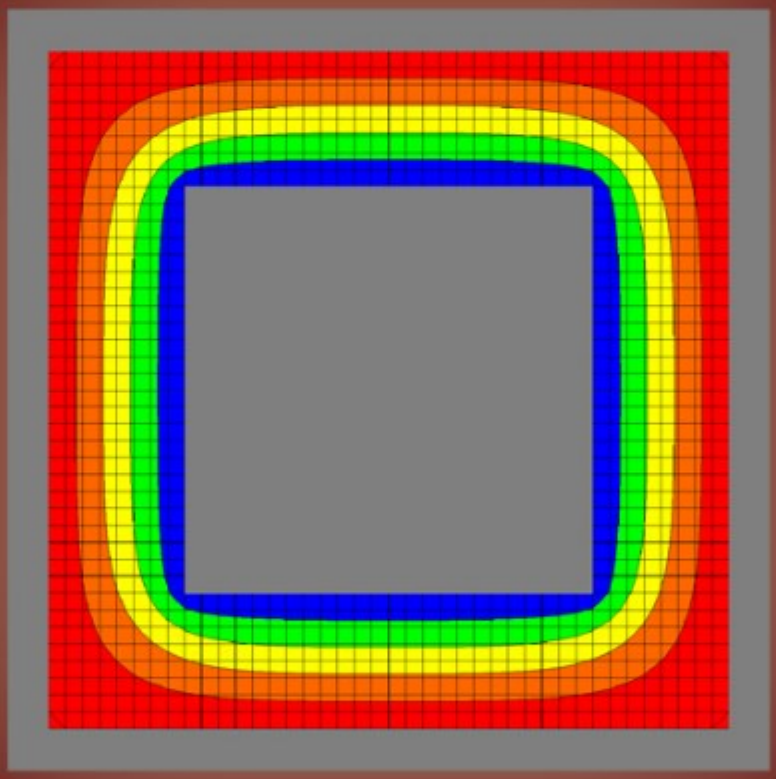
Worked Examples In Electromagnetism

Richard Carter



Richard G. Carter

Electromagnetism for Electronic Engineers – Examples

Electromagnetism for Electronic Engineers – Examples

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Contents

Preface

This is a companion volume to *Electromagnetism for Electronic Engineers* (3rd edn.) (Ventus, 2009). It contains the worked examples, together with worked solutions to the end of chapter examples, which featured in the previous edition of the book. I have discovered and corrected a number of mistakes in the previous edition.

I hope that students will find these 88 worked examples helpful in illustrating how the fundamental laws of electromagnetism can be applied to a range of problems. I have maintained the emphasis on examples which may be of practical value and on the assumptions and approximations which are needed. In many cases the purpose of the calculations is to find the circuit properties of a component so that the link between the complementary circuit and field descriptions of a problem are illustrated.

Richard Carter
Lancaster 2010

1 Electrostatics in free space

1.1 Introduction

Electrostatic problems in free space involve finding the electric fields and the potential distributions of given arrangements of electrodes. Strictly speaking 'free space' means vacuum but the properties of air and other gases are usually indistinguishable from those of vacuum so it is permissible to include them in this section. The chief difference is that the breakdown voltage between electrodes depends upon the gas between them and upon its pressure. The calculation of capacitance between electrodes in free space is deferred until Chapter 2.

The other problems included in this chapter involve the motion of charged particles (electrons and ions) in electric fields in vacuum. This topic remains important for certain specialised purposes including high power radio-frequency and microwave sources, particle accelerators, electron microscopes, mass spectrometers, ion implantation and electron beam welding and lithography.

1.2 Summary of the methods available

Note: This information is provided here for convenience. The equation numbers in the companion volume *Electromagnetism for Electronic Engineers* are indicated by square brackets.

Symbol	Signifies	Units
ε_0 (*epsilon*)	The primary electric constant	8.854×10^{-12} F.m^{-1}
Q	Electric charge	C
q	Electric line charge	C.m^{-1}
σ (*sigma*)	Surface charge density	C.m^{-2}
ρ (*rho*)	Volume charge density	C.m^{-3}
$\mathbf{E}$	Electric field	V.m^{-1}
V	Electric potential	V
∇ (*del*)	The vector differential operator	
$\hat{\mathbf{r}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$	Unit vectors	

Inverse square law of force between charges in free space

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad [1.1]$$

- Definition of the electric field of a charge in free space

$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad [1.2]$$

- Force acting on a charge placed in an electric field

$$\mathbf{F} = Q_2\mathbf{E} \qquad [1.3]$$

- Gauss' Theorem

The flux of $\mathbf{E}$ *out of any closed surface in free space is equal to the charge enclosed by the surface divided by* ε_0.

- The integral form of Gauss' Theorem

$$\oiint_S \mathbf{E}.\mathbf{dA} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dv \qquad [1.5]$$

- The differential form of Gauss' Theorem

$$\nabla.\mathbf{E} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{\rho}{\varepsilon_0} \qquad [1.9]$$

- Electrostatic potential difference

$$V_B - V_A = -\int_A^B \mathbf{E}\cdot\mathbf{dl} \qquad [1.13]$$

- Calculation of electric field from the electrostatic potential

$$\mathbf{E} = -\left(\hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}\right) = -grad\ V = -\nabla V \qquad [1.22]$$

- Poisson's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial zx^2} = -\frac{\rho}{\varepsilon_0} \qquad [1.24]$$

- Laplace's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad [1.27]$$

- The Principle of Superposition and the method of images
- The Principle of conservation of energy
- The finite difference method

Example 1.1

Find the force on an electron (charge -1.602 × 10^{-19} C) which is 1 nm from a perfectly conducting plane. What is the electric field acting on the electron?

Solution

Using the method of images the conducting plane is replaced by an image charge of +1.602 × 10^{-19} C which is 1 nm behind the position of the conducting plane

The force acting on the electron is found using the inverse square law [1.1] noting that the charges are 2 nm apart.

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} = \frac{-(1.602\times10^{-19})^2}{4\pi\times8.854\times10^{-12}\times\left(2\times10^{-9}\right)^2} = -57.7\times10^{-12}\ \text{N} \tag{1.1}$$

Force is a vector quantity so a complete answer must specify its direction. The negative sign indicates that the electron is attracted to the image charge. The force is therefore acting towards the plane and at right angles to it.

The electric field acting on the electron is found by substituting its charge and the force acting on it into [1.3]

$$E = \frac{F}{Q} = \frac{-57.7\times10^{-12}}{-1.602\times10^{-19}} = 360\ \text{MV}\cdot\text{m}^{-1} \tag{1.2}$$

The electric field is a vector quantity and the positive sign indicates that it is acting away from the plane.

Example 1.2

The surface charge density on a metal electrode is σ. Use Gauss' theorem to show that the electric field strength close to the surface is $E = \sigma/\varepsilon_0$.

Solution

Consider a small element of area of the surface *dA* such that the surface around it can be considered to be a plane. The local charge density can be considered to be constant and, from symmetry considerations, the electric field must be normal to the conducting surface. Now construct a Gaussian surface *dS*, as shown in fig. 1.1, such that it encloses the element *dA* and has sides which are normal to the surface and top and bottom faces which are parallel to the surface.

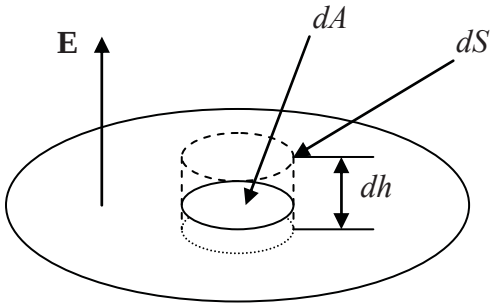

Fig. 1.1 A Gaussian surface for calculating the electric field of a surface charge.

Since **E** is parallel to the sides of *dS* the flux of **E** through the sides is zero. Also, because the electric field within a conducting material is zero when the charges are stationary, the flux of **E** through the bottom of *dS* is zero. The flux of **E** through the top of *dS* is

$$d\Phi = E\, dA \tag{1.3}$$

where E is the magnitude of E (since **E** is normal to the top of *dS*). The total charge enclosed by *dS* is

$$dQ = \sigma\, dA \tag{1.4}$$

By Gauss' theorem

$$d\Phi = \frac{dQ}{\varepsilon_0} \tag{1.5}$$

Substituting in (1.5) from (1.3) and (1.4) gives

$$E = \frac{\sigma}{\varepsilon_0} \tag{1.6}$$

Note: Because a conducting surface is always an equipotential surface when the charges are stationary **E** must always be normal to it. If the surface is curved the electric field varies over it (1.6) shows that, locally, the charge density is always proportional to the electric field.

Example 1.3

Figure 1.2 right shows a charged wire which is equidistant from a pair of earthed conducting planes which are at right angles to each other.

a) Where should image charges be placed in order to solve this problem by the method of images?
b) What difference would it make if the planes were at 60° to each other?
c) Could the method be used when the planes were at 50° to each other?

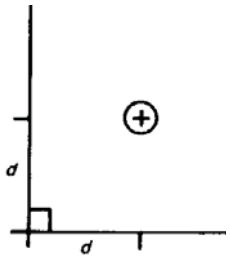

Fig. 1.2 A charged wire close to the intersection of two conducting planes

Solution

a) If Cartesian co-ordinates are used to describe the positions of the wire and of its images in the plane then the image line charges are $-q$ at $(-d, d)$ and $(d, -d)$ and $+q$ at $(-d, -d)$ as shown in fig. 1.3.

b)

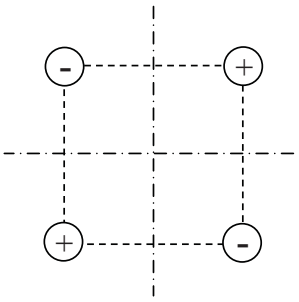

Fig. 1.3 Image charges for planes intersecting at 90°

c) When the planes are at 60° to each other five image charges are equally spaced on a circle as shown in fig. 1.4.

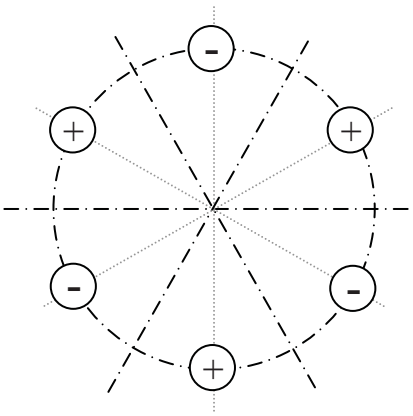

Fig. 1.4 Image charges for planes intersecting at 60°

d) No. The method can only be used when the angle between the planes divides an even number of times into 360°. Thus it will work for planes at angles of 1/4, 1/6, 1/8, 1/10 of 360° and so on.

Example 1.4

A wire l mm in diameter is placed mid-way between two parallel conducting planes 10 mm apart. Given that the planes are earthed and the wire is at a potential of 100 V, find a set of image charges that will enable the electric field pattern to be calculated.

Solution

If we were to put just one image charge on either side of the wire the field pattern could be calculated by superimposing the fields of the original wire and the image wires. The results would be as shown in fig.1.5. None of the equipotential surfaces is a plane. The solution is to use an infinite set of equally spaced wires charged alternately positive and negative, as shown in Fig. 1.6. The symmetry of this set of wires is such that there must be equipotential planes mid-way between the wires.

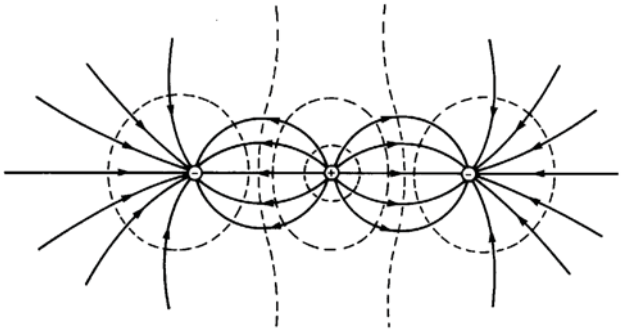

Fig. 1.5 The field pattern around a positively charged wire flanked by a pair of negatively charged wires.

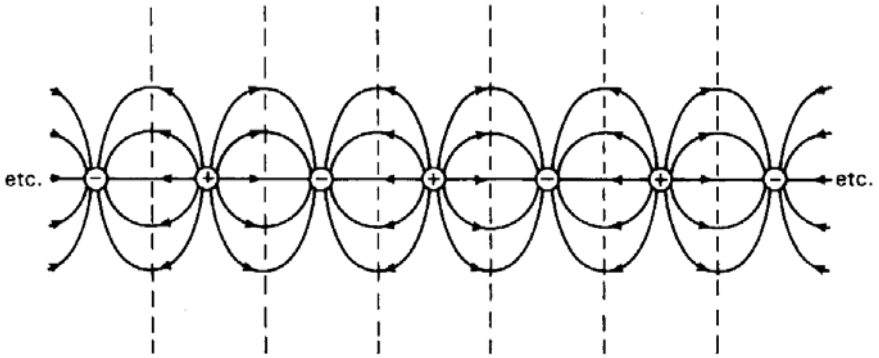

Fig. 1.6 The field pattern around a set of equispaced parallel wires charged alternately positive and negative.

Example 1.5

An air-spaced coaxial line has inner and outer conductors with radii a and b respectively as shown in fig.1.7. Show that the breakdown voltage of the line is highest when $\ln(a/b) = 1$.

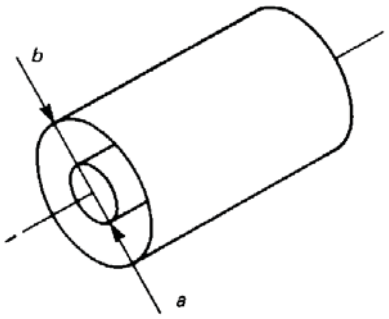

Fig. 1.7: The arrangement of an air-spaced coaxial line

Solution

For most practical purposes the properties of air are indistinguishable from vacuum. From the symmetry of the problem we note that the electric field must everywhere be radial. The field between the conducting cylinders is identical to that of a long, uniform, line charge q placed along the axis of the system.

To find the electric field of a line charge we apply the integral form of Gauss' equation to a Gaussian surface consisting of a cylinder of unit length whose radius is r and whose ends are normal to the line charge as shown in fig.1.8. We note that, from considerations of symmetry, the electric field must be acting radially outwards and depend only on the radius r.

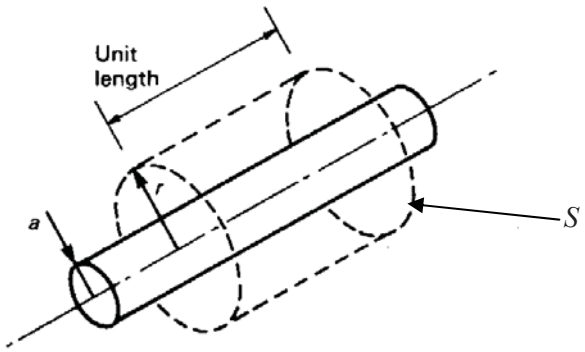

Fig. 1.8 A Gaussian surface for calculating the electric field strength around a line charge.

Let the radial component of the electric field at radius r be $E_r(r)$. On the curved surface of the cylinder the radial component of the electric field is constant and the flux is thus the product of the electric field and the area of the curved surface.

$$\oiint_S \mathbf{E}\cdot\mathbf{dA} = 2\pi r \times 1 \times E_r(r) \tag{1.7}$$

The flux of the electric field through the ends of the cylinder is zero because the electric field is parallel to these surfaces.

We apply Gauss' theorem [1.5] to find the relationship between the electric field, radius (r) and the unknown line charge q. Since S has unit length the total charge contained within it, which is denoted by the right-hand side of [1.5] is just q. Thus

$$2\pi r E_r(r) = \frac{q}{\varepsilon_0} \tag{1.8}$$

which can be rearranged to give

$$E_r(r) = \frac{q}{2\pi\varepsilon_0}\frac{1}{r} \tag{1.9}$$

Since the electric field is inversely proportional to r, it must be greatest when the radius is least, i.e. when $r = a$.

$$E_{\max} = \frac{q}{2\pi\varepsilon_0}\frac{1}{a} \tag{1.10}$$

The potential difference between the cylinders is found from the electric field using [1.13]

$$V_b - V_a = -\int_a^b E_r(r)\,dr = -\frac{q}{2\pi\varepsilon_0}\int_a^b \frac{1}{r}\,dr = -\frac{q}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) \tag{1.11}$$

The negative sign tells us that if the charge on the inner cylinder is positive then the electrostatic potential of the outer cylinder is negative with respect to the inner cylinder.

The unknown charge q can be eliminated between (1.10) and (1.11) to give the potential difference in terms of the maximum permitted electric field and the dimensions of the line.

$$V_b - V_a = -E_{\max} a \ln\frac{b}{a} \tag{1.12}$$

The condition that the potential difference should be maximum is found by differentiating the potential difference with respect to the ratio of the dimensions of the conductors and setting the result to zero. If we set $R = b/a$ the condition can be expressed as

$$\frac{d}{dR}\left(\frac{1}{R}\ln(R)\right) = -\frac{\ln(R)}{R^2} + \frac{1}{R^2} = 0 \tag{1.13}$$

or

$$\ln(R) = \ln(b/a) = 1 \tag{1.14}$$

Example 1.6

An air-spaced transmission line consists of two parallel cylindrical conductors each 2 mm in diameter with their centres 10 mm apart as shown in fig. 1.9. Calculate the maximum potential difference which can be applied to the conductors assuming that the electrical breakdown strength of air is $3\ \mathrm{MV \cdot m^{-1}}$.

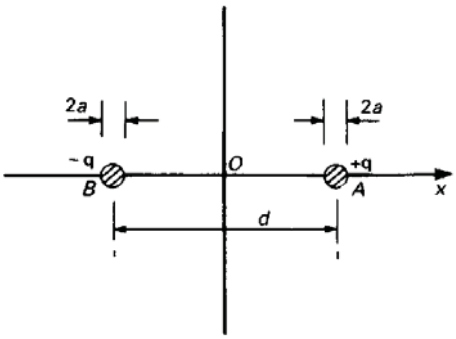

Fig. 1.9 A cross-sectional view of a parallel-wire transmission line.

Solution

Since the diameters of the wires are small compared with their separation it is reasonable to assume that close to the surface of each wire the field pattern is determined almost entirely by that wire. The equipotential surfaces close to the wires take the form of coaxial cylinders, as may be seen in Fig. 1.10. This is equivalent to assuming that the two wires can be represented by uniform line charges $\pm q$ along their axes. Note that this approximation is only valid if the diameters of the wires are small compared with the spacing between them.

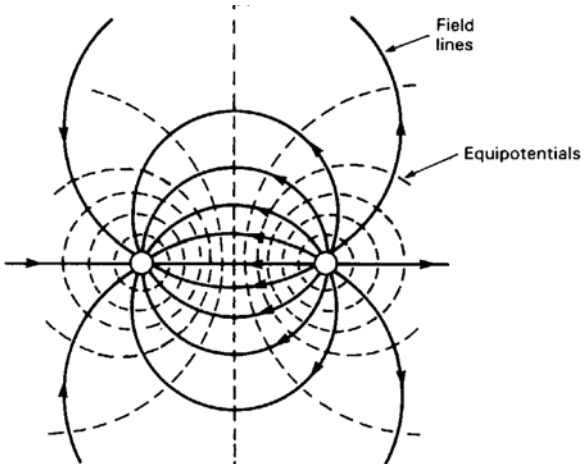

Fig. 1.10 The field pattern around a parallel-wire transmission line

The electric field of either wire is then given by Equation (1.9) (for $r \geq 1$ mm) with the appropriate sign for q. Since the strength of the electric field of each line charge is inversely proportional to the distance from the charge, the greatest electric field must occur on the plane passing through the axes of the two conductors. Using the notation of Fig. 1.9 and Equation (1.9) the electric field on the x axis between the wires is found by superimposing the fields of the two wires.

$$E_x = -\frac{q}{2\pi\varepsilon_0\left(\frac{1}{2}d - x\right)} - \frac{q}{2\pi\varepsilon_0\left(\frac{1}{2}d + x\right)} \tag{1.15}$$

It is easy to show that this expression is a maximum on the inner surfaces of the wires (as might be expected from Fig. 1.10), that is, when $x = \pm\left(\frac{1}{2}d - a\right)$. The maximum permissible charge is therefore given by

$$q_{\max} = 2\pi\varepsilon_0 E_{\max} \frac{a(d-a)}{d} \tag{1.16}$$

The potential at points on the x axis between the wires is found from (1.15) using [1.13]

$$V(x) = \frac{q}{2\pi\varepsilon_0}\int\left(\frac{q}{\left(\frac{1}{2}d - x\right)} + \frac{q}{\left(\frac{1}{2}d + x\right)}\right)dx = \frac{q}{2\pi\varepsilon_0}\ln\left(\frac{\frac{1}{2}d + x}{\frac{1}{2}d - x}\right) \quad + \quad C \tag{1.17}$$

where C is a constant of integration. It is convenient to choose $C = 0$ so that the potential is zero at the origin.

The maximum permissible potential at A is obtained by substituting the maximum charge from (1.16) into (1.17) and setting $x = \left(\frac{1}{2}d - a\right)$ to give

$$V_A = E_{\max}\frac{a(d-a)}{d}\ln\left(\frac{d-a}{a}\right) \tag{1.18}$$

The potential at B is $-V_A$ so the maximum potential difference between the wires is $2V_A$. Substituting the numbers gives the maximum voltage between the wires as 5.9 kV.

When the wires are not thin compared with their separation the method of solution is similar but, as can be seen from the equipotentials in Fig. 1.10, the equivalent line charges are no longer located at the centres of the wires.

Example 1.7

A metal sphere of radius 10 mm is placed with its centre 100 mm from a flat earthed sheet of metal. Assuming that the breakdown strength of air is 3 MV.m^{-1}, calculate the maximum voltage which can be applied to the electrode without breakdown occurring. What is then the ratio of the maximum to the mean surface-charge density on the sphere?

Solution

This problem is solved using the same procedure as the previous one. It is necessary to assume that the field is equivalent to that of a pair of point charges placed at the centre of the sphere and of its image in the plane as shown in fig.1.11. Since the diameter of the sphere is 20% of its distance from the plane this assumption should not be seriously in error. We note that the attraction between the surface charges will ensure that the charge density and the electric field are greatest at the point on each sphere lying closest to the other one.

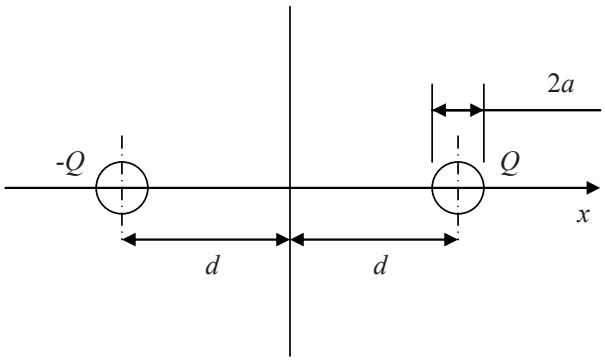

Fig. 1.11 The arrangement of the sphere and its image in the plane.

The first step is to use Gauss' theorem to find the electric field at a distance *r* from a point charge *Q*. The problem has spherical symmetry and therefore the electric field must be constant on the surface of a sphere of radius r centred on the charge and directed radially outwards. The surface area of a sphere of radius r is $4\pi r^2$ so that from [1.5]

$$4\pi r^2 E_r(r) = \frac{Q}{\varepsilon_0} \tag{1.19}$$

so that

$$E_r(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \tag{1.20}$$

Next we use [1.13] to find the potential at a distance *r* from the charge.

$$V(r) = -\int E_r(r)\,dr = -\int \frac{Q}{4\pi\varepsilon_0 r^2}\,dr = \frac{Q}{4\pi\varepsilon_0 r} \quad + \quad C \tag{1.21}$$

where *C* is a constant of integration. Now

$$r = d - x \tag{1.22}$$

so that the potential due to the first charge is

$$V_1(x) = \frac{Q}{4\pi\varepsilon_0(d-x)} + C_1 \tag{1.23}$$

Similarly the potential due to the other charge is

$$V_2(x) = \frac{-Q}{4\pi\varepsilon_0(d+x)} + C_2 \tag{1.24}$$

Superimposing the potentials of the two charges gives

$$V(x) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{d-x} - \frac{1}{d+x}\right) = \frac{Q}{2\pi\varepsilon_0}\cdot\frac{x}{(d-x)(d+x)} \tag{1.25}$$

At the surface of the first sphere $x = d - a$ and

$$V(d-a) = \frac{Q}{2\pi\varepsilon_0}\left(\frac{d-a}{a(2d-a)}\right) \quad (1.26) \tag{1.27}$$

The electric field at the surface of the sphere is found by superimposing the fields of the two charges using (1.20)

$$E(d-a)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a^2}+\frac{1}{(2d-a)^2}\right) \tag{1.28}$$

Eliminating Q between (1.27) and (1.28) gives the relationship between the breakdown field and the breakdown voltage

$$V_{\max}=2E_{\max}\left(\frac{d-a}{a(2d-a)}\right)\cdot\left(\frac{1}{a^2}+\frac{1}{(2d-a)^2}\right)^{-1} \tag{1.29}$$

Substituting the numerical values of the quantities we find that the maximum voltage is 28.3 kV.

From example 1.2 we know that the maximum surface charge density is

$$\sigma_{\max}=\varepsilon_0 E_{\max}=8.854\times10^{-12}\times3\times10^{6}=26.6\times10^{-6}\,C\cdot m^{-2} \tag{1.30}$$

The total charge on the sphere can be computed from (1.28)

$$Q=E_{\max}4\pi\varepsilon_0\left(\frac{1}{a^2}+\frac{1}{(2d-a)^2}\right)^{-1} \tag{1.31}$$

so the average charge density is

$$\sigma_{av}=E_{\max}\frac{4\pi\varepsilon_0}{4\pi a^2}\left(\frac{1}{a^2}+\frac{1}{(2d-a)^2}\right)^{-1}=\varepsilon_0 E_{\max}\left(1+\frac{a^2}{(2d-a)^2}\right)^{-1} \tag{1.32}$$

and the ratio of peak to average charge density is

$$\frac{\sigma_{\max}}{\sigma_{av}}=\left(1+\frac{a^2}{(2d-a)^2}\right)=1.003 \tag{1.33}$$

Example 1.8

An electron starts with zero velocity from a cathode which is at a potential of -10 kV and then moves into a region of space where the potential is zero. Find its velocity.

Solution

The principle of conservation of energy requires that the sum of the kinetic energy and the potential energy of the electron must be constant. Thus

$$\frac{1}{2}mv^2 + qV = 0 \tag{1.34}$$

where q is the charge on the electron and V is the potential relative to the cathode. The charge to mass ratio of an electron $q/m = -1.759\times10^{11} C.kg^{-1}$ and the region of zero potential has a potential relative to the cathode +10 kV so that, rearranging (1.34) we obtain

$$v = \sqrt{\frac{2qV}{m}} = \sqrt{2\times1.759\times10^{11}\times10^{4}} = 59.3\times10^{6} \text{ m.s}^{-1} \tag{1.35}$$

Note: For accelerating voltages much above 10 kV relativistic effects become important because the electron velocity is comparable with the velocity of light (0.2998×10^9 m s^{-1}). It is then necessary to use the correct relativistic expression for the kinetic energy of the electron, but the principle of the calculation is unchanged.

Example 1.9

An electron beam originating from a cathode at a potential of -10 kV has a current of 1 A and a radius of 10 mm. The beam passes along the axis of an earthed conducting cylinder of radius 20 mm as shown in fig. 1.12. Use Gauss' theorem to find expressions for the radial electric field within the cylinder, and calculate the potential on the axis of the system.

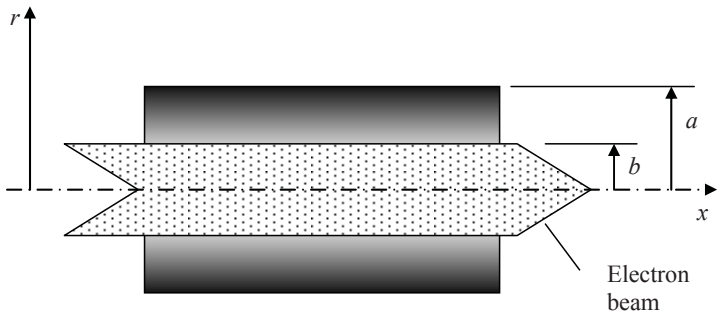

Fig. 1.12 The arrangement of an electron beam within a concentric conducting tunnel

Note: Electron beams like this are found in the high power microwave vacuum tubes used in transmitters for radar, TV broadcasting and satellite communications and for powering particle accelerators such as the Large Hadron Collider at CERN.

Solution

The velocity of the electrons is given by(1.35). The charge per unit length in an electron beam with current I and electron velocity v is given by

$$q = -\frac{I}{v} = -16.9\times10^{-9}\ \text{C.m}^{-1} \tag{1.36}$$

The negative sign arises because the direction of the conventional current is opposite to that of the electron velocity. If the radius of the beam is b and it is assumed that the current density ρ is uniform within the beam then

$$\rho = \frac{q}{\pi b^2} = -53.7\times10^{-6}\ \text{C.m}^{3} \tag{1.37}$$

Between the electron beam and the conducting cylinder (region 2) the problem is identical to that in Example 1.5 and the radial electric field is given by (1.8)

$$E_2(r) = \frac{q}{2\pi\varepsilon_0}\frac{1}{r} \tag{1.38}$$

Within the electron beam (region 1) Gauss' Theorem can be applied in exactly the same way but the charge enclosed in unit length of a Gaussian surface of radius r is now

$$q(r) = \pi r^2 \rho \tag{1.39}$$

This expression replaces q in (1.38) to give the radial field in region 1

$$E_2(r) = \frac{\rho}{2\varepsilon_0} r \tag{1.40}$$

The potential in each region is found using [1.13]. In region 1 the result is

$$V_1 = -\frac{q}{2\pi\varepsilon_0} \ln r + C_1 \tag{1.41}$$

The value of C_1 is chosen by requiring V_1 to be zero when $r = a$ so that

$$V_1 = -\frac{q}{2\pi\varepsilon_0} \ln \frac{r}{a} \tag{1.42}$$

In region 2 we have

$$V_2 = -\frac{\rho}{2\varepsilon_0} \int r \, dr = -\frac{\rho}{4\varepsilon_0} r^2 + C_2 \tag{1.43}$$

The value of C_2 is chosen by setting $V_2 = V_1$ when $r = b$.

$$V_2 = \frac{\rho}{4\varepsilon_0}\left(b^2 - r^2\right) + \frac{q}{2\pi\varepsilon_0} \ln\left(\frac{a}{b}\right) \tag{1.44}$$

On the axis $r = 0$ and

$$V_2(0) = \frac{\rho}{4\varepsilon_0} b^2 + \frac{q}{2\pi\varepsilon_0} \ln\left(\frac{a}{b}\right) = -362 \text{ V} \tag{1.45}$$

Note: This means that the electrons on the axis have a velocity slightly less than that calculated in (1.35) and electron velocity increases with radius. To obtain an accurate result it would be necessary to re-compute the electron velocities and the charge density (which now depends on r) to obtain mutually consistent values.

Example 1.10

Figure 1.13 shows a simplified form for the deflection plates for a low current electron beam. Given that the electron beam is launched from an electrode (the cathode) at a potential of -2000V and passes between the deflection plates as shown, estimate the angular deflection of the beam when the potentials of the plates are ±50 V.

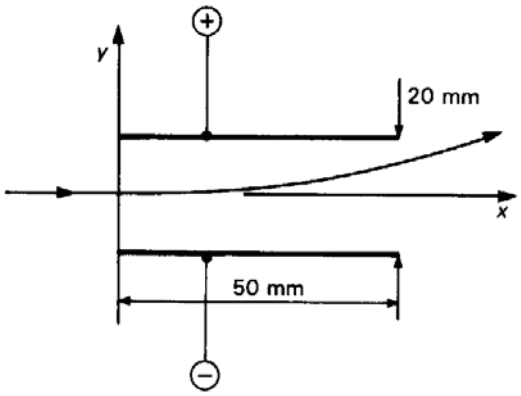

Fig. 1.13 The arrangement of a pair of electrostatic deflection plates for an electron beam.

Note: The original use of electrostatic deflection systems in cathode ray tubes for oscilloscopes is now obsolete but the same system can be used in machines for electron beam lithography, electron beam welding and scanning electron microscopes.

Solution

To make the problem easier we assume that the electric field is constant everywhere between the plates and falls abruptly to zero at the ends. Then the field between the plates is found by dividing the potential difference between the plates by their separation to be E_y = -5000V m^{-1}.

Because there is no x-component of **E**, the axial velocity of the electrons is constant and found using the principle of conservation of energy as in Example 1.8.

$$v_x = \sqrt{2\eta V} = 26.5\times10^6 \text{ m.s}^{-1} \tag{1.46}$$

where η is the charge to mass ratio of the electron. The time taken for an electron to pass along the length of the plates (L) is then

$$t = \frac{L}{v_x} = 1.89 \text{ ns} \tag{1.47}$$

The equation of motion in the y direction for an electron is

$$m\frac{d^2y}{dt^2} = -qE_y \tag{1.48}$$

where q is the magnitude of the electronic charge. The transverse acceleration of the electrons is constant and the y-component of velocity as they leave the plates is

$$v_y = -\frac{qE_y}{m}t = 1.66\times10^6 \text{ m.s}^{-1} \tag{1.49}$$

The angle of deflection is found from the ratio of the y and x components of the velocity

$$\theta = \arctan\left(\frac{v_y}{v_x}\right) = 3.6^o \tag{1.50}$$

Note: It is, of course, unrealistic to assume that the field between the plates has the idealized form chosen above. To obtain a more accurate estimate of the deflection it would be necessary to find the field distribution between the plates by solving Laplace's equation. Equation (1.48) could then be integrated using a more realistic expression for E_y.

Example 1.11

A simple thermionic diode consists of two plane parallel electrodes: the cathode and the anode. Electrons are emitted from the surface of the cathode with zero velocity and accelerated towards the anode which is maintained at a potential V_a with respect to the cathode. If the density of electrons between the electrodes is great enough the space charge alters the distribution of the electric field. Show that, in the limit of high space-charge density, the current through the diode is proportional to $V_a^{3/2}$ and independent of the rate at which electrons are supplied by the cathode.

Solution

The problem as stated is a one-dimensional problem in which the electron velocity, charge density and potential depend only on the position x. The motion of the electrons is governed by three equations: the non-relativistic velocity is found from (1.46) with the difference that V is now a function of x.

$$\dot{x} = \sqrt{2\eta V(x)} \tag{1.51}$$

The current density is related to the charge density and the velocity by

$$J = \rho \dot{x} \tag{1.52}$$

The relationship between the charge density and the potential is given by the 1-dimensional form of Poisson's equation [1.24]

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\varepsilon_0} \tag{1.53}$$

Eliminating the velocity and the charge density between these equations yields

$$\frac{d^2V}{dx^2} = -\frac{J}{\varepsilon_0\sqrt{2\eta}} V^{-\frac{1}{2}} \tag{1.54}$$

This equation can be integrated by multiplying both sides by $2(dV/dx)$ to give

$$\left(\frac{dV}{dx}\right)^2 = -\frac{4J}{\varepsilon_0\sqrt{2\eta}} V^{\frac{1}{2}} + C \tag{1.55}$$

where C is a constant.

To determine C we consider the effect of the electronic space-charge on the potential as shown in fig. 1.14. If no electrons are present in the space between the electrodes the potential varies linearly with position as shown by the dashed line. When electrons are emitted from the cathode (at $x = 0$) they are drawn towards the anode gaining velocity as they go. Because the electrons are negatively charged they depress the electrostatic potential locally as shown by the solid curve. The limit to this process arises when the slope of the solid curve is zero at the origin because the electric field is zero there and no more electrons are drawn from the cathode. The current cannot be increased beyond this limit given by setting $C = 0$ in(1.55).

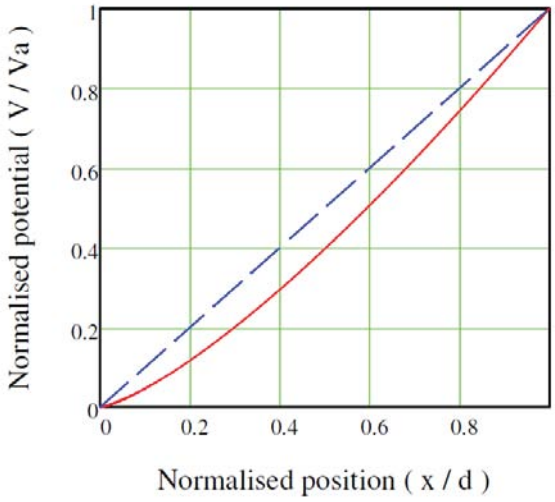

Fig. 1.14 The potential distribution in a space-charge limited diode

Equation (1.55) can then be written

$$\frac{dV}{dx} = -\sqrt{\frac{4J}{\varepsilon_0\sqrt{2\eta}}}\,V^{\frac{1}{4}} \tag{1.56}$$

which can be integrated by writing

$$\int V^{-\frac{1}{4}}\,dx = -\sqrt{\frac{4J}{\varepsilon_0\sqrt{2\eta}}}\int dx \tag{1.57}$$

Performing the integration we get

$$\frac{4}{3}V^{\frac{3}{4}} = -\sqrt{\frac{4J}{\varepsilon_0\sqrt{2\eta}}}\,x \quad + \quad C \tag{1.58}$$

Now $V = 0$ when $x = 0$ and therefore the constant of integration C is zero. At the anode $V = V_a$ and $x = d$ so that

$$V_a^{\frac{3}{2}} = -\frac{9}{4}\left(\frac{d^2}{\varepsilon_0\sqrt{2\eta}}\right)J \tag{1.59}$$

as required.

The total current flowing in the diode is obtained by multiplying the current density by the area of the cathode surface. This equation, known as the Child-Langmuir Law, is of fundamental importance in the theory of vacuum electron devices which remain the dominant technology for generation of radio waves at high power levels.

Example 1.12

Find the potential distribution between a long thin conducting strip and a surrounding rectangular conducting tube, as shown in fig. 1.15, when the potential difference between them is 100 V.

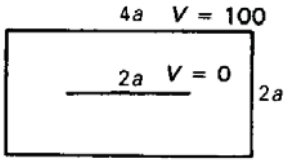

Fig. 1.15 The arrangement of conductors for this problem.

Solution

The problem may be simplified by observing that the solution is the same in each quadrant, subject to appropriate reflections about the planes of symmetry. One quadrant of the diagram is redrawn on an enlarged scale in Fig. 1.17 with a square mesh added to it. In this example we discuss the solution by hand.

To start the solution we first write down the potentials on the electrodes and estimate them at all the interior mesh points. An easy way to do this is to assume that the potential varies linearly with position. These potentials are written along-side the mesh points as shown. Next we choose a starting point such as *A* and work through the mesh, generating new values of the potentials with Equation [1.30].

$$V_0 = \frac{1}{4}\left(V_1 + V_2 + V_3 + V_4\right) \qquad [1.30]$$

where the definitions of the potentials are as shown in fig. 1.16

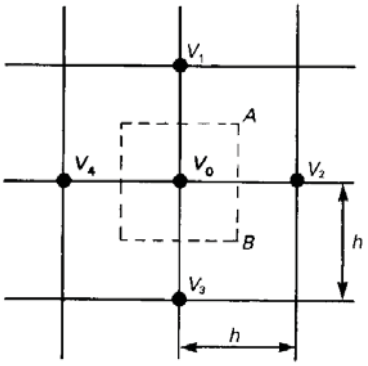

Fig. 1.16 Basis of the finite difference calculation of potential.

As each new value is calculated it is written down and the previous estimate crossed out. Figure 1.17 shows the results of the first pass through the mesh working along each row from right to left. Along the lines *PQ* and *RS* we make use of the symmetry of the field to supply the potentials at the mesh points outside the figure (i.e. $V_4 = V_2$ on *PQ* and $V_3 = V_1$ on *RS*). Check the figures for yourself and carry the process on for one more pass through the mesh to see how the solution develops. It is not necessary to retain many significant figures in the early stages of the calculation because any errors introduced do not stop the method from converging. If we work to two significant figures we can avoid the use of decimal points by choosing the electrode potentials at 0 and 100 V. The final values of the potentials can be scaled to any other potential difference if required.

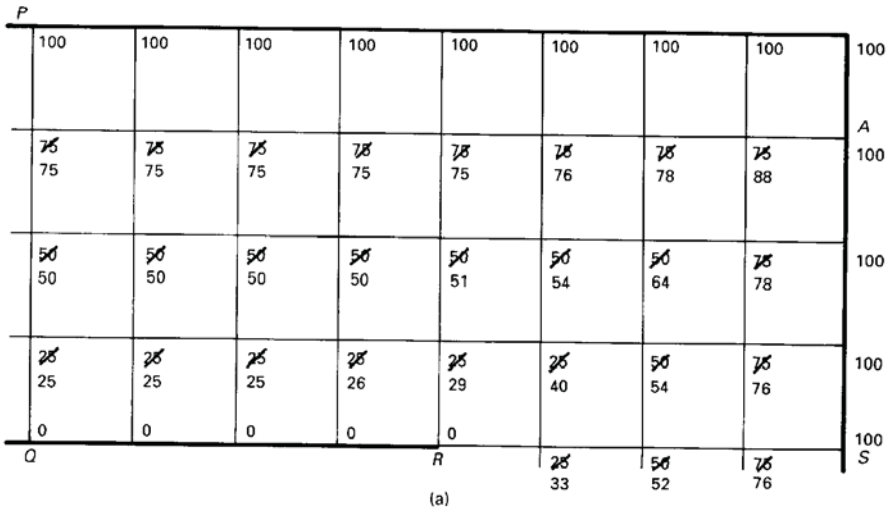

Fig. 1.17. The finite difference solution for one quadrant of the problem: The initial stages.

The process is continued until no further changes are observed in the figures to the accuracy required. The final result is shown in fig. 1.18. Evidently the accuracy could be improved by using a finer mesh.

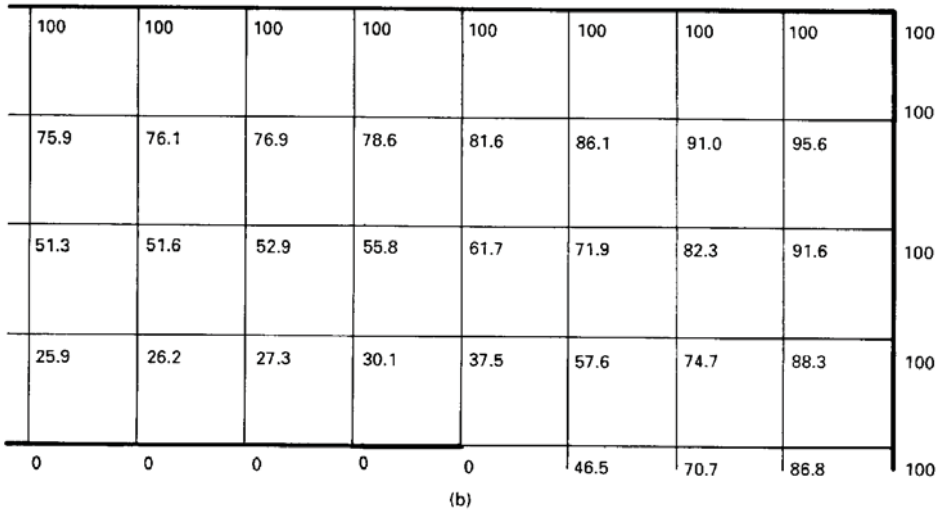

Fig. 1.18. The finite difference solution for one quadrant of the problem: The final solution.

Example 1.13

Figure 1.19 shows a square coaxial arrangement of electrodes. If the potential of the inner electrode is 5 V above that of the outer electrode estimate the maximum and minimum values of the electric field in the space between the electrodes.

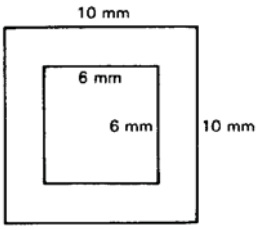

Fig. 1.19. A square coaxial arrangement of electrodes.

Solution

The finite difference method can be used to find the fields around two-dimensional arrangements of electrodes on which the potentials are specified. In this example we show how the method can be implemented on a spreadsheet.

A uniform square mesh is defined such that the electrodes coincide with mesh lines. The mesh spacing is chosen so that it is small enough to provide a reasonably detailed approximation to the fields whilst not being so small that the computational time is very large.

Cells of the spreadsheet are marked out such that one cell corresponds to each mesh point. The symmetry of the problem can be used to reduce the number of cells required. Thus, for the geometry shown above it is sufficient to find the solution for one quadrant of the problem. Special care is needed to ensure that the correct numbers of cells are used. Remember that the cells correspond to intersections between mesh lines and not to the cells enclosed by them.

The electrode potentials are entered into the cells corresponding to the electrodes and the formula in Equation [1.30] is entered into all the other cells. It is convenient to take the electrode potentials as 0 and 100 to reduce the number of digits displayed. When symmetry has been used to reduce the size of the problem the formulae in the cells along symmetry boundaries make use of the fact that the potentials on either side of the boundary are equal.

The formulae in the cells are then applied repeatedly (a process known as **iteration**) until the numbers in the cells cease to change. To do this the **calculation options** of the spreadsheet must be set to permit iteration. It is best to set the **iteration** to **manual** and to limit the number of iterations so that the evolution of the solution can be observed. The final numbers in the cells are then approximations to the potentials at the corresponding points in space.

From this solution the equipotential curves can be plotted and the field components can be calculated at any mesh point by taking the ratio of the potential difference to the mesh step. Figure 1.20 shows the final result obtained in this way. An active version of this figure is available for download as an EXCEL file. Clicking on the Potential Map tab will show you the potential map plotted using the results of the calculations. The electric field lines could be sketched in at right angles to the equipotential lines.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
1	87.6	87.6	87.6	87.7	87.7	87.8	87.9	88.1	88.4	88.8	89.3	89.9	90.7	91.6	92.7	93.8	95.0	96.3	97.5	98.8	100
2	75.2	75.2	75.2	75.3	75.4	75.6	75.8	76.2	76.7	77.4	78.4	79.6	81.2	83.1	85.3	87.6	90.1	92.5	95.0	97.5	100
3	62.7	62.7	62.8	62.9	63.0	63.3	63.6	64.1	64.8	65.8	67.2	69.0	71.4	74.4	77.7	81.3	85.0	88.8	92.5	96.3	100
4	50.2	50.3	50.3	50.4	50.6	50.8	51.2	51.8	52.6	53.8	55.5	57.9	61.1	65.2	69.9	74.9	79.9	85.0	90.1	95.0	100
5	37.7	37.7	37.8	37.9	38.0	38.3	38.6	39.2	40.0	41.3	43.1	45.9	50.0	55.5	61.8	68.3	74.9	81.3	87.6	93.8	100
6	25.2	25.2	25.2	25.3	25.4	25.6	25.9	26.3	27.0	28.1	29.8	32.7	37.5	45.0	53.4	61.8	69.9	77.7	85.3	92.7	100
7	12.6	12.6	12.6	12.7	12.7	12.8	13.0	13.2	13.6	14.3	15.4	17.6	22.1	33.6	45.0	55.5	65.2	74.4	83.1	91.6	100
8	0	0	0	0	0	0	0	0	0	0	0	0	0	22.1	37.5	50.0	61.1	71.4	81.2	90.7	100
9	0	0	0	0	0	0	0	0	0	0	0	0	0	17.6	32.7	45.9	57.9	69.0	79.6	89.9	100
10	0	0	0	0	0	0	0	0	0	0	0	0	0	15.4	29.8	43.1	55.5	67.2	78.4	89.3	100
11	0	0	0	0	0	0	0	0	0	0	0	0	0	14.3	28.1	41.3	53.8	65.8	77.4	88.8	100
12	0	0	0	0	0	0	0	0	0	0	0	0	0	13.6	27.0	40.0	52.6	64.8	76.7	88.4	100
13	0	0	0	0	0	0	0	0	0	0	0	0	0	13.2	26.3	39.2	51.8	64.1	76.2	88.1	100
14	0	0	0	0	0	0	0	0	0	0	0	0	0	13.0	25.9	38.6	51.2	63.6	75.8	87.9	100
15	0	0	0	0	0	0	0	0	0	0	0	0	0	12.8	25.6	38.3	50.8	63.3	75.6	87.8	100
16	0	0	0	0	0	0	0	0	0	0	0	0	0	12.7	25.4	38.0	50.6	63.0	75.4	87.7	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	12.7	25.3	37.9	50.4	62.9	75.3	87.7	100
18	0	0	0	0	0	0	0	0	0	0	0	0	0	12.6	25.2	37.8	50.3	62.8	75.2	87.6	100
19	0	0	0	0	0	0	0	0	0	0	0	0	0	12.6	25.2	37.7	50.3	62.7	75.2	87.6	100
20	0	0	0	0	0	0	0	0	0	0	0	0	0	12.6	25.2	37.7	50.2	62.7	75.2	87.6	100

Fig. 1.20. Finite difference calculation of the problem. Mesh step = 0.25 mm. The red and blue areas contain fixed potentials. The white area contains the standard formula and the green areas use formulae which assume symmetry at the boundaries.

The method can be applied to more complicated problems including those with curved electrodes which do not fit the mesh and three-dimensional problems. Further information can be found in the literature.

2. Dielectric materials and capacitance

2.1 Introduction

This chapter provides examples of the solution of problems involving dielectric materials and the calculation of capacitance. The methods can also be used for air-spaced and vacuum capacitors. The introduction of materials also makes it possible to discuss problems in the theory of semi-conductor devices.

2.2 Summary of the methods available

Note: This information is provided here for convenience. The equation numbers in the companion volume *Electromagnetism for Electronic Engineers* are indicated by square brackets.

Symbol	Signifies	Units
ε (*epsilon*)	Permittivity	$F.m^{-1}$
ε_r	Relative permittivity	Dimensionless
D	Electric flux density	$C.m^{-2}$
C	Capacitance	F
W	Stored energy	J

- Relationship between permittivity and relative permittivity

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

- Definition of electric flux density D

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad [2.4]$$

- The integral form of Gauss' theorem (all materials)

$$\oiint \mathbf{D} \cdot \mathbf{dS} = \iiint \rho \, dv \qquad [2.5]$$

- The differential form of Gauss' theorem (all materials)

$$\nabla \cdot \mathbf{D} = \rho \qquad [2.6]$$

- Boundary conditions
  The tangential component of **E** is continuous at a boundary
  The normal component of **D** is continuous at a boundary

- The definition of capacitance

$$Q = CV \qquad [2.14]$$

- The energy stored in a capacitor

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV \qquad [2.16]$$

- Energy stored in an electric field

$$W = \frac{1}{2}\iiint \mathbf{D}\cdot\mathbf{E}\, dv \qquad [2.18]$$

- Finite difference method
- Estimation of capacitance using energy methods

Example 2.1

A MOS transistor is essentially a parallel-plate capacitor comprising a silicon substrate, a silicon dioxide insulating layer, and an aluminium gate electrode as shown in fig. 2.1. The silicon dioxide has relative permittivity 3.85 and dielectric strength 6.0×10^8 V.m^{-1}, the insulating layer is 0.1 μm thick, and the area of the gate electrode is 0.02 mm^2. Estimate the capacitance between the gate and the substrate and the maximum voltage which can be applied to the gate electrode.

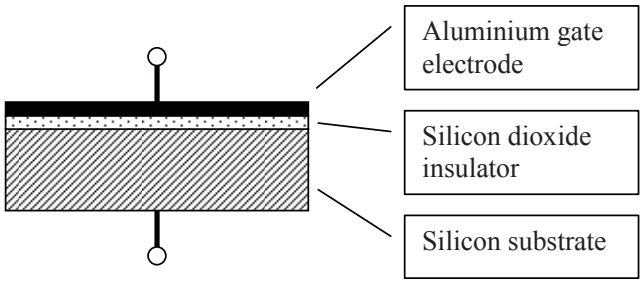

Fig. 2.1 Arrangement of layers in a MOS transistor.

Solution

When fringing fields are ignored the capacitance of a parallel plate capacitor can be calculated by using (1.5) in the form

$$D = \sigma \qquad (2.1)$$

If the potential difference between the electrodes is V and their separation is d then

$$D = \varepsilon_r \varepsilon_0 E = \varepsilon_r \varepsilon_0 \frac{V}{d} \tag{2.2}$$

The total charge on either plate of the capacitor is

$$Q = A\sigma = AD = \frac{\varepsilon_r \varepsilon_0 A}{d} V \tag{2.3}$$

where A is the area of one plate. The capacitance of a parallel plate capacitor is therefore, from [2.14]

$$C = \frac{\varepsilon_r \varepsilon_0 A}{d} = 6.8 \text{ pF} \tag{2.4}$$

when the numbers given in the question are inserted. The maximum permissible voltage difference between the gate and the substrate is the product of the dielectric strength (breakdown field) of the silicon dioxide and the thickness of the insulating layer (see [1.13]). The result is 60 V.

Example 2.2

Using the results of Example 2.1 calculate the maximum charge per unit area which can be induced in the semiconductor material. If there are 2.0×10^{18} atoms per square metre in the first layer of the silicon crystal, what proportion can be ionized by applying a voltage to the gate which is one sixth of the breakdown voltage?

Solution

The maximum charge per unit area is obtained from (2.3) by setting $A = 1$ m^2 and $V = 60$ V

$$\sigma_{\max} = \frac{\varepsilon_0 \times 3.85 \times 60}{0.1 \times 10^{-6}} = 0.02 \text{ C.m}^{-2} \tag{2.5}$$

If the applied voltage is 10 V then the number of electrons per square metre corresponding to the surface charge is

$$n = \frac{\sigma}{q} = \frac{\sigma_{\max}}{6q} = 2.13 \times 10^{16} \text{ m}^{-2} \tag{2.6}$$

where q is the charge on an electron. If we assume that this charge is represented by ionisation of atoms in the first layer of the silicon substrate then, dividing n by the number of atoms per square metre, we find that 1.06% of them are ionised

Example 2.3

A variable capacitor comprised a set of fixed plates, A, and a set of moving plates, B, as shown in Fig. 2.2. The capacitor is used to tune the frequency of a resonant circuit which varies inversely as the square root of the capacitance. Assuming that the effects of fringing fields can be neglected, find the shape which the moving plates must have if the frequency is to be proportional to the angle θ in the range 20-160° and 500-1500 kHz.

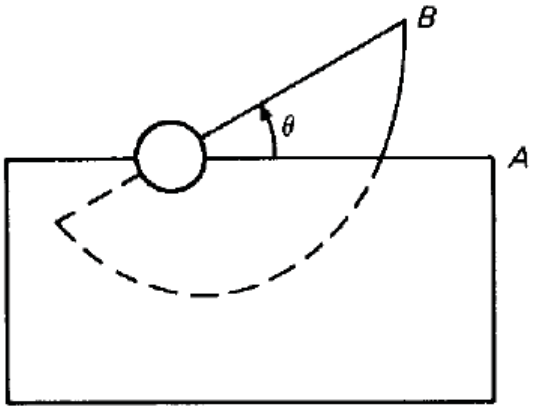

Fig. 2.2 Schematic diagram of a variable capacitor. A set of moving plates B rotates within a parallel set of fixed plates A.

Solution

This capacitor is a special example of a parallel plate capacitor. The separation between the plates is fixed so we know from (2.4) that the capacitance is proportional to the area of overlap between the plates if fringing effects are neglected. Since the frequency must be linearly related to the angle let

$$f = a + b\theta \tag{2.7}$$

If f is in kHz and θ is in degrees then, substituting the extreme values given

$$500 = a + 20b \quad \text{and} \quad 1500 = a + 160b \tag{2.8}$$

The solution of the pair of simultaneous equations (2.8) is $b = a/50$ so (2.7) becomes

$$f = a\left(1 + \frac{\theta}{50}\right) \tag{2.9}$$

Now the capacitance, and therefore the overlap of the plates, is proportional to the inverse square of the frequency so we may write the area of overlap as

$$A(\theta) = \frac{A_0}{\left(1 + \theta/50\right)^2} \tag{2.10}$$

where A_0 is a constant. If the plates are moved through a small angle $d\theta$ then the change in the area of overlap is

$$dA = -\frac{1}{2} r^2 d\theta \tag{2.11}$$

so that

$$r = \sqrt{-2\frac{dA}{d\theta}} = \sqrt{(-2)\frac{-2\,A_0/50}{\left(1 + \theta/50\right)^3}} \tag{2.12}$$

Therefore the dependence of r on θ which is required is

$$r \propto \left(1 + \theta/50\right)^{-1.5} \tag{2.13}$$

where θ is in degrees.

Example 2.4

Show that the capacitance per unit length between the parallel wires shown in Fig. 2.3 is given by $C = \dfrac{\pi\varepsilon_0}{\ln\left((d/a)-1\right)}$ if $d >> a$. Calculate the capacitance per unit length if d = 20 mm and a = 1 mm.

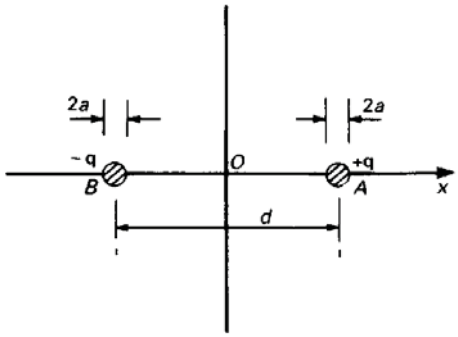

Fig. 2.3 Cross-sectional view of a parallel-wire transmission line

Solution

From Example 1.6 we know that the electric field on the x-axis is given by

$$E_x = -\frac{q}{2\pi\varepsilon_0\left(\frac{1}{2}d - x\right)} - \frac{q}{2\pi\varepsilon_0\left(\frac{1}{2}d + x\right)} \tag{1.15}$$

The electrostatic potential difference between the wires is found using [1.13]

$$\Delta V = -\frac{q}{2\pi\varepsilon_0}\int_{-\frac{1}{2}d+a}^{\frac{1}{2}d-a}\left(\frac{1}{\left(\frac{1}{2}d - x\right)} + \frac{1}{\left(\frac{1}{2}d + x\right)}\right)dx = \frac{q}{2\pi\varepsilon_0}\ln\left(\frac{\frac{1}{2}d - x}{\frac{1}{2}d + x}\right)\Bigg|_{-\frac{1}{2}d+a}^{\frac{1}{2}d-a} = \frac{q}{\pi\varepsilon_0}\ln\left(\frac{d}{a} - 1\right) \tag{2.14}$$

Then from the definition of capacitance [2.14] we obtain the result required. This result is only valid if $d >> a$ because we have assumed that the equipotential surfaces are circles centred on the wires. Substituting the dimensions given we find that

$$C = \frac{\pi\varepsilon_0}{\ln(19)} = 9.4\ \text{pF.m}^{-1} \tag{2.15}$$

Example 2.5

The parallel wires of Example 2.4 are placed so that each is 10 mm from a large flat sheet of metal. How does this affect the capacitance? Calculate the capacitance per unit length between the wires in the presence of the metal sheet. Represent the result by an equivalent circuit which shows the effect of the presence of the sheet.

Solution

The effect of the metal sheet on the potential distribution around the wires is found by using the method of images as in Example 1.3(a). Let us number the line charges as shown in fig. 2.5.

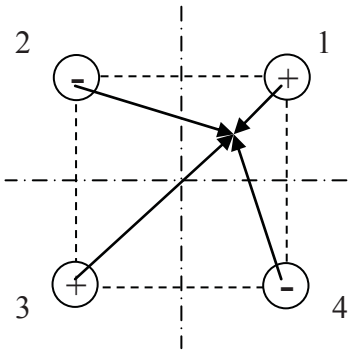

Fig. 2.4 Arrangement of charges and image charges for example 2.5

The potential at a point distance r from a line charge q is given by

$$V = -\frac{q}{2\pi\varepsilon_0}\ln(r) \quad + C \tag{2.16}$$

where C is a constant of integration. Summing for all the charges we find that

$$V = \frac{q}{2\pi\varepsilon_0}\ln\left(\frac{r_2 r_4}{r_1 r_3}\right) \tag{2.17}$$

where the sum of the constants of integration has been set to zero so that the zero of potential is on the mid-planes between the wires (shown as chain-dotted lines in fig. 2.4). Provided that $d >> a$ we may write for a point on the surface of wire 1 on a line passing through the origin $r_1 = a$, $r_2 = r_4 = \left(d - a/\sqrt{2}\right)$ and $r_3 = \sqrt{2}d - a$ at a point on the surface of wire 1. Thus

$$V_1 = \frac{q}{2\pi\varepsilon_0}\ln\left(\frac{19.3\times 19.3}{1\times 27.3}\right) = \frac{q}{2\pi\varepsilon_0}\times 2.61 \tag{2.18}$$

The potential on the surface of wire 2 is $-V_1$ so the capacitance between wires 1 and 2 is therefore

$$C = \frac{\pi\varepsilon_0}{2.61} = 10.6\ pF.m^{-1} \tag{2.19}$$

which is an increase of 13% compared with the previous result. This result can be represented by an equivalent circuit which includes the parasitic capacitance between the wires and the metal sheet as shown in fig. 2.5.

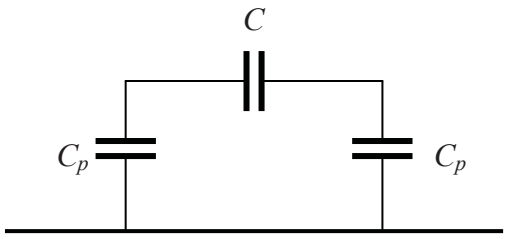

Fig. 2.5 Equivalent circuit for example 2.5

The capacitance added by the plane comprises the two parasitic capacitors in series with each other and in parallel with the capacitance between the wires. Thus C_p = 2.4 pF.

Example 2.6

A capacitor consists of two parallel conducting plates whose area is large enough compared with their spacing for edge effects to be negligible. The sheets are 0.1 mm apart and the space between them is partially filled with polythene sheet of thickness 0.09 mm, relative permittivity 2.25 and breakdown strength 30 MV.m^{-1} as shown in fig. 2.6. Assuming that the breakdown strength of air is 3 MV.m^{-1}, calculate the maximum voltage which can be applied to the capacitor. What difference does it make if the polythene sheet completely fills the space between the plates?

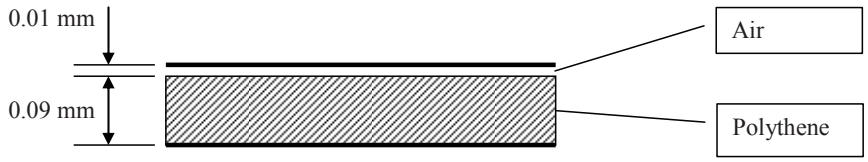

Fig. 2.6 Capacitor with mixed dielectrics

Solution

Let the surface charge on the plates of the capacitor be σ then by [2.4] and (1.5) $D = \sigma$. Since the normal component of D is continuous at the boundary between the dielectrics it follows that D is the same everywhere. The electric fields in the two dielectrics are given by [2.4] as

$$E_{air} = \frac{D}{\varepsilon_0} \tag{2.20}$$

and

$$E_{poly} = \frac{D}{\varepsilon_0 \varepsilon_r} \tag{2.21}$$

The potential drops across the two materials are found by multiplying the electric field by the thickness. The total potential drop is therefore

$$V = E_{\text{air}} \left(t_{\text{air}} + \frac{t_{\text{poly}}}{\varepsilon_r} \right) \tag{2.22}$$

From (2.20) and (2.21) we see that the electric field in the air is greater than that in the polythene. Therefore the air will break down before the polythene. The maximum permissible voltage across the capacitor is obtained by substituting the breakdown strength of air into (2.22) to give 150 V.

If the polythene completely fills the space between the plates then the maximum voltage is the product of the breakdown strength of the polythene and its thickness: 3.0 kV. This example shows the importance of avoiding air gaps in high voltage insulation because they seriously reduce its effectiveness.

Example 2.7

Figure 2.7 shows a coaxial cable with two layers of dielectric material. Find an expression for the electric field at any point within the dielectric.

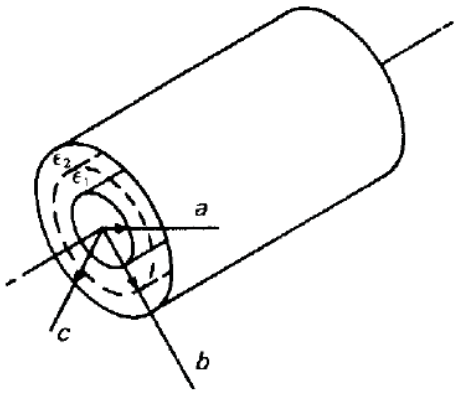

Fig. 2.7 A coaxial cable with two concentric layers of dielectric material between the conductors

Solution

As in the simpler case with a single dielectric described in section 2.2 of the textbook we assume a charge q per unit length on the inner conductor. Applying Gauss' theorem gives

$$D_r = \frac{q}{2\pi r} \qquad [2.7]$$

From this point on we must consider the two dielectric regions separately. If the radial components of the electric field are E_1 and E_2 in the inner and the outer region, respectively, then

$$E_1 = \frac{q}{2\pi\varepsilon_1 r} \tag{2.23}$$

and

$$E_2 = \frac{q}{2\pi\varepsilon_2 r} \tag{2.24}$$

The potential difference across the inner layer is found using [1.13]

$$V_{ba} = -\int_a^b \frac{q}{2\pi\varepsilon_1 r}\, dr = \left(\frac{q}{2\pi\varepsilon_1}\right)\ln\left(\frac{a}{b}\right) \tag{2.25}$$

In the same way the potential difference across the outer region is

$$V_{cb} = \left(\frac{q}{2\pi\varepsilon_2}\right)\ln\left(\frac{b}{c}\right) \tag{2.26}$$

The overall potential difference is the sum of these two potential differences

$$V_{ca} = \frac{q}{2\pi}\left(\frac{1}{\varepsilon_1}\ln(b/a) + \frac{1}{\varepsilon_2}\ln(c/b)\right) \tag{2.27}$$

The expressions for the electric field can then be found as eliminating q between (2.23), (2.24) and (2.27).

$$E_1 = \frac{V_{ca}}{\varepsilon_1 r}\left(\frac{1}{\varepsilon_1}\ln(b/a) + \frac{1}{\varepsilon_2}\ln(c/b)\right)^{-1} \tag{2.28}$$

and

$$E_2 = \frac{V_{ca}}{\varepsilon_2 r}\left(\frac{1}{\varepsilon_1}\ln(b/a) + \frac{1}{\varepsilon_2}\ln(c/b)\right)^{-1} \tag{2.29}$$

Although the conditions for continuity of the field components have not been involved explicitly on this occasion, it is easy to show that they are satisfied by this solution. The symmetry of the problem ensures that the electric field is entirely radial, so the tangential field is zero on each side of the boundary. The continuity condition for the radial component of the electric flux density is satisfied automatically in the application of Gauss' theorem.

Example 2.8

Find the capacitance per unit length of the coaxial system shown in Fig. 2.7.

Solution

Using (2.27) and [2.14]

$$C = 2\pi\left(\frac{1}{\varepsilon_1}\ln\left(b/a\right)+\frac{1}{\varepsilon_2}\ln\left(c/b\right)\right)^{-1} \tag{2.30}$$

Example 2.9

A coaxial cable is to be made with two dielectric layers as shown in Fig. 2.7. The inner layer is made from a high-quality but expensive material which has dielectric strength E_1 and the outer layer from a cheaper material of dielectric strength E_2. Find an expression for the outer radius, b, of the inner layer if the cable is to be made as cheap as possible without reducing its maximum working voltage.

Solution

From (2.23) and (2.24) we know that the maximum electric field in each region occurs at the inner surface (i.e. the minimum radius). Thus the radius of the boundary between the two regions cannot be reduced beyond the point at which the two regions reach their breakdown fields simultaneously. From (2.23) and (2.24) this requires

$$\frac{E_1}{E_2} = \frac{\varepsilon_2 b}{\varepsilon_1 a} \tag{2.31}$$

where E_1 and E_2 are the breakdown fields in the two dielectrics. Thus the minimum value of b is

$$b = \frac{E_1 \varepsilon_1}{E_2 \varepsilon_2} a \tag{2.32}$$

Example 2.10

Calculate the stored energy per unit length in the coaxial cable shown in Fig. 2.8 when the voltage between the electrodes is V_0

a) by using the expression for the energy stored in a capacitor, and

b) by integrating the energy stored in the electric field,

and compare the results.

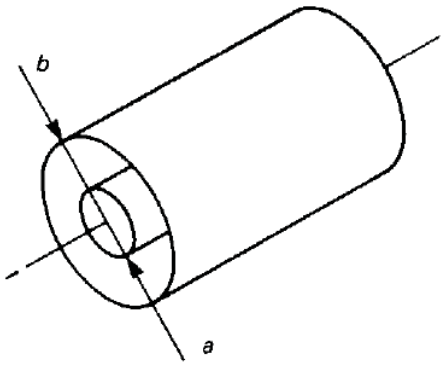

Fig. 2.8 A coaxial cable

Solution

The capacitance per unit length of a cable with a single dielectric layer can be deduced from (2.30) as

$$C = \frac{2\pi\varepsilon}{\ln(b/a)} \tag{2.33}$$

Then, using Equation [2.16], the stored energy per unit length is

$$W = \frac{1}{2}CV_0^2 = \frac{\pi\varepsilon V_0^2}{\ln(b/a)} \tag{2.34}$$

Alternatively, the electric field at radius r is given by eliminating q between (2.23) and (2.25)

$$E = \frac{V_0}{r\ln(b/a)} \tag{2.35}$$

The stored energy in a thin cylinder of thickness dr at radius r is then

$$dW = \frac{1}{2}D\,E\,2\pi\varepsilon\,r\,dr = \frac{\pi\varepsilon V_0^2}{\left(\ln(b/a)\right)^2}\frac{dr}{r} \tag{2.36}$$

Integrating with respect to r from $r = a$ to $r = b$ gives

$$W = \frac{\pi\varepsilon V_0^2}{\ln(b/a)} \tag{2.37}$$

as before.

Example 2.11

Figure 2.9 shows the cross-section of two adjacent tracks on a printed circuit board. If the tracks run parallel to each other for 50 mm, estimate the capacitance between them. The relative permittivity of the material of the board is 6.0.

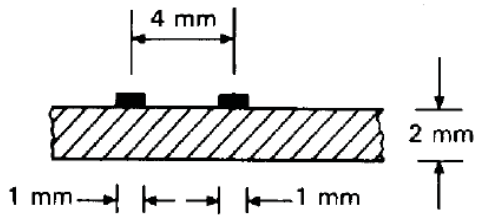

Fig. 2.9. The arrangement of conducting tracks on a printed circuit board.

Solution

The exact solution to this problem would be rather difficult, but an adequate estimate can be obtained by elementary methods. Consider first the approximation of the system by two parallel cylindrical wires as shown in fig. 2.3. We should not be too far out if we take the separation of the conductors to be the same in each case and the diameters of the wires to be equal to the widths of the strips on the printed circuit board.

From Example 2.4 the capacitance per metre of the parallel wires in free space is

$$C = \frac{\pi\varepsilon_0}{\ln\left((d/a)-1\right)} \tag{2.38}$$

So, for the dimensions given, C = 0.7 pF. If the wires were wholly embedded in the dielectric then this capacitance would be multiplied by the relative permittivity to give 4.2 pF.

Now the capacitance between the conductors can be regarded as two capacitances in parallel, one for the part of the diagram above the board (C_1) and the other (C_2) for the part within and below it. Since capacitances in parallel add, C_1 must be just half the capacitance between the wires in free space, so that C_1 = 0.35 pF.

The value of C_2 must lie somewhere between this figure and half of the capacitance between the wires when they are wholly embedded in the dielectric because of the finite thickness of the dielectric board. Thus 0.35 pF < C_2 < 2.1 pF. Combining these estimates for C_1 and C_2 gives for the total capacitance 0.7 pF < C< 2.5 pF. Although this estimate is rather crude it is probably accurate enough for the purpose for which it is required.

Example 2.12

Calculate the capacitance per unit length for the coaxial system of electrodes shown in fig. 2.10

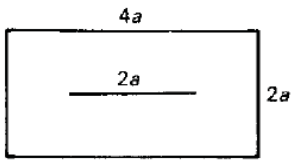

Fig. 2.10. Rectangular coaxial electrodes.

Solution

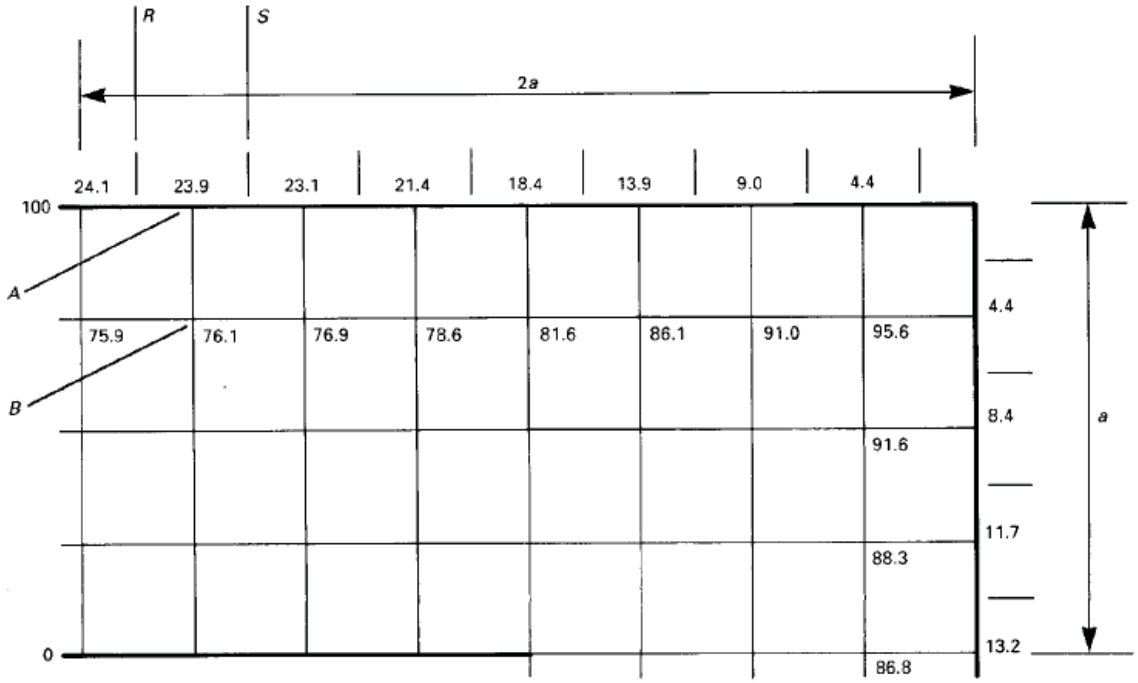

Fig. 2.11. The electrostatic potential distribution close to the outer conductor of the arrangement of electrodes shown in fig. 2.10. (see Fig. 1.18).

Figure 2.11 shows the potentials at the mesh points close to the outer electrode taken from Fig. 1.18. The potential difference across *AB* is 23.9 V so the electric field strength is

$$E_{AB} = \frac{23.9}{a/4} \text{ V.m}^{-1} \tag{2.39}$$

The charge density at *A* is therefore

$$\sigma = \frac{23.9\varepsilon_0}{a/4} \quad \text{C.m}^{-2} \tag{2.40}$$

from (1.6). If we assume that this is the mean charge density between *R* and *S*, then the total charge on a strip of the outer conductor, of unit length, lying between *R* and *S* is

$$q = \sigma\frac{a}{4} = 23.9\varepsilon_0 \quad \text{C.m}^{-1} \tag{2.41}$$

The charge on each section of the outer conductor can be calculated in this way. The results of the calculations are shown in Fig. 2.11. Summing these charges gives the total charge on the outer conductor: $629\varepsilon_0$ C.m^{-1} for a potential difference of 100 V between the electrodes. The capacitance per unit length is therefore $6.3\varepsilon_0$ F.m^{-1}.

When this figure is compared with the exact value (5.87 ε_0) it is found to be in error by about 7%. This is accurate enough for many purposes but greater accuracy can be obtained, if required, by using a finer mesh.

Example 2.13

Use approximations to the equipotential surfaces and field lines to obtain estimates of the capacitance per unit length of the electrodes shown in fig. 2.10.

Solution

From the finite difference solution to this problem shown in Fig. 1.18 we can plot the equipotentials, field lines and charge distributions as shown in Fig. 2.12.

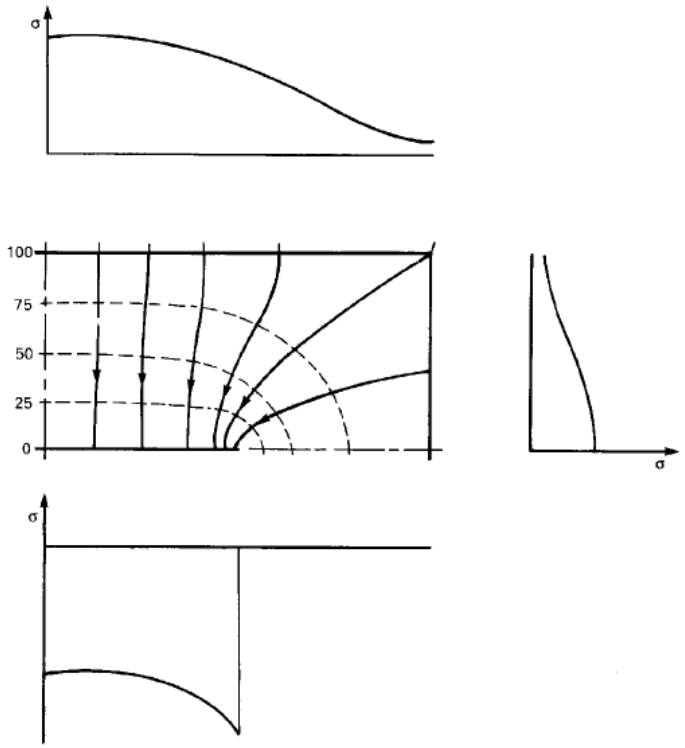

Fig. 2.12. The field map and charge density distributions derived from Fig. 1.18.

If conducting sheets were placed along the equipotentials the field pattern would remain the same. It follows that the whole capacitance can be thought of as being made up of a set of capacitors in series, with each capacitor being bounded by a pair of adjacent equipotentials. This idea can be used to estimate the capacitance from a set of approximate equipotential surfaces. These surfaces must be chosen so that the electrodes form part of the set and match the boundary conditions on the electrodes. Figure 2.13(a) shows one possible way of satisfying this requirement.

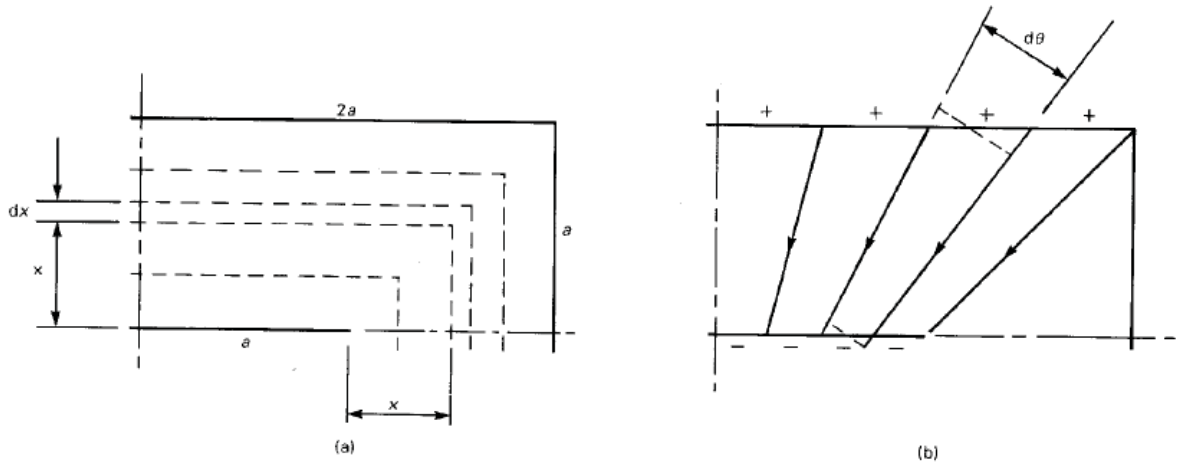

Fig. 2.13. (a) Approximate equipotentials and (b) approximate field lines for the problem

Now consider the elementary capacitor which is formed by the surfaces which are x and $(x + dx)$ from the inner electrode. The perimeter of this capacitor is $(4a + 8x)$ so that its contribution to the reciprocal of the capacitance per unit length between the electrodes is

$$d\left(\frac{1}{C}\right)=\frac{dx}{\varepsilon_0\left(4a+8x\right)} \tag{2.42}$$

When this is integrated from $x = 0$ to $x = a$ the result is

$$\frac{1}{C}=\frac{\ln\left(3\right)}{8\varepsilon_0} \tag{2.43}$$

so that $C = 7.3\ \varepsilon_0$.

Comparison between Figures 2.12 and 2.13(a) shows that the potential gradient close to the outer electrode is greater in the latter case. Thus the total charge associated with the approximate equipotentials is greater than the true value. We therefore conclude that the actual capacitance must be less than the figure above.

In the second approach we imagine the charges and their associated field lines being rearranged as shown in Fig. 2.13(b). Because the charges are crowded more closely together than in their equilibrium state we expect the potential difference between the electrodes to be increased so that the estimate of the capacitance will be lower than the actual value. In this case we can imagine the whole capacitance as being made up of a number of wedge-shaped capacitors in parallel.

Each elementary capacitor is bounded by a pair of field lines and can be thought of as a sector of a coaxial capacitor. The boundaries at the ends of the element are small arcs which do not coincide exactly with the electrode surfaces but, provided that $d\theta$ is small, the error is not very large. The capacitance of the element is readily deduced from (2.33) to be

$$dC = \frac{\varepsilon_0}{\ln\left(r_2/r_1\right)} d\theta \tag{2.44}$$

where r_1 and r_2 are the inner and outer radii of the element, respectively. Since the charge distributions on the electrodes have been assumed to be uniform it follows from the geometry of the problem that $r_2/r_1 = 2$ for every element. Equation (2.40) is then readily integrated from $\theta = 0$ to $\theta = \pi/4$ to find the capacitance of one quadrant. The whole capacitance is therefore

$$C = \frac{\pi\varepsilon_0}{\ln\left(2\right)} = 4.5\,\varepsilon_0 \tag{2.45}$$

If we take the arithmetic mean of the two values of capacitance as the best estimate, we conclude that

$$C = \left(5.9 \pm 1.4\right)\varepsilon_0 \quad \text{F.m}^{-1}$$

This figure is about 0.5% greater than the exact value (5.87 ε_0) and is actually much closer to it than the result of the finite difference calculation. It should not be expected that the mean of the upper and lower bounds calculated in this way will always be as accurate as this. Nevertheless, the method usually gives accuracy which is remarkable considering the crudity of the assumptions made.

Example 2.14

Calculate the capacitance per unit length of the square coaxial arrangement of electrodes shown in fig.2.14:

a) by making use of the results of the finite difference solution obtained in Example 1.13, and
b) by energy methods

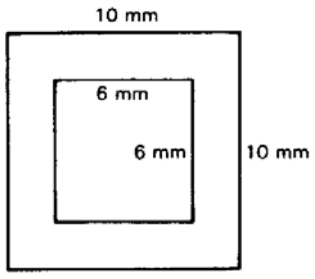

Fig. 2.14. Square coaxial electrodes

Solution (a)

The capacitance between the electrodes is found in exactly the same manner as described in Example 2.12. Details of the calculation of capacitance using the spreadsheet of the finite difference solution are shown in fig. 2.15. An active version of this figure is available for download as an EXCEL file. The potential differences between the mesh points are calculated and stored in a row above the matrix and a column at the right-hand side. These figures are summed (note that the figure corresponding to the central plane must be halved to avoid counting it twice). The sum for the top edge and the right-hand side is 179.0 as would be expected from the symmetry of the problem. Thus the capacitance is

$$C = \frac{8 \times 179}{100} \varepsilon_0 = 14.3 \varepsilon_0 \quad \text{F.m}^{-1} \tag{2.46}$$

The details of the calculation and the potential map can be viewed by double-clicking on the figure.

Solution (b)

Consider an elementary capacitor in the form of a square tube of unit length, side 2*x*, and thickness *dx*.

$$d\left(\frac{1}{C}\right) = \frac{dx}{8\varepsilon_0 x} \tag{2.47}$$

Integrating from $x = 3$ to $x = 5$

$$\frac{1}{C} = \frac{1}{8\varepsilon_0} \ln(10/6) \tag{2.48}$$

By the same argument as before this must be an upper bound. i.e. $C \leq 15.7\varepsilon_0$ F.m^{-1}

Now consider an elementary capacitor in the form of a wedge of angle $d\theta$. As before

$$dC = \frac{\varepsilon_0}{\ln(10/6)} d\theta \tag{2.49}$$

Integrating from 0 to 2π gives the lower bound.

$$C = \frac{2\pi \varepsilon_0}{\ln(10/6)} = 12.3\varepsilon_0 \tag{2.50}$$

Taking the arithmetic mean of the two bounds gives the final value of the capacitance as

$C = 14.0\varepsilon_0 \quad \text{F.m}^{-1} \quad \pm 12\%$

Again we note that this is very close to the finite difference solution.

10.7	10.1	9.3	8.4	7.3	6.2	5.0	3.7	2.5	1.2		SUM = 179.0
10	11	12	13	14	15	16	17	18	19	20	
100	100	100	100	100	100	100	100	100	100	100	Delta V
89.3	89.9	90.7	91.6	92.7	93.8	95.0	96.3	97.5	98.8	100	1.2
78.4	79.6	81.2	83.1	85.3	87.6	90.1	92.5	95.0	97.5	100	2.5
67.2	69.0	71.4	74.4	77.7	81.3	85.0	88.8	92.5	96.3	100	3.7
55.5	57.9	61.1	65.2	69.9	74.9	79.9	85.0	90.1	95.0	100	5.0
43.1	45.9	50.0	55.5	61.8	68.3	74.9	81.3	87.6	93.8	100	6.2
29.8	32.7	37.5	45.0	53.4	61.8	69.9	77.7	85.3	92.7	100	7.3
15.4	17.6	22.1	33.6	45.0	55.5	65.2	74.4	83.1	91.6	100	8.4
0	0	0	22.1	37.5	50.0	61.1	71.4	81.2	90.7	100	9.3
0	0	0	17.6	32.7	45.9	57.9	69.0	79.6	89.9	100	10.1
0	0	0	15.4	29.8	43.1	55.5	67.2	78.4	89.3	100	10.7
0	0	0	14.3	28.1	41.3	53.8	65.8	77.4	88.8	100	11.2
0	0	0	13.6	27.0	40.0	52.6	64.8	76.7	88.4	100	11.6

Fig. 2.15. Calculation of capacitance from the finite difference solution.

Example 2.15

Figure 2.16 shows the distribution of fixed electric charges in the depletion region of a p-n junction in equilibrium. Assuming that the silicon has a permittivity ε, show that the potential difference between the two sides of the junction is given by:

$$V_0 = \frac{q}{2\varepsilon_r}\left(N_D d_n^2 + N_A d_p^2\right)$$

where q is the magnitude of each charge.

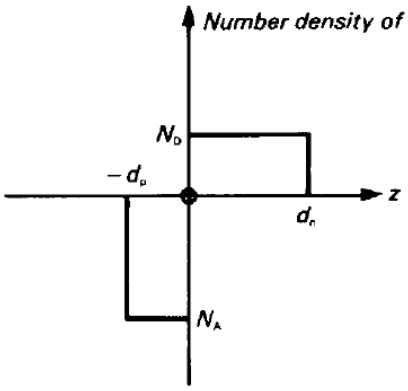

Fig. 2.16. The distribution of ionized donor and acceptor states in the depletion layer of an idealized p-n junction.

Solution

The depletion region is created by the transfer of mobile charges from one side of the junction to the other. Thus the total number of exposed ionised charge is the same on both sides of the junction and

$$N_A d_p = N_D d_n \tag{2.51}$$

Consider first a plane at position z in the n region. The charge per unit area of the junction between this plane and $z = d_n$ is $qN_D\left(d_n - z\right)$ where q is the charge on each ionised atom.. Applying Gauss' Theorem to this charge we find that

$$E_z\left(z\right) = -\frac{q}{\varepsilon} N_D\left(d_n - z\right) \tag{2.52}$$

because the electric field is zero outside the junction. The potential difference across the part of the depletion region in the n-type material is

$$V_n = \frac{qN_D}{\varepsilon}\int_0^{d_n}\left(d_n - z\right)dz = \frac{qN_D d_n^2}{2\varepsilon} \tag{2.53}$$

The potential difference across the part of the depletion region in the p-type material is found in the same way and the contact potential difference is then

$$V_0 = \frac{q}{2\varepsilon}\left(N_D d_n^2 + N_A d_p^2\right) \tag{2.54}$$

as required.

Example 2.16

The formula obtained in Example 2.15 is still valid when a reverse-bias voltage is applied to the junction if V_0 is replaced by the sum of V_0 and the reverse bias V. If a small a.c. signal dV is superimposed on the bias voltage, the transition capacitance of the junction is given by

$$C_T = \frac{dQ}{dV} \tag{2.55}$$

where Q is the total charge on either side of the junction. The net charges on the two sides of the junction are always equal to each other. Find an expression for the capacitance C_T in terms of V, V_0 and the constants of the material.

Solution

The effect of the applied reverse bias voltage is to increase the thickness of the depletion region by exposing more fixed ionised atoms. Since the total charge on each side of the junction must be the same we may substitute for d_p in (2.54) from (2.51) to give

$$V + V_0 = \frac{q}{2\varepsilon}\left(1 + \frac{N_D}{N_A}\right) N_D \left(d_n + x\right)^2 \tag{2.56}$$

where V is the externally applied reverse bias and x is the increase in the thickness of the depletion region on the n-type side of the junction. Now the total charge exposed in the n-type material is

$$Q = qN_D A\left(d_n + x\right) \tag{2.57}$$

where A is the area of the junction. Differentiating (2.56) and (2.57) with respect to x gives

$$\frac{dV}{dx} = \frac{q}{\varepsilon}\left(1 + \frac{N_D}{N_A}\right) N_D \left(d_n + x\right) \tag{2.58}$$

and

$$\frac{dQ}{dx} = qN_D A \tag{2.59}$$

Dividing (2.59) by (2.58) yields the expression for the transition capacitance

$$C_T = \frac{\varepsilon A}{\left(1+\frac{N_D}{N_A}\right)\left(d_n + x\right)} \tag{2.60}$$

But, from (2.56)

$$\left(1+\frac{N_D}{N_A}\right)\left(d_n + x\right) = \sqrt{\frac{V+V_0}{\frac{qN_D}{2\varepsilon}\left(1+\frac{N_D}{N_A}\right)}} \tag{2.61}$$

Substituting in (2.60) gives

$$C_T = \varepsilon A\sqrt{\frac{qN_D}{2\varepsilon}\left(1+\frac{N_D}{N_A}\right)}\frac{1}{\sqrt{V+V_0}} \tag{2.62}$$

We note that in (2.60) the denominator is the total thickness of the depletion layer (W) so that the transition capacitance may be written

$$C_T = \frac{\varepsilon A}{W} \tag{2.63}$$

which is the same as the capacitance of a parallel plate capacitor (2.4) whose thickness is equal to the thickness of the depletion layer.

3. Steady electric currents

3.1 Introduction

This chapter provides examples of the solution of problems involving resistive materials and the calculation of resistance. The examples include both uniform and non-uniform electric current densities and show how localised heating results from concentrations of current density.

3.2 Summary of the methods available

Note: This information is provided here for convenience. The equation numbers in the companion volume *Electromagnetism for Electronic Engineers* are indicated in square brackets.

Symbol	Signifies	Units
μ (*mu*)	Mobility	$m^2.V^{-1}.s^{-1}$
J	Current density	$A.m^{-2}$
σ (*sigma*)	Conductivity	$S.m^{-1}$
ρ (*rho*)	Resistivity	$\Omega.m$
R	Resistance	Ω
$\mathcal{E}$	Electromotive force	V

- Drift velocity of charge carriers in an electric field

$$\mathbf{v}_d = \mu \mathbf{E} \qquad [3.1]$$

- Current density in a current-carrying conductor

$$\mathbf{J} = nqv_d = nq\mu\mathbf{E} = \sigma\mathbf{E} \qquad \text{or} \qquad \mathbf{E} = \rho\mathbf{J} \qquad [3.3]$$

- Ohm's law

$$V = \left(\frac{\rho l}{A}\right) I = RI \qquad [3.6]$$

- Power dissipated in a resistor

$$P = VI = I^2R = V^2/R \qquad [3.7]$$

- Density of power dissipation in a current-carrying conductor

$$\delta P = \mathbf{E} \cdot \mathbf{J}\, \delta v \qquad [3.11]$$

- Principle of conservation of charge

$$\oiint_S \mathbf{J} \cdot \mathbf{dA} = -\frac{\partial}{\partial t} \iiint_V \rho\, dv \qquad [3.13]$$

- The continuity equation (differential form of [3.13])

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad [3.14]$$

- Electromotive force

$$\int_C \mathbf{E} \cdot \mathbf{dl} = \mathcal{E} \qquad [3.17]$$

Example 3.1

A current of 1 A flows in a copper wire l mm in diameter. Given that there are about 10^{29} conduction electrons per cubic metre in copper, calculate their *rms* drift velocity. If the current is the *rms* value of a 50 Hz alternating current, how far do the electrons move along the wire? What is the power dissipated per metre in the wire? ($\rho = 17.7 \times 10^{-9}\ \Omega$.m).

Solution

Using [3.3] with wire of radius $r = 0.5$ mm carrying current I

$$v_d = \frac{J}{nq} = \frac{I}{nq\pi r^2} = 79.5\ \mu\text{m.s}^{-1} \tag{3.1}$$

where I is the *rms* current. The instantaneous drift velocity is

$$v_d(t) = \frac{\sqrt{2}I\sin(\omega t)}{nq\pi r^2} \tag{3.2}$$

so the distance travelled by an electron in half a cycle is

$$d = \frac{\sqrt{2}I}{nq\pi r^2}\frac{1}{\omega}\int_0^{\pi}\sin\theta\, d\theta = \frac{\sqrt{2}I}{nq\pi r^2}\frac{2}{\omega} = 0.716\ \mu\text{m} \tag{3.3}$$

where $\theta = \omega t$ and $\omega = 100\pi\ s^{-1}$.

Using [3.6] the resistance per metre of the wire is

$$R = \frac{\rho}{\pi r^2} = 0.023\quad \Omega.\text{m}^{-1} \tag{3.4}$$

and from [3.7] the power dissipated per metre is

$$P = I^2R = 23\ \text{mW.m}^{-1} \tag{3.5}$$

Example 3.2

A coaxial cable has insulation made of a slightly conducting material of resistivity ρ as shown in fig. 3.1. Given that the inner and outer radii of the insulation are a and b, respectively, find an expression for the leakage resistance per unit length between the inner and outer conductors.

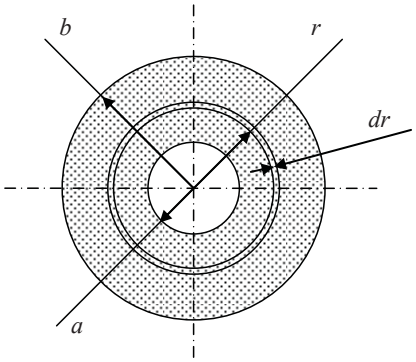

Fig. 3.1. Arrangement of a coaxial cable.

Solution

From the symmetry of the cable the flow of the leakage current must be radial. Consider a thin tube of radius r and thickness dr as shown. The resistance of unit length of the tube to a radial current is, from [3.6], the product of the resistivity and the thickness of the tube in the direction of the current flow divided by the area of the tube normal to the current flow

$$dR = \frac{\rho}{2\pi r} dr \tag{3.6}$$

The total leakage resistance per unit length is found by adding the resistances of all such elementary tubes connected in series

$$R = \frac{\rho}{2\pi}\int_a^b \frac{1}{r}\,dr = \frac{\rho}{2\pi}\ln\left(b/a\right) \tag{3.7}$$

Example 3.3

A diffused resistor in an integrated circuit is made by diffusing a thin layer of p-type impurity into an n-type isolation island. Assuming that the conductivity of the p-type layer varies linearly from σ_0 at the surface to zero at the interface with the n-type layer, and that the interface is d below the surface, find an expression for the resistance between opposite edges of a square section of p-type layer whose transverse dimensions are much greater than its thickness.

Solution

Consider an elementary sheet of the p-type layer of thickness dz lying z below the surface, as shown in fig. 3.2.

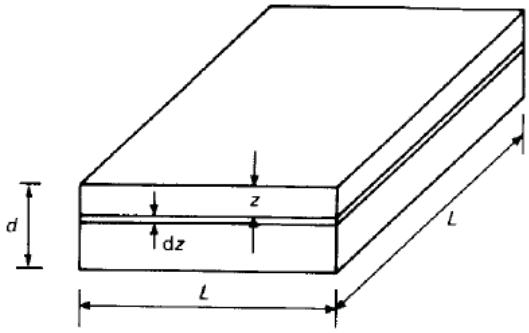

Fig. 3.2 Calculation of the sheet resistance of an integrated-circuit diffused resistor.

Because the current flow is along the length of the sheet we shall want to add up the resistances of all such sheets in parallel and it is best to use the conductance $G = 1/R$ in the analysis. The conductivity of the sheet as a function of z is given by

$$\sigma\left(z\right) = \sigma_0\left(1 - z/d\right) \tag{3.8}$$

so the conductance of the sheet is, from the inverse of [3.6] the product of the conductivity and the cross-sectional area normal to the current flow divided by the length in the direction of the current flow

$$dG = \sigma_0\left(1 - z/d\right)\left(\frac{L\,dz}{L}\right) = \sigma_0\left(1 - z/d\right)dz \tag{3.9}$$

The conductance of the whole layer is thus

$$G = \sigma_0 \int_0^d (1 - z/d)\, dz = \sigma_0 d/2 \tag{3.10}$$

so the resistance in ohms is

$$R = \frac{1}{G} = \frac{2}{\sigma_0 d} \tag{3.11}$$

Note that the resistance is independent of the size of the square chosen. This is a general result applying to any sheet of resistive material. For this reason the resistances of resistive sheets are usually quoted in 'Ohms per square` (ohms/□). In this example we have, in effect, assumed unit potential difference across the square and calculated the sum of the currents flowing in the elementary sheets.

Example 3.4

Figure 3.3 shows the arrangement of a pinch resistor in an integrated circuit. If the p regions (shown shaded) are earthed and the n region is always positive, the two p-n junctions are reverse biased. A layer of the n region adjacent to each junction is depleted of conduction electrons and is, effectively, an insulator. The thickness of the depletion layer is given by

$$t = a\sqrt{\frac{V + V_{GS}}{V_p}} \tag{3.12}$$

where V_p is a constant and V is the local potential in the channel referred to S. Given that the n channel has width w (into the page) and its other dimensions are as shown in Fig. 3.3 find expressions for the current in the channel:

a) when the current through the resistor is small, so that the voltage V_{DS} is much less than V_{GS}, and
b) when the current through the resistor is not small.

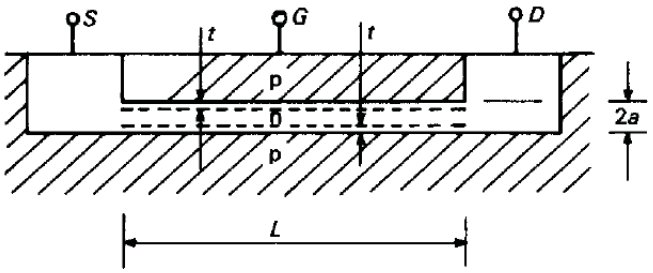

Fig. 3.3 Arrangement of a pinch resistor for an integrated circuit.

Solution (a)

If $V_{DS} << V_{GS}$ then, to a first approximation $V = 0$ and the channel thickness is constant along its length and equal to $2(a - t)$. The conductance of the channel, using the inverse of [3.6], is

$$G = \frac{2\sigma_n w(a-t)}{L} = \frac{2\sigma_n wa}{L}\left(1 - \sqrt{\frac{V_{GS}}{V_p}}\right) \tag{3.13}$$

where σ_n is the conductivity of the n-type material. The current in the channel is then

$$I_D = \frac{2\sigma_n wa}{L}\left(1 - \sqrt{\frac{V_{GS}}{V_p}}\right)V_{DS} \tag{3.14}$$

If $V_{GS} = V_p$ then $t = a$, the thickness of the channel is zero and the current falls to zero. The resistor is then said to be *pinched off*.

Solution (b)

When V_{DS} is not small then the thickness of the channel is not constant but the continuity equation requires that the current is constant everywhere. Consider a small length dx of the channel which is distant x from S. Its contribution to the resistance of the channel is

$$dR = \frac{dx}{2\sigma_n w(a-t)} = \frac{1}{2\sigma_n wa}\left(1-\sqrt{\frac{V+V_{GS}}{V_P}}\right)^{-1} dx \tag{3.15}$$

The potential difference across the element is

$$dV = I_D\, dR = \frac{I_D}{2\sigma_n wa}\left(1-\sqrt{\frac{V+V_{GS}}{V_P}}\right)^{-1} dx \tag{3.16}$$

Rearranging (3.16) and integrating along the length of the channel we have

$$\frac{I_D}{2\sigma_n wa}\int_0^L dx = \int_0^{V_{DS}}\left(1-\sqrt{\frac{V+V_{GS}}{V_P}}\right) dV \tag{3.17}$$

Performing the integrations we obtain the required expression for the current

$$I_D = \frac{2\sigma_n wa}{L}\left\{ V_{DS} + \frac{2}{3}\sqrt{\frac{V_{GS}^3}{V_P}}\left[1 - \left(1+\frac{V_{DS}}{V_{GS}}\right)^{\frac{3}{2}}\right]\right\} \tag{3.18}$$

When $V_{DS} << V_{GS}$ it can be shown that (3.18) reduces to (3.14) by expanding the term in $\left(1+V_{DS}/V_{GS}\right)$ using the binomial theorem and ignoring powers of (V_{DS}/V_{GS}) greater than one.

Example 3.5

A burglar alarm system works by detecting the change in the resistance of a thin conducting film on a window when the window is broken. A window is typically 1 m square, and contact with the film is made by conducting strips at the top and bottom of the window. Estimate the percentage change in the resistance which would have to be detected by the electronic circuits if a burglar cut a hole just large enough to pass an arm through. What difference does it make where the hole it cut?

Solution

When the window is broken the current flux lines must deviate to pass around the hole. If the total current flowing across the window is held constant then the power dissipated in the resistance of the window increases and therefore the resistance of the window increases. Since we wish to know the minimum change of resistance which must be detected it is sufficient to find a lower bound to the resistance of the broken window. It is shown in [3.20] that this is achieved by employing approximate equipotentials.

Let us suppose that the hole made by the burglar is 100 mm square. Now divide the whole window into equal squares this size, as shown in fig. 3.4, and let the resistance of each square be R. The resistance of one horizontal strip of such squares is that of ten squares in parallel, i.e. $0.1R$. The resistance of the whole window before it is broken is the sum of ten such strips in series, i.e. R. Note that this result is expected because the resistance of a square of conducting sheet is independent of its size.

When the window is broken let us assume that the equipotentials remain as equally spaced horizontal straight lines. The resistance of the window is then just the sum of the resistances between the equipotentials. When one of the strips has a hole in it the resistance of the strip is now that of 9 squares in parallel, i.e. $R/9 = 0.11R$ and the resistance of the window is $9 \times 0.1R + 0.11R = 1.01R$ which is a lower bound.

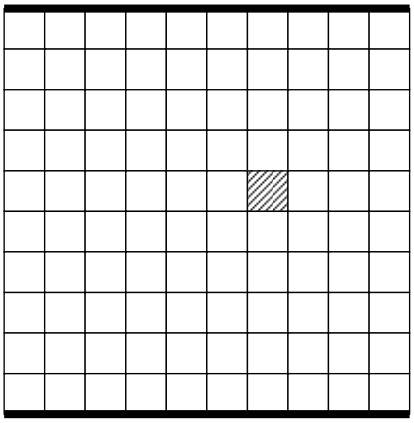

Fig. 3.4 Division of the window into equal squares.

Thus, if the electronic circuits can detect a change in the resistance of the window of at least +1%, they will certainly be able to detect the hole made by the burglar. At this level of approximation it does not matter where in the window the hole is made. A higher value of the lower bound could be found by using a more complex approximation to the equipotentials but it is unlikely that the improved estimate would justify the effort involved.

Example 3.6

A right-angled bend is formed in a conducting bar of square section. Estimate the length of straight bar which would have the same resistance as the corner.

Solution

Let the width and thickness of the bar be a and its resistivity be ρ. Then the resistance of unit length of a straight section of the bar is

$$R = \frac{\rho}{a^2} \tag{3.19}$$

To find an upper bound for the resistance of the bend we assume that the current flux lines are as shown by the broken lines in fig. 3.5.

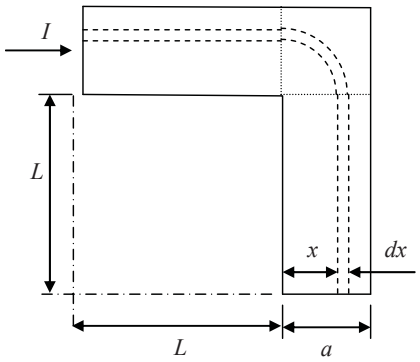

Fig. 3.5 Approximate current flow lines.

Well away from the corner we expect the current density to be uniform. Now consider an elementary strip of the conductor bounded by the dashed lines. The resistance of such a strip is

$$dR = \frac{\rho(2L + \pi x/2)}{a\,dx} \tag{3.20}$$

The current in the strip is

$$dI = \frac{dx}{a} I \tag{3.21}$$

The power dissipated in the strip is

$$dW' = dI^2 dR = \frac{I^2}{a^2}\left[\frac{\rho(2L+\pi x/2)}{a}\right]dx \tag{3.22}$$

The power dissipated in the whole bar is

$$W' = \frac{\rho I^2}{a^3}\int_0^a \left(2L+\frac{\pi}{2}x\right)dx = \frac{\rho I^2}{a^2}\left(2L+\frac{\pi a}{4}\right) \tag{3.23}$$

Now the power calculated using approximate current flux lines must always be greater than or equal to the actual power dissipated $W = I^2 R$ where R is the resistance of the bar. Thus

$$R \le \frac{W'}{I^2} = \frac{\rho}{a^2}\left(2L+\frac{\pi a}{4}\right) \tag{3.24}$$

To find the upper bound of the resistance of the corner we subtract the resistance of the straight sections of bar

$$R_c \le \frac{\rho}{a^2}\left(2L+\frac{\pi a}{4}\right)-\frac{2L\rho}{a^2} = \frac{\pi\rho}{4a} = 0.785\frac{\rho}{a} \tag{3.25}$$

We note that this is independent of the choice of L as might be expected from an examination of fig. 3.5.

To seek for a lower bound to the resistance of the corner we employ approximate equipotential surfaces as shown in fig. 3.6. Far away from the corner these surfaces will be at right angles to the sides of the bar.

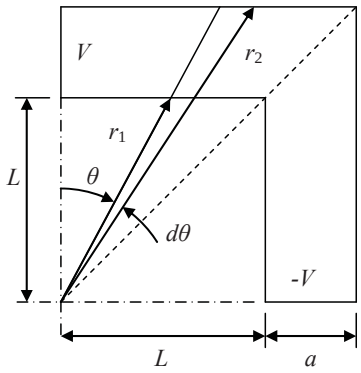

Fig. 3.6 Approximate equipotential surfaces

Consider an element such as the one shown which can be treated as a section of a circular current flow. The resistance contributed by a sub-element of thickness *dr* at radius *r* is

$$dR = \frac{\rho\, r\, d\theta}{a\, dr} \tag{3.26}$$

If the potentials at the ends of the bar are $\pm V$ then the potential difference across the element is $\left(4V d\theta/\pi\right)$ so that the power dissipated in the whole element is

$$dW'' = \left(\frac{4V d\theta}{\pi}\right)^2 \int_{r_1}^{r_2} \frac{a}{\rho\, r\, d\theta}\, dr = \frac{16V^2 a\, d\theta}{\pi^2 \rho} \ln\left(r_2/r_1\right) \tag{3.27}$$

From the geometry of the figure we observe that

$$\frac{r_2}{r_1} = \frac{L+a}{L} \tag{3.28}$$

for all elements. The angle θ ranges from 0 to $\pi/2$ so the resistance of the whole bar is

$$W'' = \frac{16V^2 a}{\pi^2 \rho} \ln\left(1+a/L\right) \int_0^{\frac{\pi}{2}} d\theta = \frac{8V^2 a}{\pi \rho} \ln\left(1+a/L\right) \tag{3.29}$$

This power must be greater than or equal to the actual power dissipated in the bar given by $W = \left(2V\right)^2 / R$ so that

$$R \geq \frac{4V^2}{W''} = \frac{\pi \rho}{2a \ln\left(1+a/L\right)} \tag{3.30}$$

Subtracting the resistance of the straight sections of bar we find that the resistance of the corner is

$$R_c \geq \frac{\pi \rho}{2a \ln\left(1+a/L\right)} - \frac{2\rho L}{a^2} = \frac{\rho}{a}\left(\frac{\pi}{2\ln\left(1+a/L\right)} - \frac{2L}{a}\right) \tag{3.31}$$

This function does depend on *L* so we wish to find the value of *L* which maximises it. The best way to do this is to plot the function using a spreadsheet. It is then found that the function has a maximum value $R_c = 0.477\left(\rho/a\right)$ when $a/L = 4.56$ and this must be a lower bound for *R*.

Finally we take the geometric mean of the upper and lower bounds as our best estimate of the resistance of the corner so that the length of straight bar equivalent to the corner is $0.61a \pm 28\%$.

Example 3.7

Find the resistance between AA and BB of the conducting strip shown in Fig. 3.7.

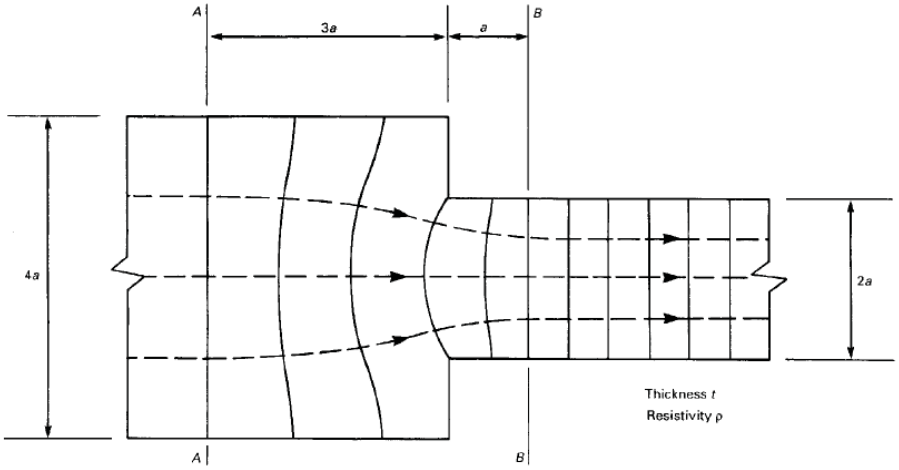

Fig. 3.7 Electric field lines and equipotential surfaces for a current flowing through a conducting strip with a step change in width

Solution

The solution can be found using the finite difference method because when a steady current flows through a uniform resistive medium, the electrostatic potential satisfies Laplace's equation. Figure 3.8 shows the implementation of the finite difference method using a spreadsheet.

One half of the problem has been modelled and the step size in each direction has been chosen to be $a/5$. The red and blue regions are specified as fixed potentials of 100V and 0V respectively and the white region uses the standard finite difference formula. In the green regions it is assumed that the equipotential lines are normal to the boundary and that the potentials at points which are mirror images of each other in the boundary are equal. This procedure forces the current flow to be tangential to the boundary.

An active version of this figure is available for download as an EXCEL file. Tabs showing the calculations and the potential map can be selected. Note that the potential map is upside down compared with the calculations and that the bottom right-hand corner of the map lies outside the conducting area and should be ignored.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Delta V
0	100	96.4	92.7	89.1	85.5	82.0	78.5	75.1	71.8	68.6	65.6	62.9	60.6	58.8	57.6	57.2						3.64
1	100	96.4	92.7	89.1	85.5	82.0	78.5	75.0	71.7	68.5	65.5	62.7	60.4	58.5	57.2	56.8						3.64
2	100	96.3	92.7	89.1	85.5	81.9	78.3	74.9	71.5	68.2	65.1	62.2	59.6	57.5	56.1	55.5						3.65
3	100	96.3	92.7	89.0	85.4	81.8	78.2	74.6	71.1	67.7	64.5	61.3	58.4	55.9	54.0	53.1						3.67
4	100	96.3	92.6	88.9	85.3	81.6	78.0	74.3	70.7	67.2	63.7	60.2	56.9	53.7	50.8	48.9						3.69
5	100	96.3	92.6	88.9	85.2	81.4	77.7	74.0	70.3	66.6	62.8	59.0	55.2	51.1	46.7	41.0	30.8	22.5	14.8	7.4	0	3.71
6	100	96.3	92.5	88.8	85.0	81.3	77.5	73.7	69.8	66.0	62.0	57.9	53.6	49.0	43.8	37.6	29.9	22.2	14.7	7.3	0	3.73
7	100	96.2	92.5	88.7	84.9	81.1	77.3	73.4	69.5	65.4	61.3	57.0	52.4	47.4	42.0	35.8	28.9	21.7	14.5	7.2	0	3.76
8	100	96.2	92.5	88.7	84.8	81.0	77.1	73.2	69.2	65.0	60.7	56.3	51.5	46.4	40.8	34.7	28.2	21.3	14.2	7.1	0	3.77
9	100	96.2	92.4	88.6	84.8	80.9	77.0	73.0	69.0	64.8	60.4	55.8	51.0	45.8	40.2	34.2	27.7	21.0	14.1	7.1	0	3.78
10	100	96.2	92.4	88.6	84.8	80.9	77.0	73.0	68.9	64.7	60.3	55.7	50.8	45.6	40.0	34.0	27.6	20.9	14.0	7.0	0	3.79
																				SUM		37.11

Fig. 3.8 Finite difference solution for the problem shown in Fig. 3.7.

To find the resistance between the ends of the strip we note that each mesh square has resistance $R = \rho/t$. We assume that the current through each square can be calculated from the mean potential difference across it. The best potentials to use for this purpose are those along the left-hand edge of the mesh because the field is most uniform there. The current through each square is obtained using Ohm's law and the currents are summed across the strip. The result for the half problem is

$$I = 37.1\frac{t}{\rho} \tag{3.32}$$

The total current is found by doubling this figure. The resistance of the strip is then the ratio of the potential difference (100 V) to the current:

$$R = \frac{100}{2\times 37.1}\frac{\rho}{t} = 1.35\frac{\rho}{t} \tag{3.33}$$

Example 3.8

Use energy methods to obtain an approximate value for the resistance of the conducting strip shown in Fig. 3.7.

Solution

To obtain the upper bound for R we must choose approximate current flux lines. Figure 3.9 shows one possibility, no current flows at all in the shaded area of the diagram and the total current is assumed to be uniformly distributed across any plane perpendicular to the axis. It is convenient to divide the problem into two parts as shown and to calculate the resistances R_1 and R_2 separately.

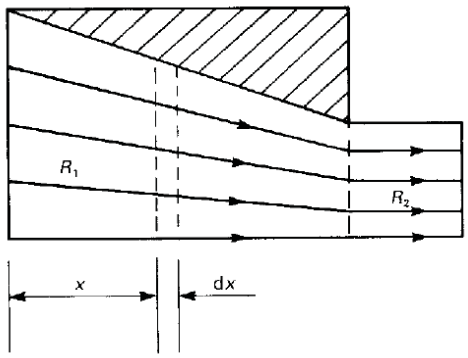

Fig. 3.9 Approximate current flow lines to find an upper bound for the resistance of the strip shown in Fig. 3.7.

The total width of the strip dx is found from the geometry of the problem to be

$$w = 4a - \frac{2}{3}x \tag{3.34}$$

If the total current is I, then the power dissipated the strip is the product of the resistance of the strip and the square of the current.

$$dW' = I^2 \frac{3\rho\, dx}{2t(6a - x)} \tag{3.35}$$

Integrating from $x = 0$ to $x = 3\,a$ gives the total power dissipated in the left-hand part of the strip

$$W' = I^2 \frac{3\rho}{2t} \int_0^{3a} \frac{dx}{(6a - x)} = I^2 \frac{3\rho}{2t} \ln 2 \tag{3.36}$$

so that

$$R_1 \leq \frac{W'}{I^2} = \frac{3\rho}{2t}\ln 2 = 1.04\frac{\rho}{t} \tag{3.37}$$

The resistance of the right-hand part of the strip is found by elementary methods to be

$$R_2 = 0.5\frac{\rho}{t} \tag{3.38}$$

Thus the upper bound of the resistance of the strip is $1.54\,\rho/t$.

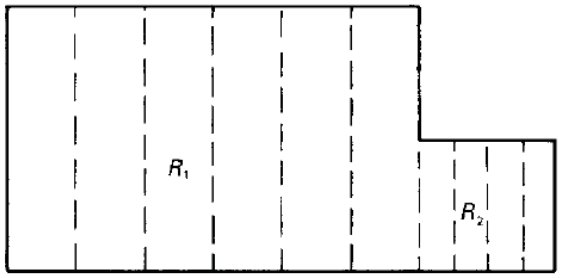

Fig. 3.10 Approximate equipontential surfaces to find a lower bound for the resistance of the strip shown in Fig. 3.7.

To obtain the lower bound to R we assume that the equipotential surfaces are planes perpendicular to the axis, as shown in Fig. 3.10. The resistances R_1 and R_2 can be calculated by elementary methods:

$$R_1 = 0.75\frac{\rho}{t} \tag{3.39}$$

and

$$R_2 = 0.5\frac{\rho}{t} \tag{3.40}$$

giving the lower bound

$$R \geq 1.25\frac{\rho}{t} \tag{3.41}$$

We note that the result found in Example 3.7 does indeed lie between these two bounds. The resistance of the strip is taken to be the geometric mean of the two bounds so that the best estimate of R using this method is

$$R = 1.4\frac{\rho}{t} \pm 11\% \tag{3.42}$$

which is very close to the result obtained with greater effort by the finite difference method.

4. The magnetic effects of electric currents

4.1 Introduction

This chapter illustrates the methods of calculating the magnetic flux density produced by given arrangements of current-carrying conductors. The methods are appropriate for all problems in which no ferromagnetic materials are present. For practical purposes all other materials can be regarded as free space.

4.2 Summary of the methods available

Note: This information is provided here for convenience. The equation numbers in the companion volume *Electromagnetism for Electronic Engineers* are indicated in square brackets.

Symbol	Signifies	Units
μ_0 (*mu*)	Primary magnetic constant	$H.m^{-1}$
B	Magnetic flux density	T, $Wb.m^{-2}$
υ	Magnetic scalar potential	A

- The law of force between moving charges

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} + \frac{\mu_0 Q_1 Q_2}{4\pi r^2}\left(\mathbf{v_2} \wedge \left(\mathbf{v_1} \wedge \hat{\mathbf{r}}\right)\right) = \mathbf{F_e} + \mathbf{F_m} \qquad [4.1]$$

- Magnetic flux density due to a moving charge

$$\mathbf{B} = \frac{\mu_0 Q_1}{4\pi r^2}\left(\mathbf{v_1} \wedge \hat{\mathbf{r}}\right) \qquad [4.3]$$

- Magnetic force on a moving charge

$$\mathbf{F_m} = Q_2\left(\mathbf{v_2} \wedge \mathbf{B}\right) \qquad [4.4]$$

- Magnetic flux density due to a current element

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi r^2}\left(\mathbf{dl} \wedge \hat{\mathbf{r}}\right) \qquad [4.6]$$

- The Biot-Savart Law

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint \frac{\mathbf{dl} \wedge \hat{\mathbf{r}}}{r^2} \qquad [4.7]$$

- The magnetic circuit law in free space

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 \iint \mathbf{J} \cdot \mathbf{dA} \qquad [4.12]$$

- Magnetic scalar potential

$$\upsilon_Q - \upsilon_P = -\frac{1}{\mu_0}\int_P^Q \mathbf{B} \cdot \mathbf{dl} \qquad [4.13]$$

$$\mathbf{B} = -\mu_0 \operatorname{grad} \upsilon = -\mu_0 \nabla \upsilon \qquad [4.14]$$

$$\operatorname{div}\left(\operatorname{grad} \upsilon\right) = \nabla^2 \upsilon = 0 \qquad [4.17]$$

- The integral form of Gauss' theorem

$$\oiint \mathbf{B} \cdot \mathbf{dA} = 0 \qquad [4.15]$$

- The differential form of Gauss' theorem

$$\text{div}\mathbf{B} = \nabla \cdot \mathbf{B} = 0 \qquad [4.16]$$

- Force on a current element

$$\mathbf{F} = I\left(\mathbf{dl} \wedge \mathbf{B}\right) \qquad [4.18]$$

Example 4.1

Figure 4.1 shows a single circular loop of wire carrying current *I*. Find an expression for the magnetic flux density at the point *P* on the axis of the loop.

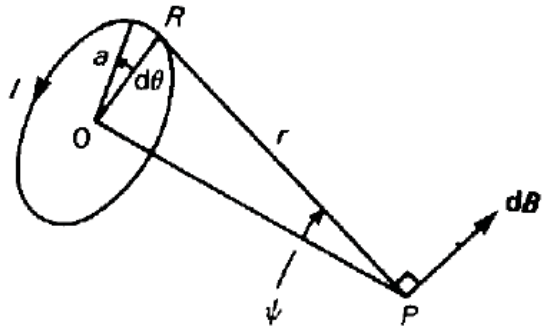

Fig. 4.1. Calculation of the magnetic flux density on the axis of a circular current loop.

Solution

Consider the flux density at *P* due to the current element at *R*. The current element has length $a\,d\theta$ and its distance from *P* along the line *RP* is *r*. From the triangle *ORP*,

$$r = a/\sin\psi \qquad (4.1)$$

The flux density produced by the current element is represented by the vector **dB**, which must be normal to both *RP* and the current element according to the rule for vector products. The vector **dB**, therefore, lies in the plane *ORP* at right angles to *RP*, as shown in Fig. 4.1. Because *RP* is at right angles to the current element it follows from equation [4.6] that

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi r^2} a\,d\theta = \frac{\mu_0 I}{4\pi a}\sin^2\psi\; d\theta \qquad (4.2)$$

This can be resolved into its radial and axial components

$$dB_z = dB\sin\psi \quad \text{and} \quad dB_r = dB\cos\psi \qquad (4.3)$$

Integrating around the loop we find that the total *z* component is

$$B_Z = \frac{\mu_0 I}{2a}\sin^3\psi \tag{4.4}$$

while $B_r = 0$, as would be expected from symmetry.

This method can be extended to a very wide range of problems involving currents in wires. Some examples of its application are given in the examples in this the chapter. The solution of problems involving several current-carrying conductors in free space can be made easier by using the principle of superposition. This is possible because the flux density is proportional to the current producing it, as can be seen from Equation [4.7].

Real coils usually have many turns of wire and one should, in theory, integrate over the cross-sectional area of the winding. But, if the cross-section of the winding is small compared with the diameter of the coil, it is possible to make an approximation. The real coil is replaced by an equivalent current loop having a diameter equal to the mean diameter of the coil. The error in the calculated values of the flux density on the axis is of the order of a few per cent when this approximation is made. The calculation of the off-axis flux density is trickier and involves elliptic integrals.

Example 4.2

A pair of circular coils have the same axis and radius a. They are separated by a distance $2d$. Find the value of d which makes the magnetic flux density on the axis mid-way between the coils as uniform as possible.

Solution

The field of the pair of coils is found from (4.4) using the principle of superposition. If the currents are in the same direction in both coils then the fields are added together. The field variation along the axis close to the mid-point can be expressed as a power series in z. From the symmetry of the problem we note that this series can only contain even powers of z.

The problem as stated has only one variable, namely the ratio d/a. In order to make the field as uniform as possible we choose the value of this ratio so that the term in z^2 is zero. The field expansion then contains only a constant term and terms in even powers of z from the fourth power upwards. This choice is equivalent to requiring that the second derivative of the field of each coil with respect to z vanishes at the mid-point.

Differentiating (4.4) with respect to z

$$\frac{dB_Z}{dz} = \frac{\mu_0 I}{2a}\frac{d}{d\psi}\left(\sin^3\psi\right)\left(\frac{d\psi}{dz}\right) \tag{4.5}$$

Now

$$a = z\tan\psi \tag{4.6}$$

Differentiating (4.6) with respect to ψ gives

$$0 = \frac{dz}{d\psi}\tan\psi + z\frac{1}{\cos^2\psi} \tag{4.7}$$

Substituting for z from (4.6) in (4.7) and rearranging we obtain

$$\frac{dz}{d\psi} = -\frac{a}{\sin^2\psi} \tag{4.8}$$

And, substituting for $\left(d\psi/dz\right)$ in (4.5) gives

$$\frac{dB_Z}{dz} = -\frac{\mu_0 I}{2a^2}3\sin^4\psi\cos\psi \tag{4.9}$$

Differentiating a second time with respect to z

$$\frac{d^2B_Z}{dz^2} = -\frac{3\mu_0 I}{2a^2}\frac{d}{d\psi}\left(\sin^4\psi\cos\psi\right)\left(\frac{d\psi}{dz}\right) = \frac{3\mu_0 I}{2a^2}\left(4\sin^3\psi\cos^2\psi - \sin^5\psi\right)\left(\frac{\sin^2\psi}{a}\right) \tag{4.10}$$

which is zero when

$$\tan^2\psi = 4 \tag{4.11}$$

or

$$\frac{a}{d} = 2 \tag{4.12}$$

Therefore the field midway between the coils is as uniform as possible when the distance between them is equal to the radius of each coil. Coils arranged in this way are known as Helmholtz coils. They are sometimes used to provide uniform magnetic fields for scientific experiments.

Example 4.3

Find an expression for the magnetic flux density at the centre of a loop of wire in the shape of a regular polygon with $2N$ sides. Check your answer by showing that it tends to the expression for the flux density at the centre of a circular loop when $N \to \infty$.

Solution

The magnetic flux density at a point distant R from a length of straight wire carrying current I is

$$B = \frac{\mu_0 I \sin\alpha}{2\pi R} \tag{4.9}$$

where the wire is perpendicular to the line joining it to the point and it subtends an angle 2α there. The polygon is made up of $2N$ such pieces of wire so

$$\alpha = \frac{\pi}{2N} \tag{4.13}$$

Substituting for α in [4.9] and multiplying by the number of sides

$$B = \frac{\mu_0 I \sin\left(\pi/2N\right)}{2\pi R}\cdot 2N = \frac{\mu_0 I}{2R}\cdot\frac{\sin\left(\pi/2N\right)}{\pi/2N} \tag{4.14}$$

when $N \to \infty$, $\sin(\pi/2N) \to \pi/2N$ and

$$B \to \frac{\mu_0 I}{2R} \tag{4.15}$$

which is the flux density at the centre of a circular loop of radius *R* as given by (4.4) when $\psi = \pi/2$.

Example 4.4

Figure 4.2 shows the arrangement of a short solenoid. Find expressions for the magnetic flux density
a) at the point *P*,
b) at the centre of a solenoid whose diameter is equal to its length, and
c) at the centre of a solenoid whose diameter is small compared with its length.

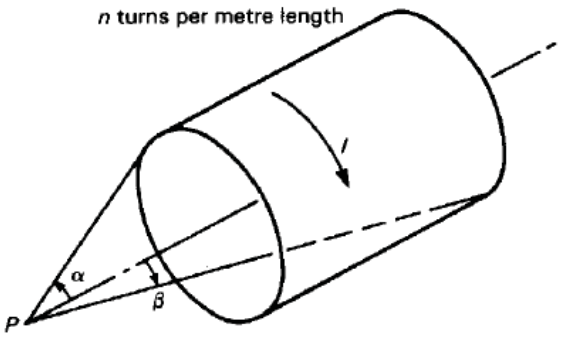

Fig. 4.2. A short solenoid.

Solution

The solutions are obtained from (4.4) using the principle of superposition. Consider an element of the solenoid of length *dz* at a distance z from *P*. The current flowing in the element is $nI\,dz$ so its contribution to the flux density at *P* is

$$dB_Z = \frac{\mu_0 nI}{2a}\sin^3\psi \; dz \tag{4.16}$$

Substituting for *dz* using (4.8)

$$dB_Z = \frac{\mu_0 nI}{2a}\sin^3\psi\left(\frac{-a}{\sin^2\psi}\right)d\psi = -\frac{\mu_0 nI}{2}\sin\psi \; d\psi \tag{4.17}$$

a) The flux density at *P* is found by integrating (4.17) with respect to ψ over the length of the solenoid

$$B_Z = \int_{\alpha}^{\beta} -\frac{\mu_0 nI}{2}\sin\psi \; d\psi = \frac{\mu_0 nI}{2}(\cos\beta - \cos\alpha) \tag{4.18}$$

b) At the centre of a solenoid of length $2a$ $\beta = \pi/4$ and $\alpha = 3\pi/4$. Substituting these angles into (4.18)

$$B_Z = \frac{\mu_0 nI}{\sqrt{2}} \tag{4.19}$$

c) At the centre of a solenoid whose diameter is small compared with its length $\alpha \to \pi$ and $\beta \to 0$. Substituting these angles in (4.18)

$$B_Z = \mu_0 nI \tag{4.20}$$

Example 4.5

A square coil of side a carries a current I as shown in fig. 4.3. Show that the magnetic flux density at a point on the axis of the coil and distance z from its plane is given by

$$B_z = \frac{2\sqrt{2}\mu_0 I}{\pi} \frac{a^2}{\left(a^2 + 4z^2\right)\sqrt{a^2 + 2z^2}}$$

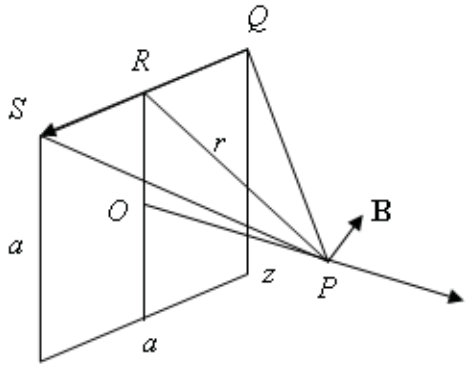

Fig. 4.3. A square coil

Solution

The field at P due to the current in QS is

$$B = \frac{\mu_0 I \sin\alpha}{2\pi r} \qquad [4.9]$$

where r is the perpendicular distance between the wire QS and the point P. From the geometry of the problem

$$r = \sqrt{\left(\tfrac{1}{2}a\right)^2 + z^2} = \tfrac{1}{2}\sqrt{a^2 + 4z^2} \qquad (4.21)$$

and α is the angle RPQ so that

$$\sin\alpha = \frac{RQ}{QP} = \frac{\tfrac{1}{2}a}{\sqrt{\left(\tfrac{1}{2}a\right)^2 + r^2}} = \frac{a}{\sqrt{2a^2 + 4z^2}} \qquad (4.22)$$

So that, substituting for sin α in [4.9],

$$B = \frac{\mu_0 I}{2\pi r} \frac{a}{\sqrt{2a^2 + 4z^2}} \qquad (4.23)$$

The z components of the fields of the four sides of the coil add together whilst the components normal to the z axis cancel each other out. Thus the field at P is

$$B_z = 4B\sin\left(\angle OPR\right) = B\frac{2a}{r} \tag{4.24}$$

Substituting for B and r in (4.24) gives

$$B_z = \frac{\mu_0 I}{2\pi r}\frac{2a}{r}\frac{a}{\sqrt{2a^2+4z^2}} = \frac{2\sqrt{2}\mu_0 I}{\pi}\frac{a^2}{\left(a^2+4z^2\right)\sqrt{a^2+2z^2}} \tag{4.25}$$

as required. This result can be checked by setting $z = 0$ and comparing the result with that obtained from (4.14) when $N = 2$ and $R = a/2$.

Example 4.6

Figure 4.4 shows the arrangement of a set of saddle coils. Find an expression for the magnitude and direction of the magnetic flux density at the point P.

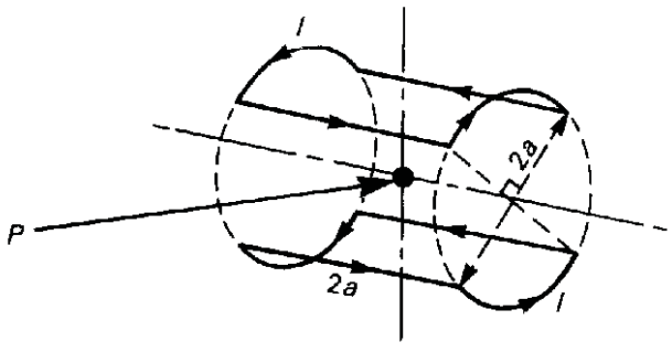

Fig. 4.4. The arrangement of a pair of saddle coils.

Solution

The direction of the magnetic flux due to a current in a wire is given by the right hand corkscrew rule. If the thumb of the right hand points in the direction of the current then the curled fingers point in the direction of the flux. From the symmetry of the problem the flux density at P must therefore be directed vertically upwards. The flux density at P is found by adding together the flux densities due to the individual wires.

The flux density due to one of the straight wires is given by [4.9] in which the perpendicular distance between P and the wire $R = a$ and the angle $\alpha = \pi/4$. The angle between **B** and the vertical is also $\pi/4$ so that the contribution to the total flux density is

$$B_{\perp} = \frac{\mu_0 I \sin(\pi/4)\cos(\pi/4)}{2\pi a} = \frac{\mu_0 I}{4\pi a} \tag{4.26}$$

The radial component of the flux density due to an element one of the curved wires is given by (4.2) and (4.3)

$$dB_r = \frac{\mu_0 I}{4\pi a}\sin^2\psi\cos\psi\, d\theta \tag{4.27}$$

where $\psi = \pi/4$. Resolving this in the vertical direction and integrating over the length of the wire

$$dB_{\perp} = \frac{\mu_0 I}{4\pi a}\cdot\frac{1}{2\sqrt{2}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi a}\cdot\frac{1}{2\sqrt{2}}\cdot\frac{2}{\sqrt{2}} = \frac{\mu_0 I}{8\pi a} \tag{4.28}$$

Finally we sum the contributions to the flux density from all the wires to give

$$B_{\perp} = 4\left(\frac{\mu_0 I}{4\pi a} + \frac{\mu_0 I}{8\pi a}\right) = \frac{3\mu_0 I}{2\pi a} \tag{4.29}$$

Example 4.7

Figure 4.5 shows a non-magnetic ring having inner radius R_1 and outer radius R_2 on which is wound a uniform coil of N turns carrying current I. Find the magnetic flux density at any point within the ring.

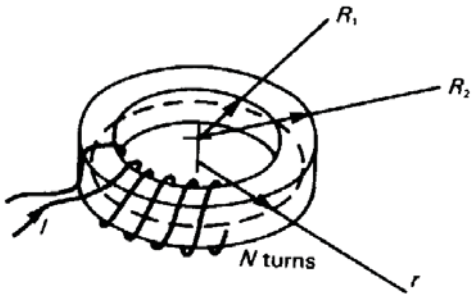

Fig 4.5. A coil wound on a ring-shaped former.

Solution

From considerations of symmetry, the flux density within the winding must be in the tangential direction and vary only with radius. Applying the magnetic circuit law [4.12] to the circular path of radius r shown by the broken line in the diagram gives

$$\oint \mathbf{B} \cdot \mathbf{dl} = 2\pi r B_\theta = \mu_0 \iint \mathbf{J} \cdot \mathbf{dA} = \mu_0 NI \tag{4.30}$$

Hence

$$B_\theta = \frac{\mu_0 NI}{2\pi r} \tag{4.31}$$

within the winding. Any surface spanning a path of integration lying outside the winding is crossed by equal numbers of conductors carrying currents in each direction so that the magnetic flux density outside the ring is apparently zero.

More careful thought shows that if the winding is a single layer, then current I must encircle the ring once in the tangential direction. The magnetic flux density is therefore given by Equation (4.4) at points outside the ring and remote from it. If it is necessary to produce a ring with no flux outside it then a winding with an even number of layers is used. Each layer is wound in the opposite direction around the ring from the one beneath it so that the tangential component of the current is cancelled.

Example 4.8

Find an expression for the magnetic flux density at the centre of a long solenoid by considering the limit of a toroidal coil (the coil-former is a ring with circular cross-section) whose diameter is large compared with the diameter of the winding.

Solution

Let the number of turns per unit length of the winding be *n*. Then, for a toroidal winding with a total of *N* turns and an inner radius R_1

$$n = \frac{N}{2\pi R_1} \tag{4.32}$$

As the radius of the torus increases while the radial thickness is kept constant the inner radius, the mean radius and the outer radius tend to the same value, thus $r \to R_1$. Also the azimuthal component of the flux density in a toroidal coil becomes the axial field component in a long solenoid. Substituting for *N* and *r* in (4.31) from (4.32)

$$B_z = \mu_0 n I \tag{4.33}$$

which is identical to the result obtained in (4.20) by a different method.

Example 4.9

Two straight wires are parallel to each other and 10 mm apart. Find the force exerted per unit length on each wire when the wires carry currents of ±1 A.

Solution

Let us suppose that the current in the first wire is in the *x* direction and that the wires lie in the *x*-*y* plane. To find the flux density in the region of one wire caused by the current in the other we allow α to tend to $\pi/2$ in [4.9] so that

$$B_z = \frac{\mu_0 I}{2\pi r} \tag{4.34}$$

where *r* is the distance between the wires. (Note: the same result can be obtain by using the magnetic circuit law [4.12]. The flux density is at right angles to the wires and in the right-hand corkscrew sense with respect to the current in the first wire.

The force on the second wire is given by

$$\mathbf{F} = -I\left(\mathbf{dl} \wedge \mathbf{B}\right) = -I\, dl_x\, B_z\, \hat{\mathbf{y}} \tag{4.18}$$

where $\hat{\mathbf{y}}$ is a unit vector in the *y* direction. Substituting for B_z we obtain

$$F_y = -\frac{\mu_0 I^2}{2\pi r} = \frac{4\pi \times 10^{-7} \times 1.0^2}{2\pi \times 0.01} = 20\ \mu N \tag{4.35}$$

Example 4.10

An electron moving with velocity v enters a region of space in which the magnetic flux density is B at right angles to the velocity of the electron, as shown in Fig. 4.6. Given that the magnetic field is constant for a distance L in the direction of the initial motion of the electron and then falls abruptly to zero, find an expression for the angular deflection of the electron motion by the magnetic field.

Solution

The force on a moving charged particle is given by [4.4]. This force always acts at right angles to the direction of motion of the particle and therefore does no work on it. The magnetic force acting on a particle can therefore change its direction of motion but not its speed. The simplest case is that of a particle moving at right angles to the direction of a uniform magnetic field, as shown in Fig. 4.6. Because the particle is acted on by a constant force which is perpendicular to its direction of motion, it must move in an arc of a circle, as shown.

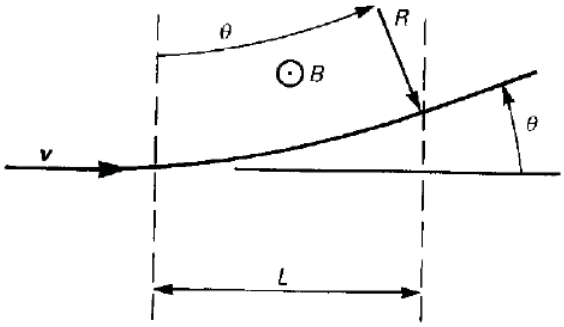

Fig. 4.6. The deflection of a stream of electrons by a transverse magnetic flux.

The radial acceleration is provided by the magnetic force, so that

$$mR\omega^2 = qvB = qR\omega B \tag{4.36}$$

whence

$$\omega = \frac{q}{m}B \tag{4.37}$$

This frequency is known as the *cyclotron frequency*, from its application in the particle accelerator of that name. For an electron in a magnetic flux density of 0.1 Tesla the cyclotron frequency is about 2.8 GHz.

If the particle has an initial component of velocity parallel to the vector **B**, then that component is unaffected by the magnetic field. The motion of the particle is then helical from the combination of the steady motion in the direction of **B** with the circular motion in the direction perpendicular to it.

Since $v = R\omega$ the radius of the circular path of the particle is from (4.36)

$$R = \frac{mv}{qB} \tag{4.38}$$

But

$$R\sin\theta = L \tag{4.39}$$

so the angular deflection is given by

$$\theta = \arcsin\left(\frac{qBL}{mv}\right) \tag{4.40}$$

Example 4.11

An electric current of density J_x flows through a sample of semiconductor material in the x direction and a uniform magnetic field B_y is imposed in the y direction as shown in fig. 4.7. The resulting deflection of the charge carriers produces a surface charge on the sample and an electric field E_z in the z direction which balances the magnetic force on the charges in the steady state. Find an expression for the electric field strength in terms of J_x, B_y, the charge on each charge carrier and the density of charge carriers.

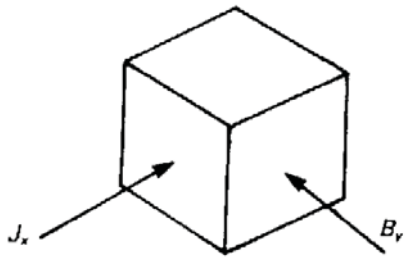

Fig. 4.7. Current density and flux density for Example 4.11

Solution

In equilibrium no current flows in the z direction and the net force on a charge carrier in the z direction must be zero. Therefore

$$qE_z + qv_xB_y = 0 \tag{4.41}$$

where q is the charge on a charge carrier (assumed positive) and v_x is the mean drift velocity of the carriers. Now

$$J_x = nqv_x \tag{3.3}$$

where n is the number density of the carriers, so from (4.41)

$$E_z = -v_xB_y = -\frac{J_xB_y}{nq} \tag{4.42}$$

The electric field E_z is associated with a potential difference between the faces of the sample and can be measured by a high-impedance voltmeter. As the product J_xB_y is positive, the magnitude of the voltage is a measure of the density of charge carriers in the sample and its sign shows their sign. This effect, known as the *Hall Effect*, is used routinely to measure the properties of samples of semiconductor material. It is also used as a means of measuring magnetic flux density.

5. The magnetic effects of iron

5.1 Introduction

This chapter illustrates the calculation of the fields of systems of current-carrying wires in the presence of iron pole-pieces using the method of images and the principle of superposition. It also shows the application of the magnetic circuit law to problems involving soft iron and permanent magnets.

5.2 Summary of the methods available

Note: This information is provided here for convenience. The equation numbers in the companion volume *Electromagnetism for Electronic Engineers* are indicated in square brackets.

Symbol	Signifies	Units
μ (*mu*)	Permeability	$H.m^{-1}$
μ_r	Relative permeability	-
H	Magnetic field	$A.m^{-1}$
$\mathcal{M}$	Magneto-motive force	A
$\mathcal{R}$	Reluctance	$A.Wb^{-1}$

- Definition of magnetic field

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \qquad [5.5]$$

- The general form of the Magnetic Circuit Law

$$\oint \mathbf{H} \cdot \mathbf{dl} = \iint \mathbf{J} \cdot \mathbf{dA} \qquad [5.7]$$

- Boundary conditions
  The normal component of **B** is continuous across a boundary
  The tangential component of **H** is continuous across the boundary

- Definition of Magneto-Motive Force (MMF) in a magnetic circuit

$$\mathcal{M} = NI$$

- Definition of the reluctance of a uniform section of a magnetic circuit

$$\mathcal{R} = \frac{L}{\mu A}$$

- Simplified form of the magnetic circuit law applied to iron circuits

$$\mathcal{M} = \mathcal{R}\Phi \qquad [5.15]$$

Example 5.1

Figure 5.1 shows a long straight wire parallel to the surface of a large flat sheet of iron and at a distance *d* from it. Given that the wire carries a current *I*, find expressions for the *x* and *y* components of the magnetic flux density at the point *Q*(*x*, *y*).

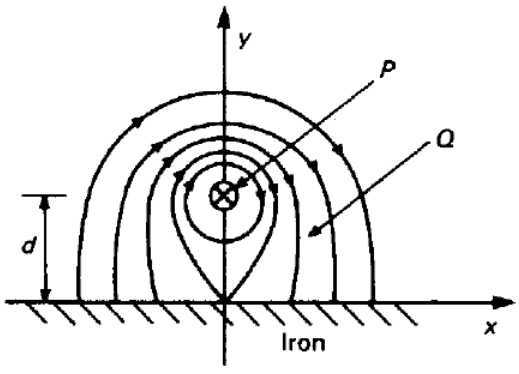

Fig. 5.1 The magnetic flux lines when a current-carrying conductor *P* lies parallel to the surface of a sheet of iron.

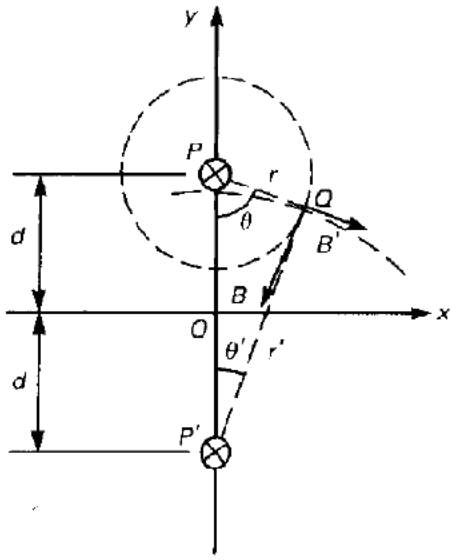

Fig. 5.2 The calculation of the flux density at *Q* for the arrangement shown in Fig. 5.1 by replacing the iron sheet with an image conductor at *P′*.

Solution

Assuming that the iron can be regarded as a perfect magnetic conductor, its effect can be represented by an image wire at *P′* as shown in Fig. 5.2. Notice carefully that, in order to make the flux lines meet the iron surface at right angles, the current in the image wire must be in the same direction as that in the original wire. The flux density at *Q* can now be found by considering each wire in turn and superimposing the results. Let the angle *OPQ* be θ. Since the field around a long straight wire is circularly symmetrical about the wire we can apply the magnetic circuit law to give

$$B = \frac{\mu_0 I}{2\pi r} \tag{5.1}$$

The field components in the coordinate directions are then

$$B_x = -\frac{\mu_0 I}{2\pi r}\cos\theta \tag{5.2}$$

and

$$B_y = -\frac{\mu_0 I}{2\pi r}\sin\theta \tag{5.3}$$

If the position of *Q* with respect of *P′* is given by θ' and r', then the field components due to the image wire are

$$B'_x = \frac{\mu_0 I}{2\pi r'}\cos\theta' \tag{5.4}$$

and

$$B'_y = -\frac{\mu_0 I}{2\pi r'}\sin\theta' \tag{5.5}$$

The total field is thus

$$B_x = \frac{\mu_0 I}{2\pi}\left(\frac{\cos\theta'}{r'} - \frac{\cos\theta}{r}\right) \tag{5.6}$$

and

$$B_y = -\frac{\mu_0 I}{2\pi}\left(\frac{\sin\theta'}{r'} + \frac{\sin\theta}{r}\right) \tag{5.7}$$

Finally, using some simple trigonometry, we obtain

$$B_x = \frac{\mu_0 I}{2\pi}\left(\frac{y+d}{x^2+(y+d)^2}+\frac{y-d}{x^2+(y-d)^2}\right) \tag{5.8}$$

and

$$B_y = -\frac{\mu_0 I}{2\pi}\left(\frac{x}{x^2+(y+d)^2}+\frac{x}{x^2+(y-d)^2}\right) \tag{5.9}$$

Notice that B_x is zero when $y = 0$ as it should be.

Strictly speaking, the imaging plane should not quite coincide with the iron surface, because iron is not a perfect conductor of flux, but the difference is not usually important. Because the flux density in iron is limited by saturation effects it is possible to regard an iron surface as a magnetic equipotential only if the flux density is below the saturation level.

Example 5.2

A solenoid of the form shown in Fig. 5.3 may be idealized as a set of circular current loops at the centres of the windings. If each coil has 100 turns, mean diameter 200mm, and carries a current of 1A calculate the maximum and minimum values of the magnetic flux density on the axis of the solenoid. What is the effect of changing the coil spacing to $2a$?

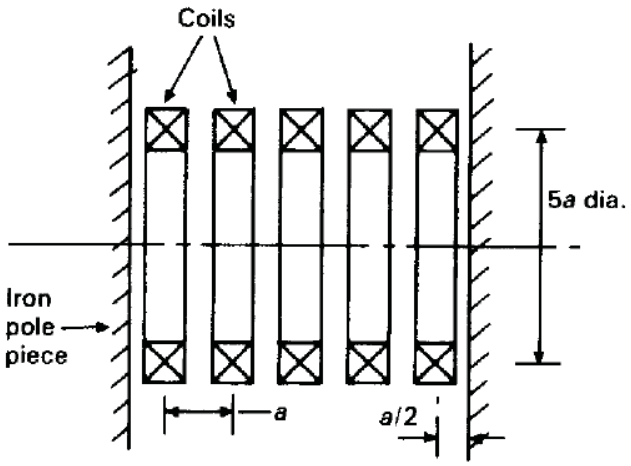

Fig. 5.3 The arrangement of a solenoid having a number of coils and iron pole pieces at either end.

Solution

The iron pole-pieces are at symmetry planes so the coils are multiply reflected in them with the result that the field within the solenoid is identical to that of an infinite set of coils at the same spacing. The field on the axis of an individual coil is found by substituting the current, the number of turns and the radius of the coil in (4.4)

$$B_Z = \frac{100\mu_0}{0.2}\sin^3\psi \tag{5.10}$$

From the symmetry of the problem we expect that the flux density will be a maximum in the plane of each coil and a minimum midway between the coils. We recall that ψ is the angle subtended by a coil at the point at which the flux density is determined and superimpose the fields of all the coils at that point to find the total flux density.

a) The angle subtended at the centre of a coil by one which is distant *na* from it is

$$\psi_n = \arctan\left(\frac{5a}{2na}\right) \tag{5.11}$$

The total flux density at the centre of a coil is therefore

$$B_Z = 500\mu_0\left[1+2\sum_{n=1}^{\infty}\sin^3\psi_n\right] \tag{5.12}$$

b) The angle subtended at a point midway between two coils by one distant $(n-1/2)a$ from it is

$$\psi_n = \arctan\left(\frac{5a}{(2n-1)a}\right) \tag{5.13}$$

and the total flux density at that point is given by

$$B_Z = 1000\mu_0 \sum_{n=1}^{\infty} \sin^3 \psi_n \tag{5.14}$$

The simplest way to evaluate this is to use a spreadsheet as shown in fig. 5.4.

pi	3.141592654		In plane of coil		Midway between coils	
μ_0	1.25664E-06	n	ψ_n	B_n	ψ_n	B_n
Constant	0.000628319					
		0	1.571	0.00063		
		1	1.190	0.00101	1.373	0.00118
		2	0.896	0.00060	1.030	0.00079
		3	0.695	0.00033	0.785	0.00044
		4	0.559	0.00019	0.620	0.00025
		5	0.464	0.00011	0.507	0.00014
		6	0.395	0.00007	0.427	0.00009
		7	0.343	0.00005	0.367	0.00006
		8	0.303	0.00003	0.322	0.00004
		9	0.271	0.00002	0.286	0.00003
		10	0.245	0.00002	0.257	0.00002
	Total Flux Density (Tesla)			**0.003056**		**0.003048**

Fig. 5.4. Calculation of the flux density produced by the coils

The flux density when the coil spacing is increased to 2a (and the distance from the pole-pieces to a) is calculated in exactly the same manner. The maximum and minimum flux densities are 1.564mT and 1.554mT. An active version of this figure showing the details of the calculations is available for download as an EXCEL file.

Example 5.3

Figure 5.5 shows the arrangement of a magnetic circuit. Given that the coil has 1000 turns and carries a current of 10 mA, and that the dimensions of the circuit are:

L = 50 mm g = 2 mm

w = 10 mm $\mu = 1000\mu_0$

calculate the flux density in the air gap.

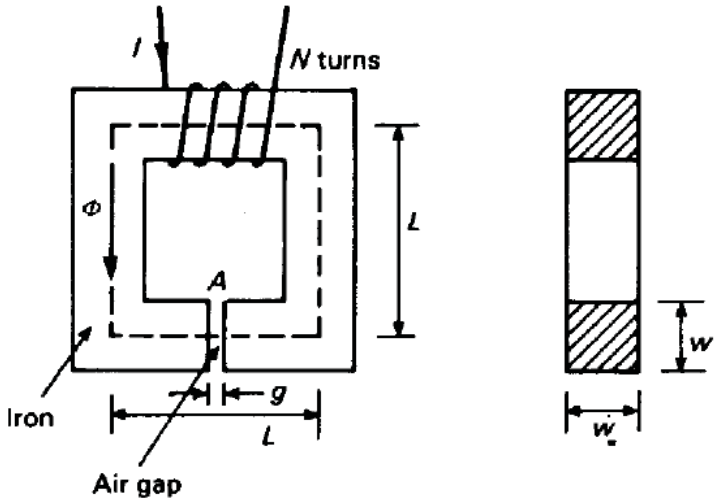

Fig. 5.5. A simple magnetic circuit made up of an iron core with an air gap in it. The magneto-motive force is supplied by a winding of *N* turns of wire

Solution

The problem can be represented approximately by the equivalent circuit shown in Fig. 5.6.

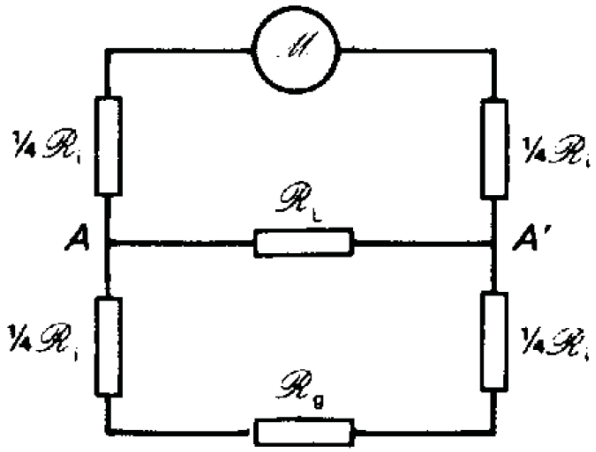

Fig. 5.6. A network representing the magnetic circuit of Fig. 5.5 including the effects of flux leakage.

The magnetomotive force is given by

$$\mathcal{M} = NI = 10 \text{ A} \tag{5.15}$$

The reluctances of the iron and the air gap are, from the two terms of equation [5.17]

$$\mathcal{R}_i = \left(\frac{4L}{\mu w^2} \right) = \frac{2}{\mu_0} \tag{5.16}$$

$$\mathcal{R}_g = \left(\frac{g}{\mu_0 (w+g)^2} \right) = \frac{1000}{72\mu_0} \tag{5.17}$$

To estimate the leakage reluctance across A-A' we regard the path as a large air gap. The pole-face areas are (40 × 10) mm^2 and the gap is 40 mm. To allow for the fringing of the flux let us suppose that the effective cross-sectional area is double the pole-face area. Then the leakage reluctance is given by

$$\mathcal{R}_L = \frac{1}{\mu_0} \cdot \frac{0.04}{2 \times 0.04 \times 0.01} = \frac{50}{\mu_0} \tag{5.18}$$

Some simple network analysis then yields the flux in the air gap

$$\Phi_g = 0.618\mu_0 \text{ Wb} \tag{5.19}$$

The very crude approximation to $\mathcal{R}_L$ can be justified by considering the effect of ignoring leakage altogether. In that case $\Phi = 0.629\mu_0$ Wb — an error of 2%. An even more drastic approximation is to regard the iron as having negligible reluctance so that $\Phi = 0.72\mu_0$ Wb — an error of about 17%. The best estimate of the flux density in the air gap is obtained by putting in the numerical value of μ_0 and dividing Φ by the effective area of the air gap: $B_g = 5.4$ mT.

The exact solution of this problem would require a very large computer. Fortunately such high accuracy is not needed in most practical cases. The worked example shows how figures which are accurate to a few per cent can be obtained by some quite crude approximations. Even the most accurate calculation made using magnetic circuit theory is likely to be in error by a few per cent because of the non-linear properties of the iron.

Example 5.4

Figure 5.7 shows a 'pot core' made of Ferroxcube B4. Cores like this are used to make radio-frequency inductors. Given that the coil has 100 turns, estimate the maximum a.c. current which can be passed through it without the behaviour of the inductor becoming non-linear. The initial magnetization curve of Ferroxcube B4 is shown in Fig. 5.8.

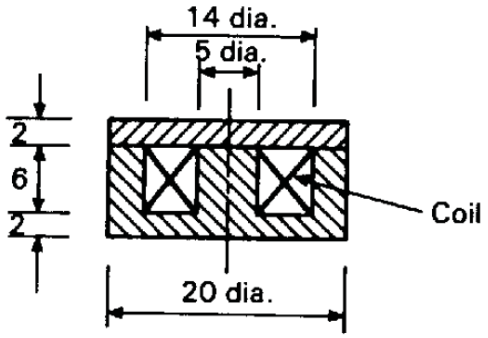

Fig. 5.7. The arrangement of a ferrite pot core for a radio frequency inductor. The dimensions are in millimetres.

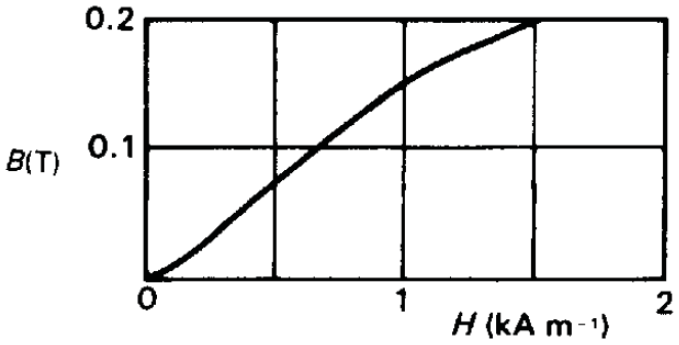

Fig. 5.8. The initial magnetization curve for the ferrite material Ferroxcube B4.

Solution

Since the flux circulating in the core is constant the flux density is greatest where the cross-sectional area is least. Two parts of the magnetic circuit must be examined:

- Axial flux density in the central core: Area $= \pi \times 2.5^2 = 19.6 \text{ mm}^2$
- Radial flux density at the end of the central core: Area $= \pi \times 5 \times 2 = 31.4 \text{ mm}^2$

Evidently the flux density is greatest in the central core. The magnetisation curve is approximately linear up to a flux density of 0.15 T so the maximum flux in the core is

$$\Phi = 19.6 \times 10^{-6} \times 0.15 = 2.95 \times 10^{-6} \text{ Wb} \tag{5.20}$$

To calculate the maximum current in the winding using [5.7] we must compute the line integral of H around a closed path. It is not possible to do this exactly without knowledge of the detailed fields but a useful approximation can be found by breaking the magnetic circuit up into a number of sections as shown in fig. 5.9. The Magneto-motive force needed to drive the flux through each of these sections can be computed using elementary methods. This approach requires the assumption that the bunching of the flux as it turns the corner from one section to another can be represented satisfactorily by including some extra length in one or more of the sections as shown.

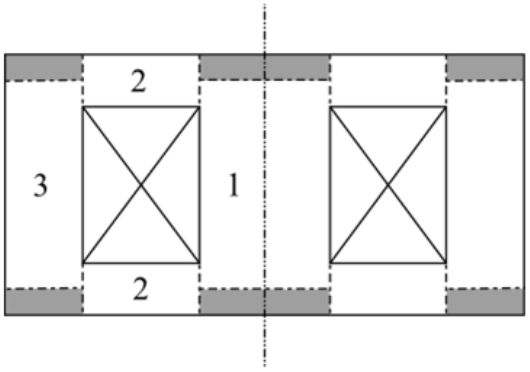

Fig. 5.9. The sections of the magnetic circuit.

We note that, in the linear region of the curve in fig. 5.8 the relative permeability can be calculated using [5.5] as

$$\mu = \frac{B}{\mu_0 H} = \frac{0.15}{4\pi \times 10^{-7} \times 1000} = 119 \tag{5.21}$$

In region 1 the flux density is the maximum flux density so H = 1000 A.m^{-1} and the magneto-motive force is

$$\mathcal{M}_1 = 1000 \times 8 \times 10^{-3} = 8 \text{ A} \tag{5.22}$$

In region 2 the flux density and the magnetic field decreases inversely as the radius as the flux spreads out radially. Thus the flux density at radius r is

$$B_2(r) = \frac{\Phi}{2\pi \times 0.002} \frac{0.0025}{r} \tag{5.23}$$

Then

$$H_2(r) = \frac{B_2(r)}{\mu\mu_0} = \frac{1}{\mu\mu_0}\frac{2.95\times10^{-6}}{2\pi\times0.002}\frac{0.0025}{r} = \frac{1.563}{r} \tag{5.24}$$

and the magneto-motive force is

$$\mathcal{M}_2 = \int_{r_1}^{r_2} H_2(r)\,dr = 1.563\ln\left(\frac{7}{2.5}\right) = 1.609\text{ A} \tag{5.25}$$

In region 3

$$H_3 = H_1\left(\frac{2.5^2}{10^2 - 7^2}\right) = 122.5\,\text{A.m}^{-1} \tag{5.26}$$

so that

$$\mathcal{M}_3 = 122.5\times8\times10^{-3} = 0.98\text{ A} \tag{5.27}$$

The total magneto-motive force is therefore

$$M = 8 + 2\times1.61 + 0.98 = 12.2\text{ A} \tag{5.28}$$

Dividing by the number of turns in the coil we find that the peak current should not exceed 120 mA if the behaviour of the inductor is to remain linear.

Example 5.5

Figure 5.10 shows a magnetic recording head made of Permalloy D. Given that the magnetic material has a relative permeability of 500, estimate the number of turns needed to produce a flux density of 0.05 T in the air gap from a current of 10 mA.

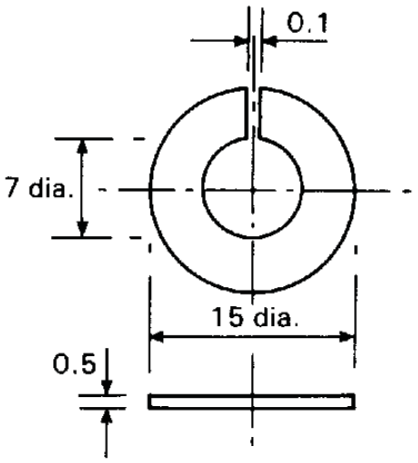

Fig. 5.10. The magnetic core for a recording head (dimensions in millimetres).

Solution

Magnetic recording is not now as important as it once was for video and audio recordings but it is still used for archiving large amounts of computer data. The magnetic field in the air-gap of the magnetic core magnetises a small region of magnetic tape placed against it.

If fringing of the magnetic field in the air gap is ignored then the flux density is constant throughout the circuit. The magneto-motive force is the sum of the MMF's for the core and the air gap and is equal to the current in the coil multiplied by the number of turns. We take the path length in the core to be the perimeter of a circle having the mean diameter (11 mm). The thickness of the core is not needed for this calculation. Thus

$$N = \frac{1}{I}\left(\frac{B_{air}}{\mu_0} l_{air} + \frac{B_{iron}}{\mu\mu_0} l_{iron}\right) = \frac{1}{0.01}\left(0.1\times10^{-3} + \frac{\pi\times11\times10^{-3}}{500}\right)\frac{0.05}{4\pi\times10^{-7}} = 673 \text{ turns} \qquad (5.29)$$

Example 5.6

A magnetic circuit like that shown in Fig. 5.5 has the following parameters

L = 50 mm	g = 0.2 mm	w = 10 mm
N = 100 turns	I = 1.0 A	

Given that the core is made of a magnetic material having the initial magnetization curve shown in fig. 5.10, estimate the flux density in the air gap.

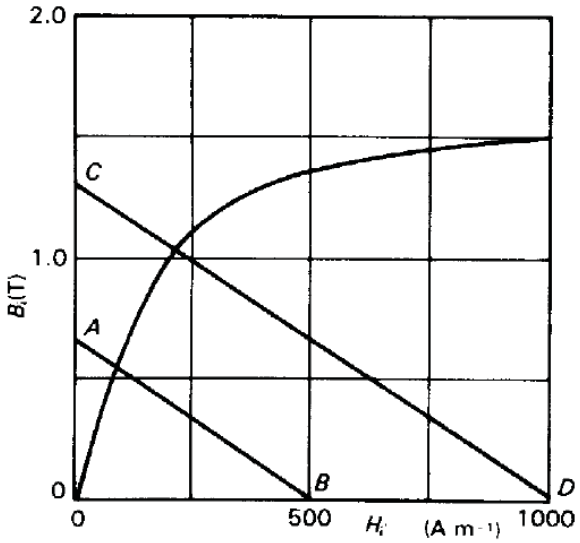

Fig. 5.11. The graphical solution of Example 5.6, showing that doubling the magnetomotive force in a magnetic circuit does not necessarily produce a doubling of the flux density.

Solution

We assume that for this material the initial magnetization curve is an adequate approximation to the hysteresis loop. Allowing for the fringing around the air gap,

$$B_{iron} = \left(\frac{w+g}{w}\right)^2 B_{air} = 1.04 B_{air} \tag{5.30}$$

so that

$$H_i = \left(NI - \frac{gB_a}{\mu_0} \right) \frac{1}{4L} = 500 - 765 B_{iron} \tag{5.31}$$

or

$$B_{iron} = \frac{1}{765}\left(500 - H_{iron}\right) \tag{5.32}$$

A second relationship between B_{iron} and H_{iron} is given by the initial magnetisation curve in fig. 5.11. The solution to this pair of non-linear simultaneous equation is found by plotting the load line (5.32) plotted as AB on Fig. 5.11. The working point is

$$H_{iron} = 85 \text{ A.m}^{-1}, \qquad B_{iron} = 0.55 \text{ T}$$

and the flux density in the air gap is

$$B_{air} = 0.53 \text{ T}$$

B_{air} is less than B_{iron} because of the fringing field of the gap.

If the current in the coil is doubled so that the load line is CD, then the working point is

$$H_{iron} = 210 \text{ A.m}^{-1}, \qquad B_{iron} = 1.04 \text{ T}$$

and

$$B_{air} = 1.0 \text{ T}$$

Doubling the current in the coil does not result in a doubling of the flux density in the air gap. This shows clearly the non-linear behaviour of the material.

Example 5.7

When a magnetic circuit is being designed it is likely that the reverse problem to that in Example 5.6 has to be solved. The flux is specified and the ampere turns to produce it have to be calculated. Calculate the ampere turns needed to produce a flux of 0.14 mWb in the magnetic circuit of Example 5.6.

Solution

The flux density in the iron is

$$B_{iron} = \frac{0.14 \times 10^{-3}}{(0.01)^2} = 1.4\,T \tag{5.33}$$

The magnetic field strength in the iron is read from Fig. 5.11: $H_{iron} = 660$ A.m^{-1}. Neglecting fringing, $B_{air} = B_{iron}$ and

$$H_{air} = \frac{B_{air}}{\mu_0} = 1.114 \times 10^{-6}\,\text{A.m}^{-1} \tag{5.34}$$

Applying the magnetic circuit law [5.7], the total number of ampere turns is

$$NI = H_{iron}\,l_{iron} + H_{air}\,l_{air} = 660 \times 0.2 + 1.1 \times 10^{-6} \times 0.2 \times 10^{-3} = 355\ \text{Ampere} - \text{turns} \tag{5.35}$$

The number of turns used would depend upon the current available.

Example 5.8

A magnetic circuit consists of a soft iron yoke, which may be assumed to have infinite permeability, a permanent magnet 60 mm long and 2400 mm^2 cross-sectional area, and an air gap 5 mm long and 3600 mm^2 in cross-sectional area. The permanent magnet is made from Columax, which has the demagnetisation curve shown in Fig. 5.12. Initially a keeper made of soft magnetic material is inserted into the air gap and the magnet magnetised to a residual flux density of 1.35 T by means of a coil wound on the iron yoke.

Determine:

a) the flux density in the air gap when the keeper is removed if fringing around the gap can be neglected, and
b) the value of the flux density in the permanent magnet if the keeper is replaced in the air gap, assuming that the recoil permeability of Columax is $1.8\mu_0$.

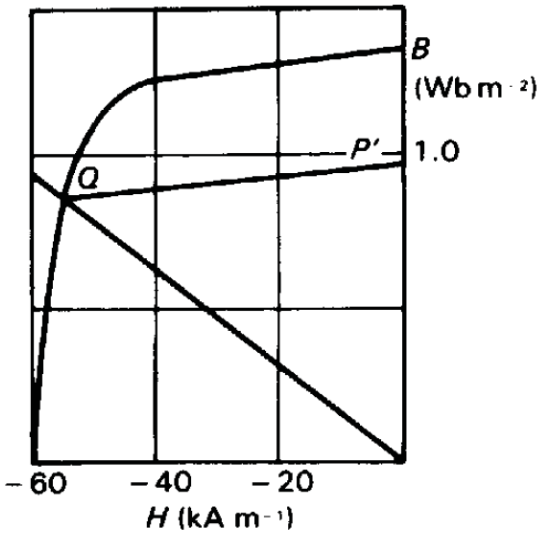

Fig. 5.12. The demagnetization curve for Columax.

Solution

a) When the keeper is removed the flux density in the air gap is related to that in the iron by

$$B_{air} = \frac{2400}{3600} B_{iron} \tag{5.36}$$

No MMF is supplied by a current in a winding, therefore

$$H_{iron}\, l_{iron} + H_{air}\, l_{air} = H_{iron}\, l_{iron} + \frac{B_{air}}{\mu_0} l_{air} = 0 \tag{5.37}$$

Substituting for B_{air} in (5.36) from (5.35) and for the lengths of the iron and air paths

$$H_{iron} + \frac{B_{iron}}{\mu_0} \frac{5}{60} \frac{3600}{2400} = 0 \tag{5.38}$$

so that the load line is

$$B_{iron} = -2.262 \times 10^{-5} \times H_{iron} \tag{5.39}$$

The working point is found from the intersection of this line with the demagnetisation curve as shown by the dashed line in fig. 5.13. The working point is

B_{iron} = 1.12 T H_{iron} = -49 kA.m^{-1}

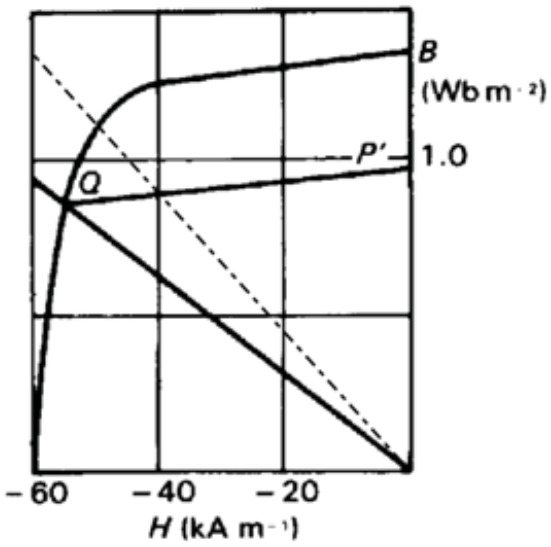

Fig. 5.13. The graphical solution to part (a) of example 5.8.

b) When the keeper is replaced the magnet recoils to $H_{iron} = 0$ along a line whose slope is $1.8\mu_0$. Thus the increase in B_{iron} is

$$\Delta B_{iron} = 1.8 \times \mu_0 \times 49{,}000 = 0.11\,\text{T} \tag{5.40}$$

Thus the flux density becomes 1.23 T.

Example 5.9

Figure 5.14(b) shows a cross-sectional view of a cylindrical loudspeaker magnet. Estimate the flux density in the air gap.

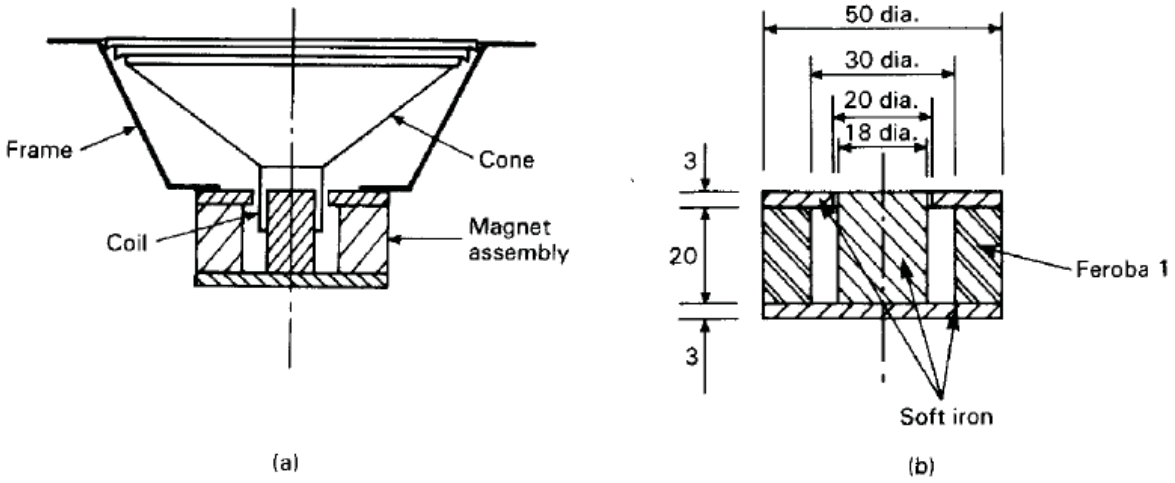

Fig. 5.14. (a) A cross-sectiona| view of a moving-coil loudspeaker. The coil is placed in the air gap of the magnetic circuit so that it experiences an axial force when a current is passed through it. The movement of the coil is transferred to the air through the motion of the paper cone. (b) An enlarged view of the magnet assembly.

Solution

To get a rough estimate of the flux density in the air gap we neglect fringing and the reluctance of the soft iron parts of the circuit, and we assume that the magnet is operating on its main demagnetization curve. From the magnetic circuit law [5.7] we know that

$$H_{iron} = -\left(\frac{l_{air}}{l_{iron}}\right) H_{air} = -\frac{1}{20} H_{air} \tag{5.41}$$

The condition for the continuity of flux gives

$$B_{iron} = \left(\frac{A_{air}}{A_{iron}}\right) B_{air} = \frac{20 \times 3\pi}{\left(25^2 - 15^2\right)\pi} B_{air} = \frac{3}{20} B_{air} \tag{5.42}$$

But $B_{air} = \mu_0 H_{air}$ so that

$$B_{iron} = -3\mu_0 H_{iron} = -3.8 \times 10^{-6} H_{iron} \tag{5.43}$$

Figure 5.15 shows the demagnetization curve for Feroba l with the load line plotted on it. The working point of the magnet is $H_{iron} = 40$ kA.m^{-1}, B = 0.15 T. The flux density in the air gap is 1.0 T.

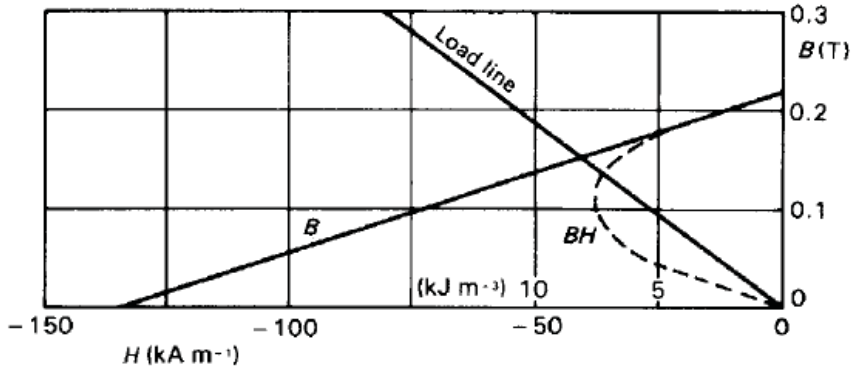

Fig. 5.15 The demagnetization curve for the ferrite permanent magnet material Feroba 1. The energy-product curve is shown dotted.

Example 5.10

Redesign the magnetic circuit shown in Fig. 5.14(b) so that the magnetic material is used as efficiently as possible.

Solution

Figure 5.15 shows the demagnetization curve of Feroba 1 with the curve of BH against B added. The energy product is a maximum when B_{iron} = 0.11 T and H_{iron} = 68 kA.m^{-1}. The flux density in the air gap is to be 1.0 T as before, so the cross-sectional area of the magnet must be 9 times that of the air gap, thus $A_{iron} = 9 \times 60\pi$ mm^2. Assuming that the inner diameter of the magnet is to remain unchanged, it can be shown that the outer diameter must be 55 mm, to give the correct area. Similarly the length of the magnet must be

$$l_{iron} = \left(\frac{B_{air}}{\mu_0 H_{iron}} \right) l_{air} = \frac{1 \times 0.001}{\mu_0 \, 68000} = 12\,\text{mm} \tag{5.44}$$

Example 5.11

Figure 5.16 shows a cross-sectional view of a cylindrical loudspeaker magnet. The permanent magnet is made from Columax (Fig. 5.12) and operates on its demagnetization curve. Neglecting fringing fields, leakage flux, and the reluctance of the soft iron pole pieces, estimate the flux density in the air gap. Calculate the optimum magnet dimensions for this flux density. If the magnet were already in production would you consider this design change justified?

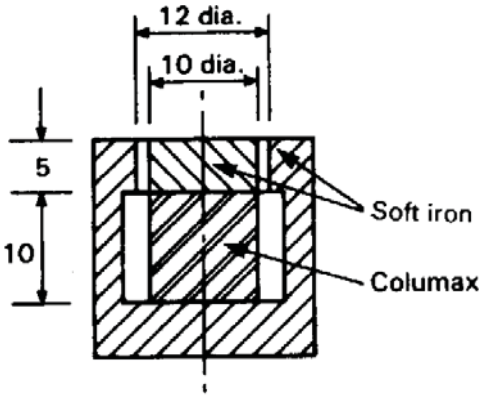

Fig. 5.16. The arrangement of a Columax loudspeaker magnet. Notice how the use of a magnet material with a higher energy product allows the permanent magnet to be put at the centre instead of outside as shown in Fig. 5.13(b).

Solution

The solution proceeds in exactly the same manner as that of Example 5.9. The load line is

$$B_{iron} = -2.765 \times 10^{-5} H_{iron} \tag{5.45}$$

The working point is B_{iron} = 1.23 T, H_{iron} = -43 kA.m^{-1}. Thus the flux density in the air gap is

$$B_{air} = \frac{A_{iron}}{A_{air}} B_{iron} = \frac{5^2 \pi}{5 \times 11 \pi} 1.23 = 0.56 \text{ T} \tag{5.46}$$

The maximum *BH* product for Columax is about 56 kJ.m^{-3}. This can be compared with the *BH* product at the working point: 53 kJ.m^{-3}. The saving in magnet volume by redesigning the magnet is about 6% and therefore the change is probably not worth making.

Example 5.12

Figure 5.17 shows an Alnico magnet for a magnetron, which is operated at a flux density of 0.4 T. How much weight could be saved by replacing it with a samarium cobalt magnet? The demagnetization curves of the two materials are shown in Fig. 5.18. The density of Alnico is 7300 kg.m^{-3} and that of samarium cobalt is 8100 kg.m^{-3}.

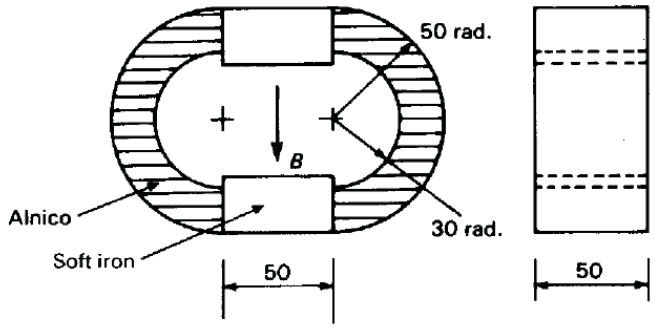

Fig. 5.17. The arrangement of an Alnico magnet for a magnetron. This magnet is typical of those used earlier generations of these radar transmitter tubes (dimensions in millimetres).

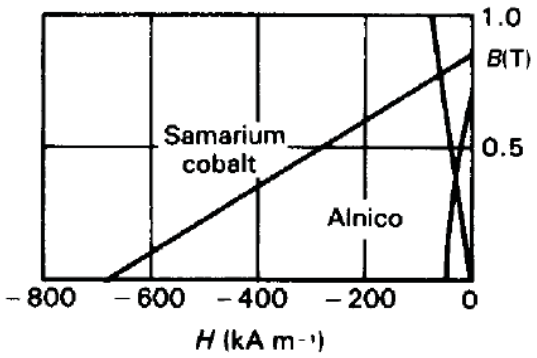

Fig. 5.18. The demagnetization curves for Alnico and samarium cobalt ($SmCo_5$).

Solution

The Alnico magnet is operated at B = 0.4 T, H = - 30 kA.m^{-1}. Therefore its MMF is

$$Hl = 30\times10^{3}\times2\times40\times10^{-3}\pi = 7540 \text{ A} \tag{5.47}$$

The energy product of samarium cobalt is maximum when B = 0.4 T and H = -350 kA.m^{-1}. The two magnets must supply the same MMF and, therefore, the length of the samarium cobalt magnet is

$$l_S = \frac{7540}{350\times10^{3}} = 21.5 \text{ mm} \tag{5.48}$$

Both magnets have the same flux and, as it happens, the same flux density, so the cross-sectional area of the magnet is unchanged. The weights of the two magnets are

$$W_A = \pi\left(0.05^2 - 0.03^2\right)\times0.05\times7300 = 1.83 \text{ kg} \tag{5.49}$$

$$W_S = 0.05\times0.02\times0.0215\times8100 = 0.17 \text{ kg} \tag{5.50}$$

Thus the change to the more modern material leads to a substantial saving in both the size and the weight of the magnetron. The actual saving in weight would be less than appears at first sight because additional soft iron would be needed to complete the magnetic circuit with the samarium cobalt magnet.

6. Electromagnetic induction

6.1 Introduction

This chapter shows the solution of problems involving time-varying currents using Faraday's Law of Electromagnetic Induction. The problems include examples both with and without magnetic materials. The results of the calculations are typically self and mutual inductances.

6.2 Summary of the methods available

Note: This information is provided here for convenience. The equation numbers in the companion volume *Electromagnetism for Electronic Engineers* are indicated in square brackets.

Symbol	Signifies	Units
$\mathcal{E}$	Electro-motive force (EMF)	V
Λ (*Lambda*)	Flux linkage	Wb
L	Self inductance	H
M	Mutual inductance	H
W	Stored magnetic energy	J
w	Magnetic energy density	$J.m^{-3}$

- Faraday's Law of Electromagnetic Induction

$$\mathcal{E} = -\frac{d\Lambda}{dt} \qquad [6.14]$$

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\iint \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dA} \qquad [6.18]$$

- Definition of flux linkage (single turn)

$$\Lambda = \int_S \mathbf{B} \cdot \mathbf{dA} \qquad [6.15]$$

- Definition of flux linkage (N turns)

$$\Lambda = N\Phi \qquad [6.24]$$

- Definition of self inductance

$$L_{11} = \frac{\Lambda_{11}}{I_1} \qquad [6.20]$$

- Definition of mutual inductance

$$M = L_{12} = L_{21} = \frac{\Lambda_{21}}{I_1} = \frac{\Lambda_{12}}{I_2} \qquad [6.19]$$

- Energy stored in an inductor

$$W = \frac{1}{2} L I^2 \qquad [6.38]$$

- Energy storage in a magnetic field

$$w = \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \qquad [6.45]$$

$$W = \frac{1}{2} \iiint \mathbf{B} \cdot \mathbf{H}\, dv \qquad [6.46]$$

- Energy storage in iron

$$w = \int_{B_1}^{B_2} H\, dB \qquad [6.68]$$

Example 6.1

Two coaxial circular wire loops, with radii 0.01 m and 0.5 m, lie in the same plane. Estimate their mutual inductance. If the smaller loop is rotated about a diameter, how does the mutual inductance depend upon the angle between the planes of the two loops?

Solution

The flux density on the axis at the centre of the larger loop is given by (4.4) with $\psi = \pi/2$. Because the small loop is small compared with the large one we can assume that the magnetic flux density is uniform everywhere within it. Thus

$$B_z = \frac{\mu_0 I_1}{2r_1} \tag{6.1}$$

where r_1 is the radius of the larger loop. Then the flux linkage to the smaller loop is

$$\Lambda_{21} = \pi r_2^2 B_z = \pi r_2^2 \frac{\mu_0 I_1}{2r_1} \tag{6.2}$$

From [6.19]

$$M = \frac{\Lambda_{21}}{I_1} = \mu_0 \pi \frac{r_2^2}{2r_1} = 0.395 \text{ nH} \tag{6.3}$$

If the smaller loop is rotated about a diameter so that its plane makes and angle α with the plane of the larger loop then the component of flux density normal to the small loop is

$$B_\perp = B_z \cos\alpha \tag{6.4}$$

The mutual inductance is proportional to the normal component of the flux and therefore varies as $\cos\alpha$.

Example 6.2

Figure 6.1 shows a cross-section of adjacent tracks on a printed circuit board. Given that the tracks run parallel to each other for 50 mm, estimate the mutual inductance between the circuit comprising A and B and that comprising C and D.

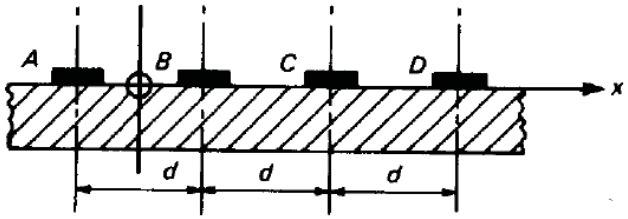

Fig. 6.1. A cross-sectional view of the conductors on a printed circuit board.

Solution

Take the origin of coordinates to be mid-way between A and B as shown. The flux density at a distance r from a long straight wire carrying current I is given by (4.30)

$$B = \frac{\mu_0 I}{2\pi r} \tag{6.5}$$

Assume that equal and opposite currents I flow in A and B. Then, applying the magnetic circuit law to A and B separately and superimposing the results, the net flux density at a point lying between C and D and x from the origin is from (6.5).

$$B = \frac{\mu_0 I}{2\pi}\left(\frac{1}{x-\frac{1}{2}d} - \frac{1}{x+\frac{1}{2}d}\right) \tag{6.6}$$

The magnetic flux is normal to the plane containing the conductors. The flux linked to the second circuit is found by integrating **B** over the area of circuit *C-D*. Since the circuit has only one turn the result is

$$\Lambda = \frac{\mu_0 I L}{2\pi} \int_{\frac{3d}{2}}^{\frac{5d}{2}} \left(\frac{1}{x - \frac{1}{2}d} - \frac{1}{x + \frac{1}{2}d} \right) dx = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{4}{3}\right) \tag{6.7}$$

where L is the length over which the two circuits are coupled by the magnetic field. Putting in the numbers shows that the mutual inductance between the circuits is 2.9 nH.

Whether this mutual inductance will lead to appreciable crosstalk between the circuits of the kind illustrated by Fig. 6.1 depends upon such factors as their impedances and signal levels, Note that in solving the problem we have assumed that the source currents can be regarded as line currents and that the dimensions of the conductors can be neglected. These assumptions are not strictly valid, but the result obtained by making them has an accuracy which is quite adequate for most purposes. In cases like this the real question is whether the mutual inductance is likely to be big enough to cause trouble.

Example 6.3

The centres of the line and neutral conductors of a 30 A, 50 Hz, ring main cable are 5 mm apart, Estimate the maximum e.m.f. induced in a circuit enclosing an area of 0.1 m^2 which is 2 m from the cable.

Solution

The amplitudes of the currents in the two wires are $\pm 30\sqrt{2}$ A and their distances from the circuit under investigation are 2.005 m and 1.995 m. Thus the maximum flux density at the circuit is obtained using (6.6) as

$$B = \frac{\mu_0 30\sqrt{2}}{2\pi}\left(\frac{1}{1.995} - \frac{1}{2.005}\right) = 21\ \text{nT} \tag{6.8}$$

The amplitude of the EMF induced in the circuit is obtained from [6.14]

$$\left|\mathcal{E}\right| = \frac{d\Lambda}{dt} = \omega A B = 2\pi \times 50 \times 0.1 \times 21 \times 10^{-9} = 0.67\ \mu\text{V} \tag{6.9}$$

where A is the area of the circuit.

Example 6.4

Figure 6.2 shows an iron core which may be assumed to have a constant relative permeability of 1000, a cross-sectional area of 100 mm^2, and a mean path length around the magnetic circuit of 150 mm. Two coils of wire are wound on the core to form a simple transformer. Coil 1 has 100 turns and coil 2 1000 turns. Calculate the self-inductances of the two windings and the mutual inductance between them.

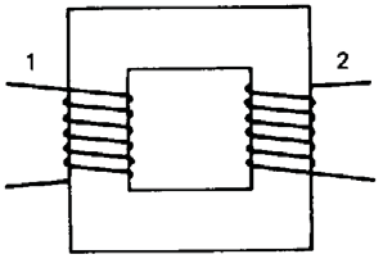

Fig.6.2. The arrangement of a simple transformer

Solution

Assume that a current I_1 flows in coil 1. Applying the magnetic circuit law [5.7] the magnetic field in the core is

$$H_1 = \frac{N_1 \times I_1}{0.15} = 667 \times I_1 \text{ A.m}^{-1} \tag{6.10}$$

Then the flux in the core is

$$\Phi_1 = A\mu_r\mu_0 H_1 = 84\ \mu\text{Wb} \tag{6.11}$$

so that from [6.20] the self and mutual inductances are

$$L_1 = \frac{N_1\Phi_1}{I_1} = 8.4\ \text{mH} \tag{6.12}$$

$$M = \frac{N_2\Phi_1}{I_1} = 84\ \text{mH} \tag{6.13}$$

If now a current I_2 is assumed to flow in coil 2 we get

$$H_2 = \frac{N_2 \times I_2}{0.15} = 6670 \times I_2 \text{ A.m}^{-1} \tag{6.14}$$

$$\Phi_2 = A\mu H_2 = 840\ \mu\text{Wb} \tag{6.15}$$

so that

$$L_2 = \frac{N_2\Phi_2}{I_2} = 840 \text{ mH} \tag{6.16}$$

and

$$M = \frac{N_1\Phi_2}{I_2} = 84 \text{ mH} \tag{6.17}$$

We have shown, incidentally, that Equation [6.22] is valid for this case and that $M^2 = L_1L_2$ as would be expected from Equation [6.28] for the perfect coupling between the coils which has been assumed here.

Example 6.5

Figure 6.3 shows the arrangement of a ferrite pot core inductor for use at radio-frequencies. Given that the relative permeability of the ferrite is 130, what is the inductance of the coil? If the ferrite begins to saturate at a flux density of 0.15 T, what is the maximum current in the coil for linear operation?

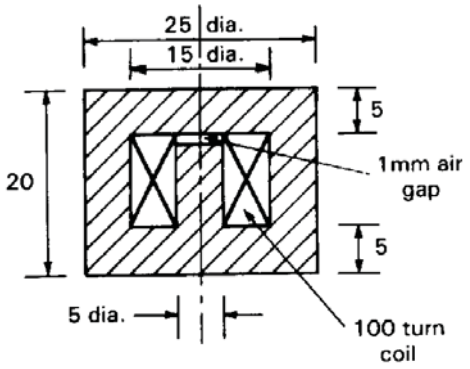

Fig. 6.3. A cross-sectional view of an inductor made with a cylindrical pot core of magnetic material.

Solution

The magnetic circuit in this problem is similar to that in example 5.4 and can be treated using similar methods. We will assume that the reluctance attributable to the bending of the flux around the corners can be represented by extending the lengths of the inner and outer cylindrical sections of the core by 5 mm. The reluctance of a section of the path in which the flux density is uniform is given by

$$\mathcal{R} = \frac{L}{\mu A} \tag{6.18}$$

The reluctances of the air gap and the inner and outer cylindrical regions are therefore

$$\mathcal{R}_g = \frac{0.001}{\mu_0 \pi \times 0.0025^2} = \frac{50.9}{\mu_0} \tag{6.19}$$

$$\mathcal{R}_1 = \frac{0.014}{\mu_0 \pi \times 130 \times 0.0025^2} = \frac{5.5}{\mu_0} \tag{6.20}$$

$$\mathcal{R}_3 = \frac{4 \times 0.015}{\mu_0 \pi \times 130 \times \left(0.025^2 - 0.015^2\right)} = \frac{0.4}{\mu_0} \tag{6.21}$$

The reluctances of the top and bottom regions, where the flux is radial, are given by

$$\mathcal{R}_2 = \frac{1}{\mu}\int_r^{r_2} \frac{1}{2\pi r h}\,dr = \frac{1}{2\pi\mu h}\ln\left(\frac{r_2}{r_1}\right) = \frac{1}{2\pi\mu_0 \times 130 \times 0.005}\ln(3) = \frac{0.3}{\mu_0} \tag{6.22}$$

The total reluctance of the circuit is therefore

$$\mathcal{R} = \mathcal{R}_g + \mathcal{R}_1 + 2\mathcal{R}_2 + \mathcal{R}_3 = \frac{57.3}{\mu_0} \tag{6.23}$$

If the current in the winding is I then, from [5.14] the flux is

$$\Phi = \frac{NI}{\mathcal{R}} = \frac{100\mu_0 I}{57.3} \tag{6.24}$$

The self inductance is found from [6.20] as

$$L = \frac{N\Phi}{I} = \frac{100^2}{57.3}\mu_0 = 0.22 \text{ mH} \tag{6.25}$$

To find the maximum current we note that the flux density is greatest in the region where the cross-sectional area is least namely the inner cylindrical region. Thus the maximum flux is

$$\Phi_{\max} = \pi r_1^2 B_{\max} = \pi \times 0.0025^2 \times 0.15 = 2.9 \text{ μWb} \tag{6.26}$$

And the maximum current is, from [5.14]

$$I_{\max} = \frac{\mathcal{R}\Phi_{\max}}{N} = \frac{57.3 \times 2.9 \times 10^{-6}}{100 \times \mu_0} = 1.3 \text{ A} \tag{6.27}$$

Example 6.6

A toroidal transformer core is made by winding 20 turns of Permalloy D strip 10 mm wide and 0.5 mm thick onto a 40 mm diameter former. A toroidal primary winding of 100 turns is wound uniformly over the core and a secondary winding of 500 turns is wound uniformly over the primary. Calculate the self-inductances of the two windings and the mutual inductance between them, given that Permalloy D has a relative permeability of 4000.

Solution

From the dimensions given the finished core has an inner radius of 20 mm, an outer radius of 30 mm and a thickness of 10 mm. If the current in the primary winding is *I* then the magnetic field at radius *r* in the core can be found using the magnetic circuit law [5.7]

$$H(r) = \frac{N_1 I}{2\pi r} \tag{6.28}$$

Thus the flux in the core is

$$\Phi = h\int_{r_1}^{r_2} B(r)\, dr = \frac{\mu N_1 I h}{2\pi}\int_{r_1}^{r_2} \frac{1}{r}\, dr = \frac{\mu N_1 I h}{2\pi} \ln\left(\frac{r_2}{r_1}\right) \tag{6.29}$$

where *h* is the thickness of the core. The self inductance is given by [6.20]

$$L_1 = \frac{N_1\Phi}{I} = \frac{\mu N_1^2 h}{2\pi}\ln\left(\frac{r_2}{r_1}\right) = \frac{4000\mu_0 \times 100^2 \times 0.01}{2\pi}\ln\left(\frac{30}{20}\right) = 32\ \text{mH} \tag{6.30}$$

Similarly the self inductance of the secondary winding is given by

$$L_2 = \frac{\mu N_2^2 h}{2\pi}\ln\left(\frac{r_2}{r_1}\right) = \frac{4000\mu_0 \times 500^2 \times 0.01}{2\pi}\ln\left(\frac{30}{20}\right) = 811\ \text{mH} \tag{6.31}$$

And the mutual inductance is

$$M = \frac{\mu N_1 N_2 h}{2\pi}\ln\left(\frac{r_2}{r_1}\right) = \frac{4000\mu_0 \times 100 \times 500 \times 0.01}{2\pi}\ln\left(\frac{30}{20}\right) = 162\ \text{mH} \tag{6.32}$$

Example 6.7

Figure 6.4 shows an iron ring of square cross-section. An inductor is made by winding *N* turns of wire uniformly on the ring as shown. Assuming that the permeability of the iron is constant, calculate the energy stored in the inductor when a current I is flowing in the winding,

a) using equation [6.38], and
b) using equation [6.46].

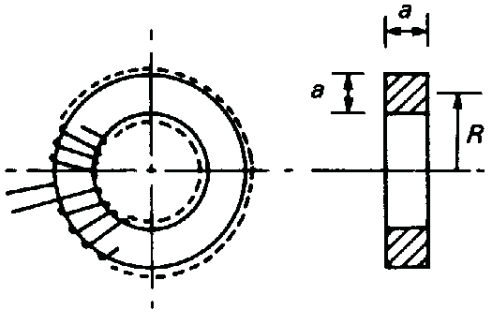

Fig. 6.4. The arrangement of a toroidal inductor.

Solution

a) Using equation [6.38]:
When a current I is flowing in the winding the magnetic field at radius r is given by (6.28) and the flux circulating within the iron is, from (6.29)

$$\Phi = \frac{\mu N I a}{2\pi} \ln\left(\frac{R+\frac{1}{2}a}{R-\frac{1}{2}a}\right) \tag{6.33}$$

From (6.30) the inductance is

$$L_1 = \frac{\mu N^2 a}{2\pi} \ln\left(\frac{R+\frac{1}{2}a}{R-\frac{1}{2}a}\right) \tag{6.34}$$

and, from [6.38], the stored energy is

$$W = \frac{\mu N^2 I^2 a}{4\pi} \ln\left(\frac{R+\frac{1}{2}a}{R-\frac{1}{2}a}\right) \tag{6.35}$$

b) Using equation [6.46]:
Taking a volume element in the shape of a ring of radius *r*, thickness *dr* and width *a*, the stored energy in the element is

$$dW = \frac{1}{2}\mu(2\pi r a)H(r)^2\, dr \tag{6.36}$$

Substituting for H(r) from (6.28) and integrating over r from $\left(R-\frac{1}{2}a\right)$ to $\left(R+\frac{1}{2}a\right)$ gives

$$W=\pi\mu a\int_{R-\frac{1}{2}a}^{R+\frac{1}{2}a}\left(\frac{NI}{2\pi r}\right)^2 r\,dr=\frac{\mu N^2I^2a}{4\pi}\ln\left(\frac{R+\frac{1}{2}a}{R-\frac{1}{2}a}\right) \tag{6.37}$$

as before. It is important to realize that this calculation has been made possible by assuming that μ is a constant so that the system is linear. When the non-linearity of the iron has to be taken into account the inductance of the coil is not a constant and the energy stored depends upon the previous history of the iron.

Example 6.8

Figure 6.5 shows the arrangement of a transformer used to measure the current in a high-voltage cable. Calculate the open circuit voltage induced in the toroidal winding if the material of the ring has a relative permeability of 130 and the current in the central conductor is 10 A *rms* at l kHz. Does this answer change if the primary conductor is not on the axis of the transformer?

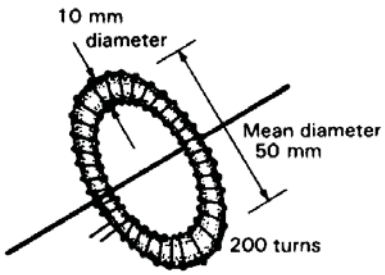

Fig. 6.5. A toroidal coil can be used as a transducer to measure the current in a conductor which passes through it.

Solution

When the conductor passes along the axis of the ring the magnetic field is given by the magnetic circuit law as

$$H=\frac{I}{2\pi r} \tag{6.38}$$

If the thickness of the ring ($2a$) is small compared with its diameter ($2R$) we may assume that the magnetic field is constant within it and equal to the value at the mean radius R. Then the *rms* flux in the ring due to the central conductor is

$$\Phi = \mu H \pi a^2 = \frac{\mu I a^2}{2R} = \mu_0 \frac{130 \times 10 \times 0.005^2}{0.5} = 0.41\,\mu\text{Wb} \tag{6.39}$$

The *rms* EMF induced in the toroidal winding is given by

$$\mathcal{E} = \frac{d\Lambda}{dt} = \omega N \Phi = 0.51\,\text{V} \tag{6.40}$$

When the wire is not in the centre of the ring the MMF remains the same even though the magnetic field is no longer uniform around the path of integration. Provided that the behaviour of the material of the core is linear the flux and the EMF induced will also be independent of the position of the central wire. This principle is used in current transformers. The flux density in the core may be detected, alternatively, by a Hall Effect sensor (see Example 4.11) and this method is used in clip-on ammeters.

Example 6.9

Estimate the inductance per unit length of the coaxial line shown in Fig. 6.6.

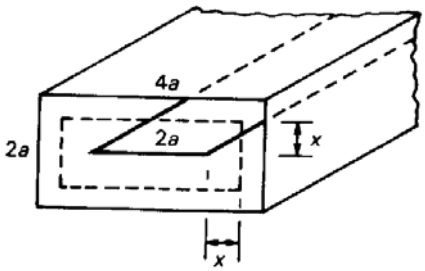

Fig. 6.6. A rectangular coaxial line with a strip centre conductor.

Solution

Upper and lower bounds to the inductance of the line can be obtained using approximate equipotential and flux distributions in a manner analogous to that demonstrated for capacitance in Example 2.13.

To obtain a lower bound to the inductance we assume approximate flux lines as shown by the dashed line in Fig. 6.6. Assuming that H is everywhere parallel to that line and applying the magnetic circuit law to this path we find that

$$H(x) = \frac{I}{4a + 8x} \tag{6.41}$$

The flux contained in unit length of tube of thickness dx is

$$\Phi = \mu_0 H\, dx = \frac{\mu_0 I}{4a + 8x}\, dx \tag{6.42}$$

so that the self-inductance of the element dx is

$$dL = \frac{\mu_0}{4a + 8x}\, dx \tag{6.43}$$

The inductances of the elements are effectively in series with each other so the total self-inductance is found by integrating (6.43) from $x = 0$ to $x = a$. The result is

$$L \geq \frac{\mu_0}{8}\ln(3) = 0.14\mu_0 \tag{6.44}$$

An upper bound to the inductance is obtained by assuming an approximate current distribution. One possibility, as shown in fig. 6.7, is to distribute the current uniformly over the broad faces of the outer conductor and over the whole of the inner conductor. The approximate magnetic equipotential surfaces associated with this distribution of current are also shown in fig. 6.7. Now the inductance of the region between each pair of equipotentials is associated with part of the total current flow. The total inductance is, therefore, approximately that of all such elements in parallel with each other.

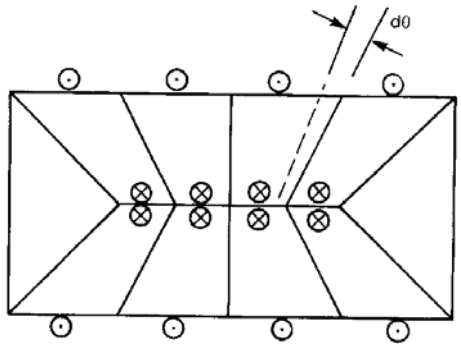

Fig. 6.7. An approximate current distribution in the rectangular coaxial line with a strip centre conductor.

The reciprocal of the inductance of an element subtending an angle $d\theta$ is obtained from [6.37]

$$d\left(\frac{1}{L}\right)=\frac{d\theta}{\mu_0 \ln\left(b/a\right)} \tag{6.45}$$

Since the outer conductor is twice the width of the inner conductor it follows that $b/a = 2$ for all the elements. Equation (6.45) is integrated over all the strips to give

$$L \leq \frac{\mu_0}{\pi}\ln\left(2\right) = 0.22\mu_0 \tag{6.46}$$

as the upper bound of L. It will be noticed that the triangular regions at each side of the diagram have been omitted from the calculations so that (6.45) is integrated over π rather than 2π. The inductances of these regions are large because the ratio $b/a \to \infty$. Thus they have the effect of reducing the estimate of L slightly and so the figure given above is certainly an upper bound on L.

The best estimate of L is given by the average of the upper and lower bounds, so

$$L=\left(0.18\pm 0.04\right)\mu_0 \tag{6.47}$$

The exact figure is $0.17\mu_0$ so the approximate value is in error by about 6%.

Example 6.10

Show how the *LCRZ* analogy can be used to deduce the results of Examples 2.13 and 6.9 from each other.

Solution

In Example 2.13 it was shown that

$$\frac{\pi\varepsilon_0}{\ln(2)} \leq C \leq \frac{8\varepsilon_0}{\ln(3)} \tag{6.48}$$

so the geometrical parameter Γ must lie in the range

$$\frac{\pi}{\ln(2)} \leq \Gamma \leq \frac{8}{\ln(3)} \tag{6.49}$$

Now, using the LCRZ analogy

$$L = \frac{\mu}{\Gamma} \tag{6.56}$$

so the inductance must lie in the range

$$\frac{\mu_0}{8}\ln(3) \leq L \leq \frac{\mu_0}{\pi}\ln(2) \tag{6.50}$$

which agrees exactly with the results of Example 6.9.

Example 6.11

Estimate the inductance per unit length of the square coaxial arrangement of conductors shown in fig. 6.8.

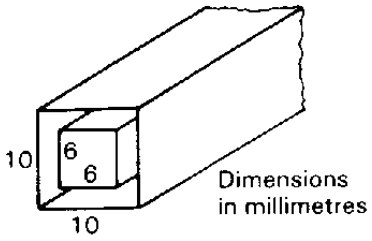

Fig.6.8. A square coaxial arrangement of electrodes

Solution

The solution to this problem proceeds in exactly the same manner as that of example 6.9.

An upper bound is obtained from (6.45) by noting that each side of the line subtends an angle of 90° on the axis and that the ratio of the width of the outer sides to the inner sides is $5/3$. Thus

$$L \leq \frac{\mu_0}{2\pi}\ln(5/3) = 0.081\mu_0 \tag{6.51}$$

A lower bound is obtained in the same manner as before. The element of inductance is

$$dL = \frac{\mu_0}{4(a+2x)}\,dx \tag{6.52}$$

where a is the length of one side of the inner conductor. Integrating from $x = 0$ to $x = a/3$ gives

$$L \geq \frac{\mu_0}{8}\ln(5/3) = 0.064\mu_0 \tag{6.53}$$

The inductance per unit length is therefore

$$L = (0.073 \pm 0.008)\,\mu_0 \tag{6.54}$$

7. Transmission lines

7.1 Introduction

This chapter shows the solution of problems involving pulses and waves on transmission lines. It is shown how the finite velocity of propagation produces distortion of pulses and changes in the apparent impedance as a result of the reflection of waves.

7.2 Summary of the methods available

Note: This information is provided here for convenience. The equation numbers in the companion volume *Electromagnetism for Electronic Engineers* are indicated in square brackets.

Symbol	Signifies	Units
k	Propagation constant	m^{-1}
λ (*lambda*)	Wavelength	m
Z_0	Characteristic impedance	Ω
Γ (*Gamma*)	Voltage reflection coefficient	-
S	Voltage standing wave ratio	-

- Phase velocity of a wave

$$v_p = \frac{\omega}{k} = f\lambda \qquad [7.10]$$

- Phase velocity on an LC transmission line

$$v_p = \frac{1}{\sqrt{LC}} \qquad [7.11]$$

- Relationship between propagation constant and wavelength

$$k = \frac{2\pi}{\lambda} \qquad [7.12]$$

- Definition of characteristic impedance

$$Z_0 = \frac{V}{I} = \frac{\omega L}{k} = \sqrt{\frac{L}{C}} \qquad [7.17]$$

- Definition of the voltage reflection coefficient

$$\Gamma = \frac{V_r}{V_i} = \frac{Z - Z_0}{Z + Z_0} \qquad [7.25]$$

- Time-average power flow on a transmission line

$$\overline{W} = \frac{|V_i|^2}{2Z_0} \qquad [7.27]$$

- Definition of voltage standing wave ratio (VSWR)

$$S = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \qquad [7.32]$$

- Apparent impedance on a line terminated by impedance Z_L

$$\frac{Z'_L}{Z_0} = \frac{Z_L + jZ_0 \tan(kl)}{Z_0 + jZ_L \tan(kl)} \qquad [7.42]$$

- Phase velocity of a TEM wave

$$v_p = \frac{1}{\sqrt{\mu_0 \varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} \qquad [7.46]$$

- Wave impedance of a TEM wave

$$Z_w = \frac{E_r}{H_\theta} = \sqrt{\frac{\mu_0}{\varepsilon}} \qquad [7.51]$$

Example 7.1

A transmission line of characteristic impedance Z_0 terminated by a resistor $R = \frac{1}{2} Z_0$ as shown in fig.7.1 is connected to a matched source by a switch for a short time τ to produce a pulse on the line. If the propagation time along the line is T, where $T \gg \tau$, investigate the variation of voltage with time at each end of the line.

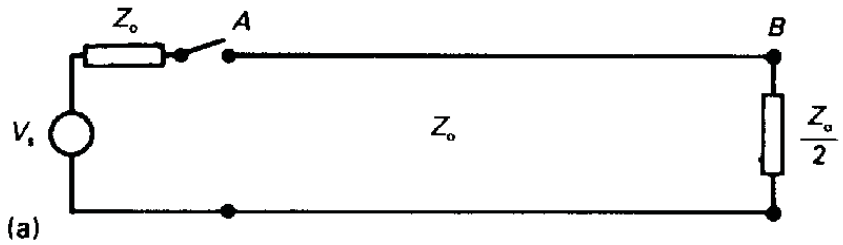

Fig. 7.1. A matched source is connected to a transmission line of impedance Z_0 for a short time τ.

Solution

When the switch is closed the uncharged line appears to the source as an impedance equal to its characteristic impedance. The source voltage is, therefore, divided equally between the source impedance and the line impedance and a voltage pulse of amplitude $\frac{1}{2}V_S$ is launched on the line. After time T this pulse reaches the other end of the line where the reflection coefficient from Equation [7.25] is

$$\Gamma_B = \frac{\frac{1}{2}Z_0 - Z_0}{\frac{1}{2}Z_0 + Z_0} = -\frac{1}{3} \tag{7.1}$$

After time $2T$ the pulse reflected at B returns to A. Since $T \ll \tau$ the switch is now open and the reflection coefficient is

$$\Gamma_A = \frac{\infty - Z_0}{\infty + Z_0} = 1 \tag{7.2}$$

The multiple reflection of pulses can be represented by the diagram in fig. 7.2. Each time a pulse is reflected its amplitude is multiplied by the appropriate reflection coefficient.

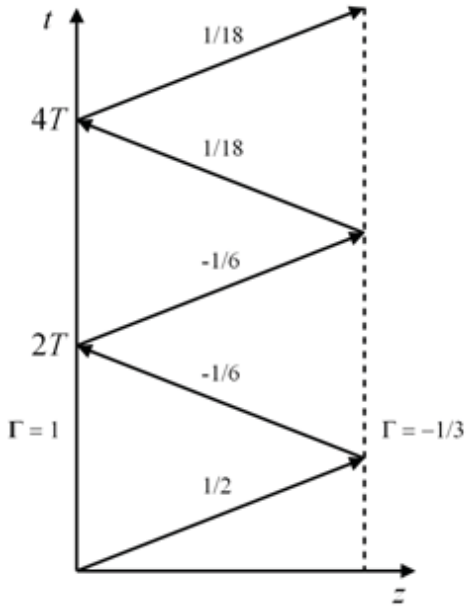

Fig. 7.2. Time of flight diagram for the circuit in fig. 7.1.

During each reflection of the pulse the voltage appearing across the line is equal to the sum of the amplitudes of the incident and reflected waves. Figure 7.3 shows the voltages across the line at A and B as a function of time. The result of closing the switch for a short time is to produce not just one pulse at B but a whole string of pulses with gradually decreasing amplitudes. If the line is short so that $T << \tau$, then this effect shows itself as a blurring of the edges of the pulse. Evidently care has to be taken with the matching of lines for the digital transmission of data if errors in the information received are not to occur.

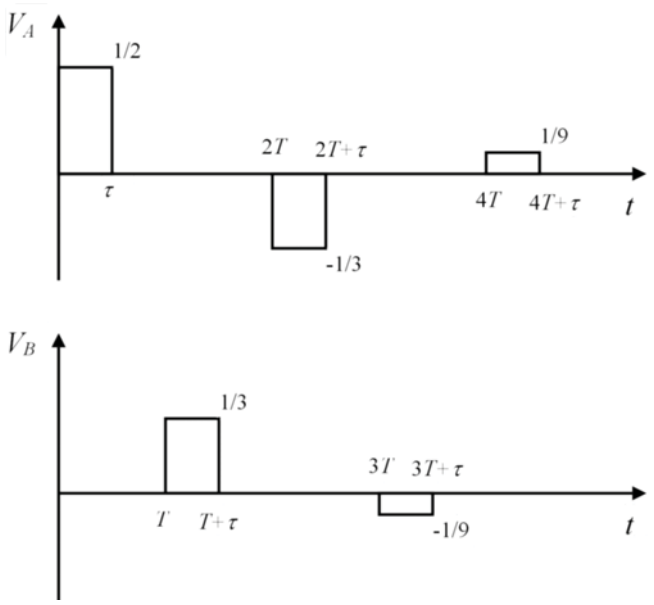

Fig. 7.3 Voltages across the line at *A* and *B* as a function of time for example 7.1

Example 7.2

Sketch graphs of the variation of voltage with time at A and B in the circuit of Fig. 7.1 if the source and load impedances are $2Z_0$ and the switch is closed when $t \geq 0$.

Solution

When the switch is first closed the amplitude of the voltage pulse is $\frac{1}{3}V_S$. The reflection coefficients at the source and the load are given by [7.25]

$$\Gamma = \frac{2Z_0 - Z_0}{2Z_0 + Z_0} = \frac{1}{3} \tag{7.3}$$

The method of solution is exactly the same as in the previous example. Figure 7.4 shows the time of flight diagram for this case. Because the pulse persists after it has been switched on the voltages at A and B accumulate. Thus

$$\begin{aligned} t = T: &\qquad V_B = \frac{1}{3} + \frac{1}{9} = 0.444 \\ t = 2T: &\qquad V_A = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = 0.482 \\ t = 3T: &\qquad V_B = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = 0.494 \end{aligned} \tag{7.4}$$

and so on.

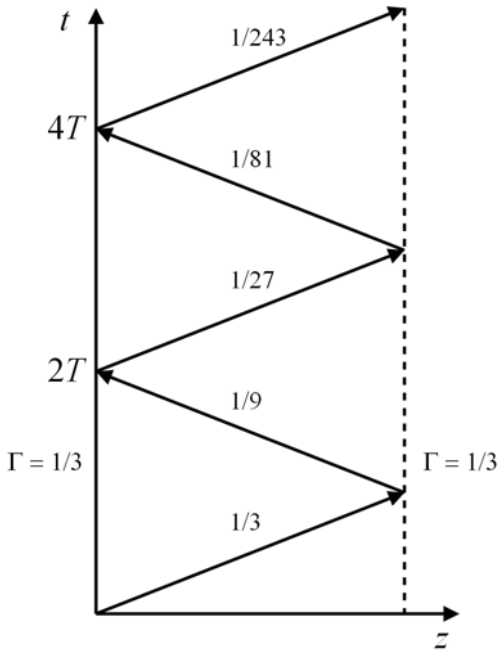

Fig. 7.4. Time of flight diagram for example 7.2

Figure 7.5 shows the variation of the voltages at A and B with time. As $t \to \infty$ the voltages at A and B tend to 0.5 which is the value which would be obtained from d.c. analysis which ignores the finite propagation time of the pulses.

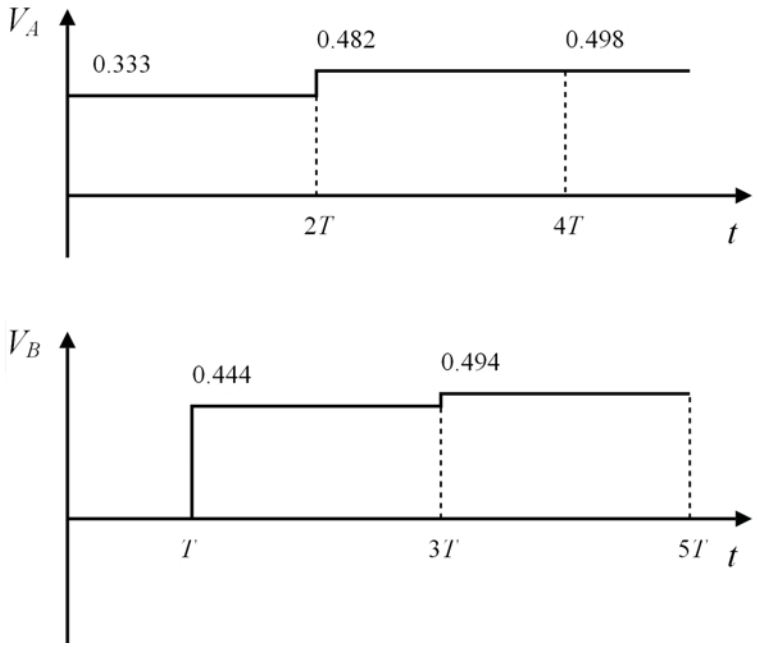

Fig. 7.5. Voltages across the line at *A* and *B* as a function of time for example 7.2

Example 7.3

A solenoid is wound uniformly at 318 turns per metre on an insulating cylindrical former 100 mm in diameter and 2.5 m long. Calculate its inductance per unit length. Ten tapping points are made on the solenoid at regular intervals, and each is connected to earth through a 0.001 μF capacitor. The resulting network is a cascade of symmetrical tee sections as shown in Fig. 7.6. This line is used as the pulse-forming network for a high-power radar transmitter in the following manner: The line is charged to a potential difference of 20 kV with both ends open-circuited; a matched load is then connected across one end and the line is discharged through it. Assuming that the line can be treated as a uniform transmission line, calculate the amplitude and duration of the voltage pulse supplied to the load. Calculate also the current flowing in the load during the pulse, and the total pulse energy.

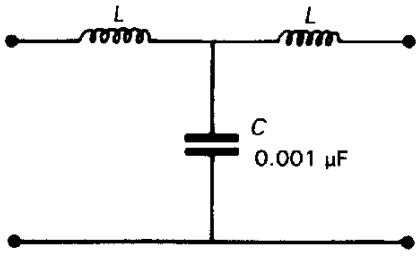

Fig. 7.6. One section of the pulse-forming network

Solution

The flux density within a long solenoid having *n* turns per metre is given by

$$B = \mu_0 n I \tag{4.29}$$

The flux linked to unit length of the solenoid is

$$\Lambda = \mu_0 n^2 I \left(\pi a^2 \right) \tag{7.5}$$

where *a* is the radius of the former on which it is wound. Then the self inductance per unit length is

$$L = \frac{\Lambda}{I} = \mu_0 n^2 \left(\pi a^2 \right) = 1 \text{ mH.m} \tag{7.6}$$

From the information given the shunt capacitance per unit length is

$$C = \frac{10 \times 1 \times 10^{-9}}{2.5} = 0.004 \ \mu F.m^{-1} \tag{7.7}$$

Thus, from [7.11] and [7.17] we know that the phase velocity and characteristic impedance are

$$v_p = \frac{1}{\sqrt{LC}} = 0.5 \times 10^6 \text{ m.s}^{-1} \tag{7.8}$$

$$Z_0 = \sqrt{\frac{L}{C}} = 500 \ \Omega \tag{7.9}$$

When the line is discharged it behaves as a source whose impedance is equal to the characteristic impedance until the discharge pulse has travelled to the other end of the line and back. This is the reverse of the charging sequence shown in fig.7.5 of the textbook. Thus the amplitude of the voltage pulse applied to the load is 10 kV and its duration is

$$\tau = \frac{2l}{v_p} = \frac{2 \times 2.5}{0.5 \times 10^6} = 10 \ \mu\text{s} \tag{7.10}$$

The current flowing into the load during the pulse is found using Ohm's law

$$I = \frac{10 \times 10^3}{Z_0} = 20 \text{ A} \tag{7.11}$$

If it is assumed that the line is loss-less then the total pulse energy is equal to the energy stored in the capacitors when the line is fully charged

$$W = \frac{1}{2}CV^2 = \frac{10 \times 0.001 \times 10^{-6} \times \left(20 \times 10^3\right)^2}{2} = 2\ \text{J} \tag{7.12}$$

Example 7.4

A voltage step generator whose source impedance is 50 Ω is connected to an oscilloscope by 1 m of 50 Ω, polythene-insulated, coaxial cable. If the input impedance of the oscilloscope is 1 MΩ in parallel with 10 pF, find the waveforms appearing at the input of the oscilloscope and at the source.

Solution

The relative permittivity of polythene is 2.25 so, from [7.46], the phase velocity of waves on the line is

$$v_p = \frac{c}{\sqrt{\varepsilon_r}} = 0.2 \times 10^9\ \text{m.s}^{-1} \tag{7.13}$$

and the transit time for signals is $T = 5$ ns.

Initially the source sees the cable as a matched load so the incident wave amplitude $V_+ = \frac{1}{2}V_S$ where V_S is the amplitude of the source voltage step. The amplitude of this wave remains constant because the cable is matched to the source and any reflected wave is absorbed by the source. Because the input resistance of the oscilloscope is much greater than the impedance of the source we can make the approximation that the effective input resistance of the oscilloscope is infinite.

At the input to the oscilloscope the current is related to the voltage by

$$I = C\frac{dV}{dt'} \tag{7.14}$$

where $t' = t + T$. The current and voltage can be expressed in terms of the amplitudes of the incident and reflected waves so that

$$I_+ - I_- = C\frac{d}{dt}\left(V_+ + V_-\right) \tag{7.15}$$

Then, substituting for the amplitudes of the currents in terms of the corresponding voltages and rearranging (7.15)

$$\frac{dV_-}{dt'} = \frac{1}{Z_0 C}\left(V_+ - V_-\right) \tag{7.16}$$

because V_+ is constant. Thus

$$\int \frac{dV_-}{\left(V_+ - V_-\right)} = \frac{1}{Z_0 C}\int dt' \tag{7.17}$$

Performing the integrations we have

$$-\ln\left(V_+ - V_-\right) = \frac{t'}{Z_0 C} \quad + \quad \text{const} \tag{7.18}$$

Now the voltage across a capacitor cannot change instantaneously because, as (7.14) shows, that would require an infinite current to flow. Therefore $V_+ + V_- = 0$ when $t' = 0$ so that the constant of integration is $-\ln\left(2V_+\right)$. Substituting this expression into (7.18) and rearranging gives the equation for the amplitude of the reflected wave as a function of time

$$V_- = V_+\left(1 - 2\exp\left(-\frac{t'}{Z_0 C}\right)\right) \tag{7.19}$$

we see that as $t' \to \infty, V_- \to V_+$ so that the steady-state voltage at the oscilloscope terminals is V_S

When the system eventually reaches equilibrium the oscilloscope input voltage will be equal to V_S because the input resistance of the oscilloscope is much greater than the characteristic impedance of the cable. The time constant of the exponential is

$$\tau = Z_0 C = 0.5 \text{ ns} \tag{7.20}$$

Thus the wave-forms at the oscilloscope input and at the source are as shown in Fig. 7.7.

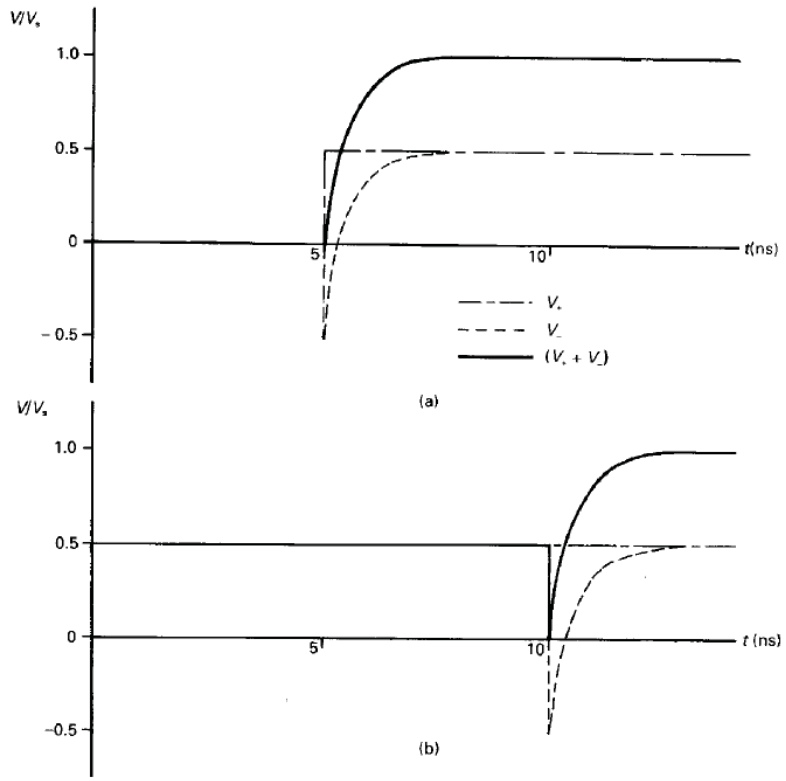

Fig. 7.7. Waveforms: (a) at the oscilloscope, and (b) at the source

The response of the system to short pulses could be found by superimposing positive and negative steps at appropriate time intervals.

Example 7.5

A transmission line is terminated by a resistive load which is less than the characteristic impedance and the VSWR on the line is 2.0. Calculate the voltage reflection coefficient of the load, the ratio of the load impedance to the characteristic impedance of the line, and the ratio of the reflected wave to the incident wave, expressing your answer in decibels.

Solution

By re-arranging [7.32] we find that the voltage reflection coefficient is

$$|\Gamma| = \frac{S-1}{S+1} = 0.33 \tag{7.21}$$

Note that because the load is resistive the voltage reflection coefficient is real. Also, because the load Z is real and less than Z_0, Γ must be negative. Similarly, rearranging [7.25]

$$\frac{Z}{Z_0} = \frac{1+\Gamma}{1-\Gamma} = \frac{2/3}{4/3} = \frac{1}{2} \tag{7.22}$$

Note that this ratio is equal to $1/S$ if Z is real and less than Z_0. If Z is real and greater than Z_0 the magnitude of the voltage reflection coefficient is the same but the ratio of impedances is equal to S. The ratio of the amplitude of the reflected wave to that of the incident wave is

$$20\log|\Gamma| = -9.5 \text{ dB} \tag{7.23}$$

This is known as the Return Loss.

Example 7.6

A transmission line has a characteristic impedance of 50 Ω, and the wavelength on the line is 150 mm at a frequency of l GHz. Given that the line is 20 mm long and is terminated by a 100 Ω load, find the impedance which would be measured at the input of the line at a frequency of 1 GHz.

Solution

The electrical length of the line is

$$kl = \frac{2\pi}{0.15} \times 0.02 = 0.838 \text{ radians} \tag{7.24}$$

so

$$\tan(kl) = 1.11 \tag{7.25}$$

From Equation [7.42] the input impedance of the line is

$$Z_L' = \frac{Z_L + jZ_0\tan(kl)}{Z_0 + jZ_L\tan(kl)} Z_0 = \left(\frac{100 + j50 \times 1.11}{50 + j100 \times 1.11}\right) \times 50 = (37.6 - 28.1j)\,\Omega \tag{7.26}$$

Example 7.7

A coaxial cable with a characteristic impedance of 50Ω is insulated with a dielectric whose relative permittivity is 2.7. Calculate the wavelength of signals on the line at 500 MHz, 1 GHz and 2 GHz. Given that a 1 metre length of this cable is terminated in a 75Ω resistor, calculate the input impedance of the cable at each of the three frequencies given above.

Solution

The phase velocity of waves on this transmission line is found from [7.46] to be

$$v_p = \frac{1}{\sqrt{\mu_0 \varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = 182 \times 10^6 \text{ m.s}^{-1} \tag{7.27}$$

The wavelengths at the three frequencies are found from [7.10] to be 365mm, 182mm and 91 mm respectively. The input impedances at the three frequencies are calculated in exactly the same manner as in the previous example to give: (33.4-1.7j) Ω; (73.7+7.3j) Ω; (70.1+13.4j) Ω respectively.

Example 7.8

A polythene-insulated coaxial cable has a characteristic impedance of 50Ω and an inner conductor 1 mm in diameter. Calculate the diameter of the outer conductor and the wavelength on the line at a frequency of 1 GHz.

Solution

The relative permittivity of polythene is 2.25. The characteristic impedance of a coaxial cable is given by

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon}}\ln\left(\frac{b}{a}\right) \qquad [7.47]$$

where a and b are the diameters of the inner and outer conductors respectively and ε is the permittivity of the dielectric separating them. Hence

$$\ln\left(\frac{b}{a}\right) = 50 \times 2\pi \times \sqrt{2.25} \times \sqrt{\frac{\varepsilon_0}{\mu_0}} = 1.25 \qquad (7.28)$$

so the diameter of the outer conductor should be

$$2b = 2a \times e^{1.25} = 3.5 \text{ mm} \qquad (7.29)$$

From [7.46] the phase velocity is

$$v_p = \frac{c}{\sqrt{2.25}} = 0.2 \times 10^9 \text{ m.s}^{-1} \qquad (7.30)$$

At 1 GHz the wavelength is given by [7.10] as

$$\lambda = \frac{v_p}{f} = 0.2 \text{ m} \qquad (7.31)$$

Example 7.9

A 50Ω air-spaced coaxial cable has an outer conductor with 10 mm inside diameter. Calculate the diameter of the inner conductor. A quarter-wave transformer is needed to match this cable to a 75Ω load at l GHz. Assuming that the outer diameter of the transformer is the same as that of the cable, calculate the dimensions of the transformer.

Solution

Using [7.47] we find that

$$\ln\left(\frac{b}{a}\right) = 50 \times 2\pi \times \sqrt{\frac{8.854 \times 10^{-12}}{4\pi \times 10^{-7}}} = 0.834 \tag{7.32}$$

Hence the diameter of the inner conductor is

$$10 \times e^{-0.834} = 4.34 \text{ mm} \tag{7.33}$$

From [7.43] the characteristic impedance of the quarter-wave transformer must be

$$Z_0 = \sqrt{50 \times 75} = 61.2\ \Omega \tag{7.34}$$

Thus, from (7.32),

$$\ln\left(\frac{b}{a}\right) = 0.834 \times \frac{61.2}{50} = 1.021 \tag{7.35}$$

Hence the diameter of the inner conductor is

$$10 \times e^{-1.021} = 3.6 \text{ mm} \tag{7.36}$$

At 1 GHz the free-space wavelength is 300 mm so the transformer should be 75 mm long.

Example 7.10

Show that an air-spaced coaxial cable is correctly matched if a sheet of resistive card having a surface resistance of 377 ΩΩ per square is placed across its end with a short circuit behind it. Find the position of the short-circuit.

Solution

The resistance to radial flow of electric current of a sheet having inner and outer radii *a* and *b* and sheet resistance R_S is calculated by considering the resistance of a ring of radial thickness *dr* and integrating from *a* to *b*.

$$R = \int_a^b \frac{R_S}{2\pi r}\, dr = \frac{R_S}{2\pi} \ln\left(\frac{b}{a}\right) \tag{7.37}$$

Now the characteristic impedance of an air-spaced line is given by

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\ln\left(\frac{b}{a}\right) \qquad [7.47]$$

so, if R in (7.37) is equal to Z_0 in [7.47] it follows that

$$R_S = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega \qquad (7.38)$$

Thus the resistive card forms a shunt resistance equal to the characteristic impedance of the line. If the line were merely continued beyond the card then the input resistance of the continuing line would be in parallel with the resistance of the card and the termination would not be matched. To obtain a matched termination the continuation of the transmission line must present an open circuit at the plane of the card. Therefore, from [7.43] a short circuit must be placed a quarter-wavelength behind the card. Note that this arrangement will only be matched exactly at the frequency at which this condition holds.

Example 7.11

Find the phase velocity and characteristic impedance of the air-spaced two-wire transmission line shown in fig.7.8.

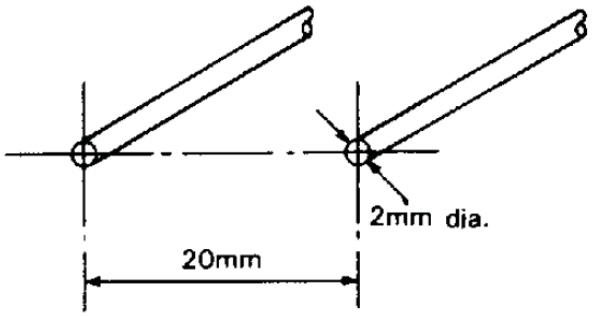

Fig. 7.8. Air-spaced two wire transmission line

Solution

All uniform air-spaced transmission lines propagate TEM waves as the velocity of light.

To find the characteristic impedance it is sufficient to determine either the inductance per unit length or the capacitance per unit length by analysis of the static fields. The capacitance per unit length of a transmission line with the dimensions shown in fig. 7.8 was found in Example 2.4 to be 9.4 $pF.m^{-1}$. Thus, from [7.11] and [7.17]

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{C}\sqrt{LC} = \frac{1}{v_p C} = 355\ \Omega \tag{7.39}$$

Example 7.12

Find the phase velocity and characteristic impedance for the square coaxial transmission line shown in fig. 7.9 if the space between the conductors is filled with a material having a relative permittivity of 4.0.

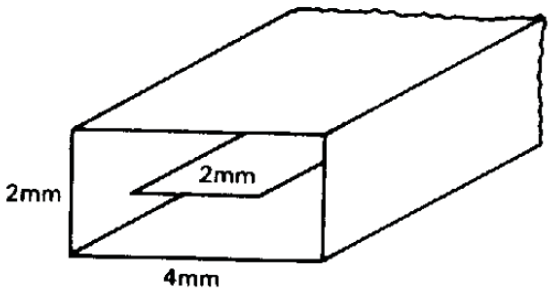

Fig. 7.9. A square coaxial transmission line

Solution

The phase velocity of TEM waves on all uniform two-wire transmission lines embedded in a uniform dielectric is equal to the velocity of light divided by the square root of the relative permittivity of the dielectric (see [7.46]). Thus

$$v_p = \frac{c}{2} = 0.15 \times 10^9 \text{ m.s}^{-1} \tag{7.40}$$

To find the characteristic impedance it is sufficient to find either the capacitance or inductance per unit length. The capacitance per unit length of an air-spaced line having the form shown in fig.7.9 was found in Example 2.12 to be $6.29\varepsilon_0$ F.m^{-1}. Note that this figure depends only on the shapes of the conductors and not on their absolute dimensions. The capacitance per unit length when the space between the conductors is filled with a dielectric is found by replacing the primary electric constant ε_0 by the permittivity of the dielectric. From (7.39)

$$Z_0 = \frac{1}{v_p C} = \frac{\sqrt{\varepsilon_r}}{6.29\,\varepsilon_r \varepsilon_0\, c} = \frac{1}{6.29 \times 2}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{377}{6.29 \times 2} = 30\ \Omega \tag{7.41}$$

8. Maxwell's equations and electromagnetic waves

8.1 Introduction

The main purpose of Chapter 8 in the textbook is to summarise the laws of electromagnetism as a preparation for advanced study of the subject. A few examples are introduced here to illustrate some of the simpler applications of those laws to electromagnetic waves in free space.

8.2 Summary of the methods available

Note: This information is provided here for convenience. The equation numbers in the companion volume *Electromagnetism for Electronic Engineers* are indicated in square brackets.

Symbol	Signifies	Units
S	The Poynting vector	$W.m^{-2}$

- The integral forms of Maxwell's equations

$$\iint \mathbf{D}\cdot\mathbf{dS} = \iiint \rho\, dv \qquad [8.20]$$

$$\oiint \mathbf{B}\cdot\mathbf{dA} = 0 \qquad [8.21]$$

$$\oint \mathbf{H}\cdot\mathbf{dl} = \iint \left(\mathbf{J} + \frac{d\mathbf{D}}{dt}\right)\cdot\mathbf{dA} \qquad [8.22]$$

$$\oint \mathbf{E}\cdot\mathbf{dl} = -\iint \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{dA} \qquad [8.23]$$

- The differential forms of Maxwell's equations

$$\nabla\cdot\mathbf{D} = \rho \qquad [8.24]$$

$$\nabla\cdot\mathbf{B} = 0 \qquad [8.25]$$

$$\nabla\wedge\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \qquad [8.26]$$

$$\nabla\wedge\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad [8.27]$$

- The constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad [8.29]$$

$$\mathbf{J} = \sigma \mathbf{E} \qquad [8.30]$$

$$\mathbf{B} = \mu \mathbf{H} \qquad [8.31]$$

- The phase velocity of plane waves in free space

$$v_p = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad [8.37]$$

- The wave impedance of plane waves in free space

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega \qquad [8.42]$$

- The Poynting vector

$$\mathbf{S} = \mathbf{E} \wedge \mathbf{H} \qquad [8.45]$$

- Poynting's theorem

  The integral of $(\mathbf{E} \wedge \mathbf{H})$ over a closed surface is equal to the instantaneous flow of electromagnetic power out of the volume enclosed by that surface.

Example 8.1

Show that plane electromagnetic waves can propagate in a uniform dielectric material and find their phase velocity and wave impedance.

Solution

The solution follows the method used in section 8.6 of the textbook. We assume that waves propagate in the *z*-direction and that the electric and magnetic fields have only *x* and *y* components respectively. Then, using Maxwell's equations, we obtain as before

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad [8.33]$$

and

$$-\frac{\partial H_y}{\partial z}=\frac{\partial D_x}{\partial t} \qquad [8.34]$$

Making use of [8.29] and [8.31] with $\mu=\mu_0$ we find that [8.35] is replaced by

$$\frac{\partial^2 E_x}{\partial z^2}=-\mu_0\frac{\partial^2 H_y}{\partial z\,\partial t}=\varepsilon\mu_0\frac{\partial^2 E_x}{\partial t^2} \qquad (8.1)$$

This is the wave equation for waves whose phase velocity is given by

$$v_p=\frac{1}{\sqrt{\varepsilon\mu_0}}=\frac{c}{\sqrt{\varepsilon_r}} \qquad (8.2)$$

Assuming propagation as $\exp j(\omega t-kz)$ and substituting in [8.33] we find that

$$jkE_0=j\omega\mu_0 H_0 \qquad (8.3)$$

Thus the wave impedance is

$$\frac{E_0}{H_0}=\frac{\omega\mu_0}{k}=\mu_0 v_p=\sqrt{\frac{\mu_0}{\varepsilon}}=\frac{1}{\sqrt{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad (8.4)$$

Example 8.2

The international guidelines for exposure of the general public to electromagnetic radiation in the range 400 to 2000 MHz state that it should not exceed a power density of f / 200 W.m^{-2} where the frequency f is in MHz. Find the maximum electric and magnetic field strengths permissible at 2000 MHz.

(See: Guidelines for Limiting Exposure to Time-Varying Electric, Magnetic, and Electro-magnetic Fields (Up To 300 GHz), Table 7, International Commission on Non-Ionizing Radiation Protection, 1998; available at http://www.icnirp.de/documents/emfgdl.pdf)

Solution

The time-average power density in an electromagnetic wave is given by

$$\overline{S}=\tfrac{1}{2}E_0H_0 \qquad (8.5)$$

where E_0 and H_0 are the amplitudes of the electric and magnetic fields.

Substituting for H_0 from [8.42]

$$E_0 = \sqrt{2\bar{S} \times 377} = \sqrt{2 \times 10 \times 377} = 87 \text{ V.m}^{-1} \tag{8.6}$$

Substituting for E_0 in [8.42]

$$H_0 = \frac{E_0}{377} = \frac{87}{377} = 0.23 \text{ A.m}^{-1} \tag{8.7}$$

Note: This information needs careful interpretation in practical cases because radio-frequency antennas do not radiate power uniformly in all directions and the fields at distances closer to an antenna than around ten wavelengths can not be approximated by plane waves

Example 8.3

Calculate the wave impedances of electromagnetic waves travelling in free-space, polystyrene $(\varepsilon_r = 2.7)$ and alumina $(\varepsilon_r = 8.9)$. Find the time-average power density in each case if the amplitude of the electric field is 100 V.m^{-1}.

Solution

The wave impedances calculated from (8.4) are 377 Ω, 229 Ω and 126 Ω respectively. The power density may be calculated using (8.5) and (8.4)

$$\overline{S} = \tfrac{1}{2} E_0 H_0 = \frac{E_0^2}{2Z_w} \tag{8.8}$$

where Z_w is the wave impedance. Substitution of the given field strength and the wave impedances gives 13.3, 21.8 and 39.6 W.m^{-2} respectively.

Example 8.4

The reflection and transmission of plane electromagnetic waves at plane dielectric boundaries normal to the direction of propagation can be treated by using transmission line theory in which the electric and magnetic field strengths are represented by the voltage and current and the wave impedance is represented by the characteristic impedance. Use this method to estimate the attenuation of an electromagnetic wave at 2 GHz as it passes through a brick wall 120 mm thick assuming that the relative permittivity is 4.0 and that losses can be neglected.

Solution

In air the wavelength of radio waves at 2 GHz is 150 mm and the wave impedance is 377 Ω. Making use of (8.2) and (8.4) the wavelength in brick is 75 mm and the wave impedance is 189 Ω. The electrical thickness of the brick is

$$kl = \frac{2\pi l}{\lambda} = \frac{2\pi \times 120}{75} = 10.1 \text{ radians} \tag{8.9}$$

The problem can therefore be represented by the equivalent transmission line network shown in fig. 8.1 in which the source is taken to be at the left of the diagram. Waves which pass through the wall are assumed to be absorbed in a matched load.

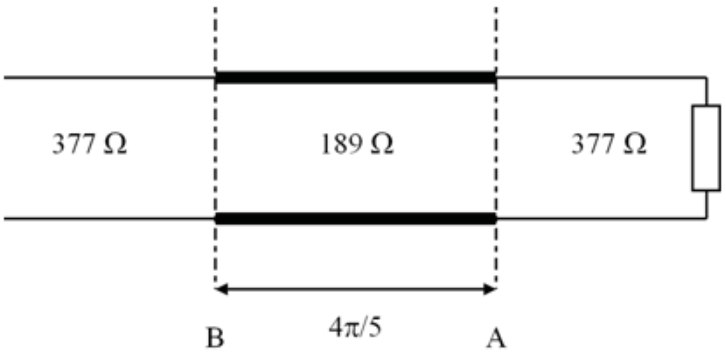

Fig. 8.1. Transmission line equivalent circuit for example 8.4

The impedance terminating the 189 Ω line at *A* is 377 Ω. The input impedance to the line at *B* can be found from [7.42]

$$Z_B = \frac{377 + j189\tan(10.1)}{189 + j377\tan(10.1)} \times 189 = 183 - j131 \tag{8.10}$$

Then the voltage reflection coefficient at B is, from [7.25]

$$\Gamma = \frac{186 - j131 - 377}{186 - j131 + 377} = -0.28 - j0.30 \tag{8.11}$$

For unit incident power the reflected power is

$$|\Gamma|^2 = 0.165 \tag{8.12}$$

Therefore the amplitude of the transmitted wave in decibels relative to the incident wave is

$$A = 10\log\left(1 - |\Gamma|^2\right) = 10\log(0.835) = -0.79 \text{ dB} \tag{8.13}$$

Example 8.5

A long straight cylindrical wire carries a current *I*. Given that the wire has radius *a* and resistance *R* per unit length, calculate the power dissipated per unit length both directly and by integration of the Poynting vector.

Solution

(a) *By direct methods*. The power dissipated per unit length is $I^2 R$.

(b) *By Poynting's theorem*. Just outside the surface of the wire, the magnetic circuit law gives

$$H_\theta = \frac{I}{2\pi a} \tag{8.14}$$

Electric field is the voltage drop per unit length

$$E_z = I R \tag{8.15}$$

The Poynting vector on the surface of the wire is

$$S_r = -E_z H_\theta = \frac{I^2 R}{2\pi a} \tag{8.16}$$

The fig.8.2 shows the directions of E_z and H_θ. It is clear that the Poynting vector is directed into the surface of the wire as represented by the minus sign in (8.16). The power flow into unit length of the wire is found by integrating the Poynting vector over a cylinder just outside the wire which has radius a and unit length. The total power flow into the wire is

$$W = -2\pi a S_r = I^2 R \tag{8.17}$$

as before. Thus the ohmic power loss in the wire can be considered the result of a steady flow of energy into the wire from the surrounding electromagnetic field. The circuit and field approaches are alternative descriptions of the same situation.

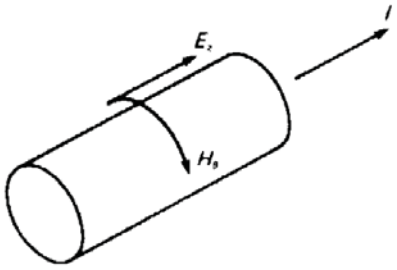

Fig. 8.2. Electric and magnetic fields associated with current flow in a resistive wire