *dark* solitary wave because it describes a *depression* in a non-zero background state; it is not relevant in water-wave problems when there is no disturbance at infinity.]

Q4.11 *NLS+ equation: the Ma solitary wave.* Show that the NLS+ equation in Q4.9 has a solution

$$u(x, t) = a\exp(\mathrm{i}a^2 t)\left\{1 + \left(\frac{2m(m\cos\theta + \mathrm{i}n\sin\theta)}{n\cosh(ma\sqrt{2}x) + \cos\theta}\right)\right\},$$

for all real a and m, where $n^2 = 1 + m^2$ and $\theta = 2mna^2 t$. What is the behaviour of this solution as $|x| \to \infty$?

[Note that this solution does not represent a travelling wave; see Ma(1979), Peregrine (1983) and Figure 4.3.]

Q4.12 *A rational-cum-oscillatory solution.* Show that the NLS+ equation in Q4.9 has the solution

$$u(x, t) = \mathrm{e}^{\mathrm{i}t}\{1 - 4(1 + 2\mathrm{i}t)/(1 + 2x^2 + 4t^2)\}.$$

[This solution contains no free parameters, but see Q4.14 and Q4.15; this is not a travelling wave, as Figure 4.3 makes clear.]

Q4.13 *Behaviour of the Ma solitary wave.* Obtain the asymptotic behaviour of the Ma solitary wave (Q4.11) as $m \to \infty$ at fixed a. Retain terms of O(1) and O(m), and regard $mx = \mathrm{O}(1)$.

Q4.14 *A normalised Ma solution.* Show that the solution in Q4.11 can be 'normalised' by the removal of the amplitude a, under the transformation $x \to x/a$, $t \to t/a^2$, $u \to au$. Further, confirm that the NLS equation is invariant under this same transformation; see Q4.16.

Q4.15 *Ma $\to$ rational-cum-oscillatory.* For the solution given in Q4.11, set $a = 1$ and choose $n = -\sqrt{1 + m^2}$. Now let $m \to 0$ (for x and t fixed) and hence recover the solution in Q4.12. Repeat the calculation for arbitrary a, and compare your result with the general property described in Q4.14.

Q4.16 *Similarity solution of the NLS equation.* Show that the equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0, \quad \varepsilon = \pm 1,$$

is invariant under each of the group transformation (a) $t \to t + \lambda$, $x \to x$, $u \to u$; (b) $t \to t$, $x \to x + \lambda$, $u \to u$; (c) $t \to \lambda^2 t$, $x \to \lambda x$, $u \to \lambda^{-1} u\,(\lambda \neq 0)$. Now use the property in (c) to obtain a similarity solution in the form $u(x, t) = t^m f(xt^n)$, for suitable m and n, and write down the equation for f.

Q4.17 *Normalised NLS± equations.* Use the results of Q4.8 to write equation (4.58):

$$\mathrm{i}\alpha(l - m)u_t + (l + m)u_{xx} \pm \frac{2}{lm}(l - m)(l^2 - m^2)u|u|^2 = 0,$$

in normalised form.

Q4.18 *Solution of the matrix Marchenko equation I.* Obtain the equations for c and d from equation (4.60), corresponding to equations (4.61) and (4.62) for a and b. Follow the same route as for a and b, and hence find the solutions for c and d.

Q4.19 *Solution of the matrix Marchenko equation II.* See Q4.18; impose the condition $c = -u^*$ and hence deduce that $g_0 = -f_0$ (for real f_0). Show, for the choice $c = u^*$ (which corresponds to the NLS– equation), that a solution of the form used in Q4.18 does not exist.

Q4.20 *NLS equation: bilinear form.* Show that the NLS equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon \text{ real constant})$$

can be written in the bilinear form

$$(\mathrm{i}D_t + D_x^2)(g \cdot f) = 0; \quad D_x^2(f \cdot f) = \varepsilon |g|^2$$

where $u = g/f$ and f is a real function.

Q4.21 *Generalised NLS equation.* Show that the equation

$$\mathrm{i}u_t + \beta u_{xx} + \mathrm{i}\gamma u_{xxx} + 3\mathrm{i}\delta|u|^2 u_x + \varepsilon u|u|^2 = 0,$$

where β, γ, δ and ε are real constants such that $\beta\delta = \gamma\varepsilon$, can be written in the bilinear form

$$(\mathrm{i}D_t + \beta D_x^2 + \mathrm{i}\gamma D_x^3)(g \cdot f) = 0; \quad \gamma D_x^2(f \cdot f) = \delta|g|^2$$

where $u = g/f$ and f is a real function. Show how the equations used in Q4.20 can be recovered from these equations.

Q4.22 *Solitary-wave solution.* Obtain the solitary-wave solution of the generalised NLS equation (Q4.21) by seeking an appropriate solution of the bilinear form. (Follow the method described in Section 4.2.2.)

Q4.23 *NLS+ equation: a bi-soliton solution.* Seek a solution of the bilinear equations given in Q4.20 (for $\varepsilon = +1$), in the form of power series in the parameter δ, with

$$f = 1 + \sum_{n=1}^{\infty} \delta^{2n} f_{2n}; \quad g = \sum_{n=1}^{\infty} \delta^{2n-1} g_{2n-1},$$

which terminate (cf. Section 3.3.3). In particular obtain the solution

$$g_1 = 4\sqrt{2}(\mathrm{e}^{\mathrm{i}t+x} + 3\mathrm{e}^{9\mathrm{i}t+3x})$$

and hence determine corresponding expressions for g_3, f_2, and f_4; show that this solution terminates, so that $f_6 = f_8 = \ldots = 0$ and $g_5 = g_7 = \ldots = 0$. Finally, set $\delta = 1$ and write down a solution of the NLS+ equation.

[The confirmation that this is a bi-soliton solution is obtained by comparing it with the result of Q4.31, which provides a more general solution; this special bi-soliton solution is a standing wave.]

Q4.24 *DS equations → NLS equation I.* Seek a solution of the Davey–Stewartson equations, (4.40) and (4.41), which depend on ζ and Y only through the combination $(l\zeta + mY)$, for arbitrary constants l and m. Show that the resulting plane oblique waves satisfy a Nonlinear Schrödinger equation.

Q4.25 *DS equations → NLS equation II.* Repeat the calculation of Q4.24 (or start with the results of that calculation) to give the corresponding results for long waves; see Q4.6 and equations (4.83), (4.84).

Q4.26 *DS equations: solitary wave.* Use the results of Q4.25 and Q4.9 to find the solitary-wave solution of the Davey–Stewartson equations for long waves.

Q4.27 *Long-wave DS equations: bilinear form.* Show that the equations

$$\mathrm{i}A_t + A_{xy} + 2A(Z + Z^*)_y = 0; \quad Z_x + \mathrm{i}Z_y = |A|^2$$

(see (4.88)) can be written in the bilinear form

$$(\mathrm{iD}_t + \mathrm{D}_x\mathrm{D}_y)(g \cdot f) = 0; \quad (\mathrm{D}_x^2 + \mathrm{D}_y^2)(f \cdot f) = 2|g|^2$$

where $A = g/f$ and

$$Z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)\ln f,$$

for f real; see equations (4.89).

Q4.28 *DS bilinear form: solution.* Obtain the solitary-wave solution of the pair of bilinear equations given in Q4.27; see equation (4.90).

Q4.29 *Long-wave DS equations: solitary wave.* Show that your solutions obtained in Q4.28 and Q4.26 are equivalent.

Q4.30 *NLS+ equation: 2-soliton solution.* Use the bilinear form of the NLS+ equation (given in Q4.20) to obtain the 2-soliton solution of that equation; see equations (4.79), (4.80), *et seq.*

Q4.31 *NLS+ equation: bi-soliton solution.* From the 2-soliton solution obtained in Q4.30, construct the bi-soliton (or bound soliton) solution by choosing the two speeds to be equal (that is, $c_1 = c_2$); see equation (4.81) *et seq.*, and Figure 4.5.

Q4.32 *NLS equation: two conservation laws.* Show that the NLS equation

$$\mathrm{i}u_t + u_{xx} + \varepsilon u|u|^2 = 0 \quad (\varepsilon = \pm 1),$$

possesses conserved quantities

$$\int_{-\infty}^{\infty} (|u_x|^2 - \frac{1}{2}\varepsilon|u|^4)\mathrm{d}x, \qquad \int_{-\infty}^{\infty} (uu^*_{xxx} + \frac{3}{2}\varepsilon|u|^2uu^*_x)\,\mathrm{d}x.$$

Q4.33 *NLS equation: a special conservation law.* Show that the NLS equation in Q4.32 has the conservation law

$$\int_{-\infty}^{\infty} \left\{\mathrm{i}x|u|^2 - t(u^*u_x - uu^*_x)\right\}\mathrm{d}x = \text{constant}.$$

Q4.34 *NLS equation: conservation of momentum.* Show that the conservation law

$$\int_{-\infty}^{\infty} (uu^*_x - u^*u_x)\,\mathrm{d}x = \text{constant}$$

corresponds to the leading term in the expression for the conservation of momentum in the wave motion; see equations (4.96) and (3.92).

Q4.35 *DS equations: special conservation laws.* Given the DS equation written as

$$f_{yy} + \lambda f_{xx} = \mu(|A|^2)_x;$$

$$-\mathrm{i}\alpha A_t + \beta A_{xx} - \gamma A_{yy} + \delta A|A|^2 + Af_x = 0,$$

where λ, μ, α, β, γ and δ are real constants, and given that

$$\int_{-\infty}^{\infty} |A|^2 \,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} |A|^2 \,\mathrm{d}y$$

are constant, consider the following:

(a) What are the forms of

$$\int_{-\infty}^{\infty} (A^* A_y - AA_y^*)\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} (A^* A_x - AA_x^*)\,\mathrm{d}y?$$

(b) Are there conditions under which the two expressions in (a) are constants?

(c) What is the form of $\int_{-\infty}^{\infty} f\,\mathrm{d}y$?

(d) Are there conditions under which the expression in (c) is constant? If so, what is this constant?

Q4.36 *DS equations: conservation laws.* Show, provided solutions of the equations given in Q4.35 decay sufficiently rapidly as $x^2 + y^2 \to \infty$, that two constants of the motion are

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (AA_x^* - A^* A_x)\,\mathrm{d}x\,\mathrm{d}y; \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (AA_y^* - A^* A_y)\,\mathrm{d}x\,\mathrm{d}y.$$

Q4.37 *A 2D NLS equation.* A two-dimensional NLS+ equation is

$$\mathrm{i}u_t + u_{xx} + u_{yy} + u|u|^2 = 0;$$

obtain the plane solitary-wave solution of this equation; see Q4.9.

[This equation is a natural two-dimensional variant of the NLS equation; for more details in this direction, see Hui & Hamilton (1979) and Yuen & Lake (1982).]

Q4.38 *2D NLS equation: conservation laws.* Show that, with suitable decay conditions at infinity, solutions of the two-dimensional NLS+ equation given in Q4.37 possess the following conserved quantities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^2\,\mathrm{d}x\,\mathrm{d}y; \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{|u_x|^2+|u_y|^2-\frac{1}{2}|u|^4\right\}\mathrm{d}x\,\mathrm{d}y.$$

Q4.39 *NLS equation: moment of inertia.* For the NLS equation given in Q4.32, define the moment of inertia

$$I=\int_{-\infty}^{\infty} x^2|u|^2\,\mathrm{d}x$$

and hence show that

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2}=8\int_{-\infty}^{\infty}\left[|u_x|^2-\frac{\varepsilon}{4}|u|^4\right]\mathrm{d}x,$$

which is *not* a constant of the motion (see Q4.32).

Q4.40 *Another solution of the NLS equation.* Obtain a solution of the equation

$$\mathrm{i}u_t+u_{xx}+\varepsilon u|u|^2=0 \quad (\varepsilon=\pm 1)$$

in the form

$$u(x,t)=A(x)\exp(\mathrm{i}\omega t),$$

where ω is a real constant. Write down the equation for $A(x)$ and hence obtain, under suitable conditions that should be stated, the solution for which

(a) A is a *sech* function if $\varepsilon=+1$;
(b) A is a *tanh* function if $\varepsilon=-1$.

[See, for example, Hasimoto & Ono (1972).]

Q4.41 *Another representation of the NLS equation.* Seek a solution of the NLS equation given in Q4.40 in the form

$$u(x,t)=A(x,t)\exp\left(\mathrm{i}\int^{x} k(x',t)\,\mathrm{d}x'\right),$$

where both A and k are real functions. Obtain the two (real) equations that together describe A and k, and show that each can be written in conservation form. Explain how one of these equations gives a result consistent with the first conservation law, equation (4.93).

Q4.42 *Set-down and mean drift.* Show that the harmonic-wave solution,

$$\eta \sim \eta_0 + \varepsilon \eta_1,$$

of Section 4.1.1, generates a non-oscillatory component in η_1. Find this component, and confirm that one contribution corresponds to the set-down of the Stokes wave (given in Section 2.5) and the other to the mean drift (given in Q4.4).

Q4.43 *Derivation of a dispersion relation.* Find the solution of the pair of equations

$$2Kc_p\theta_t + \alpha a_{xx} + 2\beta|A|^2 a = 0; \quad -2Kc_p a_t + \alpha\theta_{xx} = 0,$$

where K, c_p, α, β and $|A|^2$ are real constants, for which both θ and a are proportional to

$$\exp\{\mathrm{i}(kx - \omega t)\} \quad (+\text{c.c.}).$$

Show that this solution exists provided ω and k satisfy a certain dispersion relation; what is it?

Q4.44 *Phase speed in the absence of shear.* For the choice $U(z) = 0$, obtain $P(z)$ from equations (4.114, 4.115), and hence determine c_p from the generalised Burns condition, (4.116). Confirm that your expression for c_p is the anticipated result.

Q4.45 *Classical result for c_g.* Show that the group speed, c_g, and the phase speed, c_p, are related by the classical identity

$$c_g = \frac{\mathrm{d}}{\mathrm{d}k}(kc_p),$$

where c_p and c_g are the expressions given for the NLS equation with shear; see equations (4.116, 4.117).

[Hint: formulate the problem for $\partial P/\partial k$, on the assumption that $P(z; k)$ may be differentiated with respect to k, where P satisfies equation (4.114).]

Q4.46 *Modulated waves over a shear: long wave limit.* Obtain the solution for $P(z)$ (defined by equations (4.114), 4.115)) as $\delta \to 0$, retaining terms as far as $\mathrm{O}(\delta^2)$. Hence show that

$$I_{21} \sim 1 + (\delta k)^2 J, \quad \delta \to 0;$$

see equation (4.119) *et seq.* for the notation adopted here.

Q4.47 *NLS equation for variable depth.* Obtain equation (4.121) from equation (4.120), where $B = A\sqrt{c_g/\omega}$.

Q4.48 *NLS equation for slow depth variation.* Seek a solution of equation (4.121) for which $B = B(\zeta, X, \sigma X)$, as $\sigma \to 0$, where $D = D(\sigma X)$; cf. equation (3.150) *et seq.* for the corresponding KdV problem. Write down the solitary-wave solution of the leading order NLS equation obtained as $\sigma \to 0$.

5

Epilogue

Is this the end? Is this the end?
In Memoriam A.H.H. XII

So many worlds, so much to do,
So little done, such things to be.
In Memoriam A.H.H. LXXIII

In the earlier chapters we have described the mathematical background – and the mathematical details – of many classical linear and nonlinear water-wave phenomena. In addition, in the later chapters, we have presented many of the important and modern ideas that connect various aspects of soliton theory with the mathematical theory of water waves. However, much that is significant in the practical application of theories to real water waves – turbulence, random depth variations, wind shear, and much else – has been omitted. There are two reasons for this: first, most of these features are quite beyond the scope of an introductory text, and, second, the modelling of these types of phenomena follows a less systematic and well-understood path. Of course, that is not meant to imply that these approaches are unimportant; such studies have received much attention, and with good reason since they are essential in the design of man-made structures and in our endeavours to control nature.

What we have attempted here, in a manner that we hope makes the mathematical ideas transparent, is a description of some of the current approaches to the *theory* of water waves. To this end we have moved from the simplest models of wave propagation over stationary water of constant depth (sometimes including the effects of surface tension), to more involved problems (for example, with 'shear' or variable depth), but then only for gravity waves. It is our intention, in this short concluding chapter, to give an indication of how the effects of viscosity – the friction inherent in any flow of water – manifest themselves in our mathematical description. The approach that we adopt is based on following a rather systematic and precise route, rather than invoking any *ad hoc* modelling of the phenomena. Nevertheless, careful and wise modelling can often provide quick, neat and accurate results, even if this is possible only by a skilled practitioner. Here, we shall restrict our discussion to that of gravity waves (although, in the case of the linear theory, the application to short waves will also provide an estimate for the damping of capillary waves). We shall first examine linear harmonic waves, and obtain a measure of their damping due to the viscosity of the water. Then we shall discuss the attenuation of the solitary wave and, finally, provide two descriptions of the undular bore (which requires some viscous contribution for its existence). In all but one of these calculations we shall consider only one-dimensional (plane) surface waves moving over stationary water of constant depth.

5.1 The governing equations with viscosity

We consider plane waves that propagate in the x-direction, so our governing equations (written in original physical variables) are, from Appendix A (equations (A.2)),

$$\begin{aligned} u_t + uu_x + wu_z &= -\frac{1}{\rho}P_x + \nu(u_{xx} + u_{zz}); \\ w_t + uw_x + ww_z &= -\frac{1}{\rho}P_z - g + \nu(w_{xx} + w_{zz}); \\ u_x + w_z &= 0. \end{aligned}$$

These equations describe an incompressible fluid with a kinematic viscosity, ν. The boundary conditions (given in Appendix B) are chosen to be those relevant to a gravity wave (so $\Gamma = 0$, but see Q5.4) in the absence of any wind shear. Thus equation (B.1), the normal stress condition, gives

$$P - 2\mu\{h_x^2 u_x - h_x(u_z + w_x) + w_z\}/(1 + h_x^2) = P_a,$$

where P_a is the (constant) pressure in the atmosphere and μ is the coefficient of Newtonian viscosity; equation (B.3), one of the two tangential stress conditions, likewise gives

$$2h_x(u_x - w_z) + (h_x^2 - 1)(u_z + w_x) = 0.$$

These conditions apply on the free surface $z = h(x, t)$ (and we note that equation (B.2) is redundant for plane waves moving only in the x-direction). On the bottom (taken as $z = 0$) we use the boundary conditions

$$u = w = 0 \quad \text{on} \quad z = 0,$$

and, finally, we have the familiar kinematic condition

$$w = h_t + uh_x \quad \text{on} \quad z = h.$$

Our first task is to obtain the nondimensional version of these equations; to this end we use the scheme introduced in Section 1.3.1 (and see Q1.35), namely

$$x \to \lambda x, \quad z \to h_0 z, \quad t \to (\lambda/\sqrt{gh_0})t,$$
$$u \to \sqrt{gh_0}u, \quad w \to (h_0\sqrt{gh_0}/\lambda)w$$

with

$$h = h_0 + a\eta \quad \text{and} \quad P = P_a + \rho g(h_0 - z) + \rho g h_0 p.$$

The equations of motion then become

$$\begin{aligned} u_t + uu_x + wu_z &= -p_x + \frac{1}{\delta R}(u_{zz} + \delta^2 u_{xx}); \\ \delta^2(w_t + uw_x + ww_z) &= -p_z + \frac{\delta}{R}(w_{zz} + \delta^2 w_{xx}); \\ u_x + w_z &= 0, \end{aligned}$$

with

$$\left.\begin{aligned} p - \eta - \frac{2\delta}{R}\{w_z - \varepsilon\eta_x(u_z + \delta^2 w_x) \\ + \varepsilon^2\delta^2\eta_x^2 u_x\}/(1 + \varepsilon^2\delta^2\eta_x^2) = 0; \\ (1 - \varepsilon^2\delta^2\eta_x^2)(u_z + \delta^2 w_x) + 2\varepsilon\delta^2(w_z - u_x)\eta_x = 0; \\ w = \varepsilon(\eta_t + u\eta_x) \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta$$

and

$$u = w = 0 \quad \text{on} \quad z = 0.$$

We have introduced our familiar parameters, $\varepsilon = a/h_0$ and $\delta = h_0/\lambda$, and the Reynolds numbers is $R = h_0\sqrt{gh_0}/\nu$ (which uses only the scale length h_0 in its definition); for many problems in fluid mechanics we are interested in the case of $R \to \infty$.

The small-amplitude limit of these equations, described by $\varepsilon \to 0$, is obtained by employing the further transformation

$$(u, w, p) \to \varepsilon(u, w, p).$$

This gives the set of equations and boundary conditions

$$u_t + \varepsilon(uu_x + wu_z) = -p_x + \frac{1}{\delta R}(u_{zz} + \delta^2 u_{xx}); \tag{5.1}$$

$$\delta^2\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z + \frac{\delta}{R}(w_{zz} + \delta^2 w_{xx}); \tag{5.2}$$

$$u_x + w_z = 0 \tag{5.3}$$

with

$$\left.\begin{aligned} & p - \eta - \frac{2\delta}{R}\{w_z - \varepsilon\eta_x(u_z + \delta^2 w_x) \qquad (5.4)\\ & \qquad + \varepsilon^2\delta^2\eta_x^2 u_x\}/(1 + \varepsilon^2\delta^2\eta_x^2) = 0; \\ & (1 - \varepsilon^2\delta^2\eta_x^2)(u_z + \delta^2 w_x) + 2\varepsilon\delta^2(w_z - u_x)\eta_x = 0; \qquad (5.5)\\ & w = \eta_t + \varepsilon u\eta_x \qquad (5.6) \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta$$

and

$$u = w = 0 \quad \text{on} \quad z = 0. \tag{5.7}$$

These equations, or a simple variant of them, will be discussed in the following sections, where we shall describe the construction of appropriate asymptotic solutions. (It is easily seen that these equations recover our earlier versions for one-dimensional motion when we take $R \to \infty$, with equation (5.5) now redundant and (5.7) becoming simply $w = 0$ on $z = 0$.)

5.2 Applications to the propagation of gravity waves

All the problems that we have examined in this text can, in principle at least, be re-examined with the appropriate contribution from the viscous effects included. Many of these problems rapidly become very involved indeed, so we choose to look at a few of the simpler ones (although even these, as we shall see, are considerably more complicated than their inviscid counterparts). In addition we shall also describe, via two different models, a phenomenon that requires some viscous contribution in the equations in order for an appropriate solution to exist. This is the undular bore, a special – and rather weak – version of the bore that was described using a discontinuity in Section 2.7. It turns out to be fairly straightforward to *write down* a model equation which contains the essential characteristics of the undular bore, but it is far from a routine calculation to *derive* such an equation.

5.2.1 Small amplitude harmonic waves

The first problem that we tackle is that of harmonic gravity waves moving on the surface of a stationary viscous fluid; the inviscid problem has been described in Section 2.1. From Section 5.1 we obtain the governing equations, after imposing the small-amplitude limit $\varepsilon \to 0$, in the form

$$u_t = -p_x + \frac{1}{\delta R}(u_{zz} + \delta^2 u_{xx}); \quad \delta^2 w_t = -p_z + \frac{\delta}{R}(w_{zz} + \delta^2 w_{xx}); \tag{5.8}$$

$$u_x + w_z = 0 \tag{5.9}$$

with

$$p - \eta - \frac{2\delta}{R} w_z = 0; \quad u_z + \delta^2 w_x = 0; \quad w = \eta_t \quad \text{on} \quad z = 1 \tag{5.10}$$

and

$$u = w = 0 \quad \text{on} \quad z = 0. \tag{5.11}$$

We consider here the most general *linear* problem, in that we treat the parameters δ and R as fixed (as $\varepsilon \to 0$). The solution that we seek (cf. Section 2.1) is to take the form

$$\eta = E, \quad u = U(z)E, \quad w = W(z)E, \quad p = P(z)E \tag{5.12}$$

with

$$E = \exp\{\mathrm{i}(kx - \omega t\},$$

where k is the (real) wave number; we anticipate that the presence of the terms associated with R will produce an imaginary contribution to the frequency ω. The real part of ω will – as before – give the phase speed of the gravity waves. The solution described in (5.12) has been written, for convenience, with the amplitude of η as $A = 1$ (which could be reinstated as A by writing AE for E throughout); the four expressions in (5.12) must be combined with their complex conjugates in order to produce a real solution-set.

The choice described by equations (5.12) is substituted into equations (5.8) and (5.9) to give

$$-\mathrm{i}\omega U = -\mathrm{i}kP + \frac{1}{\delta R}(U'' - \delta^2 k^2 U); \quad -\mathrm{i}\omega\delta^2 W = -P' + \frac{\delta}{R}(W'' - \delta^2 k^2 W); \tag{5.13}$$

$$\mathrm{i}kU + W' = 0. \tag{5.14}$$

The boundary conditions, (5.10), yield

$$P-1-\frac{2\delta}{R}W'=0; \quad U'+\mathrm{i}k\delta^2W=0; \quad W=-\mathrm{i}\omega \quad \text{on} \quad z=1 \quad (5.15)$$

and from (5.11) we obtain

$$U=W=0 \quad \text{on} \quad z=0. \quad (5.16)$$

The simplest manoeuvre that leads to a suitable single equation (for $W(z)$), is to substitute for U from (5.14) into the first equation of (5.13), and to differentiate this equation once with respect to z. Then P' can be eliminated between the pair of equations resulting from (5.13), to give

$$\omega(W''-\delta^2k^2W)=\frac{\mathrm{i}}{\delta R}(W^{\mathrm{iv}}-2\delta^2k^2W''+\delta^4k^4W). \quad (5.17)$$

The boundary conditions (5.15), after using (5.14) to eliminate U and the first of (5.13) to eliminate P, give

$$\left.\begin{aligned} k^2-\mathrm{i}\omega W'-\frac{1}{\delta R}(W'''-3\delta^2k^2W')=0; \\ W''+\delta^2k^2W=0; \quad W=-\mathrm{i}\omega \end{aligned}\right\} \text{ on } z=1, \quad (5.18)$$

and from (5.16) we have simply

$$W=W'=0 \quad \text{on} \quad z=0. \quad (5.19)$$

Equation (5.17), which is a linear equation for $W(z)$ with constant coefficients, has solutions of the form $W=\exp(\lambda z)$, where

$$\begin{aligned}\omega(\lambda^2-\delta^2k^2)&=\frac{\mathrm{i}}{\delta R}(\lambda^4-2\delta^2k^2\lambda^2+\delta^4k^4)\\ &=\frac{\mathrm{i}}{\delta R}(\lambda^2-\delta^2k^2)^2.\end{aligned}$$

Thus

$$\lambda=\pm\delta k \quad \text{or} \quad \lambda^2=\delta^2k^2-\mathrm{i}\omega\delta R,$$

so the general solution can be written as

$$W(z)=A\sinh\delta kz+B\cosh\delta kz+C\sinh\mu z+D\cosh\mu z$$

for arbitrary constants A, B, C and D, with

$$\mu^2=\delta^2k^2-\mathrm{i}\omega\delta R. \quad (5.20)$$

The boundary conditions on $z=0$, (5.19), require that $D=-B$ and that $C=-A\delta k/\mu$, so

$$W = A(\sinh \delta kz - \frac{\delta k}{\mu}\sinh \mu z) + B(\cosh \delta kz - \cosh \mu z).$$

This expression for W is used in the boundary conditions on $z=1$, (5.18); the second of these gives

$$A\left\{2\delta^2k^2 \sinh \delta k - \frac{\delta k}{\mu}(\delta^2k^2+\mu^2)\sinh \mu\right\}$$
$$= -B\{2\delta^2k^2\cosh \delta k - (\delta^2k^2+\mu^2)\cosh \mu\},$$

and the third yields simply

$$A(\sinh \delta k - \frac{\delta k}{\mu}\sinh \mu) + B(\cosh \delta k - \cosh \mu) = -\mathrm{i}\omega.$$

These two equations are solved for A and B, and then, finally, the complete expression for W is used in the first boundary condition in (5.18). After $\mathrm{i}\omega$ is eliminated by using (5.20), we obtain the dispersion relation between δk, μ and R:

$$\begin{aligned}&\delta k(\delta k \cosh \delta k \sinh \mu - \mu \sinh \delta k \cosh \mu)\\ &+\frac{1}{R^2}\{4\mu\delta^2k^2(\mu^2+\delta^2k^2) + 4\mu\delta^3k^3(\mu \sinh \delta k \sinh \mu - \delta k \cosh \delta k \cosh \mu)\\ &\qquad + (\mu^2+\delta^2k^2)^2(\delta k \sinh \delta k \sinh \mu - \mu \cosh \delta k \cosh \mu)\} = 0; \quad (5.21)\end{aligned}$$

the details of this straightforward but rather lengthy calculation are left to the reader. (Expression (5.21), written using a slightly different notation, can be found in Kakutani & Matsuuchi (1975).) The interpretation of equation (5.21) is that it determines the complex frequency, ω (via (5.20)), for given (real) values of δk and R.

The involved nature of the dispersion relation is quite evident; indeed, even a numerical study of it is far from routine. We shall quote a few relevant observations about the (asymptotic) solutions for ω, the details of which are to be found in the exercises (Q5.1–Q5.3); the essential character of the complex frequency is presented in Figure 5.1. Here we produce a representation of both the real and imaginary parts of (ω/k), for various R, where these curves are based on the asymptotic behaviours of the solution of the dispersion relation. Figure 5.1 is intended to give only an idea of the variation of (ω/k) with δk, rather than accurate numerical estimates. What we see is that the real part of (ω/k), which is the speed of the harmonic wave, is very nearly unity for all δk not too small and R

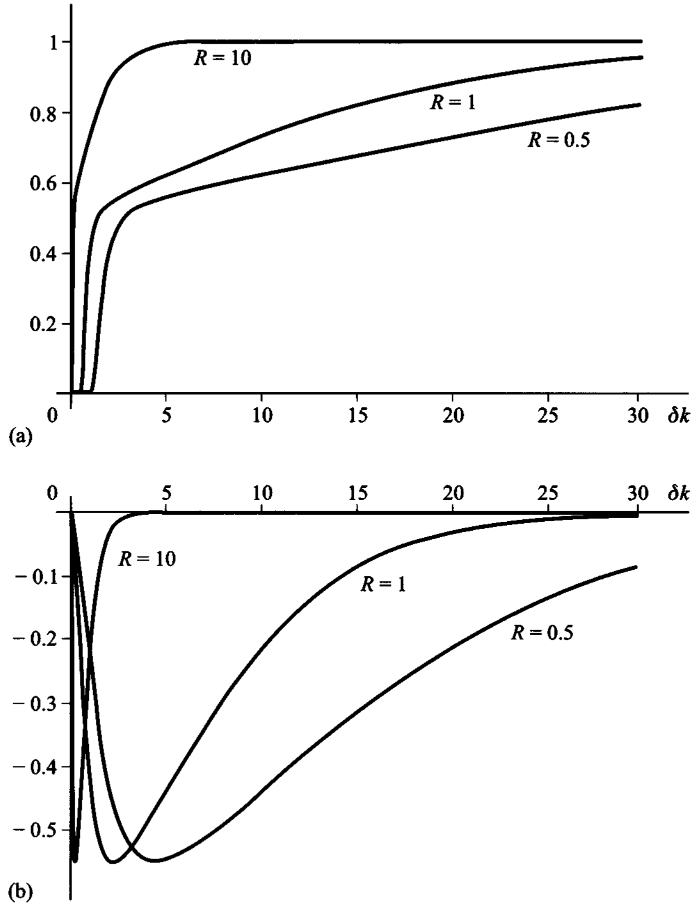


Figure 5.1. Plots of (a) the real part of ω/k and (b) the imaginary part of ω/k, for the values of Reynolds number $R = 0.5$, 1, 10, based on the asymptotic behaviours of the dispersion relation, (5.21).

increasing – and even for moderate R; see also equation (5.22). On the other hand, the damping of the wave (and it is always damped, since $\mathscr{I}m(\omega) < 0\ \forall \delta k \neq 0$) varies quite significantly with δk, although this variation is restricted to a narrow band as R increases. For large R and δk not too small (actually the critical size is $\delta k = \mathrm{O}(1/R)$; see Q5.2), the damping is very small indeed, which again is evident in equation (5.22) below.

For conventional gravity waves, the Reynolds number (R) is typically quite large: anywhere from about 10^3 upwards, and for deep water this could be much larger. Thus the approximation of interest to us is described by $R \to \infty$; under this limiting process we find (Q5.1) that

$$\omega \sim k\left\{\sqrt{\frac{\tanh \delta k}{\delta k}} - \frac{(1+\mathrm{i})}{2\sqrt{2R}}\,\frac{(\delta k)^{1/4}}{\cosh^{5/4}\delta k \sinh^{3/4}\delta k}\right\}, \tag{5.22}$$

where we have chosen the waves to be right-running (and hence the positive square root is taken). The leading term is our very familiar result for the propagation speed of (inviscid) gravity waves, first given in equation (2.13). The viscous contribution in (5.22), which is provided by the term in $1/\sqrt{R}$, possesses both real and imaginary parts and therefore affects the speed of the wave as well as its attenuation. The decay of the harmonic wave, in this approximation, is controlled by the negative exponent proportional to

$$(\delta k)^{5/4}\operatorname{sech}^{5/4}\delta k \operatorname{cosech}^{3/4}\delta k; \tag{5.23}$$

this function is plotted in Figure 5.2. It is clear that long waves, described by $\delta k \to 0$, are very weakly damped, but that shorter waves (δk increasing) have much higher damping rates. (The exponential decay of the expression in (5.23), as δk increases indefinitely, is not to be relied

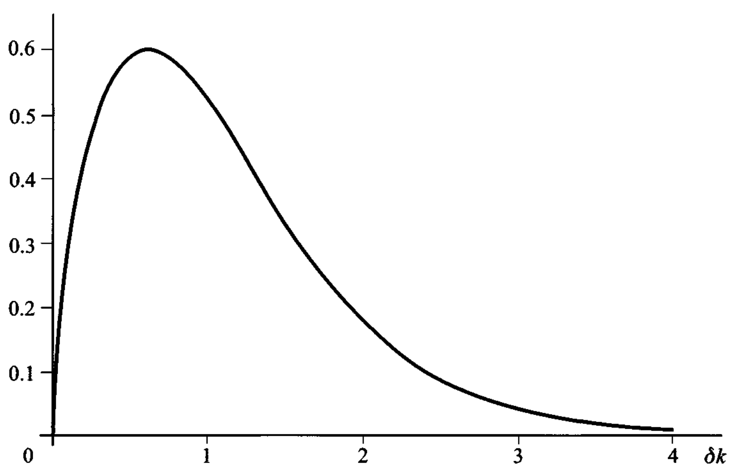


Figure 5.2. The function, (5.23), which provides the dominant contribution, for large Reynolds number, to the damping of harmonic waves.

upon, since the argument underpinning (5.22) was $R \to \infty$ at δk fixed; a different asymptotic structure appears for $\delta k \to \infty$ and, indeed, probably we should then include the surface tension contribution; see Q5.4.) The damping rate of shorter, as compared with longer, waves provides the explanation for the limited distances over which capillary waves are seen to survive, as compared with gravity waves (as we commented in Section 2.1.2). Other approximations and interpretations of the dispersion relation, (5.21), can be found in the exercises at the end of this chapter.

5.2.2 *Attenuation of the solitary wave*

In Section 2.9 we quoted Russell's description of his chase, on horseback, of a solitary wave; his evidence, and much that has been collected since his time, indicates that the solitary wave is only very weakly affected by viscosity. We shall study the way in which the viscous effects, as described by the Navier–Stokes equations, provide a slow evolution of the solitary wave. This we shall do using the *method of multiple scales*, the scales being associated with the propagation of the wave, the nonlinear evolution of the wave, and the evolution on a viscous scale.

We start with the equations given in Section 5.1, (5.1)–(5.7), but introduce the transformation which describes the scales on which a KdV-type balance occurs, as $\varepsilon \to 0$ for *arbitrary* δ; these are (cf. equations (3.10), (3.11))

$$x \to \frac{\delta}{\varepsilon^{1/2}} x, \quad t \to \frac{\delta}{\varepsilon^{1/2}} t, \quad w \to \frac{\varepsilon^{1/2}}{\delta} w.$$

The equations are therefore

$$\begin{aligned} u_t + \varepsilon(uu_x + wu_z) &= -p_x + \frac{1}{R\sqrt{\varepsilon}}(u_{zz} + \varepsilon u_{xx}); \\ \varepsilon\{w_t + \varepsilon(uw_x + ww_z)\} &= -p_z + \frac{\sqrt{\varepsilon}}{R}(w_{zz} + \varepsilon w_{xx}); \\ u_x + w_z &= 0, \end{aligned}$$

with

$$\left.\begin{aligned} p - \eta - 2\frac{\sqrt{\varepsilon}}{R}\{w_z - \varepsilon\eta_x(u_z + \varepsilon w_x) + \varepsilon^3\eta_x^2 u_x\}/(1 + \varepsilon^3\eta_x^2) = 0; \\ (1 - \varepsilon^3\eta_x^2)(u_z + \varepsilon w_x) + 2\varepsilon^2(w_z - u_x)\eta_x = 0; \\ w = \eta_t + \varepsilon u\eta_x, \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta$$

and

$$u = w = 0 \quad \text{on} \quad z = 0.$$

Now, in the derivation of the KdV equation, we introduced the variables

$$\xi = x - t, \qquad \tau = \varepsilon t;$$

see equations (3.17) and (3.18). Here we follow essentially the same route, but include evolution on a suitable (slow) viscous scale and also allow the nonlinear contribution to the speed of the wave to vary on this same scale. (We have already seen that the speed of harmonic waves is altered by the presence of a viscous ingredient; see equation (5.22).) Thus, anticipating a KdV-type of equation with independent variables τ and ξ in the absence of viscosity, we introduce a slow evolution of this system in the form

$$\tau = \varepsilon t, \quad T = \Delta\tau = \varepsilon\Delta t, \quad \xi = x - t - \frac{1}{\Delta}\int_0^T c(T')\mathrm{d}T', \tag{5.24}$$

where we shall treat ξ, τ and T as independent variables (the method of multiple scales), and where Δ is yet to be chosen. Different problems require different choices of Δ, in terms of

$$\varepsilon\ (\to 0) \quad \text{and} \quad r = 1/R\sqrt{\varepsilon}\ (\to 0),$$

which we treat as independent parameters. Under this transformation our governing equations become

$$\varepsilon u_\tau + \varepsilon\Delta u_T - (1 + \varepsilon c)u_\xi + \varepsilon(uu_\xi + wu_z) = -p_\xi + r(u_{zz} + \varepsilon u_{\xi\xi}); \tag{5.25}$$

$$\varepsilon\left\{\varepsilon w_\tau + \varepsilon\Delta w_T - (1 + \varepsilon c)w_\xi + \varepsilon(uw_\xi + ww_z)\right\} = -p_z + \varepsilon r(w_{zz} + \varepsilon w_{\xi\xi}); \tag{5.26}$$

$$u_\xi + w_z = 0, \tag{5.27}$$

with

$$\left.\begin{aligned} p - \eta - 2\varepsilon r\left\{w_z - \varepsilon\eta_\xi(u_z + \varepsilon w_\xi) + \varepsilon^3\eta_\xi^2 u_\xi\right\}/(1 + \varepsilon^3\eta_\xi^2) = 0; \\ (1 - \varepsilon^3\eta_\xi^2)(u_z + \varepsilon w_\xi) + 2\varepsilon^2(w_z - u_\xi)\eta_\xi = 0; \\ w = \varepsilon\eta_\tau + \varepsilon\Delta\eta_T - (1 + \varepsilon c)\eta_\xi + \varepsilon u\eta_\xi, \end{aligned}\right\} \text{ on } z = 1 + \varepsilon\eta \tag{5.28}$$

and

$$u = w = 0 \quad \text{on} \quad z = 0. \tag{5.29}$$

We seek an asymptotic solution of these equations in the form

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi, \tau, T, z; \Delta, r), \qquad \eta \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \tau, T; \Delta, r),$$

for $\varepsilon \to 0$, where q (and correspondingly q_n) represents u, w and p. Each function q_n and η_n ($n = 0, 1, 2, \ldots$) is, in turn, regarded as possessing an appropriate asymptotic representation as $r \to 0$, $\Delta \to 0$; this is equivalent to seeking a multiple asymptotic expansion in terms of, for example, the asymptotic sequence $\{\varepsilon^n r^m\}$, $n = 0, 1, 2, \ldots$, for a suitable set of values of m and some chosen $\Delta(\varepsilon, r)$. Further, special problems can always be posed for any choice $\Delta = \Delta(\varepsilon)$ and $r = r(\varepsilon)$; that is, $R = R(\varepsilon)$. On physical grounds, such a procedure could be criticised since ε and R are clearly *independent* parameters; however, some of the mathematical problems that are generated in this way enable us to obtain some insight into the structure of these equations and their solutions. We shall comment on this again later, but we note here that an *ab initio* choice of $R = R(\varepsilon)$ reduces the problem to an expansion in one parameter – say ε – only. To proceed, the method of solution that we follow here is, in its general outline, that employed for the derivation of the Korteweg–de Vries equation (as described in Section 3.2.1).

The leading-order equations, as $\varepsilon \to 0$, obtained from equations (5.25)–(5.27), are

$$-u_{0\xi} = -p_{0\xi} + r u_{0zz}; \quad p_{0z} = 0; \quad u_{0\xi} + w_{0z} = 0. \tag{5.30}$$

The boundary conditions, from (5.28) and (5.29), yield

$$p_0 = \eta_0; \quad u_{0z} = 0; \quad w_0 = -\eta_{0\xi} \quad \text{on} \quad z = 1 \tag{5.31}$$

and

$$u_0 = w_0 = 0 \quad \text{on} \quad z = 0. \tag{5.32}$$

It is clear that equations (5.30), for $r \to 0$, possess a solution which admits a *boundary layer*, presumably near $z = 0$ in the light of the no-slip boundary condition on $z = 0$; see Q5.5 and Q5.6. (We might expect a boundary layer to be required also near $z = 1$, in order to accommodate the shear stress condition there. However, as we shall see, the problem of no wind shear does not give rise to a surface boundary layer at the order of approximation to which we shall be working.) We therefore seek, in the first instance, a solution of equations (5.30)–(5.32), in the limit r $\to 0$ but valid away from the boundary layer near $z = 0$. This first approximation in r is denoted by an additional zero suffix, so we obtain

$$u_{00\xi} = p_{00\xi}; \quad p_{00z} = 0; \quad u_{00\xi} + w_{00z} = 0,$$

with

$$p_{00} = \eta_{00}; \quad u_{00z} = 0; \quad w_{00} = -\eta_{00\xi} \quad \text{on} \quad z = 1.$$

This set produces the familiar solution (see Section 3.2.1)

$$p_{00} = \eta_{00}; \quad u_{00} = \eta_{00}; \quad w_{00} = -z\eta_{00\xi}, \quad 0 < z \leq 1, \tag{5.33}$$

which satisfies the shear stress condition ($u_{00z} = 0$) on $z = 1$, but which cannot satisfy the bottom boundary condition ($u_{00} = 0$ on $z = 0$). Solution (5.33), when previously derived for the KdV equation, was valid for $0 \leq z \leq 1$; here it is not valid near $z = 0$, although both p_{00} and w_{00} (at this order) would appear to be uniformly valid on [0,1]. In fact solution (5.33) satisfies the *full* equations valid away from the boundary layer; see Q5.7.

The equations that define the O(ε) problem, from equations (5.25)–(5.29), are

$$u_{0\tau} + \Delta u_{0T} - u_{1\xi} - cu_{0\xi} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi} + r(u_{1zz} + u_{0\xi\xi});$$
$$-w_{0\xi} = -p_{1z} + rw_{0zz}; \quad u_{1\xi} + w_{1z} = 0,$$

with

$$\left.\begin{aligned} &p_1 + \eta_0 p_{0z} - \eta_1 - 2rw_{0z} = 0; \quad u_{1z} + w_{0\xi} = 0 \\ &w_1 + \eta_0 w_{0z} = \eta_{0\tau} + \Delta\eta_{0T} - \eta_{1\xi} - c\eta_{0\xi} + u_0\eta_{0\xi} \end{aligned}\right\} \text{ on } z = 1$$

and

$$u_1 = w_1 = 0 \quad \text{on} \quad z = 0.$$

This time we start by retaining the terms in r and Δ (cf. Q5.7), but we use the solution previously found, which is valid outside the boundary layer; in particular, we note that

$$u_{0z} = 0; \quad p_{0z} = 0; \quad w_{0z} = -\eta_{0\xi},$$

since $(u_0, p_0, \eta_0) \sim (u_{00}, p_{00}, \eta_{00})$ to all (algebraic) orders in r, as we mentioned above. Thus we obtain the equations

$$\eta_{0\tau} + \Delta\eta_{0T} - u_{1\xi} - c\eta_{0\xi} + \eta_{0\xi} + \eta_0\eta_{0\xi} = -p_{1\xi} + r(u_{1zz} + \eta_{0\xi\xi}); \tag{5.34}$$

$$p_{1z} = -z\eta_{0\xi}; \quad u_{1\xi} + w_{1z} = 0 \tag{5.35}$$

with

$$\left.\begin{array}{l} p_1 = \eta_1 - 2r\eta_{0\xi}; \quad u_{1z} - \eta_{0\xi\xi} = 0; \\ w_1 - \eta_0\eta_{0\xi} = \eta_{0\tau} + \Delta\eta_{0T} - \eta_{1\xi} - c\eta_{0\xi} + \eta_0\eta_{0\xi} \end{array}\right\} \text{ on } z = 1, \qquad (5.36)$$

for the problem outside the boundary layer. We obtain directly from equations (5.35) and (5.36) (cf. equation (3.26))

$$p_1 = \frac{1}{2}(1 - z^2)\eta_{0\xi\xi} + \eta_1 - 2r\eta_{0\xi}$$

and then from equation (5.34)

$$\begin{aligned} &\eta_{0\tau} + \Delta\eta_{0T} - c\eta_{0\xi} + \eta_0\eta_{0\xi} + w_{1z} \\ &\qquad = -\eta_{1\xi} + 2r\eta_{0\xi\xi} - \frac{1}{2}(1 - z^2)\eta_{0\xi\xi\xi} + r(u_{1zz} + \eta_{0\xi\xi}) \end{aligned}$$

so that

$$\begin{aligned} w_1 = (c\eta_{0\xi} - \eta_{0\tau} - \Delta\eta_{0T} - \eta_0\eta_{0\xi} + 3r\eta_{0\xi\xi} - \frac{1}{2}\eta_{0\xi\xi\xi} - \eta_{1\xi})z \\ + \frac{1}{6}z^3\eta_{0\xi\xi\xi} + ru_{1z} + f_0(\xi, \tau, T; \Delta, r) \quad (5.37) \end{aligned}$$

where f_0 is an arbitrary function of integration. Finally, the kinematic condition on $z = 1$, in (5.36), yields

$$\begin{aligned} &c\eta_{0\xi} - \eta_{0\tau} - \Delta\eta_{0T} - \eta_0\eta_{0\xi} + 4r\eta_{0\xi\xi} - \frac{1}{3}\eta_{0\xi\xi\xi} - \eta_{1\xi} + f_0 - \eta_0\eta_{0\xi} \\ &\qquad = \eta_{0\tau} + \Delta\eta_{0T} - \eta_{1\xi} - c\eta_{0\xi} + \eta_0\eta_{0\xi}, \end{aligned}$$

or

$$2(\eta_{0\tau} + \Delta_{0T} - c\eta_{0\xi}) + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} + 4r\eta_{0\xi\xi} = f_0, \qquad (5.38)$$

which is to be compared with our conventional KdV equation (3.28):

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0.$$

It is left as a simple exercise to confirm that the surface shear stress condition (in (5.36)) is automatically satisfied since, away from $z = 0$, $u_{1zz} + \eta_{0\xi\xi} = 0$ (to within exponentially small terms as $r \to 0$); see Q5.8.

The final stage of this calculation involves the construction of the solution in the boundary layer and hence, via matching, the determination of the function $f_0(\xi, \tau, T; \Delta, r)$. Once this is done we may return to our KdV-type equation, (5.38), and consider the size and rôle of the various new terms that have appeared. The boundary layer, as is evident from equations (5.30), is in the region defined by $z = O(r^{1/2})$, $r \to 0$, and then $w_0 = O(r^{1/2})$ (from the equation of mass conservation in (5.30)). Thus we introduce new variables

$$z = r^{1/2} Z, \quad w_0 = r^{1/2} W_0(\xi, \tau, T, Z; r) \tag{5.39}$$

and, correspondingly,

$$u_0 = U_0(\xi, \tau, T, Z; r), \quad p_0 = P_0(\xi, \tau, T, Z; r);$$

of course, η_0 is unchanged since it is not a function of z. Equations (5.30) and (5.32) therefore become

$$-U_{0\xi} = -P_{0\xi} + U_{0ZZ}; \quad P_{0Z} = 0; \quad U_{0\xi} + W_{0Z} = 0 \tag{5.40}$$

with

$$U_0 = W_0 = 0 \quad \text{on} \quad Z = 0.$$

We see immediately that

$$P_0 = \eta_0, \quad Z \geq 0,$$

in order to match to the solution $p_0 = \eta_0$ (which merely restates the uniform validity that we have previously noted). The equation for U_0 then becomes

$$\left.\begin{aligned} &U_{0ZZ} + U_{0\xi} = \eta_{0\xi}, \\ \text{with} \\ &U_0 = 0 \text{ on } Z = 0; \quad U_0 \to \eta_0 \text{ as } Z \to \infty, \end{aligned}\right\} \tag{5.41}$$

this latter condition ensuring that U_0 and $u_0\,(= \eta_0)$ match.

The problem posed in (5.41) is conveniently reformulated by writing

$$U_0 = \eta_0 + \mathscr{U}_0 \quad \text{and} \quad \zeta = -\xi$$

to give

$$\mathscr{U}_{0ZZ} = \mathscr{U}_{0\zeta}$$

with

$$\mathscr{U}_0 = -\eta_0 \text{ on } Z = 0; \quad \mathscr{U}_0 \to 0 \text{ as } Z \to \infty.$$

When we set

$$\eta_0(\xi, \tau, T) = \eta_0(-\zeta, \tau, T) = -H_0(\zeta, \tau, T),$$

the solution (following Duhamel's method, Q5.9) can be written as

$$\mathscr{U}_0 = \frac{2}{\sqrt{\pi}} \int_0^\infty H_0\left(\zeta - \frac{Z^2}{4y^2}, \tau, T\right) \exp(-y^2)\, \mathrm{d}y, \tag{5.42}$$

so

$$U_0 = \eta_0 - \frac{2}{\sqrt{\pi}} \int_0^\infty \eta_0\left(\xi + \frac{Z^2}{4y^2}, \tau, T\right) \exp(-y^2)\, \mathrm{d}y. \tag{5.43}$$

So that we can match, we require the solution for W_0 which, from equations (5.40) and (5.43) becomes

$$W_0 = -Z\eta_{0\xi} + \frac{2}{\sqrt{\pi}} \int_0^Z \left\{ \int_0^\infty \eta_{0\xi}\left(\xi + \frac{Z^2}{4y^2}, \tau, T\right) \exp(-y^2)\, \mathrm{d}y \right\} \mathrm{d}Z, \tag{5.44}$$

satisfying $W_0 = 0$ on $Z = 0$. The matching is then between (5.44) as $Z \to \infty$, and

$$w \sim w_0 + \varepsilon w_1 \tag{5.45}$$

as $z \to 0$; in particular, written in boundary-layer variables, $z = r^{1/2}Z$ and $w = r^{1/2}W$, (5.45) becomes (from (5.33) and (5.37))

$$W \sim -Z\eta_{0\xi} + \frac{\varepsilon}{r^{1/2}} f_0$$

so matching to (5.44), as $Z \to \infty$, requires

$$\frac{\varepsilon}{r^{1/2}} f_0 = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \eta_{0\xi}\left(\xi + \frac{Z^2}{4y^2}, \tau, T\right) \exp(-y^2)\, \mathrm{d}y\, \mathrm{d}Z. \tag{5.46}$$

(Notice that the first term, $-Z\eta_{0\xi}$, automatically matches, confirming the uniform validity of the solution w_0.) The appearance of ε in the definition of f_0, through (5.46), is not consistent with our formulation (since we have already expanded in terms of ε^n). This is simply telling us that a *precise* balance of terms will require a choice $r = r(\varepsilon)$, and then the calculation repeated with this choice in place; we shall write more of this shortly. It is left as an exercise (Q5.10) to demonstrate that, from equation (5.46), we may write

$$f_0 = \frac{1}{\varepsilon} \sqrt{\frac{r}{\pi}} \int_\xi^\infty \eta_{0\xi'}(\xi', \tau, T) \frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}}. \tag{5.47}$$

Thus, finally, we have a KdV-type equation which incorporates the dominant effects of (laminar) viscosity, provided $r \to 0$ (that is, if the Reynolds number is large enough); our equation is, from (5.38),

$$2(\eta_{0\tau} + \Delta\eta_{0T} - c\eta_{0\xi}) + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = \frac{1}{\varepsilon}\sqrt{\frac{r}{\pi}}\int_{\xi}^{\infty} \eta_{0\xi'}\frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}}. \tag{5.48}$$

Here, we have retained only the dominant terms associated with $\Delta\,(\to 0)$ and $r\,(\to 0)$; clearly the term $4r\eta_{0\xi\xi}$ (in equation (5.38)) is much smaller, as $r \to 0$, than that associated with $\sqrt{r}/\varepsilon$ (although the term $\eta_{0\xi\xi}$ will figure in a later calculation). A number of different and important choices can be made that describe diverse problems, each leading to an appropriate balance of terms; we shall return to equation (5.48) in the next section, but let us here examine the problem we first posed: the slow modulation of the solitary wave.

The solitary wave, in the absence of any modulation, is a steady solution of the Korteweg–de Vries equation (that is, $\eta_0 = \eta_0(\xi)$ only), so

$$-2c\eta_{0\xi} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0,$$

with the solution

$$\eta_0 = 2c\,\mathrm{sech}^2\left(\xi\sqrt{\frac{3c}{2}}\right).$$

However, we now incorporate a slow modulation of this solution, on the scale T, by virtue of the weak viscous contribution. Thus we choose $\Delta = \sqrt{r}/\varepsilon$, and hence obtain

$$-2c\eta_{0\xi} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = \Delta\left\{-2\eta_{0T} + \frac{1}{\sqrt{\pi}}\int_{\xi}^{\infty} \eta_{0\xi'}\frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}}\right\}, \tag{5.49}$$

with $\Delta \to 0$ and where

$$\eta_0 \sim 2c\,\mathrm{sech}^2\left(\xi\sqrt{\frac{3c}{2}}\right), \quad c = c(T). \tag{5.50}$$

The model that we have in mind here is represented in Figure 5.3. The solitary wave is moving into stationary water and as it does so a (thin) boundary layer is initiated near the front of the wave. This boundary layer then grows behind the solitary wave; in the frame at rest relative to

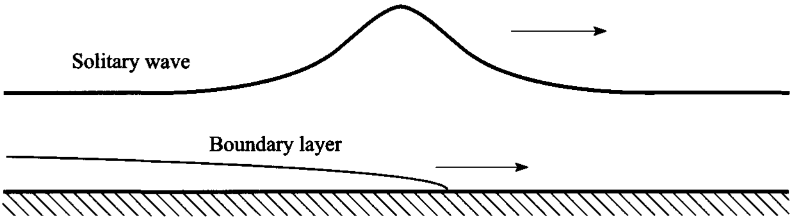


Figure 5.3. Sketch of a solitary wave moving into stationary water with a viscous boundary layer (on the bottom) growing back from the front of the wave.

the wave, the flow is from right to left and the boundary layer is stationary in this frame, and growing to the left.

The most direct way to obtain the details of the modulation is to invoke the condition

$$\eta_0 \to 0 \quad \text{as} \quad |\xi| \to \infty$$

and to form the integral over all ξ of equation (5.49), to give

$$[-2c\eta_0 + \frac{3}{2}\eta_0^2 + \frac{1}{3}\eta_{0\xi\xi}]_{-\infty}^{\infty} = \Delta\left\{-2\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{\infty}\eta_0 \mathrm{d}\xi + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{\xi}^{\infty}\eta_{0\xi'}\frac{\mathrm{d}\xi'}{\sqrt{\xi'-\xi}}\mathrm{d}\xi\right\}$$

and then

$$2\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{\infty}\eta_0 \mathrm{d}\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{\xi}^{\infty}\eta_{0\xi'}\frac{\mathrm{d}\xi'}{\sqrt{\xi'-\xi}}\mathrm{d}\xi.$$

This identity provides an equation for $c(T)$, which can be obtained by introducing (5.50) to give

$$2\frac{\mathrm{d}}{\mathrm{d}T}\left\{2c\int_{-\infty}^{\infty}\operatorname{sech}^2\left(\xi\sqrt{\frac{3c}{2}}\right)\mathrm{d}\xi\right\} = -4c\sqrt{\frac{3c}{2\pi}}\int_{-\infty}^{\infty}\int_{\xi}^{\infty}\operatorname{sech}^2\left(\xi'\sqrt{\frac{3c}{2}}\right)\tanh\left(\xi'\sqrt{\frac{3c}{2}}\right)\frac{\mathrm{d}\xi'}{\sqrt{\xi'-\xi}}\mathrm{d}\xi.$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}T}\left\{c\sqrt{\frac{2}{3c}}\int\limits_{-\infty}^{\infty}\operatorname{sech}^2 y\,\mathrm{d}y\right\}$$
$$=-\frac{c}{\sqrt{\pi}}\left(\frac{2}{3c}\right)^{1/4}\int\limits_{-\infty}^{\infty}\int\limits_{y}^{\infty}\operatorname{sech}^2 y'\tanh y'\frac{\mathrm{d}y'}{\sqrt{y'-y}}\mathrm{d}y. \quad (5.51)$$

The precise values of the constants that appear here are not particularly significant; the form of equation (5.51) is simply

$$\frac{\mathrm{d}}{\mathrm{d}T}(\sqrt{c}) = -2\mu c^{3/4} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}T}(c^{-1/4}) = \mu,$$

where $\mu\,(>0)$ is a constant (whose value turns out to be approximately 0.08), and hence

$$\frac{c}{c_0} = (1+\mu c_0^{1/4}T)^{-4}, \quad T \geq 0, \qquad (5.52)$$

where $c = c_0$ at $T = 0$. Equation (5.52) is our main result here (first obtained by Keulegan (1948)); it describes the attenuation of the amplitude of the solitary wave, and there is some experimental evidence to suggest that its general form is not too wide of the mark. Certainly we must not expect close agreement, mainly because our simple theory does not even attempt to represent a (probably) turbulent flow moving over a rough bed. Further discussion of results of this type can be found in some of the references at the end of this chapter.

We shall return to the KdV equation, with its viscous contribution as represented in (5.48), in the next section, but first we must describe another phenomenon in water waves: the undular bore.

5.2.3 *Undular bore – model I*

In Section 2.7 we introduced and discussed the hydraulic jump, as well as its counterpart, which moves relative to the physical frame: the bore. These phenomena were modelled as a discontinuity, although in reality there is usually a fairly narrow region over which the flow properties change markedly. This transition is observed to occur through a region of highly turbulent motion, which takes the form of a continually breaking wave. However, a river flow can sometimes support a change of flow properties that is far more gradual, without – or almost without – any sign of extensive turbulence. This happens if the change in levels is not too great; then it is often observed that behind the *smooth* transition there is a train of waves. This phenomenon is called the *undular bore*; see Figure 5.4. The interpretation of what is seen is that, rather than a considerable dissipation of energy at the front (as in the bore), the undular bore structure allows all (or most) of the energy loss to occur by transporting the energy away in the wave motion. We can expect this to occur when the amount of energy to be lost is quite small – so we have a 'weak' bore; a model for the energy loss can then be provided by a fairly small amount of (laminar) viscous dissipation. This is the essential idea behind the model for the undular bore that we describe here and, under slightly different assumptions, in the next section. Furthermore, we anticipate that the surface wave itself is a nonlinear object, so the oscillatory part of the profile is also likely to be nonlinear: for example, a cnoidal wave (discussed in Q2.67). Thus we look for a KdV-type of equation, which incorporates some appropriate viscous contribution – but this is precisely what we did in the previous section.

The calculation that produces our governing equation is not repeated here. It is precisely that described in Section 5.2.2, except that now we do not require the modulational ingredient (which was required in order to discuss the evolution of the solitary wave). Thus we dispense with the scale T and with $c(T)$, which were introduced in equations (5.24): we use only ξ and τ. Further, because our aim here is not to develop a slow modulation, the most convenient approach is to make the special choice $r = O(\varepsilon^2)$ (so that the Reynolds number is such that $R^{-1} = O(\varepsilon^{5/2})$). The problem now involves the single parameter ε, for $\varepsilon \to 0$, and it is then a simple exercise to confirm that our previous calculation goes through, resulting in the equation for the surface wave:

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = \frac{1}{\sqrt{\pi\mathscr{R}}}\int_{\xi}^{\infty} \eta_{0\xi'}(\xi', \tau)\,\frac{d\xi'}{\sqrt{\xi' - \xi}}. \tag{5.53}$$

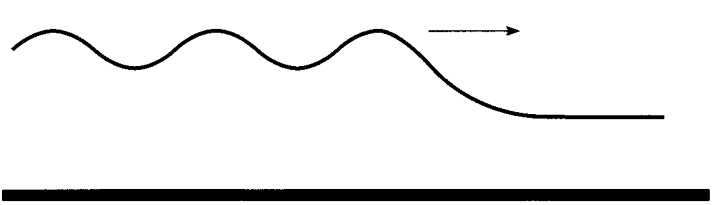


Figure 5.4. A sketch of the undular bore.

We have written $r = \varepsilon^2/\mathscr{R}$ (that is, $R = \varepsilon^{-5/2}\mathscr{R}$), and otherwise we have quoted from equation (5.48). Equation (5.53), or variants of it, have been obtained by Ott & Sudan (1970), Byatt-Smith (1971) and Kakutani & Matsuuchi (1975); the work of Byatt-Smith, in particular, is directed towards a description of the undular bore.

The equation for $\eta_0(\xi, \tau)$ represents the action of a thin viscous boundary layer – remember that $R^{-1} = \mathrm{O}(\varepsilon^{5/2})$ as $\varepsilon \to 0$ – which grows from near the wavefront; this is the mechanism which provides the dissipation of energy. Now, provided we restrict attention to regions not too far behind the front, we may seek steady solutions of equation (5.53). Clearly, far enough behind the front, the boundary layer will have grown sufficiently large that it can no longer be treated as thin: the boundary layer will then interact with, and disrupt, the surface wave. When this happens we shall not be able to sustain a steady solution. With this caveat in mind, Byatt-Smith (1971) discusses the nature of the steady solution given by

$$-2c\eta_0' + 3\eta_0\eta_0' + \frac{1}{3}\eta_0''' = \frac{1}{\sqrt{\pi\mathscr{R}}}\int_{\xi}^{\infty} \eta_0'(\xi' - c\tau)\frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}},$$

where $\eta_0 = \eta_0(\xi - c\tau)$ and the prime on η_0 denotes the derivative with respect to $(\xi - c\tau)$. It is convenient to rewrite the integral with

$$\xi' = \xi + \zeta'$$

and then to set $\xi - c\tau = \zeta$; this yields

$$-2c\eta_0' + 3\eta_0\eta_0' + \frac{1}{3}\eta_0''' = \frac{1}{\sqrt{\pi\mathscr{R}}}\int_{0}^{\infty} \eta_0'(\zeta + \zeta')\frac{\mathrm{d}\zeta'}{\sqrt{\zeta'}},$$

which is integrated once with respect to ζ, with the condition

$$\eta_0 \to 0 \quad \text{as} \quad \zeta \to +\infty.$$

Thus we obtain a nonlinear, ordinary integro-differential equation

$$-2c\eta_0 + \frac{3}{2}\eta_0^2 + \frac{1}{3}\eta_0'' = \frac{1}{\sqrt{\pi\mathscr{R}}}\int_{0}^{\infty} \eta_0(\zeta + \zeta')\frac{\mathrm{d}\zeta'}{\sqrt{\zeta'}}, \tag{5.54}$$

which describes steady solutions $\eta_0(\zeta)$, for various $\mathscr{R}$ and c. Equation (5.54) was integrated numerically by Byatt-Smith, for quite large values of $\mathscr{R}$ $(= 10^5/\pi$ and $10^6/\pi)$ and two values of c. (Our equation (5.54), although not identical to that derived by Byatt-Smith, is precisely equivalent to it.) An example of the form of solution (5.54) is shown in Figure 5.5, which makes clear that the essential character of the undular bore is recovered. A detailed numerical integration of this equation shows that: (a) the amplitude of the waves increases as c increases (completely consistent with the nonlinear character of solutions of the KdV equation); (b) the period of the oscillation increases as $\mathscr{R}$ increases.

There can be no doubt that equation (5.53), and then equation (5.54) for steady solutions, embody a mechanism which would appear to provide a perfectly reasonable model of the undular bore. However, there are some features of this approach to the problem which, although not of great significance taken individually, add up to a slightly unsatisfactory description. This model uses a well-structured and continually evolving boundary layer on $z = 0$, but a more realistic river flow is likely to have such a boundary layer completely disrupted (by an uneven bottom, for example). Indeed, we might expect that some dissipation – perhaps the major contribution – occurs near the front and that any further energy loss in the flow behind is insignificant; the excess is still propagated away. The boundary layer, as we have already commented, necessarily produces an unsteady profile (which, certainly, is not an important consideration for large $\mathscr{R}$). Nevertheless, a model that admits a completely steady solution (on the scales employed) would be a slight improvement. Finally, the equation itself, (5.53) or (5.54), is a nonlinear

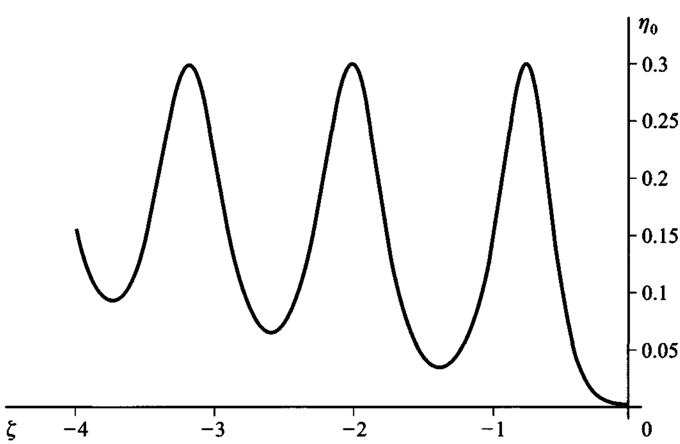


Figure 5.5. A numerical solution of equation (5.54), for $R = 10^4/\pi$ and $c = 0.1$.

integro-differential equation which is therefore not readily analysed; a simpler equation (which still embodies the relevant physics) would be an advantage. Thus, if we can find a model that addresses most of these points, we will have produced a useful alternative description of the undular bore. Of course, in the context of the theory of water waves, this might also prove to be an instructive mathematical exercise.

5.2.4 Undular bore – model II

It is surprisingly simple to *write down* an equation that should contain the essential features seen in the undular bore. This equation is to admit solutions that describe a smooth transition from one depth to another (like the Burgers equation), together with an oscillatory (dispersive) wave; see Q1.55 and Q2.67. Such an equation might take the form

$$u_t + uu_x + u_{xxx} = u_{xx}, \tag{5.55}$$

where we have set all coefficients to unity. But, no matter how attractive this appears, it must be treated as useful only if it can be shown to arise (from the relevant governing equations) under some consistent limiting process. This is what we shall now demonstrate, and then we present a brief discussion of the solutions of the resulting equation (which *is* essentially (5.55)).

The governing equations that we start from here are those given in Section 5.1 (and then as transformed in Section 5.2.2 to remove the parameter δ) but with an important addition. The undisturbed flow is no longer stationary; it is a fully developed Poiseuille channel flow moving under gravity, and so we introduce gravity components $(g\sin\alpha, -g\cos\alpha)$ and replace εu by $U(z)+\varepsilon u$; see Figure 5.6. The resulting equations then become

$$u_t + (U+\varepsilon u)u_x + (U'+\varepsilon u_z)w = -p_x + \beta + \frac{1}{R\varepsilon\sqrt{\varepsilon}}(U''+\varepsilon u_{zz}+\varepsilon^2 u_{xx}); \tag{5.56}$$

$$\varepsilon\{w_t + (U+\varepsilon u)w_x + \varepsilon w w_z\} = -p_z + \frac{\sqrt{\varepsilon}}{R}(w_{zz}+\varepsilon w_{xx}); \tag{5.57}$$

$$u_x + w_z = 0, \tag{5.58}$$

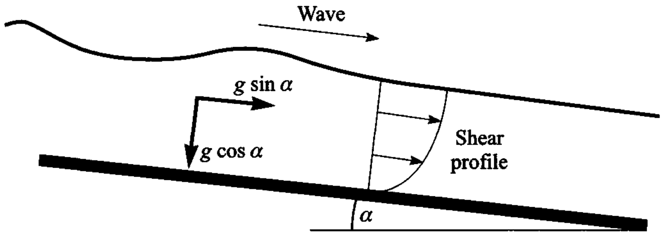


Figure 5.6. A sketch of the fully developed (Poiseuille) velocity profile for the flow moving under gravity, and the surface wave.

with

$$\left.\begin{aligned} p-\eta-\frac{2\sqrt{\varepsilon}}{R}\{w_z-\eta_x(U'+\varepsilon u_z+\varepsilon^2 w_x)& \\ +\varepsilon^3\eta_x^2 u_x\}/(1+\varepsilon^3\eta_x^2)=0;& \\ (1-\varepsilon^3\eta_x^2)(U'+\varepsilon u_z+\varepsilon^2 w_x)+2\varepsilon^3(w_z-u_x)\eta_x=0;& \\ w=\eta_t+(U+\varepsilon u)\eta_x,& \end{aligned}\right\}\quad \text{on } z=1+\varepsilon\eta \tag{5.59}$$

and

$$U+\varepsilon u=0, \quad w=0 \quad \text{on} \quad z=0. \tag{5.60}$$

The constant β is defined as

$$\beta=\frac{\tan\alpha}{\varepsilon\sqrt{\varepsilon}}, \tag{5.61}$$

and this expresses the required component of gravity down the channel which is needed in order to maintain the flow $U(z)$. In the absence of any surface wave, the equations (5.56)–(5.60) become simply

$$\beta+\frac{U''}{R\varepsilon\sqrt{\varepsilon}}=0; \quad U'(1)=0; \quad U(0)=0,$$

so we have

$$U(z)=U_0(2z-z^2), \tag{5.62}$$

the Poiseuille profile, where

$$2U_0=R\beta\varepsilon\sqrt{\varepsilon}=R\tan\alpha,$$

and we treat U_0 (which is essentially a Froude number for this flow) as O(1). This choice of U_0 obviously gives $U(z)=\mathrm{O}(1)$, and provides the balance between R and α which is required to produce the fully developed flow at leading order.

For sufficiently large Reynolds number, R, the equations (5.56)–(5.60) admit solutions which represent waves that move at speeds determined by the Burns condition, when $U(z)$ is given by (5.62), and also nonlinear waves that move over this shear flow; see Sections 3.4.1 and 3.4.2. However, our governing equations here contain a viscous contribution, and this enables another type of wave to exist. To see how this arises, let us initially transform

$$X = \Delta x, \quad T = \Delta t, \quad w = \Delta W, \tag{5.63}$$

and take the limiting process $\Delta \to 0$, at fixed ε and R, with $U(z)$ given by (5.62). (The wave that we are about to describe exists even for $\Delta = O(1)$, but it is easier to see the appropriate balance with $\Delta \to 0$.) The leading equations, as $\Delta \to 0$, are then

$$\left.\begin{aligned} &u_{zz} = 0; \quad p_z = 0; \quad u_X + W_z = 0, \\ &\text{with} \\ &p = \eta; \quad u_z + U''\eta = 0; \quad W = \eta_T + U\eta_X \quad \text{on} \quad z = 1 \\ &\text{and} \\ &u = W = 0 \quad \text{on} \quad z = 0, \end{aligned}\right\} \tag{5.64}$$

where we have written $U'(1 + \varepsilon\eta) = U'(1) + \varepsilon\eta U''(1)\ldots$. The solution of the set (5.64) is immediately

$$p = \eta, \quad u = 2U_0\eta z, \quad W = -U_0\eta_X z^2$$

with

$$\eta_T + 2U_0\eta_X = 0; \quad \text{that is,} \quad \eta = \eta(X - 2U_0 T).$$

Thus there exists a surface wave which moves to the right at a speed $2U_0$, which is twice the surface speed (U_0) of the underlying Poiseuille flow.

In consequence, the linear equations for the surface wave allow three possible waves: two waves whose speed is determined by the Burns condition (see Q3.45(b)), one of which satisfies $c_1 > 0$ and the other gives $c_2 < 0$, and one which moves at a speed $2U_0$ (> 0). It can be shown (following on from Q3.46(b)) that

$$c_2 < 2U_0 < c_1$$

and, further, that these three waves form a wave hierarchy of the type discussed in Q1.51 and Q1.52. In particular, it follows that the waves which move at the Burns speeds (c_1, c_2) decay – they are called *dynamic* waves – leaving the main disturbance to move at the speed $2U_0$ (which is called the *kinematic* wave). A discussion of kinematic and dynamic waves can be found in Lighthill & Whitham (1955), Whitham (1959, 1974); the application of these ideas in the current context, and to the undular bore, is given in Johnson (1972). It is sufficient for our purposes here to investigate more fully the nature of the propagation at the speed $2U_0$, on the assumption that the dynamic waves decay and therefore, eventually, play no rôle.

The model that we are employing represents a flowing river – so it is more realistic – with a surface wave that propagates forwards ('downhill') at a speed greater than the surface speed of the undisturbed flow. The wave moves into undisturbed conditions ahead, and we wish to determine how this wave evolves and whether a change in depth (with undulations) is possible. The approach that we adopt is the familiar one of following the wave (which moves at the speed $2U_0$), constructing its evolution on a suitable long time scale and, here, also making an appropriate choice of the Reynolds number.

To this end we introduce

$$\xi = x - 2U_0 t, \quad \tau = \varepsilon t \tag{5.65}$$

and choose

$$R = \sqrt{\varepsilon}\mathscr{R}, \tag{5.66}$$

although other scales exist, involving appropriate combinations of ε, δ and R. The choice made here is the simplest that produces the required balance of terms. The equations and boundary conditions, (5.56)–(5.60), then become

$$\begin{aligned}\varepsilon u_\tau + (U - 2U_0 + \varepsilon u)u_\xi + (U' + \varepsilon u_z)w \\ = -p_\xi + \left(\beta + \frac{U''}{\varepsilon^2 \mathscr{R}}\right) + \frac{1}{\varepsilon\mathscr{R}}(u_{zz} + \varepsilon u_{\xi\xi});\end{aligned} \tag{5.67}$$

$$\varepsilon\{\varepsilon w_\tau + (U - 2U_0 + \varepsilon u)w_\xi + \varepsilon w w_z\} = -p_z + \frac{1}{\mathscr{R}}(w_{zz} + \varepsilon w_{\xi\xi}); \tag{5.68}$$

$$u_\xi + w_z = 0, \tag{5.69}$$

with

$$\left.\begin{aligned} p-\eta-\frac{2}{\mathscr{R}}\{w_z-\eta_\xi(U'+\varepsilon u_z+\varepsilon^2 w_\xi) \\ +\varepsilon^3\eta_\xi^2 u_\xi\}/(1+\varepsilon^3\eta_\xi^2)=0; \\ (1-\varepsilon^3\eta_\xi^2)(U'+\varepsilon u_z+\varepsilon^2 w_\xi)+2\varepsilon^3(w_z-u_\xi)\eta_\xi=0; \\ w=\varepsilon\eta_\tau+(U-2U_0+\varepsilon u)\eta_\xi, \end{aligned}\right\}\quad \text{on } z=1+\varepsilon\eta \tag{5.70}$$

and

$$U+\varepsilon u=0,\quad w=0 \quad \text{on} \quad z=0, \tag{5.71}$$

where $U(z)$ is to satisfy (5.62). We seek an asymptotic solution of these equations in the form

$$q\sim\sum_{n=0}^{\infty}\varepsilon^n q_n(\xi,\tau,z),\quad \eta\sim\sum_{n=0}^{\infty}\varepsilon^n\eta_n(\xi,\tau),\quad \varepsilon\to 0,$$

where q (and correspondingly q_n) represent u, w and p; the new Reynolds number, $\mathscr{R}$, is then held fixed as $\varepsilon\to 0$. The leading order problem from equations (5.67)–(5.71) is directly

$$u_{0zz}=0;\quad p_{0z}=\frac{1}{\mathscr{R}}w_{0zz};\quad u_{0\xi}+w_{0z}=0,$$

with

$$p_0=\eta_0+\frac{2}{\mathscr{R}}w_{0z};\quad u_{0z}+\eta_0 U''=0;\quad w_0=(U-2U_0)\eta_{0\xi}\quad \text{on}\quad z=1,$$

and

$$u_0=w_0=0 \quad \text{on} \quad z=0,$$

which are essentially equations (5.64). The only difference arises in the way in which p_0 is determined here, but since p_0 is found *after* u_0, w_0 and η_0 are fixed, this is not critical. Indeed, we see that

$$u_0=2U_0\eta_0 z,\quad w_0=-U_0\eta_{0\xi}z^2,\quad p_0=\eta_0-\frac{2U_0}{\mathscr{R}}(1+z)\eta_{0\xi}, \tag{5.72}$$

where $\eta_0(\xi,\tau)$ is an arbitrary function (at this order).

At the next order we obtain, from equation (5.67),

$$(U-2U_0)u_{0\xi}+U'w_0=-p_{0\xi}+\frac{1}{\mathscr{R}}(u_{1zz}+u_{0\xi\xi});$$

from (5.69) we get simply

$$u_{1\xi} + w_{1z} = 0.$$

The boundary conditions yield

$$\left.\begin{aligned} \eta_1 U'' + u_{1z} + w_{0\xi} &= 0; \\ w_1 + \eta_0 w_{0z} &= \eta_{0\tau} + (\eta_0 U' + u_0)\eta_{0\xi} + (U - 2U_0)\eta_{1\xi} \end{aligned}\right\} \text{ on } z = 1$$

and

$$u_1 = w_1 = 0 \quad \text{on} \quad z = 0,$$

where we have omitted the boundary condition on the pressure at the surface, and equation (5.68), at this order; these enable p_1 to be determined, but this is not required in order to find the equation for $\eta_0(\xi, \tau)$. (This has happened because p essentially uncouples from the other functions, as we alluded to above; the construction of p_1 is left as an exercise in Q5.14.) It is altogether straightforward to show that

$$u_1 = \frac{\mathscr{R}U_0^2}{6}(z^4 - 4z^3)\eta_{0\xi} + \frac{\mathscr{R}z^2}{2}\eta_{0\xi} - \frac{U_0}{3}(3z^2 + 2z^3)\eta_{0\xi\xi} + Az, \tag{5.73}$$

where $A(\xi, \tau)$ is an arbitrary function of integration; similarly

$$w_1 = -\frac{\mathscr{R}U_0^2}{30}(z^5 - 5z^4)\eta_{0\xi\xi} - \frac{\mathscr{R}}{6}z^3\eta_{0\xi\xi} + \frac{U_0}{6}(2z^3 + z^4)\eta_{0\xi\xi\xi} - \frac{z^2}{2}A_\xi. \tag{5.74}$$

The surface boundary conditions yield, first,

$$A = 2U_0\eta_1 + \mathscr{R}\left(\frac{4}{3}U_0^2 - 1\right)\eta_{0\xi} + 5U_0\eta_{0\xi\xi},$$

and then

$$\eta_{0\tau} + 4U_0\eta_0\eta_{0\xi} + 2U_0\eta_{0\xi\xi\xi} = \frac{\mathscr{R}}{3}\left(1 - \frac{8}{5}U_0^2\right)\eta_{0\xi\xi}, \tag{5.75}$$

the required *Korteweg–de Vries–Burgers* (*KVB*) *equation*.

This is the equation that we seek, but its construction is very different from the corresponding equation based on boundary-layer arguments. That equation, (5.53), recovers the classical KdV equation as the Reynolds number is increased indefinitely – a comforting property; our new equation, (5.75), is *dominated* by the effects of viscosity. We require viscous stresses to balance gravity and so provide the ambient flow, and also a special limiting process ($\varepsilon \to 0$, $\mathscr{R}$ fixed) in order to generate the appropriate internal viscous dissipation (represented by the term $\eta_{0\xi\xi}$).

The removal of the viscous contribution involves $U_0 \to 0$ and $\mathscr{R} \to \infty$, which clearly destroys the character of our KVB equation: we cannot recover the KdV equation in the way we might have expected (but it does arise if we let $\mathscr{R} \to 0$). Nevertheless, we have succeeded in our intention to find a limiting process that balances KdV nonlinearity and dispersion against Burgers nonlinearity and dissipation.

The KVB equation, (5.75), possesses a number of interesting features. First, it and our model admit a steady solution; second, the damping (or dissipative) term

$$\frac{\mathscr{R}}{3}\left(1 - \frac{8}{5}U_0^2\right)\eta_{0\xi\xi},$$

has a negative coefficient if

$$U_0^2 > \frac{5}{8}. \tag{5.76}$$

This condition implies an energy input and, presumably, we must anticipate that our model is no longer valid. In fact the speed of the surface wave is, to leading order as $\varepsilon \to 0$, $2U_0$, and all speeds have been non-dimensionalised with respect to $\sqrt{g_0 h_0}$, $g_0 = g\cos\alpha$; thus $2U_0$ is the Froude number of the wave. Thus, when we write $F = 2U_0$, condition (5.76) becomes

$$F^2 > \frac{5}{2} \quad \text{or} \quad F \gtrsim 1.58.$$

Now it is commonly observed that bores with F larger than about 1.2 have turbulent, breaking fronts; on the other hand, if F is less than this (but, of course, $F > 1$; see Section 2.7), we typically observe the undular bore. This suggests that our model has captured an important phenomenon (even though the values do not quite correspond); indeed, Dressler (1949) has shown that the condition $F^2 \geq 5/2$ heralds the formation of *roll waves*, which, locally, have the appearance of turbulent bores.

This brief discussion of the rôle of laminar viscosity in water-wave theory is brought to a close as we present a few observations on the steady solutions of the KVB equation, (5.75). We seek a solution in the form $\eta_0(\xi - c\tau)$, to give

$$-c\eta_0' + 4U_0\eta_0\eta_0' + 2U_0\eta_0''' = \frac{\mathscr{R}}{3}(1 - \frac{8}{5}U_0^2)\eta_0''$$

which, after one integration in $\zeta = \xi - c\tau$ and imposing the condition $\eta_0 \to 0$ as $\zeta \to +\infty$, yields

$$-c\eta_0 + 2U_0\eta_0^2 + 2U_0\eta_0'' = \frac{\mathscr{R}}{3}(1 - \frac{8}{5}U_0^2)\eta_0'.$$

This equation is conveniently normalised by introducing the transformation

$$\eta_0 \to \frac{c}{2U_0}\eta_0, \quad \zeta \to \sqrt{\frac{2U_0}{c}}\zeta,$$

to give

$$\eta_0^2 - \eta_0 + \eta_0'' = \lambda\eta_0', \tag{5.77}$$

where

$$\lambda = \frac{1}{3}\mathscr{R}(1 - \frac{8}{5}U_0^2)/\sqrt{2U_0c}.$$

In the form (5.77), we then have

$$\eta_0 \to 1 \quad \text{as} \quad \zeta \to -\infty,$$

if solutions exist for which this is possible, which certainly requires $\lambda > 0$ i.e. $U_0^2 < 5/8$. It is an elementary exercise (see Q5.15) to seek the asymptotic behaviours

$$\eta_0 \sim a\exp(-\alpha\zeta), \quad \zeta \to +\infty$$

and

$$\eta_0 \sim 1 - b\exp(\beta\zeta), \quad \zeta \to -\infty,$$

and to find that

$$\alpha = \frac{1}{2}\left\{\sqrt{\lambda^2 + 4} - \lambda\right\}, \quad \beta = \frac{1}{2}\left\{\lambda \pm \sqrt{\lambda^2 - 4}\right\}.$$

The choice of the sign in α ensures that $\eta_0 \to 0$ as $\zeta \to +\infty$; in β, either sign is possible (and it is clear here that we must have $\lambda > 0$), but the profile as $\zeta \to -\infty$ may be either monotonic ($\lambda \geq 2$) or oscillatory ($0 < \lambda < 2$). Thus equation (5.77) will allow either a monotonic transition through the jump (a non-turbulent classical jump) or an oscillation about $\eta_0 = 1$ (the undular bore). The interpretation is quite simply that larger λ means larger dissipation (mainly in the neighbourhood of $\zeta = 0$), so no wave is required to transport the excess energy away. For smaller λ, the wave is needed to carry the surplus energy away from the front.

A number of interesting properties of the steady KVB equation can be explored and exploited; see Q5.16–Q5.19. We conclude by presenting two solutions of the KVB equation, (5.77), in Figure 5.7; these are based on

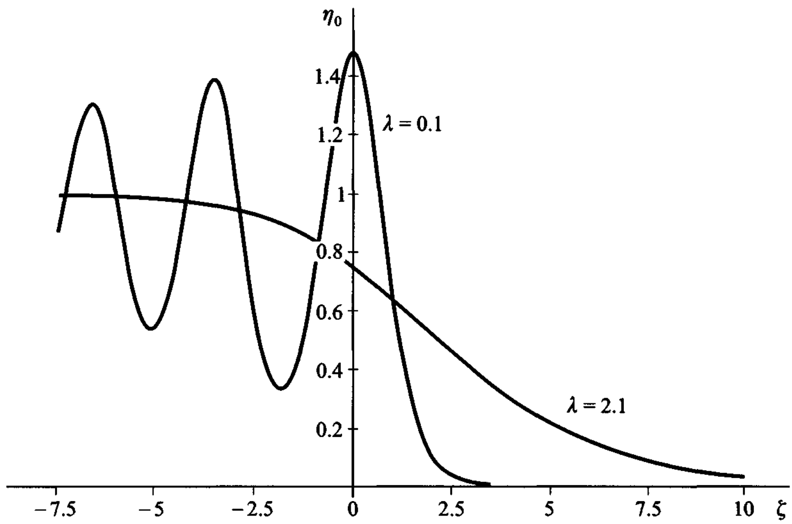


Figure 5.7. Two solutions of the steady state Korteweg–de Vries–Burgers (KVB) equation, (5.77), for $\lambda = 0.1$, 2.1.

numerical solutions of the equation. One is oscillatory ($\lambda = 0.1$), and is therefore a representation of the undular bore, and the other is a monotonic profile ($\lambda = 2.1$). The undular bore displays a front that is reminiscent of the solitary wave, and the oscillations can be described by an evolving cnoidal wave; both these properties can be formalised by examining the limit $\lambda \to 0$ (as considered in Johnson (1970)). The monotonic profile, on the other hand, takes the form of a distorted *tanh* curve, where the distortion is progressively less pronounced as $\lambda \to \infty$; see Q5.18.

Further reading

The rôle of viscosity in fluid mechanics in general, and in the theory of water waves in particular, is a very large and important subject. The fundamental effects that are encountered in the study of fluids are best addressed through standard texts on fluid mechanics (given, for example, at the end of Chapter 1). However, in addition to the references already given (including those relevant references contained therein), the reader is directed to Lighthill (1978), Craik (1988) and Mei (1989) for some useful and fairly up-to-date material on viscous dissipation in wave propagation. The text by Debnath (1994) also touches on some of these ideas.

Exercises

Q5.1 *Dispersion relation: large R.* Consider the dispersion relation, (5.21), in the limit $R \to \infty$ at fixed δk, and hence obtain result (5.22):

$$\omega \sim k\left\{\sqrt{\frac{\tanh \delta k}{\delta k}} - \frac{(1+\mathrm{i})}{2\sqrt{2R}} \frac{(\delta k)^{1/4}}{\cosh^{5/4} \delta k \sinh^{3/4} \delta k}\right\}.$$

Q5.2 *Dispersion relation: $\delta k = O(1/R)$.* Show that the result obtained in Q5.1 is not uniformly valid, as $\delta k \to 0$, where $\delta k R = \mathrm{O}(1)$. Hence obtain the equations that describe the leading approximation to the dispersion relation, (5.21), in the limit $R \to \infty$ at $\delta k R$ fixed.

[Note: these equations cannot be solved in closed form.]

Q5.3 *Dispersion relation: small δk.* Consider the dispersion relation, (5.21), in the limit $\delta k \to 0$ at fixed R, and hence show that

$$\omega \sim -\mathrm{i}\delta k^2 R/3$$

(where the real part of ω turns out to be exponentially small).

[This case, which is essentially a high-viscosity limit, shows that (at this order) there is *no propagation*, only decay. As an additional exercise, you may show that the expression obtained in Q5.2 agrees, for $\delta k \to \infty$, with that given in Q5.1 as $\delta k \to 0$, and that there is agreement between Q5.2 and Q5.3 for $\delta k \to 0$.]

Q5.4 *Dispersion relation with surface tension.* Repeat the calculation presented in Section 5.2.1, but with the surface pressure condition adjusted to accommodate the effects of surface tension, namely

$$p - \eta - \frac{2\delta}{R} w_z = -\delta^2 W_e \eta_{xx} \quad \text{on} \quad z = 1;$$

cf. Section 2.1. Hence obtain the dispersion relation, with surface tension, which corresponds to that given in equation (5.21).

Q5.5 *A model boundary layer problem.* Obtain the solution of the ordinary differential equation

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0, \quad 0 \le x \le 1,$$

that satisfies

$$y(0;\varepsilon)=0,\quad y(1;\varepsilon)=1,$$

where $\varepsilon > 0$ is a constant. Describe the character of this solution, for $\varepsilon \to 0$, in the two cases

(a) x away from $x = 0$; (b) $x = \varepsilon X$, $X = \mathrm{O}(1)$.

[The region near the boundary, measured by $x = \mathrm{O}(\varepsilon)$ is where the *boundary layer* exists; in this narrow region the solution adjusts from the value e (approximately, as $\varepsilon \to 0$) to 0 (on $x = 0$).]

Q5.6 *Asymptotic approach to a boundary layer problem.* Solve the problem given in Q5.5 by seeking two asymptotic solutions, in the form

(a) $y \sim \sum_{n=0}^{\infty} \varepsilon^n y_n(x), \quad \varepsilon \to 0,$

valid for x away from $x = 0$, and satisfying the condition on $x = 1$;

(b) set $x = \varepsilon X$ and write

$$y \sim \sum_{n=0}^{\infty} \varepsilon^n Y_n(X), \quad \varepsilon \to 0,$$

satisfying the condition on $x = 0$ (that is, on $X = 0$).

Now match the solutions obtained in (a) and (b), thereby uniquely determining the solution in (b).

[You need find only the first terms in each expansion, but a second could be found as well, if you are so minded.]

Q5.7 *Inviscid solution of the viscous equations.* Confirm that the solution given in equations (5.33) satisfies all the equations and boundary conditions in (5.30)–(5.32), with the exception of the no-slip condition on $z = 0$.

Q5.8 *Surface shear-stress condition.* Show that, away from the boundary layer (formed as $r \to 0$), a solution exists in which $u_{1zz} + \eta_{0\xi\xi} = 0$ (a term in equation (5.34)). Hence obtain an expression for $u_{1\xi z}$ and use this to confirm that the surface shear-stress condition,

$$u_{1z} - \eta_{0\xi\xi} = 0 \quad \text{on} \quad z = 1,$$

is satisfied.

[This is the counterpart, at $\mathrm{O}(\varepsilon)$, of the solution discussed in Q5.7.]

Q5.9 *Heat/diffusion equation: Duhamel's method.* The calculation of the relevant solution of the equation

$$u_t = u_{xx}, \quad t \geq 0, \quad x \geq 0,$$

is constructed in two stages.

(a) Obtain the solution for $u(x, t)$ which satisfies

$$u(x, 0) = 0, \ x > 0; \quad u(0, t) = 1, \quad t > 0, \qquad (*)$$

in the form

$$u = U(x, t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{t}} \exp(-y^2)\,\mathrm{d}y.$$

(b) The effect of raising the 'temperature' on $x = 0$ to 1 at a time $t = t'\,(> 0)$, and then reducing it to zero at $t = t' + h\,(h > 0)$, is represented by the solution

$$u = U(x, t - t') - U(x, t - t' - h).$$

Over a very short time interval this is, approximately, $h(\partial U/\partial t)$ evaluated at time $t - t'$. If, during this interval, the temperature is actually $f(t')$, the resulting temperature over all times is then

$$\hat{u} = \int_{-\infty}^{t} \left(\frac{\partial U}{\partial t}\right)\bigg|_{t-t'} f(t')\,\mathrm{d}t';$$

this is Duhamel's result. Obtain an expression for $\partial U/\partial t$, and hence write down $\hat{u}$; confirm that $u = \hat{u}$ is, indeed, a solution of equation $(*)$.

Hence rewrite this solution so that it takes the form quoted in equation (5.42).

[In (a) introduce the *similarity variable*, $x/2\sqrt{t}$. The solution obtained in (b) gives the temperature (in $x > 0$, $t > 0$) when the end, $x = 0$, is set at the variable temperature $f(t)$. Here we have used the interpretation of u as temperature, so $(*)$ is called the heat conduction equation in this context; it is also often called the diffusion equation – here the diffusion of heat.]

Q5.10 *An integral identity.* Show that

$$2\int_0^\infty\int_0^\infty f\left(x+\frac{z^2}{4y^2}\right)\exp(-y^2)\,\mathrm{d}y\,\mathrm{d}z=\int_x^\infty f(x')\frac{\mathrm{d}x'}{\sqrt{x'-x}},$$

and evaluate this integral for the choice $f(x)=\exp(-\alpha x)$, where $\alpha\,(>0)$ is a real constant.

Q5.11 *Modulation of the solitary wave.* Follow the procedure described in equation (5.49) *et seq.*, but start by multiplying this equation by η_0. Hence obtain the corresponding expression for $c(T)$.

[This derivation uses the 'conserved' density η_0^2, rather than η_0 as given in the text. You might wish to obtain numerical estimates for the integrals that appear in these two formulae; the two expressions for $c(T)$ should, of course, be identical.]

Q5.12 *Propagation of the modulated solitary wave.* The (nondimensional) speed of the solitary wave, in the characteristic frame, is

$$c(T)=c_0(1+\alpha T)^{-4},\quad T\geq 0,$$

where $\alpha\,(>0)$ is a constant. Obtain an expression for the characteristic variable (ξ) associated with the modulated solitary wave; see equations (5.24).

Q5.13 *Asymptotic behaviour of the bore.* Obtain an asymptotic solution of the equation

$$-2c\eta+\frac{3}{2}\eta^2+\frac{1}{3}\eta''=\lambda\int_0^\infty\eta(x+x')\frac{\mathrm{d}x'}{\sqrt{x'}}$$

(see equation (5.54)), in the form

$$\eta\sim ae^{-\alpha x}+be^{-2\alpha x},\quad x\to+\infty.$$

Determine the relations between c, λ, a, b and α, and compare this behaviour with equation (2.165) *et seq.* and Q2.63.

[The special case considered in Q5.10 will prove useful here. Note that a and α could be related if the front of the wave were to be like a solitary wave.]

Q5.14 *Undular bore: perturbation pressure at $O(\varepsilon)$.* Obtain the pressure term p_1 from equations (5.67)–(5.71), making use of the results given in equations (5.72)–(5.74).

Q5.15 *Asymptotic behaviours of the KVB equation.* Seek solutions of the steady KVB equation

$$\eta^2 - \eta + \eta'' = \lambda\eta' \quad (\lambda > 0,\ \text{constant})$$

in the forms

(a) $\eta \sim a\exp(-\alpha\zeta), \quad \zeta \to +\infty;$

(b) $\eta \sim 1 - b\exp(\beta\zeta), \zeta \to -\infty,$

and hence determine α and β.

Q5.16 *Steady state KVB equation: phase plane I.* Discuss the equation given in Q5.15 in the phase plane; that is, in the (η, p) plane where $p = \eta'$. Show that there are two singular points, one at $(0, 0)$ and the other at $(1, 0)$; determine their natures for various λ. In particular, include the cases: $\lambda = 0$; $0 < \lambda < 2$.

Q5.17 *Steady state KVB equation: phase plane II.* Repeat the calculation of Q5.16, but this time write $p = P/\lambda$ and include the cases $1/\lambda = 0$; $\lambda \geq 2$.

Q5.18 *KVB equation: near-Taylor profile.* Introduce the transformation $\zeta = \lambda X$ into the equation in Q5.15, and hence obtain an asymptotic solution in the form

$$\eta \sim \eta_0(X) + \lambda^{-2}\eta_1(X), \quad \lambda \to \infty.$$

[Asymptotic solutions can also be obtained for $\lambda \to 0^+$; see Johnson (1970).]

Q5.19 *KVB equation: special solution.* Show that the steady state KVB equation in Q5.15 has the exact solution

$$\eta = \frac{1}{2}\left\{1 - \tanh(\zeta/2\sqrt{6})\right\} + \frac{1}{4}\operatorname{sech}^2(\zeta/2\sqrt{6}),$$

when $\lambda = 5/\sqrt{6}$.

Appendix A

The equations for a viscous fluid

The representation of a viscous fluid requires a change only to the form of the local (short-range) force; the equation of mass conservation is unaltered. The local force is now described through the (Cartesian) *stress tensor*, σ_{ij} $(i, j = 1, 2, 3)$, which represents the i-component of the stress (force/unit area) on the surface whose outward normal is in the j-direction. If $i = j$ then σ_{ij} is a *normal stress*, and for $i \neq j$ it is a *tangential* or *shearing stress*. In order that the local forces give rise only to finite accelerations of a fluid particle, it is necessary that σ_{ij} be symmetric; that is, $\sigma_{ij} = \sigma_{ji}$. Now, symmetric tensors possess the property that, in a certain coordinate system (the *principal coordinates* or *axes*), they may be written with diagonal elements only. (Indeed, as the coordinates are transformed under rotations, the sum of the diagonal elements is unchanged.) All this leads to the choice of stress tensor for a fluid as

$$\sigma_{ij} = -P\delta_{ij} + d_{ij},$$

where P is the pressure in the fluid and δ_{ij} is the *Kronecker delta*; d_{ij} is called the *deviatoric stress tensor* and it is absent for a stationary fluid. It is this contribution which is ignored in the derivation of Euler's equation, (1.12).

It is well established empirically (and supported by arguments based on molecular transport) that, for most common fluids, d_{ij} is proportional to the velocity gradients at a point in the fluid. Thus we write

$$d_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l},$$

where A_{ijkl} is a rank four Cartesian tensor, and the *summation convention* is employed; the position vector is written as $\mathbf{x} \equiv (x_1, x_2, x_3)$ and the corresponding velocity vector is $\mathbf{u} \equiv (u_1, u_2, u_3)$. We require d_{ij} to be symmetric (because σ_{ij} is) and, further, we assume that the fluid is *isotropic*; that is, the properties are the same in all directions. These considerations lead to

$$d_{ij} = 2\mu(e_{ij} - \frac{1}{3}\delta_{ij}e_{kk})$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

is the *rate of strain tensor*, and μ is the *coefficient of (Newtonian) viscosity*. The term $e_{kk} = \partial u_k/\partial x_k$ is called the *dilatation*, and it is zero for an incompressible fluid (that is, $\nabla \cdot \mathbf{u} = 0$); we note that $d_{ii} = 0$ (since $\delta_{ii} = 3$).

The application of Newton's second law to the fluid now yields

$$\int_V \left(\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho F_i\right) \mathrm{d}v = 0$$

where $\mathbf{F} \equiv (F_1, F_2, F_3)$, and so

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} + F_i;$$

cf. equation (1.12). For an incompressible fluid with constant viscosity, this becomes

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + F_i + \frac{\mu}{\rho}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

with

$$\frac{\partial u_i}{\partial x_i} = 0.$$

Written in a vector notation, these equations are expressed as

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla P + \mathbf{F} + \nu \nabla^2 \mathbf{u}; \quad \nabla \cdot \mathbf{u} = 0, \tag{A.1}$$

where $\nu = \mu/\rho$ is the *kinematic viscosity* and $\nabla^2 \equiv \nabla \cdot \nabla$ is the *Laplace operator*. The first of these equations is the *Navier–Stokes equation* for a (classical) viscous fluid; this equation clearly reduces to Euler's equation, (1.12), for an inviscid fluid: $\mu = 0$ (so $\nu = 0$).

Finally, we write these equations in rectangular Cartesian coordinates, $\mathbf{x} \equiv (x, y, z)$, with $\mathbf{u} \equiv (u, v, w)$ and $\mathbf{F} \equiv (0, 0, -g)$, to give

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 u, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 v,\\
&\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + \nu\nabla^2 w,\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\\
&\text{and}\\
&\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},\\
&\text{with}\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{aligned}\right\} \tag{A.2}$$

These same equations, written in cylindrical coordinates, $\mathbf{x} \equiv (r, \theta, z)$ and $\mathbf{u} \equiv (u, v, w)$, are

$$\left.\begin{aligned}
&\frac{\mathrm{D}u}{\mathrm{D}t} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right),\\
&\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \theta} + \nu\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right),\\
&\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + \nu\nabla^2 w,\\
&\text{where}\\
&\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\\
&\text{and}\\
&\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},\\
&\text{with}\\
&\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0.
\end{aligned}\right\} \qquad (\mathrm{A}.3)$$

Appendix B

The boundary conditions for a viscous fluid

The inclusion of viscosity in the modelling of the fluid requires that, at the free surface, the stresses there must be known (given) and, at the bottom, that there is no slip between the fluid and the bottom boundary. The surface stresses are resolved to produce the normal stress and any two (independent) tangential stresses. The normal stress is prescribed, predominantly, by the ambient pressure above the surface, but it may also contain a contribution from the surface tension (see Section 1.2.2). The tangential stresses describe the shearing action of the air at the surface, and therefore may be significant in the analysis of the motion of the surface which interacts with a surface wind. The bottom condition is the far simpler (and familiar) one which states that, for a viscous fluid, the fluid in contact with a solid boundary must move with that boundary.

The appropriate stress conditions are derived by considering the equilibrium of an element of the surface under the action of the forces generated by the stresses. The normal and shear stresses in the fluid (see Appendix A) produce forces that are resolved normal and tangential to the free surface, although the details of this calculation will not be reproduced here. It is sufficient for our purposes (and for general reference) to quote the results – in both rectangular Cartesian and cylindrical coordinates – for the three surface stresses.

First, in rectangular Cartesian coordinates, $\mathbf{x} \equiv (x, y, z)$ and $\mathbf{u} \equiv (u, v, w)$, with the free surface given by $z = h(x, y, t)$, we obtain the normal stress condition:

$$P - 2\mu\{h_x^2 u_x + h_y^2 v_y - h_x(u_z + w_x) - h_y(v_z + w_y) \\ + h_x h_y(u_y + v_x) + w_z\}/(1 + h_x^2 + h_y^2) = P_a - \Gamma/R \qquad (\text{B.1})$$

on $z = h$ (and $\Gamma = 0$ in the absence of surface tension; $1/R$ is defined in equation (1.32)). The two tangential stress conditions (both taken to be zero; that is, no wind) are written as

$$h_x(v_z + w_y) - h_y(u_z + w_x) + 2h_x h_y(u_x - v_y) - (h_x^2 - h_y^2)(u_y + v_x) = 0; \qquad (\text{B.2})$$

$$2h_x^2(u_x - w_z) + 2h_y^2(v_y - w_z) + 2h_x h_y(u_y + v_x) \\ + (h_x^2 + h_y^2 - 1)\{h_x(u_z + w_x) + h_y(v_z + w_y)\} = 0, \qquad (\text{B.3})$$

both evaluated on $z = h$. The corresponding surface conditions, written now in cylindrical coordinates, $\mathbf{x} \equiv (r, \theta, z)$ and $\mathbf{u} \equiv (u, v, w)$, are

$$P - 2\mu\left\{h_r^2 u_r + \frac{1}{r^2}h_\theta^2\left(\frac{1}{r}v_\theta + \frac{u}{r}\right) - h_r(u_z + w_r) - \frac{1}{r}h_\theta\left(v_z + \frac{1}{r}w_\theta\right)\right.$$
$$\left. + \frac{1}{r}h_\theta h_r\left[\frac{1}{r}u_\theta + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right] + w_z\right\}\Big/\left(1 + h_r^2 + \frac{1}{r^2}h_\theta^2\right) = P_a - \Gamma/R \quad \text{(B.4)}$$

on $z = h(r, \theta, t)$, and the expression for $1/R$ is given in equation (1.34);

$$h_r\left(v_z + \frac{1}{r}w_\theta\right) - \frac{1}{r}h_\theta(u_z + w_r) + \frac{2}{r}h_\theta h_r\left\{r\frac{\partial}{\partial r}\left(\frac{u}{r}\right) - \frac{1}{r}v_\theta\right\}$$
$$- \left(h_r^2 - \frac{1}{r^2}h_\theta^2\right)\left\{\frac{1}{r}u_\theta + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\} = 0, \quad \text{(B.5)}$$

$$2h_r^2(u_r - w_z) + \frac{2}{r^2}h_\theta^2\left(\frac{1}{r}v_\theta + \frac{u}{r} - w_z\right) + \frac{2}{r}h_\theta h_r\left\{\frac{1}{r}u_\theta + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}$$
$$+ \left(h_r^2 + \frac{1}{r^2}h_\theta^2 - 1\right)\left\{h_r\left(u_z + w_r\right) + \frac{1}{r}h_\theta\left(v_z + \frac{1}{r}w_\theta\right)\right\} = 0, \quad \text{(B.6)}$$

both on $z = h$.

The bottom boundary condition is far more easily expressed. Let the bottom boundary, $z = b(\mathbf{x}_\perp, t)$, translate with the velocity $\mathbf{u}_\perp = \mathbf{U}_\perp \equiv (U, V)$, then the viscous boundary condition is

$$\left.\begin{aligned} w &= b_t + (\mathbf{U}_\perp \cdot \nabla_\perp)b \\ \text{with} \qquad \mathbf{u}_\perp &= \mathbf{U}_\perp \end{aligned}\right\} \quad \text{on } z = b. \quad \text{(B.7)}$$

Of course, if this boundary is stationary then $\mathbf{U}_\perp = \mathbf{0}$, and then if $b_t = 0$ we recover the most elementary bottom condition:

$$u = v = w = 0 \quad \text{on} \quad z = b. \quad \text{(B.8)}$$

Finally, it is clear that all the above boundary conditions reduce to those for an inviscid fluid described in Section 1.2. For $\mu = 0$, equations (B.1) and (B.4) both become equation (1.31); equations (B.2, B.3, B.5, B.6) are redundant, and equations (B.7) are just equation (1.35) (after setting $\mathbf{U}_\perp = \mathbf{u}_\perp$ where $\mathbf{u}_\perp$ is evaluated in the fluid on $z = b$).

Appendix C

Historical notes

We provide brief historical notes on some of the prominent mathematicians, scientists and engineers who have made significant contributions to the ideas that are described in this text. In some cases this contribution is a general mathematical technique, and in others it is a development in fluid mechanics or a specific idea in the theory of water waves. The selection that has been made is, of course, altogether the responsibility of the author, and it includes only those researchers who died at least 20 years ago.

Airy, Sir George Biddell (1801–92) British mathematician and physicist, who was Astronomer Royal for 46 years; he made contributions to theories of light and, of course, to astronomy, but also to gravitation, magnetism and sound, as well as to wave propagation in general and to the theory of tides in particular.

Bernoulli, Daniel (1700–82) Dutch-born member of the famous Swiss family of about 10 mathematicians (fathers, sons, uncles, nephews), best known for his work on fluid flow and the kinetic theory of gases; his equation for fluid flow first appeared in 1738; he also worked in astronomy and magnetism, and was the first to solve the Riccati equation.

Bessel, Friedrich Wilhelm (1784–1846) German mathematician who was, for many years, the director of the astronomical observatory in Königsberg; he was the first to study the equation that bears his name (which arose in some work on the motion of planets); he carried out a lengthy correspondence with Gauss on many mathematical topics.

Boussinesq, Joseph (1842–1942) French mathematician and scientist who wrote an analytical treatment of various aspects of water (and fluid) flows.

Cauchy, Baron Augustin Louis (1789–1857) French mathematician who did important work in astronomy and mechanics, but is remembered mainly as one of the founders of the modern theory of functions of a complex variable; he developed the first comprehensive theory of complex numbers and introduced a number of fundamental theorems in complex analysis which have proved very significant in both pure and applied mathematics.

D'Alembert, Jean le Rond (1717–83) French mathematician and physicist who discovered many fundamental theorems in general dynamics, and also in celestial mechanics; in addition, he made important contributions to the theory of partial differential equations.

Descartes, René du Perron (1596–1650) French philosopher and mathematician whose aim was to reduce all the physical sciences to purely mathematical principles, and in particular in terms of geometric interpretations; he is credited with the invention of analytical geometry. (His followers called themselves 'Cartesians'.)

Euler, Leonhard (1707–83) Quite outstanding Swiss mathematician who made very significant and fundamental contributions to all branches of mathematics and its applications: differential equations, infinite series, complex analysis, mechanics and hydrodynamics, and the calculus of variations; he was very influential in promoting the use and understanding of analysis.

Fermat, Pierre de (1601–75) French mathematician who regarded mathematics as a hobby (he was a lawyer by training); he made very important contributions to analytical geometry, the calculus, probability theory and, of course, to the theory of numbers (his famous Last Theorem); he investigated optics mathematically and, among other successes, formulated his Principle.

Fredholm, Erik Ivar (1866–1927) Swedish mathematician who founded the modern theory of integral equations, which was developed from his interests in differential equations and mechanics.

Froude, William (1810–79) English engineer and naval architect who founded the modern science of predicting forces on ships from experiments on small-scale models; he built the first ship-model tank at his home in Torquay.

Gauss, Karl Friedrich (1777–1855) German mathematician – one of the foremost of all mathematicians (often rated as the equal of the other two pre-eminent mathematicians and scientists: Archimedes and Newton); he had already made a number of important discoveries by the age of 17; his interests ranged over algebra, real and complex analysis, differential equations and differential geometry, as well as number theory (which remained an enduring interest throughout his life); he used his mathematical skills in the study of astronomy, electromagnetism, and theoretical mechanics.

Green, George (1793–1841) English mathematician who was self-taught (which often meant that he used unconventional methods); he made significant advances in mathematical physics (and is generally credited with laying the foundations for these studies at Cambridge University); he introduced the potential function – in particular in applications to electricity and magnetism – and his famous theorem relating single/double integrals to double/triple integrals; he also made contributions to the theory of waves, to elasticity and to theories of light.

Hamilton, Sir William Rowan (1805–65) Irish mathematician who introduced the quaternion to the mathematical community, and showed how commutativity had to be set aside in some branches of mathematics; he tackled many problems in physics and mechanics – indeed, he coined the word 'vector'; his work on mechanics that led to his 'Hamiltonian function' was started in his doctoral thesis.

Hankel, Hermann (1839–73) German mathematician who made contributions to complex and hypercomplex numbers, to the theory of quaternions and to the theory of functions; he was the first to suggest the concept of 'measure'.

Heaviside, Oliver (1850–1925) English electrical engineer who developed the operational calculus, which he applied to the equations that arise in engineering problems.

Helmholtz, Herman Ludwig Ferdinand von (1821–94) German mathematician who was the first to study the equation that is most closely associated with his name (which he encountered in a problem on the oscillation of air in a tube with an open end); he also made contributions to the classification of geometries and to the axioms of arithmetic.

Hugoniot, Pierre Henri (1851–87) French scientist whose main interests were centred on ballistics; he solved various problems in gas dynamics and found the conditions that must exist across a shock wave.

Jacobi, Karl Gustav Jacob (1804–51) German mathematician and mathematical physicist who did important work on elliptic functions (where, to some extent, he was competing with Abel), analysis, number theory, geometry and mechanics; he introduced a functional determinant (the Jacobian) and developed links between elliptic functions and number theory, methods of integration, and differential equations.

Kelvin, Lord (Sir William Thomson) (1824–1907) Irish physicist who discovered the Second Law of Thermodynamics; the study of thermodynamics was his most important work, but he also made significant contributions to the theory of telegraphy (and almost every other branch of science).

Kronecker, Leopold (1823–91) German mathematician who, though gifted, never excelled in any one specific area; he worked on number theory, algebra and elliptic functions; in linear algebra he introduced his delta symbol.

Lagrange, Joseph Louis (Comte) (1736–1813) Italian-born French mathematician who revolutionised the study of mechanics; he was recognised as having outstanding ability by the age of 16, and held a professorial chair by 19; he gave the first general solution of a problem at the heart of the calculus of variations and introduced analytical principles in the study of mechanics and fluid mechanics.

Laguerre, Edmond (1834–86) French mathematician who did significant work on both projective and Euclidean geometries; he made contributions to analysis, including integration theory and the summation of series.

Lamb, Sir Horace (1849–1934) British mathematician, generally regarded as the outstanding applied mathematician of his time; he made important contributions to the theories of hydrodynamics and sound, as well as to elasticity and mechanics (and produced one of the seminal papers in the early days of the science of seismology); he is remembered as an exceptional teacher and a first-rate author of text books, most notably his treatise on fluid mechanics: *Hydrodynamics.*

Laplace, Marquis Pierre Simon de (1749–1827) French mathematical physicist who made contributions to the study of celestial mechanics and, in particular, explained the orbits of Jupiter and Saturn; he developed ideas in the use of the potential function and orthogonal functions, and introduced his integral transform; he was also an important player in the development of the theory of probability.

Mach, Ernst (1838–1916) Austrian physicist and philosopher who had a considerable influence on 20th century scientific thought; he was a positivist who provided the basis for the Logical Positivist movement.

Navier, Claude Louis Marie Henri (1785–1836) French mathematician who did much work on mechanics; in particular he was the first to derive the equations that describe a viscous fluid (which appeared in 1821), although others later and independently obtained the same results; this work was based on experimental evidence coupled with Newton's ideas on friction.

Newton, Sir Isaac (1642–1727) English natural philosopher and mathematician, who was the pre-eminent scientist of the Age of Reason; he developed many fundamental ideas in mathematics – not least his version of the calculus – and in optics and, above all, in his discovery of the Law of Gravitation; in his Principia Mathematica, although he described all the concepts in primarily geometrical terms, he laid the foundations for our modern mathematical approach to all branches of mechanics.

Poincaré, Jules Henri (1854–1912) French mathematician and mathematical physicist who worked mainly in the area of celestial mechanics, but who also made contributions to the theory of dynamics more generally and to the theory of automorphic functions; he developed techniques now familiar in the use of asymptotic expansions and in probability theory; he also made important discoveries on the dynamics of the electron, and even produced results that pre-dated Einstein's Theory of Relativity.

Rankine, William John Macquorn (1820–72) Scottish engineer and physicist who trained originally as a civil engineer; he is regarded as one of the founders of the science of thermodynamics: the Rankine cycle is familiar to the student of heat engines; his work also included a physical and mathematical theory of shock waves.

Rayleigh, Lord (John William Strutt) (1842–1919) English physicist who received the Nobel prize for physics in 1904; he made contributions to virtually every branch of physics and mathematical physics, including the theory of sound, optics, electrodynamics, hydrodynamics (especially capillarity and viscosity) and elasticity.

Reynolds, Osborne (1842–1912) British engineer and physicist best known for his work on hydraulics and hydrodynamics; he formulated the theory of lubrication and did some classical work on the resistance of flow through parallel channels; he also investigated the transition from laminar to turbulent flow.

Riemann, Georg Friedrich Bernhard (1826–66) German mathematician who did fundamental and innovative work on geometry; his ideas in non-Euclidean geometry and topology led to significant advances in pure mathematics, and he also made important contributions to complex algebraic function theory; he extended Cauchy's work on complex functions, basing new developments on geometrical ideas, leading to the concept of Riemann surfaces; many aspects of his work found applications, after his death, in physics – especially in relativity; Riemann's life was short and, in terms of publications, he was not prolific, but he had a profound effect on many branches of mathematics and mathematical physics.

Russell, John Scott (1808–82) Scottish engineer, scientist and naval architect who was commissioned in 1834 to investigate the possibility of rapid steamboat travel on canals; this led to his extensive interest in and study of water waves, and to the design of fast ships; he founded the Institute of Naval Architects and played an important role in the design of the *Great Eastern* (with Brunel) and of HMS *Warrior*.

Schrödinger, Erwin (1887–1961) Austrian theoretical physicist who, with Dirac, was awarded the 1933 Nobel prize for physics for the outstanding work on wave mechanics and its applications to atomic strucuture.

Stokes, Sir George Gabriel (1819–1903) Irish mathematician and physicist whose most important work was concerned with wave propagation – in fluids, elastic solids and of light and sound; he also made important contributions to the theory of polarised light and to X-rays.

Taylor, Sir Geoffrey Ingram (1886–1975) English mathematician, physicist and engineer who made significant contributions to fluid mechanics (particularly theories of the atmosphere), material science and to chemical and nuclear physics (mainly in the area of explosives); he was gifted at seeing general physical principles in all manner of everyday happenings, and in devising ingenious experiments.

Weber, Wilhelm Eduard (1804–91) German physicist who made significant contributions to the theory of absolute electrical measurement and units; he wrote a treatise on waves with his brother Ernst Heinrich Weber (1795–1878), who was himself an outstanding anatomist and physiologist (who also made some contributions to concepts in psychology).

Appendix D

Answers and hints

The answer, where one is given, is designated by the prefix A; for example, the answer to Q1.1 is A1.1. In some cases a hint to the method of solution is included.

Chapter 1

A1.1 Use a subscript notation, and so consider

(a) $\dfrac{\partial}{\partial x_i}(\phi u_i)$; (b) $\varepsilon_{ijk}\dfrac{\partial}{\partial x_j}(\phi u_k)$; (c) $\varepsilon_{ijk}u_j\left(\varepsilon_{klm}\dfrac{\partial u_m}{\partial x_l}\right)$;

(d) $\varepsilon_{ijk}\dfrac{\partial}{\partial x_j}(\varepsilon_{klm}u_l v_m)$.

In (c) and (d) use $\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

A1.2 In $\int_V \nabla\cdot\mathbf{a}\,dv = \int_S \mathbf{a}\cdot\mathbf{n}\,ds$ write $\mathbf{a} = \phi\mathbf{c}$; $\mathbf{a} = \mathbf{u}\wedge\mathbf{c}$, where in each $\mathbf{c}$ is an *arbitrary* constant vector.

A1.3 Consider $\int_S A_i\mathbf{u}\cdot\mathbf{n}\,ds = \int_S (A_i\mathbf{u})\cdot\mathbf{n}\,ds$ for each i.

A1.4 Write

$$\frac{dU_i}{dt} = \frac{d}{dt}\left\{u_i(x_1(t), x_2(t), x_3(t), t)\right\} = \dot{x}_j\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial t}.$$

A1.5 Find $\partial F/\partial t + \mathbf{u}\cdot\nabla F$ directly; note that, for both, $\nabla\cdot\mathbf{u} = 0$.

A1.6 (a) Find $\mathbf{u} = d\mathbf{x}/dt$ and introduce $\mathbf{x} \equiv (x, y, z)$.

(b) Find $d\mathbf{u}/dt \equiv (4x + 16t^2x,\ -2y + 4t^2y,\ -2z + 4t^2z)$.

(c) Find $\partial\mathbf{u}/\partial t \equiv (4x, -2y, -2z)$.

(d) Follows directly.

A1.8 $\mathbf{u} \equiv (\alpha x, \beta y, \gamma z)$; $\nabla\cdot\mathbf{u} = \alpha + \beta + \gamma = 0$.

A1.9 $\nabla\cdot\mathbf{u} = 3f + rf'$ so $f(r) = A/r^3$ (A is an arbitrary constant).

A1.10 From

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla P + \mathbf{F}, \quad \nabla P \equiv \rho(-x - xt^2, -y - yt^2, 2z - 4zt^2 - g)$$

$$\text{so } P = -\frac{1}{2}\rho(x^2 + y^2)(1 + t^2) + \rho z^2(1 - 2t^2) - \rho g z + P_0(t)$$

A1.11 From $(1/\rho)\nabla P \equiv (0, 0, -g)$, then $P = P_a + \rho g(h_0 - z)$; $P(0) = P_a + \rho g h_0$.

A1.12 Stokes' Theorem gives

$$\oint_C \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell} = \int_S (\nabla \wedge \mathbf{u}) \cdot \mathbf{n} \mathrm{d}s \approx (\boldsymbol{\omega} \cdot \mathbf{n})\pi a^2 \text{ so } \boldsymbol{\Omega} \cdot \mathbf{n} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{n};$$

but $\mathbf{n}$ is arbitrary so $\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\omega}$.

NB $$\oint_C \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell} = \oint_C \mathbf{U} \cdot \mathrm{d}\boldsymbol{\ell} + \oint_C (\boldsymbol{\Omega} \wedge \mathbf{r}) \cdot \mathrm{d}\boldsymbol{\ell} = \boldsymbol{\Omega} \cdot \oint_C \mathbf{r} \wedge \mathrm{d}\boldsymbol{\ell} = \boldsymbol{\Omega} \cdot \mathbf{n} \int_0^{2\pi} a^2 \mathrm{d}\theta.$$

A1.13 $\boldsymbol{\omega} \equiv (0, U'(z), (0)$; $\boldsymbol{\omega} \equiv (0, 0, -U'(y))$.

A1.14 Use $\mathbf{u} \wedge \boldsymbol{\omega} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u}$ and

$$\frac{1}{\rho}\nabla P = \nabla\left(\int \frac{\mathrm{d}P}{\rho}\right)$$

to give

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \wedge \boldsymbol{\omega} = -\nabla\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \int \frac{\mathrm{d}P}{\rho} + \Omega\right);$$

$$\text{curl:} \frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \wedge (\mathbf{u} \wedge \boldsymbol{\omega}) = \mathbf{0}$$

and use Q1.1 (d) with $\nabla \cdot \mathbf{u} = 0$, $\nabla \cdot \boldsymbol{\omega} = 0$. In 2D, $\boldsymbol{\omega}$ is orthogonal to ∇ so $\boldsymbol{\omega} \cdot \nabla \equiv 0$; then $\mathrm{D}\boldsymbol{\omega}/\mathrm{D}t = \mathbf{0}$.

A1.15 As for A1.14, but with

$$\nabla \wedge \left(\frac{1}{\rho}\nabla P\right) = \frac{1}{\rho}\nabla \wedge (\nabla P) + \nabla(\rho^{-1}) \wedge (\nabla P),$$

and multiply by ρ^{-1}. For $P = P(\rho)$, then $\nabla(\rho^{-1}) \wedge (\nabla P) = \mathbf{0}$.

A1.16 $\boldsymbol{\omega} \equiv (u/r, \ -\theta u', \ u')$.

A1.17 $$\boldsymbol{\omega} \equiv \begin{cases} (0, 0, \omega), & 0 \le r < a; \\ \mathbf{0} \quad , & r > a \end{cases}$$

$$\frac{P}{\rho}+\Omega=\begin{cases} P_0/\rho+\frac{1}{8}\omega^2(r^2-2a^2), & 0\le r\le a\\ P_0/\rho-\frac{1}{8}\omega^2\frac{a^4}{r^2}, & r>a.\end{cases}$$

Must have

$$P_0>\frac{\rho}{8}\omega^2a^2.$$

A1.18 Consider

$$\frac{1}{\rho}\frac{\partial P}{\partial x_i}=\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho}\frac{\partial\rho}{\partial x_i}=\frac{\partial}{\partial x_i}\left(\int\frac{\mathrm{d}P}{\rho}\right).$$

A1.19 (a) $\mathbf{x}\equiv(x_0\mathrm{e}^{ct}, y_0\mathrm{e}^{-ct}, z_0)$, $xy=\text{constant}$;
(b) $\mathbf{x}\equiv(x_0\exp(t^2), y_0\exp(-t^2), z_0)$, $xy=\text{constant}$;
(c) $\mathbf{x}\equiv(1+t+(x_0-1)\mathrm{e}^t,\ y_0\mathrm{e}^{-t}, z_0)$, $y(x-t)=\text{constant}$ (at fixed t);

(d) $\mathbf{x}\equiv\left(\dfrac{x_0}{1-cx_0t},\ \dfrac{y_0}{1-cy_0t},\ z_0(1-cx_0t)^2(1-cy_0t)^2\right)$,

$y=x+Axy$
with $(1-Ax)^2=Bzx^4$ where A, B are arbitrary constants.

A1.20 (a) $\psi=cxy$; (b) $\psi=2xyt$; (c) $\psi=(x-t)y$.

A1.21 $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{v}{u}=-\psi_x/\psi_y$ so $\dfrac{\mathrm{d}}{\mathrm{d}x}\{\psi(x,y(x))\}=0$; $\psi=$constant.

A1.22 Write

$$u=\frac{1}{r}\psi_\theta,\ \ v=-\psi_r;\ \ \psi=rU\sin\theta.$$

A1.23 (a) $u=\dfrac{1}{r}\psi_z,\ w=-\dfrac{1}{r}\psi_r$; (b) $v=r\psi_z,\ w=-\psi_\theta$.

A1.24 (a) Write $u_k=b_ka_jx_j+a_kb_jx_j$ and form $\varepsilon_{ijk}\partial u_k/\partial x_j$.
Then $u_k=\partial\phi/\partial x_k$ yields

$$\phi=(\mathbf{a}\cdot\mathbf{x})(\mathbf{b}\cdot\mathbf{x})\ (+\text{constant}).$$

(b) $\phi=\dfrac{yz}{x^2+y^2}$ (+ constant).

A1.25 $\nabla\cdot\mathbf{u}=0$ yields $u=\psi_y$, $v=-\psi_x$; $\nabla\wedge\mathbf{u}=\mathbf{0}$ so $\mathbf{u}=\nabla\phi$, then $u=\phi_x$, $v=\phi_y$: $\phi_x=\psi_y$, $\phi_y=-\psi_x$. Thus $\phi+\mathrm{i}\psi=w(z)$; then

$$\frac{\partial}{\partial x}w=\frac{\partial}{\partial x}(\phi+i\psi)\text{ gives }\frac{\mathrm{d}w}{\mathrm{d}z}=u-iv.$$

Here $w = Ue^{i\alpha} = u - iv$, a uniform flow of speed $U(t)$ at $-\alpha$ to the x-axis.

A1.27 Use $\mathbf{u} = \dfrac{d\mathbf{x}}{dt} \equiv \left(\dfrac{d\mathbf{x}_\perp}{dt}, \dfrac{dz}{dt}\right) = (\mathbf{u}_\perp, w)$.

A1.28 With $P_s - P_a = \Gamma h'' = (1 + h'^2)^{3/2}$ where $P = P_s$ on $z = h(x)$ and $P_s = P_b - \rho g h$ ($P = P_b(x)$ on $z = 0$).
For equilibrium, $P_b =$ constant. Thus

$$\Gamma h'' = (P_b - P_a - \rho g h)(1 + h'^2)^{3/2}, \; -x_0 \le x \le x_0.$$

A1.29 Similar to A1.28:

$$P_b - P_a - \rho g h = \Gamma\left\{\frac{h''}{(1 + h'^2)^{3/2}} + \frac{h'}{r(1 + h'^2)^{1/2}}\right\},$$

and then for $\varepsilon \to 0$:

$$H'' + H'/R = \beta - \alpha H, \quad \beta = r_0^2(P_b - P_a)/\Gamma h_0.$$

Hence $H = \beta/\alpha + AJ_0(\sqrt{\alpha}R)$ so

$$A + \beta/\alpha = 1; \quad \beta/\alpha + (1 - \beta/\alpha)\, J_0(\sqrt{\alpha}) = 0;$$
$$H'(1) = (\sqrt{\alpha} - \beta/\sqrt{\alpha})J_0'(\sqrt{\alpha}).$$

This solution requires $0 < \beta/\alpha < 1$ with $J_0(\sqrt{\alpha}) < 0$, $J_0'(\sqrt{\alpha}) < 0$ which gives $\alpha_0 < \alpha < \alpha_1$.

A1.31 (a) $I'(x) = x^{-1}\{3\exp(x^3) - 2\exp(x^2)\}$; (b) $n = 4$.

A1.32 Euler's equation leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}\right) + \rho\mathbf{u}\cdot\{(\mathbf{u}\cdot\nabla)\mathbf{u}\} = -\mathbf{u}\cdot\nabla(P + \rho\Omega)$$

with

$$\rho\mathbf{u}\cdot\{(\mathbf{u}\cdot\nabla)\mathbf{u}\} = \nabla\cdot\left\{\frac{1}{2}(\rho\mathbf{u}\cdot\mathbf{u})\mathbf{u}\right\},$$
$$\mathbf{u}\cdot\nabla(P + \rho\Omega) = \nabla\cdot\{(P + \rho\Omega)\mathbf{u}\}$$

since $\nabla\cdot\mathbf{u} = 0$.

A1.33 (a) Write $T = \frac{1}{2}\int_V \rho\mathbf{u}\cdot\nabla\phi dv = \frac{1}{2}\int_V \nabla\cdot(\rho\phi\mathbf{u})dv$ since $\nabla\cdot\mathbf{u} = 0$.
(b) Since the conditions on S are given, either $\Phi = 0$ or $\mathbf{U} = \mathbf{0}$ on S. Thus

$$\int_V |\mathbf{U}|^2 dv = 0 \Rightarrow \mathbf{U} = \mathbf{0} \text{ in V; that is, } \mathbf{u}_1 \equiv \mathbf{u}_2.$$

A1.34 $\psi \to ch\psi$; $\phi \to c\lambda\phi$; $w \to \dfrac{ch}{\lambda} w$; $w = \phi_z \to w = \left(\dfrac{\lambda}{h}\right)^2 \phi_z$.

A1.35 The Reynolds number is $\rho\lambda\sqrt{gh_0}/\mu$.

A1.36
$$p - \varepsilon\eta = -\varepsilon\delta^2 W\Big\{[(1+\varepsilon^2\delta^2\eta_\theta^2/r^2)\eta_{rr} + (1+\varepsilon^2\delta^2\eta_r^2)(r\eta_r + \eta_{\theta\theta})/r^2$$
$$-2\varepsilon^2\delta^2\left(\eta_{r\theta} - \frac{1}{r}\eta_\theta\right)\eta_r\eta_\theta/r^2]/(1+\varepsilon^2\delta^2\eta_r^2 + \varepsilon^2\delta^2\eta_\theta^2/r^2)^{3/2}\Big\}$$

and then $p \to \varepsilon p$ with $\varepsilon \to 0$ yields

$$p - \eta = -\delta^2 W\left(\eta_{rr} + \frac{1}{r}\eta_r + \frac{1}{r^2}\eta_{\theta\theta}\right).$$

A1.37 $\phi_t + \eta + \frac{1}{2}\varepsilon(u^2 + v^2 + \delta^2 w^2) = 0$.

A1.38 $\phi_{zz} + \delta^2\nabla_\perp^2\phi = 0$; $\phi_z = \delta^2\{\eta_t + \varepsilon(\mathbf{u}_\perp \cdot \nabla_\perp)\eta\}$ on $z = 1 + \varepsilon\eta$;

$$\phi_t + \eta + \frac{1}{2}\varepsilon\left\{(\nabla_\perp\phi)^2 + \frac{1}{\delta^2}\phi_z^2\right\} = 0 \text{ on } z = 1 + \varepsilon\eta;$$
$$\phi_z = \delta^2(\mathbf{u}_\perp \cdot \nabla)b \text{ on } z = b. \text{ NB } \mathbf{u}_\perp = \nabla_\perp\phi.$$

For $\varepsilon \to 0 : \phi_{zz} + \delta^2\nabla_\perp^2\phi = 0$; $\phi_z = \delta^2\eta_t$ and $\phi_t + \eta = 0$ (or $p = \eta$) on $z = 1$; $\phi_z = \delta^2(\mathbf{u}_\perp \cdot \nabla)b$ on $z = b$. With surface tension; $p - \eta = -\delta^2 W\nabla_\perp^2\eta$ on $z = 1$.

A1.39 $u = f(x - ct) + g(x + ct)$;
$$u = \tfrac{1}{2}\{p(x - ct) + p(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} q(y)\mathrm{d}y.$$

A1.40 The right- and left-going waves no longer overlap.

A1.41 Dispersion relation is $\omega = k - k^3 - ik^2$, which is dispersive ($\mathcal{R}(\omega/k) = 1 - k^2$) and dissipative (decaying as $\exp(-k^2 t)$).

A1.42 First equation gives $\omega = k - k^3$; second gives $\omega = k/(1 + k^2)$ so $\omega = k - k^3 + O(k^5)$ as $k \to 0$ (long waves) but $\omega \sim 1/k$ as $k \to \infty$; note that $\omega > 0$ for $\forall k > 0$ in the second case, but not in the first.

A1.43

$$u(x,t) = \begin{cases} \alpha(x-t)/(1+\alpha t), & 0 \le (x-t)/(1+\alpha t) \le 1, \\ \alpha(2-x+t)/(1-\alpha t), & 1 \le (x-t-2\alpha t)/(1-\alpha t) \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

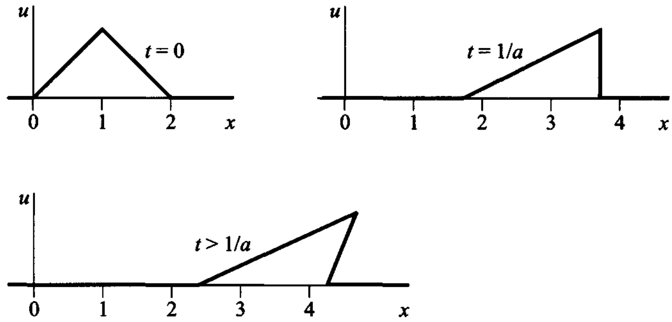


Figure A.1.

A1.44 $u = \cos\{\pi(x - ut)\}$ for $u(x, t)$;
$u_x = -\pi \sin\{\pi(x - ut)\}/[1 - \pi t \sin\{\pi(x - ut)\}]$.
The solution becomes multi-valued for $t > \pi^{-1}$.

A1.45 (a) $x = O(1)$: $f \sim 1 + \varepsilon(x^{-1} - x)$;

$$x = \varepsilon X: f \sim \left(1 - \frac{1}{1+X} + e^{-X}\right)^{-1};$$

$x = \chi/\varepsilon$: $f \sim (1 + \chi)^{-1}$.
The first and second match: $1 + 1/x$; the first and third match: $1 - \chi$.

(b) $x = O(1)$: $f \sim 1 - \frac{1}{2}\varepsilon x + \varepsilon^2(\frac{3}{8}x^2 - \frac{1}{2}x^4)$;
$x = \varepsilon^{-1/3}X$: $f \sim 1 - \frac{1}{2}\varepsilon^{2/3}(X + X^4)$;
$X = \varepsilon^{-1/6}\chi$: $f \sim (1 + \chi^4)^{-1/2} - \frac{1}{2}\varepsilon^{1/2}\chi(1 + \chi^4)^{-3/2}$.
The first and second match: $1 - \frac{1}{2}\varepsilon^{2/3}(X + X^4)$; the second and third match: $1 - \frac{1}{2}\chi^4 - \frac{1}{2}\varepsilon^{1/2}\chi$.

A1.46 (a) $f \sim 1 - \frac{1}{2}\varepsilon x - \frac{1}{2}e^{-x/\varepsilon}$
(b) $x = \varepsilon X$: $f \sim (1 - e^{-X})^{1/2} - \frac{1}{2}\varepsilon^2 X(1 - e^{-X})^{-1/2}$
(c) $x = \chi/\varepsilon$: $f \sim (1 - \chi)^{1/2} - \frac{1}{2}\varepsilon\chi^3(1 - \chi)^{-1/2}$. The first and second match: $1 - \frac{1}{2}\varepsilon^2 X - \frac{1}{2}e^{-X}$; the first and third match: $1 - \frac{1}{2}\chi$. From (c), $\chi \le \chi_0(\varepsilon)$ where $\chi_0 \sim 1$ (since $f = 0$ at $\chi = \chi_0$); try $\chi_0 \sim 1 + \varepsilon\alpha$ then $\alpha = -1$; that is, $x_0 \sim \varepsilon^{-1} - 1$.

A1.47 $2u_X + 2uu_\xi + u_{\xi\xi\xi} = 0$.

A1.48 $2u_\tau - 2uu_\zeta - u_{\zeta\zeta\zeta} = 0$.

A1.49
$$2f_\tau + 2ff_\xi + f_{\xi\xi\xi} = 0; \quad 2g_\tau - 2gg_\zeta - g_{\zeta\zeta\zeta} = 0;$$
$$\phi_{\xi\zeta} = -f_\xi g_\zeta - \tfrac{1}{2}(fg_{\zeta\zeta} + gf_{\xi\xi}).$$

Then

$$\phi = -\frac{1}{2}\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \zeta}\right)^2\left(\int f \mathrm{d}\xi\right)\left(\int g \mathrm{d}\zeta\right) \quad (= 0 \text{ if } f \equiv 0 \text{ or } g \equiv 0).$$

A1.50 $c_p^2 = 1 - k^{-2}$; $c_g = 1/c_p$; $\omega^2 = k^2 - 1$;
so $c_g = \mathrm{d}\omega/\mathrm{d}k = k/\omega = 1/c_p$;
$A_{10} = -4k^2|A_{01}|^2$ and $A_{12} = 0$.

A1.51 Introduce $\xi = x - c_1 t$, $\tau = \varepsilon^2 t$; then $u \sim u_0(\xi, \tau)$ satisfies

$$u_{0\tau} + u_0 u_{0\xi} = -\lambda u_0, \;\; \lambda = (c - c_1)/(c_2 - c_1), \;\; u_0 \to 0 \text{ as } \xi \to +\infty.$$

Thus $u_0 = \mathrm{e}^{-\lambda\tau} f(\xi + u_0/\lambda)$; exponential decay requires $\lambda > 0$.
Similarly, with $\zeta = x - c_2 t$, $\tau = \varepsilon^2 t$:

$$u_{0\tau} + u_0 u_{0\zeta} = -\mu u_0, \quad \mu = (c_2 - c)/(c_2 - c_1).$$

Thus $\lambda > 0$, $\mu > 0$ if $c_1 < c < c_2$.

A1.52 $\lambda = (c_2 - c)(c - c_1) > 0$; $\phi = \frac{1}{2}\left\{1 - \tanh\left(\frac{X - \frac{1}{2}T - X_0}{4\lambda}\right)\right\}$
where X_0 is an arbitrary constant.

A1.53 $2G_{\tau\eta} + G_\eta^2 + 2GG_{\eta\eta} - G_{\eta\eta\eta\eta} = 0$;
$f = \frac{1}{2}\int G(\eta, \tau)\mathrm{d}\eta$, $g = -\frac{1}{2}\int F(\xi, \tau)\mathrm{d}\xi$.

A1.54 $\omega^2 U_{0\theta\theta} + k^2 U_{0\theta\theta} + k^4 U_{0\theta\theta\theta\theta} + U_0 = 0$; then $\omega^2 = k^4 - k^2 + 1$.
U_1 is periodic if

$$A\omega_T + 2\omega A_T + 4k^3 A_X + 6k^2 k_X A - k_X A - 2kA_X - 3\mathrm{i}A|A|^2 = 0,$$

and use $k_T + k_X \omega'(k) = 0$. Finally

$$(\alpha^2)_T + (\omega'\alpha^2)_X = 0; \quad \beta_T + \omega'\beta_X = \frac{3}{2\omega}\alpha^2.$$

A1.55 (a) $u = -\frac{1}{2}c\,\mathrm{sech}^2\{\frac{1}{2}\sqrt{c}(x - ct - x_0)\}$;
(b) $u = \frac{1}{2}u_0\{1 - \tanh[\frac{1}{4}u_0(x - \frac{1}{2}u_0 t - x_0)]\}$ so that $c = \frac{1}{2}u_0$ (see A1.52); in both, x_0 is an arbitrary constant.

Chapter 2

A2.1 For example, find $\mathrm{d}c_p^2/\mathrm{d}\lambda$ and then sketch $y = W(t + \lambda s^2)$, $y = (t - \lambda s^2)/\lambda^2$ (where $t \equiv \tanh\lambda$, $s \equiv \mathrm{sech}\,\lambda$). Show that one point of intersection exists for $\lambda \in (0, \infty)$ provided $W < \frac{1}{3}$. As $\lambda \to \infty$, $c_p \sim \pm\sqrt{\lambda W}$.

A2.2 From A2.1, obtain

$$\left(\frac{c_p}{c_m}\right)^2 = \left(\frac{t}{t_m}\right)\left\{\frac{1}{2}(l^{-1}+l)+\frac{\lambda_m}{s_m}(l^{-1}-l)\right\};$$

for moderate λ_m (and λ), then $t/t_m \approx 1$, $\lambda_m/s_m \ll 1$, so

$$\left(\frac{c_p}{c_m}\right)^2 \approx \frac{1}{2}(l^{-1}+l).$$

The minimum is, of course, at $l = 1$, where $\lambda = \lambda_m = 1/\sqrt{W}$.

A2.3 $U(z) = A\delta\omega \cosh \delta kz/\sinh \delta k$; $P(z) = A\delta\omega^2 \cosh \delta kz/(k \sinh \delta k)$.

A2.4 In physical coords, $\hat{X}, \hat{Z}$:

$$\left(\frac{\hat{X}}{\cosh \delta kz_0}\right)^2 + \left(\frac{\hat{Z}}{\sinh \delta kz_0}\right)^2 = \text{constant}.$$

Hence approaches a circular path as $\delta k \to \infty$ (short waves).

A2.5 The problem is $\phi_{zz} + \delta^2\phi_{xx} = 0$; $\phi_z = 0$ on $z = 0$ with $\phi_z = \delta^2\eta_t$ and $\phi_t + \eta = 0$ both on $z = 1$. Write $\phi = X(x, t)Z(z)$.

A2.6 From example, $A(t) \equiv 0$, $B(t) = B_0 \sin \omega t$; then

$$\eta \propto \sin \omega t \sin kx \left(= \frac{1}{2}\cos(kx-\omega t) - \frac{1}{2}\cos(kx+\omega t)\right).$$

A2.7 It follows that $W'' - \delta^2(k^2+l^2)W = 0$, so k^2 is replaced by $k^2 + l^2$; wavefront is $kx + ly = \text{const.}$ or $\mathbf{n}\cdot\mathbf{x} = \text{constant}$, where $\mathbf{n} \propto \mathbf{k}$.

A2.8 Use Laplace's equation: $\phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0$ with boundary conditions as in A2.5 plus $\phi_y = 0$ on $y = 0$, l. Then $\alpha = n\pi/l$ and $\omega^2 = (\sigma/\delta^2)\tanh\sigma$, where $\sigma^2 = \delta^2(k^2+\alpha^2)$.

A2.9 Cf. A2.8; $\alpha = n\pi/l$, $\beta = m\pi/L$, $\omega^2 = (\sigma/\delta^2)\tanh\sigma$, $\sigma^2 = \delta^2(\alpha^2+\beta^2)$.

A2.10 $\omega^2 = (\sigma/\delta^2)\tanh\sigma$, $\sigma^2 = \delta^2(k^2+l^2+m^2+n^2)$ with $mk + nl = 0$; so $(m, n)\cdot(k, l) = 0$: the wave-number vectors are perpendicular. Wave propagates in the direction (k, l) at a speed $\omega/|\mathbf{k}|$; that is, $\delta\sqrt{k^2+l^2+m^2+n^2}/\sqrt{k^2+l^2}$, which is *faster* than in the absence of waves along the crests (for which $m = n = 0$).

A2.11 Equations are

$$u_t + u_0u_x = -p_x; \quad \delta^2(w_t + u_0w_x) = -p_z; \quad u_x + w_z = 0$$

with

$$w = 0 \quad \text{on} \quad z = 0$$

and

$$w = \eta_t + u_0\eta_x, \quad p = \eta - \delta^2 W\eta_{xx} \quad \text{on} \quad z = 1.$$

Then $(\omega - u_0 k)^2 = (1 + \delta^2 k^2 W)(\tanh \delta k)/\delta k = \sigma^2$; speed of the waves is $u_0 + \sigma/k$.

A2.12 Cf. A2.11, but with $u_t + u_0 u_x + v_0 u_y = -p_x$, etc.; then $\Omega^2 = (\omega - u_0 k - v_0 l)^2 = (1 + \sigma^2 W)(\tanh \sigma)/\sigma, \quad \sigma^2 = \delta^2(k^2 + l^2)$. Also

$$kx + ly - \omega t = \mathbf{k}\cdot\mathbf{x} - \left\{\mathbf{U} + \frac{\Omega\mathbf{k}}{|\mathbf{k}|^2}\right\}\cdot\mathbf{k}t, \ \mathbf{U} \equiv (u_0, v_0),$$

so velocity as required. Stationary implies independent of time (t), so $\mathbf{U} = -\Omega\mathbf{k}/|\mathbf{k}|^2$.

A2.13 To be valid for all t, the solution takes the form $\eta = A\exp\{\mathrm{i}(kx - \omega t)\} + R\exp\{-\mathrm{i}(k_- x + \omega t)\} + \text{c.c.}$ in $x < 0$ and $\eta = T\exp\{\mathrm{i}(k_+ x - \omega t)\} + \text{c.c.}$ in $x > 0$, where $\omega^2 = (k/\delta)\tanh(\delta k)$ for (k, δ_-), (k_-, δ_-), (k_+, δ_+) with $\delta_\pm = h_\pm/\lambda$; that is, $k_- = k$. Continuity of η gives $A + R = T$; continuity of mass flux is $u_- h_- = u_+ h_+$, where $u \propto \eta_x$, so $kh_-(A - R) = k_+ h_+ T$.
Thus $R = A(kh_- - k_+ h_+)/(kh_- + k_+ h_+)$
and $T = 2Akh_-/\,(kh_- + k_+ h_+)$.

A2.14 Stable only if ω is real, from which condition follows. The minimum is at $k = \sqrt{(1 - \lambda)/W}$ (for $k > 0$).

A2.16 (b) $$I(\sigma) \sim \int_a^{a+\varepsilon} f(x)\mathrm{e}^{\mathrm{i}\delta\alpha(x)}\mathrm{d}x = \mathrm{e}^{\mathrm{i}\sigma\alpha(a)}\int_0^{\hat{u}} \mathrm{e}^{\mathrm{i}\sigma u^2} f(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$$

$$= \mathrm{e}^{\mathrm{i}\sigma\alpha(a)}\int_0^{\hat{u}} \mathrm{e}^{\mathrm{i}\sigma u^2}\{c_0 + uF(u)\}\mathrm{d}u = \mathrm{e}^{\mathrm{i}\sigma\alpha(a)}(I_1 + I_2).$$

Then

$$I_1 = \frac{1}{2}c_0\sqrt{\frac{\pi}{\sigma}}\mathrm{e}^{\mathrm{i}\pi/4} + \mathrm{O}(\sigma^{-1}); \ I_2 = \mathrm{O}(\sigma^{-1}),$$

with $c_0 = b_1 f(a)$ and $\frac{1}{2}b_1^2\alpha''(a) = 1$; required result follows.

A2.18 $\eta(r, t) = \int_0^\infty p\bar{f}(p)\cos(tp)J_0(rp)\mathrm{d}p; \quad f(r) = \int_0^\infty p\bar{f}(p)J_0(rp)\mathrm{d}p.$

A2.19 $\eta(r, t) = \int_0^\infty p\bar{f}(p)\cos(t\sqrt{p/\delta})J_0(rp)\mathrm{d}p.$ NB: $\bar{w} = \bar{\eta}_t\mathrm{e}^{\delta p(z-1)}$.

A2.20 $\omega^2 = (\sigma\tanh\sigma)/\delta^2; \quad J_n'(\sigma a) = 0.$
If $n = 0$, then solution is independent of θ.

A2.22 $$\eta(x,t) + \int_{-\infty}^{\infty} F(k)\{\exp[ik(x - c_p t)] + \exp[ik(x + c_p t)]\}dk$$

where

$$F(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx.$$

For $f(x) = A\delta(x)$ then

$$\eta(x,t) = \frac{A}{\pi}\int_0^{\infty} \cos kx \cos \omega t \, dk \quad (\text{where } c_p(k) = \omega(k)/k).$$

A2.23 $\omega \sim k(1 - \frac{1}{6}\delta^2 k^2)$: $u_t + u_x + \frac{1}{6}\delta^2 u_{xxx} = 0$.

A2.24 $$\eta(x,t) \sim \frac{A_0}{2\pi}\int_{-\infty}^{\infty} \exp\{i[k(x-t) + \frac{1}{6}\delta^2 k^3 t]\}dk,$$

$$F(k) \sim \frac{1}{4\pi}\int_{-\infty}^{\infty} \eta_0(x)dx = \frac{A_0}{2\pi}.$$

Then

$$\eta \sim A_0\left(\frac{2}{\delta^2 t}\right)^{1/3} \mathrm{Ai}\left\{\left(\frac{2}{\delta^2 t}\right)^{1/3}(x-t)\right\}$$

where

$$\mathrm{Ai}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{i\left(ly + \frac{l^3}{3}\right)\right\}dl;$$

exponential decay ahead of $x = t$, oscillatory behind; amplitude decays like $t^{-1/3}$ at $x = t$.

A2.25 Since $\psi = 0$ on $z = 0$, then $W(\cdot, t)$ is real; that is, $\overline{W(Z,t)} = W(\bar{Z}, t)$.

A2.27 $\omega^2 = k(1 + \delta^2 k^2 W_e)/\delta$;

$$c_g = \frac{1}{2}\left(\frac{1 + 3\delta^2 k^2 W_e}{1 + \delta^2 k^2 W_e}\right).$$

A2.28 $A(X,T) = A_0\{1 + \exp[iX(X - \omega' T)]\}$.

A2.29 Waves in $(0, x)$:

$$\int_0^x k\mathrm{d}x, \text{ so } \frac{\partial}{\partial t}\int_0^x k\mathrm{d}x = \omega_0 - \omega = \text{net waves/unit time entering } (0, x).$$

$E(k), k = k(x, t)$, yields $E_t + \omega'(k)E_x = 0$.

A2.30

$$[W_0 W_{1z} - W_{0z} W_1]_0^1 + \int_0^1 W_1(W_{0zz} - \delta^2 k^2 W_0)\mathrm{d}z$$
$$= -2k\omega\delta^2 A_{0X}\int_0^1 W_0 \frac{\sinh \delta kz}{\sinh \delta k}\mathrm{d}z;$$

and

$$W_0(1) = -\mathrm{i}\omega A_0, \quad W_1(1) = -\mathrm{i}\omega A_1 + A_{0T}, \quad W_{0z}(1) = \frac{\mathrm{i}k^2}{\omega}A_0,$$
$$W_{1z}(1) = -\frac{k}{\omega}\left\{\mathrm{i}kA_1 + A_{0X} + \frac{k}{\omega}A_{0T}\right\} - \frac{k}{\omega}A_{0X},$$
$$\int_0^1 W_0 \frac{\sinh \delta kz}{\sinh \delta k}\mathrm{d}z = -\frac{-\mathrm{i}\omega A_0}{2\sinh^2 \delta k}\left\{\frac{\sinh 2\delta k}{2\delta k} - 1\right\}.$$

Finally:

$$A_{0T} + \frac{\omega}{2k}\{1 + \delta k(\coth \delta k - \tanh \delta k)\}A_{0X} = 0.$$

A2.31

$$\mathscr{E} = \int_0^{1+\varepsilon\eta}\left\{\frac{1}{2}\varepsilon^2(u^2 + v^2 + \delta^2 w^2) + z\right\}\mathrm{d}z; \quad \bar{\mathscr{E}} \sim \frac{1}{2} + \frac{1}{2}\varepsilon^2 A^2;$$

$$\mathscr{F} = \varepsilon\int_0^{1+\varepsilon\eta}(u, v)\left\{\frac{P_a}{\rho g h_0} + 1 - \varepsilon\phi_t\right\}\mathrm{d}z$$

and $$c_g = \frac{1}{2}\frac{\omega}{|\mathbf{k}|}\left(1 + \frac{2\sigma}{\sinh 2\sigma}\right), \quad \sigma^2 = \delta^2(k^2 + l^2).$$

A2.32 $\mathscr{E}_0 = \frac{1}{2}$, $\mathscr{E}_\mathrm{w} = \frac{1}{2}\varepsilon^2 A^2$, $\mathscr{F}_0 = \frac{1}{2}\frac{\varepsilon^2 A^2}{\omega}\mathbf{k}$, $\mathscr{F}_\mathrm{w} = \frac{1}{2}\varepsilon^2 A^2 c_g \frac{\mathbf{k}}{|\mathbf{k}|}$.

A2.33 $8\eta_{\xi\zeta} + \left(\dfrac{d'}{\sqrt{d}}\right)(\eta_\xi + \eta_\zeta) = 0;$

$\dfrac{2}{\sqrt{\alpha}}\{\sqrt{x_0} - \sqrt{x_0 - x}\} \pm t \quad (x \le x_0).$

A2.34 $16H_{\xi\zeta} - d^{1/4}(d'/d^{1/4})'H = 0;$
(a) $H_{\xi\zeta} = 0$; (b) $16H_{\xi\zeta} - \alpha^2 H = 0.$

A2.35 $\phi_{zz} + \delta^2\phi_{xx} = 0$; $\phi_z + \delta^2\phi_{tt} = 0$ on $z = 1$; $\phi_z = \alpha\delta^2\phi_x$ on $z = \alpha x$ (and $\eta = -\phi_t$ on $z = 1$). Set $\phi = F(x, z)\mathrm{e}^{-\mathrm{i}\omega t}$ with
$F = (A_1\mathrm{e}^{\mathrm{i}kx} + B_1\mathrm{e}^{-\mathrm{i}kx})\mathrm{e}^{\delta kz} + (A_2\mathrm{e}^{\mathrm{i}\delta kz} + B_2\mathrm{e}^{-\mathrm{i}\delta kz})\mathrm{e}^{kx}$
and $k = \delta\omega^2$, $\alpha\delta = 1$.

A2.36 As in A2.35, but with

$$F = \left(A_1\mathrm{e}^{\mathrm{i}kx} + B_1\mathrm{e}^{-\mathrm{i}kx}\right)\mathrm{e}^{\delta kz} + \left(A_2\mathrm{e}^{\mathrm{i}\delta lz} + B_2\mathrm{e}^{-\mathrm{i}\delta lz}\right)\mathrm{e}^{lx} + \left(A_3\mathrm{e}^{\mathrm{i}\delta mz} + B_3\mathrm{e}^{-\mathrm{i}\delta mz}\right)\mathrm{e}^{mx}$$

where $l, m = \frac{1}{2}(\sqrt{3} \pm \mathrm{i})k$.

A2.37 Write $F = A\mathrm{e}^{\mathrm{i}kx+\delta mz} + B\mathrm{e}^{mx+\mathrm{i}\delta kz} + \mathrm{c.c.}$
where $m = \sqrt{k^2 + l^2}$, $\delta\omega^2 = m$; cf. A2.35.

A2.38
$$\mathbf{c}_g \equiv \frac{\delta^2\omega}{2\sigma^2}\left(1 + \frac{2\sigma D}{\sinh 2\sigma D}\right)(k, l);$$
$$\omega^2 = \frac{\sigma}{\delta^2}\tanh(\sigma D), \quad \sigma = \delta\sqrt{k^2 + l^2}.$$

A2.39 $\sigma \sim c/\sqrt{D}$ as $D \to 0$; $\sigma \to c$ as $D \to \infty$ ($c =$constant).

A2.40 (a) $\Theta = c(X\cos\theta + Y\sin\theta)$ where $\cos\theta + \sin\theta = k$.
(b) Singular solution, but still of the form

$$\Theta = \frac{c}{\sqrt{2}}(X + Y).$$

A2.41 Wavefront: $l_0 Y \pm \dfrac{\sigma_0}{\delta\beta}\ln(\cosh\beta X) - \omega T =$ constant.

Ray: $\mu Y \mp \dfrac{1}{\beta\sigma_0}\ln|\sinh\beta X| =$ constant.

A2.42 Wavefront: $l_0 Y \pm \dfrac{2}{\delta}\sqrt{-\beta X} - \omega T =$ constant.

Ray: $\mu Y \mp \dfrac{2}{3\sqrt{\beta}}(-X)^{3/2} =$ constant.

Amplitude: form $A^2\dfrac{\partial\omega}{\partial k} =$ constant,
where $\omega^2 = (\sigma\tanh(\sigma D))/\delta^2$ and $\sigma = \delta\sqrt{k^2 + (\mu/\delta)^2}$.

A2.43 Rays: $X = \frac{1}{2}X_0\{1 - \sin(\text{constant} \pm \mu\sqrt{\beta}Y\}$; periodic, trapped.

A2.44 Start from

$$\frac{\mathrm{d}\Theta}{\mathrm{d}s} = 2(p^2 + q^2), \quad \frac{\mathrm{d}X}{\mathrm{d}s} = 2p, \quad \frac{\mathrm{d}Y}{\mathrm{d}s} = 2q,$$

$$\frac{\mathrm{d}p}{\mathrm{d}s} = 2cc_X, \quad \frac{\mathrm{d}q}{\mathrm{d}s} = 2cc_Y$$

where $p = \Theta_X, \; q = \Theta_Y$ (so $p^2 + q^2 = c^2$). Then

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{p}{q} \text{ and so } Y'' = \frac{c}{p^2}(c_Y - Y'c_X) \text{ and } \frac{c^2}{p^2} = 1 + (Y')^2.$$

(a) Straight path;

(b) Becomes $\left(cY' \Big/ \sqrt{1 + (Y')^2}\right)' = 0$: $cY' \propto \sqrt{1 + (Y')^2}$.

A2.45 Time $= \int_a^b c(X, Y)\sqrt{1 + (Y')^2}\mathrm{d}X = \int_a^b F(X, Y)\mathrm{d}X$;
Euler–Lagrange is

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\partial F}{\partial Y'}\right) - \frac{\partial F}{\partial Y} = 0$$

which is the required equation.

A2.46 Immediately, $Y' \Big/ \sqrt{1 + (Y')^2} = \sin\alpha$, so the result follows.

A2.47 Rays: $R = \dfrac{\mu^2}{\beta}\left\{1 + \tan^2\left[\dfrac{1}{2}\pi \pm \dfrac{1}{2}\mu^2(\theta - \theta_0)\right]\right\}$
where $R \to \infty$ as $\theta \to \theta_0$; closest approach to $R = 0$ is $R = \mu^2/\beta$ where $\theta = \theta_0 \mp \pi/\mu^2$.

A2.48 Rays: $\dfrac{1}{\sqrt{R_0}}\arctan\left(\sqrt{\dfrac{R}{R_0} - 1}\right) - \sqrt{R - R_0} = \pm\mu\sqrt{\beta}\theta + \text{constant}$;

rays cease to exist at $R = R_0$, at which point $\mathrm{d}R/\mathrm{d}\theta$ is infinite; the ray is perpendicular to the circle $R = R_0$.

A2.49 Let $c_g = \lambda c_p$; then $\sin\Theta = 1/(2/\lambda - 1)$ which increases as λ increases from $\frac{1}{2}$ to 1 (depth decreasing): the wedge angle increases; $\lambda = 3/4$ yields $\Theta = \arcsin(3/5)$.

A2.51 Let circle intersect course at Q_i $(i = 1, 2)$; limiting case is when circle touches. Join WQ_i; draw the perpendicular to WQ_i at W to intersect course at P_i'. Then, by similar triangles, $|PQ_i| = \frac{1}{2}|PP_i'|$, so circle through W, diameter Q_iP_i': two $(i = 1, 2)$ influence points for a given W.

A2.52 Let W be at (a, b), then W lies on the circle $(a - \frac{3}{4}Ut)^2 + b^2 = \frac{1}{4}U^2t^2$: quadratic in t, given a, b, U.

A2.53 Circular path: circle of radius R, centre at $(0, R)$, ship at origin. Write P' as $(X, Y) = R(\sin\alpha,\ 1 - \cos\alpha)$, $\alpha = Ut/R$, then W is $(x, y) = \{X - r\cos(\alpha + \theta),\ Y - r\sin(\alpha + \theta)\}$; cf. Figure 2.13. Condition of stationary phase is $r = \frac{1}{2}\lambda\cos^2\theta$ (equation 2.122) which gives (x, y). Often written as

$$x/R = \sin(\mu\cos\theta) - \frac{1}{2}\mu\cos^2\theta\cos(\theta + \mu\cos\theta)$$
$$y/R = 1 - \cos(\mu\cos\theta) - \frac{1}{2}\mu\cos^2\theta\sin(\theta + \mu\cos\theta)$$

where $\mu = \lambda/R$, $\alpha = \mu\cos\theta$ (equation 2.122). Straight-line course is $R \to \infty$, $R\mu = \lambda$ (fixed).

A2.54 Use $\Omega = -\delta\sqrt{W|\mathbf{k}|^3}$ and follow Section 2.4.2; roots for $\tan\theta$ always real.

A2.55 $h = H\{t - x/(3\sqrt{h} - 2c_0)\}$; $u = U\{t - x/(3u/2 + c_0)\}$ $(c_0 = \sqrt{h_0})$.

A2.56 $u =$ constant on lines $\mathrm{d}x/\mathrm{d}t = 3u/2 + c_0$ $(c_0 = \sqrt{h_0})$; consider characteristic through $t = \alpha$, $x = X(\alpha)$, then $u = X'(\alpha)$ on lines

$$x - (3X'(\alpha)/2 + c_0)(t - \alpha) = X(\alpha); \quad \text{also } h = (X'(\alpha)/2 + c_0)^2.$$

A2.57 $I = kZ$, with $c'\sqrt{1 + H} = \mp\frac{3}{2}$, gives $-k = 2k(k - 3\sqrt{k/2})$, which has the solution $k = 1$; this is the no-shear case.

A2.58 Set
$X = \xi + \eta = 2(u - \alpha t)$, $Y = \eta - \xi = 4c$, $t = (-\frac{1}{2}X + T_Y/Y)/\alpha$:

$$T_{XX} = T_{YY} + \frac{1}{Y}T_Y$$

(after one integration + decay conditions). Then $c = Y/4$, $u = T_Y/Y$, and $x = (X\hat{T}_Y/Y + \frac{1}{2}(\hat{T}_Y/Y)^2 - \frac{1}{2}\hat{T}_X)/2\alpha$ where $\hat{T} = T - \frac{1}{4}XY^2$; $T = AJ_0(\omega Y)\cos(\omega X)$, say, since shoreline is at $Y = 0$. Maximum run-up is where $u = 0$; which determines X and hence x. Far from the shoreline is $Y \to \infty$.

A2.59 First show that $J = 0$ can be written as $t_Y^2 - t_X^2 = 0$, then that $t_Y \pm t_X = A\omega^2\{J_2(\omega Y)\cos(\omega X) \pm J_1(\omega Y)\sin(\omega X)\}/Y \mp \frac{1}{2}$.
So $J = 0$, provided $A\omega^3 \geq 1$, first on $Y = 0$.

A2.60 $u^+/u^- = 2/(\sqrt{1 + 8F^2} - 1) < 1$ for $F > 1$; form $u^{+2}/h^+ = \alpha/(\sqrt{1 + \alpha} - 1)^3$, where $\alpha = 8F^2$ (> 8) where $\alpha < (\sqrt{1 + \alpha} - 1)^3$ (from, for example, $4 + \alpha > 4\sqrt{1 + \alpha}$, $\alpha > 8$). For the bore, move in the frame which brings the flow ahead of the hydraulic jump (u^-) to rest; then the speed of the bore is $U = u^-$.

A2.61 Behind approaching bore, let the depth be h_1 speed u_1; thus $u_1 = U(1 - h_0/h_1)$. After reflection, let the bore move away at speed V, depth h_2 behind; that is, in contact with the wall. Hence $V = U(h_1 - h_0)/(h_2 - h_1)$ and $h_2/h_1 = (\sqrt{1+8F^2} - 1)/2$, where $F = (V + u_1)/\sqrt{h_1}$; thus $(H^2 - 1)(H - 1) = 2HF_1^2$, where $H = h_2/h_1$ and $F_1^2 = U^2(1 - h_0/h_1)^2/h_1$.

A2.62 $R\,[\![h]\!] = [\![uh]\!]$; $\dot{R}\,[\![uh]\!] = [\![hu^2 + \frac{1}{2}h^2]\!]$.

A2.63 Requires $c^2 = 2a$, $\eta' = -1/(\delta\sqrt{3})$ (Stokes' highest wave) $= -\alpha a$, $c^2 = \tan(\alpha\delta)/\alpha\delta$; this yields $c^2 \approx 1.347$. Now

$$c^2 = 2(a+b) = \tan(\alpha\delta)/\alpha\delta, \quad a + 2b = 1/(\alpha\delta\sqrt{3}),$$
$$3(\tfrac{1}{2}a^2 + \tfrac{2}{3}ab + \tfrac{1}{4}b^2) = (c^2 - 1)(2a + b);$$

so $c^2 \approx 1.665$.

A2.64 Remember, speed of solitary wave here is $1 + \varepsilon c$; speed from general result is $\sqrt{\tan\alpha\delta/\alpha\delta}$ with $\delta \to 0$ and $K = \delta^2/\varepsilon$, which agrees at $O(\varepsilon)$. Simply write $\mathrm{i}\alpha$ for α.

A2.65 (b) $m = 1$ by direct integration, $u = \operatorname{arcsech}(\cos\phi)$.

(c) Use $\mathrm{d}/\mathrm{d}u \equiv (\mathrm{d}\phi/\mathrm{d}u)\mathrm{d}/\mathrm{d}\phi$; $\mathrm{d}(\mathrm{sn}u)/\mathrm{d}u = \mathrm{cn}u\,\mathrm{dn}u$; $\mathrm{d}(\mathrm{dn}u)/\mathrm{d}u = -m\,\mathrm{sn}u\,\mathrm{cn}u$.

A2.66 Period of $\sin\phi$, $\cos\phi$ is 2π, hence period of Jacobian elliptic functions is

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{1 - \mathrm{m}\sin^2\theta}} = 4\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} = 4K(m).$$

(b) Compare the terms in the series expansion in powers of m on each side of the equation.

(c) Or use properties of F from (b).

A2.67 $3b = 8K\alpha^2 m,\ 6a + \left(5 - \dfrac{1}{m}\right)b = 4c.$

A2.68 (a) On $z = 0$, $\mathbf{u} \equiv (\phi_\xi, 0)$ and $\mathrm{d}\boldsymbol{\ell} = (\mathbf{u}/|\mathbf{u}|)\mathrm{d}\xi$, so $C = \int_{-\infty}^{\infty} \phi_\xi \mathrm{d}\xi$.

(b) Use construction given in Figure 2.27; then by Stokes' theorem the integral all around the path is zero (since $\nabla \wedge \mathbf{u} = \mathbf{0}$). But $\mathbf{u}$ on $\xi = \pm\xi_0$ approaches zero as $\xi_0 \to \infty$, so the integral (from r to l) on the surface = integral in (a).

A2.69 With $\eta = \varepsilon\,\mathrm{sech}^2(\frac{1}{2}\sqrt{3\varepsilon}\xi)$, then $M \approx I \approx C \approx 4\sqrt{\varepsilon/3}$ and $V \approx T \approx 4/(3\varepsilon\sqrt{3\varepsilon})$.

A2.70 Variation, with integration by parts in z, yields

$$\int_D\!\!\int\left\{\left[\phi_t+\frac{1}{2}(\nabla\phi)^2+z\right]_{z=\eta}\delta\eta+\frac{\partial}{\partial t}\int_b^\eta\delta\phi\mathrm{d}z+\frac{\partial}{\partial x}\int_b^\eta\phi_x\delta\phi\mathrm{d}z\right.$$
$$+\frac{\partial}{\partial y}\int_b^\eta\phi_y\delta\phi\mathrm{d}z-\int_b^\eta[\phi_{xx}+\phi_{yy}+\phi_{zz}]\delta\phi\mathrm{d}z$$
$$-\left[(\eta_t+\phi_x\eta_x+\phi_y\eta_y-\phi_z)\delta\phi\right]_{z=\eta}$$
$$\left.+\left[(\phi_x\eta_x+\phi_y\eta_y+\phi_z)\delta\phi\right]_{z=b}\right\}\mathrm{d}x\mathrm{d}t,$$

from which all the equations follow.

Chapter 3

A3.1 Requires

$$\frac{\gamma}{A\beta}=\frac{\beta^2}{C}=-\frac{6}{\alpha B}.$$

A3.2 $2\eta_{0\tau}-3\eta_0\eta_{0\zeta}-\frac{1}{3}\eta_{0\zeta\zeta\zeta}=0.$

A3.3 $2\eta_{0\tau}+3\eta_0\eta_{0\xi}+\left(\frac{1}{3}-W\right)\eta_{0\xi\xi\xi}=0$, after writing $\delta^2=\varepsilon$.

A3.4 $$2\eta_{1\tau}+3(\eta_0\eta_1)_\xi+\frac{1}{3}\eta_{1\xi\xi\xi}=\frac{21}{4}\eta_0^2\eta_{0\xi}+\frac{31}{12}\eta_{0\xi}\eta_{0\xi\xi}+\frac{7}{6}\eta_0\eta_{0\xi\xi\xi}+\frac{1}{36}\eta_{0\xi\xi\xi\xi\xi},$$

where the KdV equation for η_0 has been used; set r.h.s. $=G'(\xi-c\tau)$, then $\left(\eta_0'^2F'\right)'=3\eta_0'G$, etc.

A3.5 $m=-\frac{2}{3}$; $n=-\frac{1}{3}$; $\lambda^2=1$.

A3.6 Try setting $u=6t^{-2/3}f\left(xt^{-1/3}\right)$ and observing that $f=-\frac{1}{3}(\log F)''$ where $F=\eta^3+12$ $\left(\eta=xt^{-1/3}\right)$.

A3.7 $2H_{0\tau}+\dfrac{1}{\tau}H_0+3H_0H_{0\xi}+\dfrac{1}{3}H_{0\xi\xi\xi}=0.$

A3.8 $a=-2b^2$; $c^2=1+4b^2$ (and so $c>0$ or $c<0$).

A3.9 Leading order becomes $\left(\pm2\eta_\tau-\frac{3}{2}(\eta^2)_\xi-\frac{1}{3}\eta_{\xi\xi\xi}\right)_\xi=0$, where $\xi=x\pm t$, $\tau=\varepsilon t$.

A3.11 $a=-2k^2$; $\omega=4k^3+3l^2/k$.

A3.15 $u \sim -2k_n^2 \operatorname{sech}^2\{k_n\xi_n \mp x_n\}$ as $t \to \pm\infty$ where $\xi_n = x - 4k_n^2 t$ $(n = 1, 2)$ and

$$\exp(2x_1) = \left|\frac{k_1 + k_2}{k_1 - k_2}\right|, \quad \exp(2x_2) = \left|\frac{k_1 - k_2}{k_1 + k_2}\right| \quad (k_1 \neq k_2).$$

A3.16 Let $k_1 < k_2$, then sech^2 at $t = 0$ only if $k_1 = 1$, $k_2 = 2$; two maxima if $\sqrt{3} > k_2/k_1 > 1$; one maximum if $k_2/k_1 \geq \sqrt{3}$.

A3.21 $a = -2k^2$; $\omega^2 = k^2 + k^4 + l^2$ (so $\omega > 0$ or $\omega < 0$).

A3.22 $$u = -2(12t)^{-2/3}\frac{\partial^2}{\partial\xi^2}\log\left(1 + kt^{-1/3}\int_\xi^\infty A_i^2(s)\mathrm{d}s\right), \quad \xi = x/(12t)^{1/3}.$$

A3.25 See Q3.17.

A3.29 Q3.26: $f = 1 + \mathrm{e}^\theta$, $\theta = kx + ly - \omega t + \alpha$, $\omega = k^3 + 3l/k$;
Q3.27: $f = 1 + \mathrm{e}^\theta$, $\theta = kx - \omega t + \alpha$, $\omega^2 = k^2 + k^4$;

Q3.28: $f = 1 + kt^{-1/3}\int_\xi^\infty A_i^2(s)\mathrm{d}s$, $\xi = x/(12t)^{1/3}$, k constant.

A3.31 Set $f = 1 + E_1 + E_2 + AE_1E_2$, $E_i = \exp(2k_ix - \varepsilon_i\omega_it)$, $\varepsilon_i = \pm 1$; then

$$A = -\frac{(\omega_1 - \omega_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4}{(\omega_1 + \omega_2)^2 - (k_1 + k_2)^2 - (k_1 + k_2)^4}.$$

A3.32 $f \sim 1 + \mathrm{e}^{\theta_2}$ (a θ_2 solitary wave at infinity); $f \sim 1 + \mathrm{e}^{\theta_1}$ (a θ_1 solitary wave at infinity); $f \sim 1 + \mathrm{e}^{\theta_1 - \theta_2}$ (a $\theta_1 - \theta_2$ solitary wave at infinity). NB

$$A = \frac{(m_1 - m_2)(n_1 - n_2)}{(m_1 + m_2)(n_1 + n_2)}.$$

A3.33 For $\mathscr{E}$, remember that $\int_{-\infty}^{\infty}\eta dx =$ constant.

A3.34 First, from energy conservation

$$\int_{-\infty}^{\infty}\left\{2\eta_0^2 + \varepsilon\left(4\eta_0\eta_1 - \frac{1}{2}\eta_0^3\right) + \mathrm{O}(\varepsilon^2)\right\}\mathrm{d}x = \text{const.};$$

second, use the equation for η_1 (A3.4) and the KdV equation η_0 to find

$$\int_{-\infty}^{\infty}\left(2\eta_0\eta_1 - \frac{1}{12}\eta_{0\xi}^2\right)\mathrm{d}\xi = \text{const.},$$

when the required result follows.

A3.36 Form

$$(xu+3tu^2)_t = \left(12tu^2 - 6tuu_{xx} + 3tu_x^2 + 3xu^2 - xu_{xx} + u_x\right)_x.$$

A3.37 Use Q3.36; the centre of mass has the x-coordinate $(\int_{-\infty}^{\infty} xu\mathrm{d}x)/(\int_{-\infty}^{\infty} u\mathrm{d}x)$. Consider two solitons (u_1, u_2) far apart, and then write $u = u_1 + u_2$.

A3.38 Write

$$u \sim -\sum_{n=1}^{N} k_n^2 \operatorname{sech}^2\{k_n(x - 4k_n^2 t - x_n)\} \text{ as } t \to +\infty,$$

then

$$\int_{-\infty}^{\infty} u\mathrm{d}x = -2\sum_{n=1}^{N} k_n; \quad \int_{-\infty}^{\infty} u^2\mathrm{d}x = \frac{4}{3}\sum_{n=1}^{N} k_n^3, \quad \text{etc.}$$

A3.40 For the third law, add $H\times$ (first equation) to $U\times$ (second equation).

A3.44 $c = \pm 1$; $I_{31} = \mp 1$; $I_{41} = 1$; $J_1 = \frac{1}{3}$.

A3.45

$$c = \frac{1}{2}\left\{U_0 + U_1 \pm \sqrt{4 + (U_1 - U_0)^2}\right\}; \quad I_{31} = \frac{U_1 + U_0 - 2c}{2(U_1 - c)^2(U_0 - c)^2};$$

$$I_{41} = \frac{U_1^2 + U_1U_0 + U_0^2 + 3c(U_0 + U_1) + 3c^2}{3(U_1 - c)^3(U_0 - c)^3}, \quad \text{etc.}$$

A3.46 (a) With critical level, c given in A3.45 is recovered for which $c > U_1$ or $c < U_0$: no critical level.

(b) With critical level, then $\alpha = c/U$ satisfies (for $\alpha < 1$)

$$1 - \frac{\alpha}{2\sqrt{1-\alpha}}\ln\left|\frac{1+\sqrt{1-\alpha}}{1-\sqrt{1-\alpha}}\right| = 2U_1^2\alpha(\alpha - 1),$$

which has one solution only, namely in $\alpha < 0$: no critical level.

A3.47 $c(c - U_1)^2 = c - dU_1$: three real roots (for $0 < d < 1$) with one satisfying $0 < c < U_1$: critical level exists.

A3.48 $\alpha = c/U_1$ satisfies (cf. A3.46)

$$1 - \frac{\alpha}{2\sqrt{1-\alpha}}\ln\left|\frac{1+\sqrt{1-\alpha}}{1-\sqrt{1-\alpha}}\right| = \frac{2}{d}\ U_1^2\alpha(\alpha - 1) + \frac{2\alpha(1-d)}{d(1-\alpha)},$$

and one solution satisfies $0 < \alpha < 1$: critical level exists. NB Interesting exercise: examine this equation for $d \to 1$; $d \to 0$.

A3.49 See equation (3.139).

A3.50 $1 + c\cos\theta_0 = U_0 \pm 1$.

A3.51 Choose $c = U_1$; $k = a\cos\theta + b(a)\sin\theta$ where $b^2 = 1 + a(U_1 - U_0) - a^2$.

A3.52 Set

$$h(p) = a(p)\cos p + b(p)\sin p$$

and

$$h'(p) = -a(p)\sin p + b(p)\cos p,$$

then $\mathrm{d}h/\mathrm{d}p \equiv h'(p)$ requires $a'\cos p + b'\sin p = 0$ (and a, b are related by equation (3.139b)). Required solution is described by the set: $a' = -b'\tan p$, $h = a\cos p + b\sin p$, equation (3.139), p derivative of equation (3.139).

A3.53 $D = (aX + b)^{9/4}$, a, b constants.

A3.54 Takes the form $\left(H_0^3\right)_X - \frac{1}{3}D^{9/4}\left(H_{0\xi}^2\right)_X + Q_\xi = 0$, for some Q.

A3.56 $\left(2F_\tau + \frac{3}{2}F_\xi^2 + \frac{1}{3}F_{\xi\xi\xi}\right)_\xi = \mathrm{O}(\varepsilon)$ and $H \sim F_\xi$.

Chapter 4

A4.1 $A(\zeta, \tau) \sim \int_{-\infty}^{\infty} f(\kappa; 0)\exp\left\{\kappa\zeta - \frac{1}{2}\kappa^2\omega''(\kappa_0)\tau\right\}\mathrm{d}\kappa$.

A4.2 (a) $F = A\cosh(\omega z) + B\sinh(\omega z) + \dfrac{z}{2\omega}\sinh(\omega z)$;

(b) $$F = A\cosh(\omega z) + B\sinh(\omega z) + \frac{1}{4\omega^2}\left\{\omega z^2\cosh(\omega z) - z\sinh(\omega z)\right\}.$$

A4.4 $u = \phi_x = \phi_\xi + \varepsilon\phi_\zeta = \varepsilon f_{0\zeta} +$ periodic terms.

A4.5 See equations (4.43).

A4.6 For $\delta \to 0$:

$$-2\mathrm{i}k\delta^2 A_{0\tau} + \delta^4 k^2 A_{0\zeta\zeta} - \delta^2 A_{0YY} + \frac{9}{2}A_0|A_0|^2 + 3\delta^2 k^2 A_0 f_{0\zeta} = 0,$$

$$\delta^2 k^2 f_{0\zeta\zeta} + f_{0YY} = -3\left(|A_0|^2\right)_\zeta;$$

for $\delta \to \infty$: with $c_p \sim 1/\sqrt{\delta k}$ then

$$-2i\sqrt{\frac{k}{\delta}}A_{0\tau} + \frac{1}{4\delta k}A_{0\zeta\zeta} - \frac{1}{2\delta k}A_{0YY} + 4k^3\delta A_0|A_0|^2 + 2k\sqrt{\frac{k}{\delta}}A_0 f_{0\zeta} = 0,$$

$$f_{0\zeta\zeta} + f_{0YY} = -2\sqrt{\delta k}\left(|A_0|^2\right)_\zeta.$$

For $\delta \to 0$:

$$-2ik\delta^2 A_{0\tau} + \delta^4 k^2 A_{0\zeta\zeta} - \frac{9}{2} A_0 |A_0|^2 = 0;$$

for $\delta \to \infty$:

$$-2\mathrm{i}\sqrt{\frac{k}{\delta}}\, A_{0\tau} + \frac{1}{4\delta k}\, A_{0\zeta\zeta} + 4k^3\delta A_0|A_0|^2 = 0.$$

A4.7 $c_{p1} = k^2/6$, $c_{g1} = k^2/2$, $A_{12} = 3A_{01}^2/2k^2$; then

$$-2ikA_{01T} + k^2 A_{01ZZ} - A_{01YY} + \frac{9}{2k^2} A_{01}|A_{01}|^2 + 3A_{01} f_{0Z} = 0,$$

$$2c_{g1} f_{0ZZ} + f_{0YY} + 3\big(|A_{01}|^2\big)_Z = 0.$$

A4.8 $t \to At$, $x \to Bx$, $u \to Cu$: $AC^2 = \alpha/\gamma$, $B^2C^2 = \beta/\gamma$.

A4.10 Oscillates like e^{int}, with amplitude $\sqrt{-n}$.

A4.11 Try $u(x, t) = a\exp(\mathrm{i}a^2 t)(1 + f + \mathrm{i}g)$ where $f(x, t)$ and $g(x, t)$ are both real. Oscillates like $\exp(\mathrm{i}a^2 t)$, with amplitude a.

A4.12 Use same approach as adopted for Q4.11.

A4.13 $u \sim 2am\,\mathrm{sech}\,(am\sqrt{2}x)\exp\{\mathrm{i}a^2(1 + 2m^2)t\}$; cf. Q4.9.

A4.16 $u(x, t) = t^{-1/2} f(\eta)$, $f'' - \frac{1}{2}\mathrm{i}(\eta f)' + \varepsilon f|f|^2 = 0$, $\eta = xt^{-1/2}$.

A4.18
$$c + g_0 \exp\{\mu(lx - mz) + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\}$$
$$+ \int_x^\infty dg_0 \exp\{\mu(ly - mz) + \mathrm{i}\mu^2(m^2 - l^2)t/\alpha\}\mathrm{d}y = 0;$$

$$d + \int_x^\infty cf_0 \exp\{\lambda(my - lz) + \mathrm{i}\lambda^2(l^2 - m^2)t/\alpha\}\mathrm{d}y = 0.$$

For example,

$$c = -g_0 \mathrm{e}^{3\mu(x - 3\mathrm{i}\mu t)} \Big/ \Big\{1 + \mathrm{e}^{3(\mu-1)x + 9\mathrm{i}(\lambda^2 - \mu^2)t}\Big\}.$$

A4.19 For $c = u^*$ then $g_0 = f_0^*$; but $f_0 g_0 = -8k^2$, so $|f_0|^2 = -8k^2$, which is impossible.

A4.20 Show that

$$\{(\mathrm{i}\mathrm{D}_t + \mathrm{D}_x^2)(g \cdot f)\}/f^2 + g\{\varepsilon|g|^2 - \mathrm{D}_x^2(f \cdot f)\}/f^3 = 0.$$

A4.21 Show that

$$\left\{\left(\mathrm{iD}_t+\beta \mathrm{D}_x^2+\mathrm{i}\gamma \mathrm{D}_x^3\right)(g\cdot f)\right\}/f^2 + 3\mathrm{i}\left\{\delta|g|^2-\gamma \mathrm{D}_x^2(f\cdot f)\right\}(fg_x-gf_x)/f^4 + g\left\{\varepsilon|g|^2-\beta \mathrm{D}_x^2(f\cdot f)\right\}/f^3=0;$$

$\gamma=0,\ \delta=0,\ \beta=1.$

A4.22 $g=\mathrm{e}^{\theta}, f=1+(\delta/2\gamma)(k+k^*)^{-2}\exp(\theta+\theta^*)$, $\theta=kx+(\mathrm{i}\beta k^2-\gamma k^3)t+\alpha$.

A4.23 $g_3=4\sqrt{2}\left(\mathrm{e}^{\mathrm{i}t+7x}+3\mathrm{e}^{9\mathrm{i}t+5x}\right);$
$f_2=4\left(\mathrm{e}^{2x}+\mathrm{e}^{6x}\right)+3\mathrm{e}^{4x}\left(\mathrm{e}^{8\mathrm{i}t}+\mathrm{e}^{-8\mathrm{i}t}\right);$
$f_4=\mathrm{e}^{8x}$ and rest are zero.

A4.24 Set $X=l\zeta+mY$, then

$$-2\mathrm{i}kc_pA_{0\tau}+\left(\alpha l^2-m^2c_pc_g\right)A_{0XX}+\left\{\beta+\frac{m^2k^2\gamma^2}{(1-c_g^2)c_p^2\{m^2+(1-c_g^2)l^2\}}\right\}A_0|A_0|^2=0.$$

A4.25 $-2\mathrm{i}k\delta^2A_{0\tau}+\delta^4k^2l^2A_{0XX}+\frac{9}{2}A_0|A_0|^2=0$ (retaining only the dominant contribution to each coefficient, but see Section 4.2.3).

A4.26
$$A_0=\frac{a\sqrt{2}}{3}\exp\left\{\mathrm{i}\left[\frac{c}{2k\delta^2}\left(\frac{X}{l}+\frac{c\tau}{2}\right)-\frac{n\tau}{2k\delta^2}\right]\right\}\times\operatorname{sech}\left\{\frac{a}{k\delta^2}\left(\frac{X}{l}+\frac{c\tau}{2}\right)\Big/\sqrt{2}\right\}$$

where $a^2=2\left(n-\frac{1}{4}c^2\right)>0$.

A4.29 Structure is evident if we write $\theta=\phi+\mathrm{i}\psi$ (ϕ,ψ real); then equation (4.90) gives

$$\left(\mathrm{e}^{\mathrm{i}\psi}/\sqrt{\lambda}\right)\Big/\left\{\sqrt{\lambda}\mathrm{e}^{\phi}+\mathrm{e}^{-\phi}/\sqrt{\lambda}\right\}=\left\{\mathrm{e}^{\mathrm{i}\psi}/(2\sqrt{\lambda})\right\}\operatorname{sech}(\phi+\phi_0)\ \text{where}\ \mathrm{e}^{\phi_0}=\sqrt{\lambda}.$$

A4.31

$$f_m=1/4(a_m^2);\quad c=1/(a_1+a_2)^2;\quad b_m=\frac{(a_1-a_2)^2}{4a_m^2(a_1+a_2)^2};$$

$$d=\frac{(a_1-a_2)^4}{16a_1^2a_2^2(a_1+a_2)^2}.$$

A4.32 For the first, form $u_{xt}u_x^* + u_x u_{xt}^*$; for the second form $u_t u_{xxx}^* + u u_{xxxt}^*$.

A4.33 Form

$$i\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} x|u|^2 \mathrm{d}x - \int_{-\infty}^{\infty} (u^* u_x - u u_x^*)\,\mathrm{d}x = 0.$$

A4.34 Integrate over one period to give

$$\int_{-\infty}^{\infty} \overline{\left(\int_0^{1+\varepsilon\eta} u \mathrm{d}z\right) \mathrm{d}\zeta} = \text{const.};$$

then with $u = \phi_\xi + \varepsilon\phi_\zeta$, the oscillatory part of ϕ_ζ yields

$$\int_{-\infty}^{\infty} (A_0^* A_{0\zeta} - A_0 A_{0\zeta}^*)\mathrm{d}\zeta = \text{const.}$$

A4.35 (a) Each a function of t only.
(b) If somewhere independent of t.
(c) $\{A + g(t)\}\zeta$ where both A (= const.) and g are arbitrary.
(d) If, as $\zeta \to +\infty$ or $-\infty$, the integral in (c) approaches a constant, then $g(t) = -A$ and the integral is zero.

A4.37 Write $u = f(kx + ly, t)$ to give $\mathrm{i}f_t + (k^2 + l^2)f_{\xi\xi} + f|f|^2 = 0$; then set $\xi = X\sqrt{k^2 + l^2}$ and follow Q4.9.

A4.38 For the first, form $\mathrm{i}(u^* u_t + u u_t^*)$; for the second form

$$\frac{\partial}{\partial t}\left(u_x u_x^* + u_y u_y^* - \frac{1}{2}u^2 u^{*2}\right).$$

A4.39 First form $\mathrm{i}\left(x^2 u u^*\right)_{tt} + x^2(u^* u_x - u u_x^*)_{xt} = 0$ and also obtain

$$\mathrm{i}(u^* u_x - u u_x^*)_t = \left\{4u_x u_x^* - (u^* u_x + u u_x^*)_x - \varepsilon |u|^4\right\}_x.$$

A4.40 (a) $A \to 0$ as $|x| \to \infty$; $\omega = a^2/2 > 0$; $A = a\,\mathrm{sech}(ax/\sqrt{2})$.
(b) $A \to \pm a$ as $x \to \pm\infty$; $\omega = -a^2 < 0$; $A = a\tanh(ax/\sqrt{2})$.

A4.41 $$(A^2)_t + 2(kA^2)_x = 0 \text{ so } \int_{-\infty}^{\infty} A^2 \mathrm{d}x \left(= \int_{-\infty}^{\infty} |u|^2 \mathrm{d}x\right) = \text{const.};$$

use this first equation in the second to give

$$\left\{\frac{1}{2}A_x^2+\frac{1}{2}k^2A^2+\frac{1}{4}\varepsilon A^4-\left(k_xA^2+2kAA_x\right)\int^x k\mathrm{d}x\right\}_x - \left(AA_x\int^x k\mathrm{d}x\right)_t = 0.$$

A4.42 Set-down is $\dfrac{-2\delta k}{\sinh 2\delta k}|A_0|^2$; mean drift is $c_g f_{0\zeta}$.

A4.43 $(2K\omega c_p)^2 = \alpha k^2(\alpha k^2 - 2\beta|A|^2)$.

A4.44 $P = (\cosh\delta kz)/\cosh\delta k$; $c_p^2 = (\tanh\delta k)/\delta k$.

A4.45 Use

$$\frac{\partial}{\partial z}\left(W^{-2}P_{zk}\right) - (\delta k/W)^2 P_k = 2\delta^2 kW^{-2}P + 2\delta^2 kc_p' W^{-3}P - 2c_p\frac{\partial}{\partial z}\left(W^{-3}P_z\right),$$

integrate in z, with the boundary conditions for P, and write

$$P_{zk}(0;k)=0; \quad P_k(1;k)=0; \quad P_{zk}(1;k) = 2\delta^2 kW_1^2 - 2\delta^2 k^2 c_p' W_1.$$

A4.46 $P \sim 1 - \delta^2k^2\int_z^1 W^2\left\{\int_0^z W^{-2}\mathrm{d}z\right\}\mathrm{d}z$.

A4.48 Coefficients of NLS equation now functions of $\hat{X} = \sigma X$; NLS then gives $B(\zeta, X; \hat{X})$. Use Q4.8, Q4.9.

Chapter 5

A5.2 See Q5.1; terms are the same size when $\delta kR = \mathrm{O}(1)$. Set $1/R = \alpha\delta k$, then $\delta k \to 0$ for α fixed, yields

$$\mu - \tan\mathrm{h}\mu + \alpha^2\mu^5 = 0 \quad \text{with} \quad \mu^2 = -\frac{\mathrm{i}}{\alpha}\left(\frac{\omega}{k}\right).$$

A5.4 See equation (5.21); first term (not involving R) is multiplied by $(1+\delta^2k^2W_\mathrm{e})$.

A5.5 $y = \left(\mathrm{e}^{-x} - \mathrm{e}^{-x/\varepsilon}\right)/\left(\mathrm{e}^{-1} - \mathrm{e}^{-1/\varepsilon}\right)$;
(a) $y \sim \mathrm{e}^{1-x}$; (b) $y \sim \mathrm{e}\left(1 - e^{-X}\right)$.

A5.6 To within exponentially small terms:
(a) $y_0 = \mathrm{e}^{1-x}$, $y_n = 0$, $n \geq 1$; (b) $Y_0 = \mathrm{e}(1-\mathrm{e}^{-X})$, $Y_1 = -X\mathrm{e}$.

A5.8 $u_{1z} = -z\eta_{0\xi\xi}$.

A5.9 (b) $\hat{u} = \dfrac{x}{2\sqrt{\pi}} \displaystyle\int_{-\infty}^{t} f(t')(t-t')^{-3/2} \exp\{-x^2/4(t-t')\}\mathrm{d}t'.$

A5.10 Change the order of the integration and introduce $x + z^2/4y^2 = x'$; integral $= \mathrm{e}^{-\alpha x}\sqrt{\pi/\alpha}$.

A5.11
$$\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{\infty} \eta_0^2 \mathrm{d}\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \eta_0 \left\{ \int_{\xi}^{\infty} \eta_{0\xi'} \frac{\mathrm{d}\xi'}{\sqrt{\xi' - \xi}} \right\} \mathrm{d}\xi,$$
$$\eta_0 = 2c\,\mathrm{sech}^2\left(\xi\sqrt{\frac{3c}{2}}\right).$$

A5.12 $\xi = x - t + \dfrac{c_0}{3\alpha\Delta}\left\{(1+\varepsilon\Delta\alpha t)^{-3} - 1\right\}$ $(\sim x - t - \varepsilon c_0 t$ as $\varepsilon\Delta \to 0)$.

A5.13 $c = \dfrac{1}{2}\left(\dfrac{1}{3}\alpha^2 - \lambda\sqrt{\pi/\alpha}\right)$; $\dfrac{3}{2}\dfrac{a^2}{b} = -\alpha^2 - \lambda\left(1 - \dfrac{1}{\sqrt{2}}\right)\sqrt{\pi/\alpha}$.

A5.14 Form
$$p_{1z} = (2U_0 - U)w_{0\xi} + \frac{1}{\mathcal{R}}(w_{1zz} + w_{0zz})$$
with
$$p_1 + \eta_0 p_{0z} - \eta_1 - \frac{2}{\mathcal{R}}\left\{w_{1z} + \eta_0 w_{0z} - \eta_{0\xi}u_{0z} - \eta_{1\xi}U'\right\} = 0 \text{ on } z = 1$$
and $U = U_0(2z - z^2)$; u_1 and w_1 as given.

A5.15 $\alpha = \frac{1}{2}\left(\sqrt{\lambda^2+4} - \lambda\right) > 0$; $\beta = \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 4}\right)$.

A5.16 Saddle at $(0,0)$; stable node at $(1,0)$ for $\lambda \geq 2$; stable spiral point for $0 < \lambda < 2$; a focus at $(1,0)$ for $\lambda = 0$.

A5.17 Stable node at $(1,0)$ for $0 \leq 1/\lambda \leq \frac{1}{2}$.

A5.18 $\eta_0 = \frac{1}{2}\{1 + \tanh(-X/2)\}$,
$\eta_1 = -\frac{1}{4}\mathrm{sech}^3(-X/2)\ln\{\mathrm{sech}^2(-X/2)\}$.

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