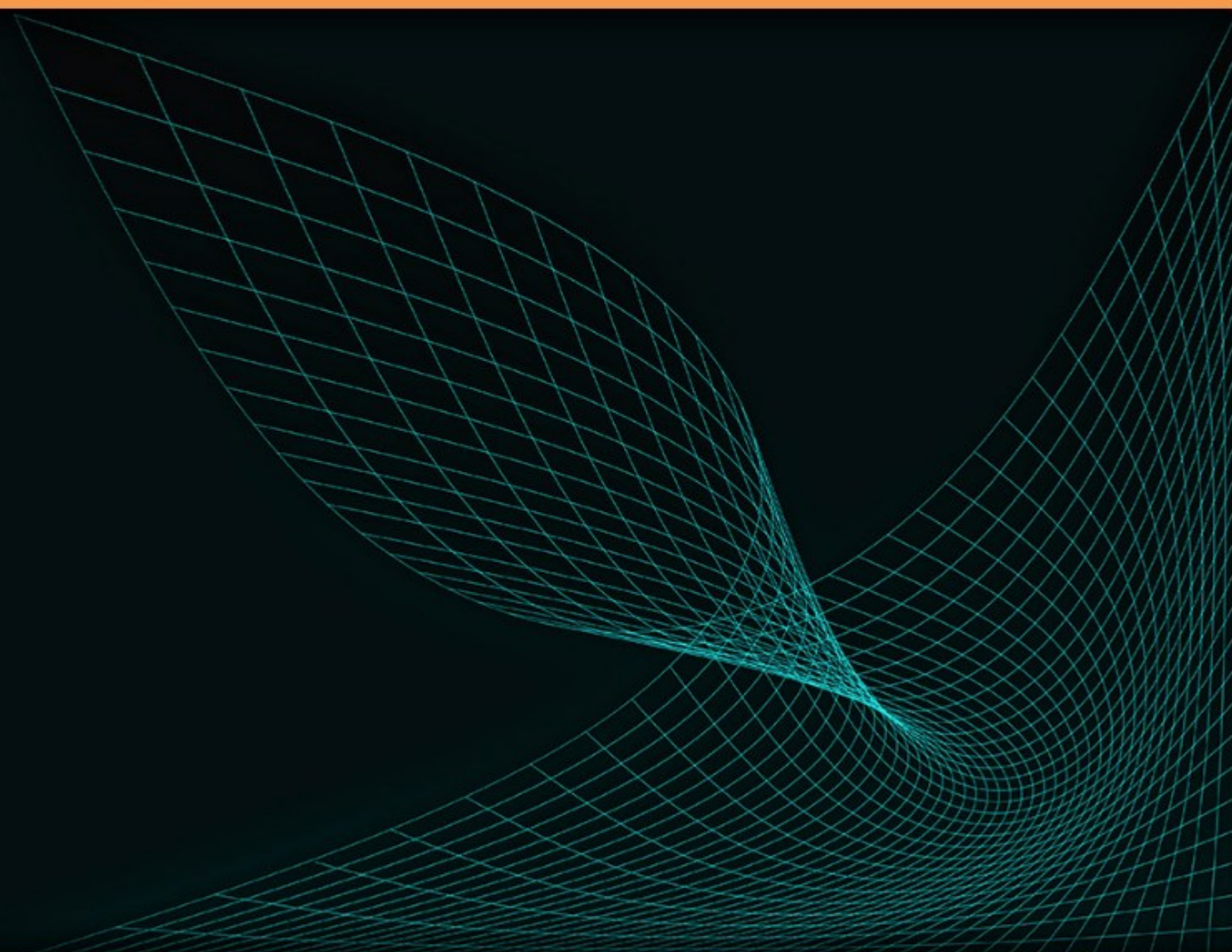


# Spectral Theory

Functional Analysis Examples c-4

Leif Mejlbro



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# Spectral Theory

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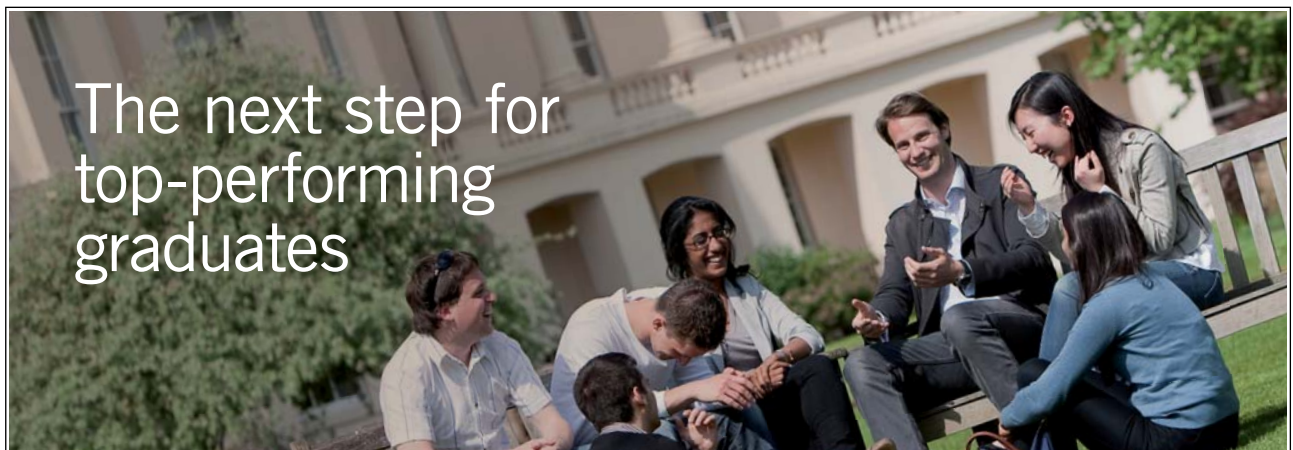
ISBN 978-87-7681-530-1

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# Contents

1.	<b>Spectrum and resolvent</b>	<b>5</b>
2.	<b>The adjoint of a bounded operator</b>	<b>28</b>
3.	<b>Self adjoint operator</b>	<b>42</b>
4.	<b>Isometric operators</b>	<b>57</b>
5.	<b>Unitary and normal operators</b>	<b>61</b>
6.	<b>Positive operators and projections</b>	<b>67</b>
7.	<b>Compact operators</b>	<b>71</b>
	<b>Index</b>	<b>80</b>

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## 1 Spectrum and resolvent

**Example 1.1** Define, for  $h \in \mathbb{R}$ , the operator  $\tau_h$  on  $L^2(\mathbb{R})$  by

$$\tau_h f(x) = f(x - h).$$

Show that  $\tau_h$  is bounded.

Obviously,  $\tau_h$  is linear, and it follows from

$$\|\tau_h f\|_2^2 = \int_{-\infty}^{+\infty} |f(x - h)|^2 dx = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

that  $\|Tf\|_2 = \|f\|_2$  for all  $f \in L^2(\mathbb{R})$ , hence  $\|T\| = 1$ .

**Remark 1.1** Here we add that  $\tau_h$  is also regular. In fact, if  $\tau_h f = 0$ , then  $f(x - h) = 0$  for all  $x \in \mathbb{R}$ , thus  $f \equiv 0$ . This shows that  $\tau_h$  is injective, hence the inverse operator exists. Then we get by the change of variable  $y = x - h$ , i.e.  $x = y + h$ , that  $\tau_h f(x + h) = f(x)$ , and we infer that

$$(\tau_h)^{-1} f(x) = f(x + h) = \tau_{-h} f(x),$$

so also  $\|(\tau_h)^{-1}\| = 1$ , and we have proved that  $\tau_h$  is regular for every  $h \in \mathbb{R}$ .  $\diamond$

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**Example 1.2** Consider in  $L^2(\mathbb{R})$  the operator  $Q$  defined by

$$Qf(x) = xf(x),$$

with

$$D(Q) = \{f \in L^2(\mathbb{R}) \mid Qf \in L^2(\mathbb{R})\}.$$

Determine  $\varrho(Q)$  and  $\sigma_p(Q)$ .

A qualified guess is that  $\varrho(Q) = \mathbb{C} \setminus \mathbb{R}$ . Let  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . We shall prove that  $Q_\lambda = Q - \lambda I$  is regular. Write  $\lambda = \xi + i\eta$ , where  $\xi, \eta \in \mathbb{R}$  and  $\eta \neq 0$ . It follows from the equation

$$Q_\lambda f(x) = Qf(x) - \lambda f(x) = (x - \lambda)f(x) = g(x),$$

that

$$Q_\lambda^{-1}g(x) = f(x) = \frac{g(x)}{x - \lambda} = \frac{g(x)}{(x - \xi) + i\eta}.$$

It follows for  $\eta \neq 0$  that

$$|Q_\lambda^{-1}g(x)|^2 = \frac{|g(x)|^2}{|(x - \xi) + i\eta|^2} \leq \frac{1}{|\eta|^2} |g(x)|^2,$$

and we infer that  $Q_\lambda^{-1}$  is defined on all of  $L^2(\mathbb{R})$ , and

$$\|Q_\lambda^{-1}\|_2 \leq \frac{1}{|\eta|} \|g\|_2.$$

Hence,

$$\|Q_\lambda^{-1}\| \leq \frac{1}{|\eta|} = \frac{1}{|\operatorname{Im} \lambda|},$$

and we have proved that  $\mathbb{C} \setminus \mathbb{R} \subseteq \varrho(Q)$ .

Then let  $\lambda \in \mathbb{R}$ . As before,  $Q_\lambda^{-1}$  is defined by

$$Q_\lambda^{-1}g(x) = \frac{g(x)}{\lambda - x},$$

only the domain is now given by

$$D(Q_\lambda^{-1}) = \left\{ g \in L^2(\mathbb{R}) \mid \frac{g(x)}{\lambda - x} \in L^2(\mathbb{R}) \right\}.$$

Due to the singularity at  $x = \lambda$ , the inverse  $Q_\lambda^{-1}$  is not defined in all of  $L^2(\mathbb{R})$ . However, it is easily seen that the subspace

$$U = \{f \in L^2(\mathbb{R}) \mid \exists \varepsilon > 0 \forall x \in [\lambda - \varepsilon, \lambda + \varepsilon] : f(x) = 0\}$$

of  $L^2(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ , so we conclude from  $U \subseteq D(Q_\lambda^{-1})$  that  $Q_\lambda^{-1}$  is densely defined and unbounded, hence  $\lambda \in \sigma_c(Q)$  for every  $\lambda \in \mathbb{R}$ . Utilizing that the splitting of the spectral sets is disjoint, we conclude that

$$\varrho(Q) = \mathbb{C} \setminus \mathbb{R}, \quad \sigma_p(Q) = \emptyset, \quad \sigma_c(Q) = \mathbb{R}, \quad \sigma_r(Q) = \emptyset.$$

**Example 1.3** Let  $(e_n)$  denote an orthonormal basis in a Hilbert space  $H$ , and consider the operator

$$T \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} a_k e_{k+1}.$$

Determine  $\|T\|$  and  $\sigma(T)$ .

It is well-known that  $T$  is called the *shift operator*. We first analyze  $T_\lambda = T - \lambda I$ , thus

$$T_\lambda x = T_\lambda \left( \sum_{k=1}^{+\infty} a_k e_k \right) = \sum_{k=1}^{+\infty} a_k e_{k+1} - \sum_{k=1}^{+\infty} \lambda a_k e_k = -\lambda a_1 e_1 + \sum_{k=2}^{+\infty} \{a_{k-1} - \lambda a_k\} e_k.$$

Hence, if  $T_\lambda x = 0$ , then

$$\lambda a_1 = 0 \quad \text{and} \quad \lambda a_k = a_{k-1}, \quad k \geq 2.$$

We have two possibilities:

- 1) If  $\lambda = 0$ , then  $a_1 = \lambda a_2 = 0$ , and  $a_{k-1} = \lambda a_k = 0$ , thus  $x = 0$ , and  $T_0 = T$  is injective, so  $\lambda = 0$  is not an eigenvalue.
- 2) If  $\lambda \neq 0$ , then  $a_1 = 0$  and  $a_k = \frac{1}{\lambda} a_{k-1}$ , hence we get by recursion that all  $a_k = 0$ , which means that  $x = 0$ . This proves that every  $T_\lambda$  is injective.

Summing up we have proved that  $T_\lambda^{-1}$  exists for every  $\lambda \in \mathbb{C}$ , sã  $\sigma_p(T) = \emptyset$ .

It follows from

$$\|Tx\|^2 = \left\| T \left( \sum_{k=1}^{+\infty} a_k e_k \right) \right\|^2 = \left\| \sum_{k=1}^{+\infty} a_k e_{k+1} \right\|^2 = \sum_{k=1}^{+\infty} |a_k|^2 = \|x\|^2$$

for all  $x$  that  $\|T\| = 1$ , hence

$$\rho(T) \supseteq \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

Let  $\lambda \neq 0$ ,  $|\lambda| < 1$  and

$$y = \sum_{k=1}^{+\infty} b_k e_k \in H.$$

We shall try to solve the equation  $T_\lambda x = y$ . It follows immediately from the above that

$$-\lambda x_1 = b_1 \quad \text{and} \quad x_{k-1} - \lambda x_k = b_k, \quad k \geq 2,$$

thus

$$x_1 = -\frac{b_1}{\lambda} \quad \text{and} \quad x_k = \frac{1}{\lambda} x_{k-1} - \frac{1}{\lambda} b_k, \quad k \geq 2,$$

from which e.g.  $x_2 = -\frac{b_1}{\lambda^2} - \frac{b_2}{\lambda}$ . Choosing in particular  $y = e_1$  we get  $x_1 = -\frac{1}{\lambda}$ ,  $x_2 = -\frac{1}{\lambda^2}$ , and in general,

$$x_n = -\frac{1}{\lambda^n}, \quad n \in \mathbb{N}.$$



From  $0 < |\lambda| < 1$  follows that  $|x_n| \rightarrow +\infty$  for  $n \rightarrow +\infty$ , so the only possible solution is

$$x = \sum_{n=1}^{+\infty} x_n e_n = - \sum_{n=1}^{+\infty} \frac{1}{\lambda^n} e_n \notin H,$$

which, however, does *not* belong to  $H$ . This shows that

$$e_1 \notin T_\lambda(D(T_\lambda)) = T_\lambda(H).$$

Hence we conclude that  $T_\lambda^{-1}$  exists, but it is unbounded, when  $0 < |\lambda| < 1$ , so

$$\{\lambda \in \mathbb{C} \mid 0 < |\lambda| < 1\} \subseteq \sigma(T).$$

The set  $\sigma(T)$  is closed, so it follows from  $\sigma(T) \cap \rho(T) = \emptyset$  that

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} \quad \text{and} \quad \rho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

**Example 1.4** Consider in  $\ell^2$  the operator

$$(x_1, x_2, x_3, \dots) \mapsto \left( x_1, \frac{1}{2}(x_1 + x_2), \frac{1}{4}(x_1 + x_2 + x_3), \dots, \frac{1}{2^{n-1}}(x_1 + x_2 + \dots + x_n), \dots \right).$$

Show that the operator is bounded and not surjective.

Let  $(e_n)$  denote an orthonormal basis in a Hilbert space  $H$ , and consider the operator

$$T \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=2}^{\infty} \sqrt{k} a_k e_{k-1}.$$

Determine the spectrum  $\sigma(T)$ , and find for each eigenvalue the corresponding eigenvectors.

Assume that

$$Tx = \left( x_1, \frac{1}{2}(x_1 + x_2), \frac{1}{4}(x_1 + x_2 + x_3), \dots \right) = (0, 0, 0, \dots).$$

Then  $x_1 = 0$ ,  $\frac{1}{2}x_2 = 0$ , thus  $x_2 = 0$ , and we get by induction that  $x_n = 0$  for all  $n \in \mathbb{N}$ . It follows that  $Tx = 0$  implies that  $x = 0$ , hence  $T$  is injective.

Then we get

$$\|Tx\|_2^2 = \sum_{n=1}^{+\infty} \frac{1}{4^{n-1}} |x_1 + x_2 + \dots + x_n|^2 \leq \sum_{n=1}^{+\infty} \frac{1}{4^{n-1}} \sum_{j=1}^n n^2 |x_j|^2 \leq \sum_{n=1}^{+\infty} \frac{n^2}{4^{n-1}} \|x\|_2^2,$$

from which we conclude that

$$\|T\| \leq \sqrt{\sum_{n=1}^{+\infty} \frac{n^2}{4^{n-1}}} < +\infty,$$



and  $T$  is bounded.

If

$$y_0 = 0 \quad \text{and} \quad y_n = \frac{1}{2^{n-1}} (x_1 + x_2 + \cdots + x_n),$$

then

$$x_1 + x_2 + \cdots + x_n = 2^{n-1}y_n, \quad \text{thus} \quad x_n = 2^{n-1}y_n - 2^{n-2}y_{n-1}, \quad n \in \mathbb{N}.$$

Choose in particular,  $y = \frac{1}{n}$ ,  $n \in \mathbb{N}$ . Then  $(y_n) \in \ell^2$  with  $\|y\| = \frac{\pi}{\sqrt{6}}$ , while

$$x_n = \frac{2^{n-1}}{n} - \frac{2^{n-2}}{n-1} = 2^{n-2} \cdot \frac{n-2}{n(n-1)} \rightarrow +\infty,$$

according to the rule of magnitudes. In particular, the necessary condition of convergence of  $\sum |x_n|^2$  is not fulfilled. We conclude that  $T$  is not surjective,  $T\ell^2 \neq \ell^2$ , hence  $T$  is singular.

Let us first find the point spectrum, i.e. let  $\lambda \in \sigma_p(T)$  be an eigenvalue. Then there exists a vector  $x \neq 0$ , such that  $Tx = \lambda x$ , which can also be written

$$T \left( \sum_{k=1}^{+\infty} x_k e_k \right) = \sum_{k=2}^{+\infty} \sqrt{k} \cdot x_k e_{k-1} = \sum_{k=1}^{+\infty} \sqrt{k-1} \cdot x_{k+1} e_k = \sum_{k=1}^{+\infty} \lambda x_k e_k.$$

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Then

$$x_{k+1} = \frac{\lambda}{\sqrt{k+1}} x_k = \cdots = \frac{\lambda^k}{\sqrt{(k+1)!}} \cdot x_1.$$

Choosing  $x_1 = 1$  we see that if  $x$  is an eigenvector with  $x_1 = 1$ , then  $x$  necessarily has the form

$$x = \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{\sqrt{k!}} e_k.$$

It only remains to check if the constructed  $x$  belongs to  $H$ . We get

$$\|x\|^2 = \sum_{k=1}^{+\infty} |x_k|^2 = \sum_{k=1}^{+\infty} \frac{|\lambda^2|^{k-1}}{k!} = \frac{1}{|\lambda|^2} \left\{ e^{|\lambda|^2} - 1 \right\},$$

because the series is convergent for all  $\lambda \in \mathbb{C}$ , and the sum function above has a removable singularity for  $\lambda = 0$ . (Notice that  $e_1$  is an eigenvector corresponding to  $\lambda = 0$ ). We infer that

$$\sigma(T) = \sigma_p(T) = \mathbb{C},$$

and the given linear operator has every complex  $\lambda \in \mathbb{C}$  as an eigenvalue.

**Example 1.5** Let  $(e_n)$  denote an orthonormal basis in a Hilbert space  $H$ . We define the sequence  $(f_k)_{k \in \mathbb{Z}}$  by

$$\begin{aligned} f_0 &= e_1, \\ f_k &= e_{2k+1} \quad \text{for } k > 0, \\ f_k &= e_{-2k} \quad \text{for } k < 0. \end{aligned}$$

In this way  $(f_k)_{k \in \mathbb{Z}}$  is an orthonormal basis. We define the double sided shift operator  $S$  by

$$S \left( \sum_{k=-\infty}^{\infty} a_k f_k \right) = \sum_{k=-\infty}^{\infty} a_k f_{k+1}.$$

Show that  $S$  is a bounded operator and show that  $S$  has no eigenvalues.

First notice that

$$\sum_{k=-\infty}^{+\infty} a_k f_k = \sum_{k=0}^{+\infty} a_k e_{2k+1} + \sum_{k=1}^{+\infty} a_{-k} e_{2k},$$

and

$$T \left( \sum_{k=-\infty}^{+\infty} a_k f_k \right) = \sum_{k=-\infty}^{+\infty} a_k f_{k+1} = \sum_{k=-\infty}^{+\infty} a_{k-1} f_k = \sum_{k=0}^{+\infty} a_{k-1} e_{2k+1} + \sum_{k=1}^{+\infty} a_{-k-1} e_{2k}.$$

From  $(f_k)_{k \in \mathbb{Z}}$  being an orthonormal basis follows that

$$\left\| T \left( \sum_{k=-\infty}^{+\infty} a_k f_k \right) \right\|^2 = \left\| \sum_{k=-\infty}^{+\infty} a_k f_{k+1} \right\|^2 = \sum_{k=-\infty}^{+\infty} |a_k|^2 = \left\| \sum_{k=-\infty}^{+\infty} a_k f_k \right\|^2,$$

from which  $\|T\| = 1$  and  $T \in B(H)$ .

Assume that the equation  $Tx = \lambda x$  is fulfilled. It follows from the above that

$$\lambda a_k = a_{k-1} \quad \text{for } k \in \mathbb{N}_0, \quad \text{and} \quad \lambda a_{-k} = a_{-k-1} \quad \text{for } k \in \mathbb{N}.$$

If  $\lambda = 0$ , then  $Tx = 0$ , and we get from  $\|Tx\| = \|x\| = 0$  that  $x = 0$ , hence  $\lambda = 0 \notin \sigma_p(T)$ .

If  $\lambda \neq 0$ , then we get by recursion,

$$a_k = \frac{1}{\lambda^{k+1}} a_{-1} \quad \text{for } k \in \mathbb{N}_0, \quad \text{and} \quad a_{-k-1} = \lambda^k a_{-1} \quad \text{for } k \in \mathbb{N}.$$

Thus, if  $a_{-1} \neq 0$ , then all possible  $a_k \neq 0$ , and we get

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} |a_k|^2 &= \sum_{k=0}^{+\infty} \frac{1}{|\lambda^{2k+1}|} |a_{-1}|^2 + \sum_{k=1}^{+\infty} |\lambda^{2k}| \cdot |a_{-1}|^2 \\ &= |a_{-1}|^2 \sum_{k=-\infty}^{+\infty} |\lambda^{2k}|, \end{aligned}$$

which of course is divergent for every  $\lambda \in \mathbb{C}$ . We conclude that  $T$  does not have eigenvalues, hence  $\sigma_p(T) = \emptyset$ .

**Example 1.6** Define, for  $h \in \mathbb{R}_+$ , the operator  $\tau_h$  on  $L^2(\mathbb{R})$  by

$$\tau_h f(x) = f(x - h).$$

Show that  $\tau_h$  has no eigenvalues and that

$$\sigma(\tau_h) \subset \{z \in \mathbb{C} \mid |z| = 1\}.$$

(It is in fact true that  $\sigma(\tau_h) = \{z \in \mathbb{C} \mid |z| = 1\}$ .)

**Remark 1.2** Note that if  $h = 0$ , then  $\tau_0 = I$ , and  $\lambda = 1$  is trivially an eigenvalue with all of  $L^2(\mathbb{R})$  as its eigenspace. For that reason we assume that  $h > 0$ .  $\diamond$

It follows from

$$\|\tau_h f\|_2^2 = \int_{-\infty}^{+\infty} |f(x - h)|^2 dx = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

that  $\|\tau_h\| = 1$ , hence

$$\sigma(\tau_h) \subseteq \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Assume that

$$\tau_h f(x) = f(x - h) = \lambda f(x), \quad \text{where } |\lambda| \leq \|\tau_h\| = 1.$$

If  $|\lambda| = 1$ , then  $|f(x-h)| = |f(x)|$ ,  $h > 0$ . Thus the function  $|f(x)|$  is periodic of period  $h > 0$ , hence

$$\|f\|_2^2 = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{+\infty} \int_0^h |f(x)|^2 dx < +\infty.$$

This is of course only possible, if  $\int_0^h |f(x)|^2 dx = 0$ , i.e. if  $f(x) = 0$  for almost every  $x \in [0, h]$ , and hence for almost every  $x \in \mathbb{R}$ . Then  $x$  is represented by the zero function, and we infer that no  $\lambda \in \mathbb{C}$  satisfying  $|\lambda| = 1$  can be an eigenvalue.

It has previously been proven in EXAMPLE 1.1 that  $(\tau_h)^{-1} = \tau_{-h}$ . Of course, this can also be proved directly,

$$\tau_{-h}\tau_h f(x) = \tau_{-h}f(x-h) = f(x-h+h) = f(x) = If(x),$$

and

$$\tau_h\tau_{-h}f(x) = \tau_h f(x+h) = f(x+h-h) = f(x) = If(x).$$

It is also obvious that  $\|(\tau_h)^{-1}\| = \|\tau_{-h}\| = 1$ , and  $(\tau_h)^{-1} \in B(H)$ . Thus if  $|\lambda| < 1$ , then

$$(\tau_h - \lambda I)^{-1} = (\tau_h)^{-1} (I - \lambda (\tau_h)^{-1})^{-1} \in B(H),$$

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because  $\|\lambda(\tau_h)^{-1}\| = |\lambda| < 1$ . Therefore,  $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subset \rho(\tau_h)$ , and thus

$$\rho(\tau_h) \supseteq \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\},$$

which implies that

$$\sigma(\tau_h) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Finally,

$$\sigma_p(\tau_h) \subseteq \sigma(\tau_h) \quad \text{og} \quad \sigma_p(\tau_h) \cap \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} = \emptyset,$$

from which follows that  $\sigma_p(\tau_h) = \emptyset$ .

**Example 1.7** Given below some closed linear operators from  $\ell^2$  into  $\ell^2$ . Check in each case if the operator is singular.

1)  $T_1x = (x_2, x_3, \dots)$ .

2)  $T_2z = \left(\frac{1}{2}x_1, \frac{1}{2^2}x_2, \frac{1}{2^3}x_3, \dots\right)$ .

3)  $T_3x = (0, x_1, x_2, \dots)$ .

4)  $T_4x = (0, x_2, x_3, \dots)$ .

A linear operator is singular, if at least one of the following three conditions is satisfied:

- 1) There exists an  $f \in D(T) \setminus \{0\}$ , such that  $Tf = 0$ .
- 2) The inverse  $T^{-1}$  exists, and  $\overline{D(T^{-1})} = \overline{TD(T)} = Y$ , while  $T^{-1}$  itself is unbounded.
- 3) The inverse  $T^{-1}$  exists, but it is not densely defined in  $Y$ , thus  $\overline{TD(T)} \neq Y$ .

We shall below check these three conditions.

- 1) It follows by choosing  $x = (1, 0, 0, \dots) \neq 0$  that  $T_1x = 0$ , hence  $T_1$  is singular of type (1). This means that  $0 \in \sigma_p(T_1)$ , i.e. 0 is an eigenvalue of  $T_1$ .
- 2) Clearly,  $T_2x = 0$  implies that  $x = 0$ , so  $T_2$  is injective and the inverse exists. Then we solve the equation  $T_2x = y$ , thus

$$T_2x = \left(\frac{1}{2}x_1, \frac{1}{2^2}x_2, \frac{1}{2^3}x_3, \dots\right) = (y_1, y_2, y_2, \dots) = y.$$

When we identify the coordinates we get  $\frac{1}{2^n}x_n = y_n$ , hence  $x_n = 2^n y_n$ , and the inverse operator  $T_2^{-1}$  is given by

$$T_2^{-1}y = (2y_1, 2^2y_2, 2^3y_3, \dots)$$

for

$$y \in D(T_2^{-1}) = \left\{ y \in \ell^2 \mid \sum_{n=1}^{+\infty} 2^{2n}|y_n|^2 < +\infty \right\} \subset \ell^2.$$

Let  $U$  be the subspace consisting of all sequences which are 0 eventually. Then clearly,

$$U \subset D(T_2^{-1}) \subset \ell^2.$$

The subspace  $U$  is dense in  $\ell^2$ , so this is also the case for the larger subspace  $D(T_2^{-1})$ . Furthermore, it follows from the definition of the inverse  $T_2^{-1}$  that it is unbounded, i.e.  $T_2$  is singular of type (2). This means that  $0 \in \sigma_c(T_2)$  lies in the continuous spectrum for  $T_2$ .

3) It is obvious that  $T_3$  is injective and that

$$T_3^{-1}y = (y_2, y_2, y_4, \dots)$$

for

$$y \in D(T_3^{-1}) \{y \in \ell^2 \mid y_1 = 0\}.$$

Clearly,  $T_3^{-1}$  is bounded, though not densely defined, so  $T_3$  is of type (3), corresponding to that  $0 \in \sigma_r(T_3)$  lies in the residual spectrum for  $T_3$ .

4) We infer from  $T_4x = 0$  for  $x = (1, 0, 0, \dots) \neq 0$  that 0 is an eigenvalue,  $0 \in \sigma_p(T_4)$ , hence  $T_4$  is singular of type (1).

**Example 1.8** Let  $V$  denote the Banach space  $(C([0, 1]), \|\cdot\|_\infty)$ , and let the operator  $T$  be given by

$$Tf(x) = \int_0^x f(t) dt, \quad f \in V.$$

Check if  $T$  is regular.

The inverse operator of  $T$  is the differential operator  $\mathcal{D}$ , given by

$$D(\mathcal{D}) = \{f \in C^1([0, 1]) \mid f(0) = 0\},$$

$$\mathcal{D}f = \frac{df}{dx} = f' \quad \text{for } f \in C^1([0, 1]), \quad f(0) = 0.$$

It is easily seen (e.g. by using Weierstraß's Approximation Theorem) that  $D(\mathcal{D})$  is dense in  $V$ . On the other hand,  $\mathcal{D}$  is unbounded. In fact, choose

$$f_n(x) = \sin(\pi nx), \quad x \in [0, 1], \quad f_n \in D(\mathcal{D}).$$

then

$$\mathcal{D}f_n(x) = \pi n \cdot \cos(\pi nx), \quad x \in [0, 1],$$

hence  $\|f_n\|_\infty = 1$  and  $\|\mathcal{D}f_n\|_\infty = \pi n$ .

**Remark 1.3** A simpler example is of course  $g_n(x) = x^n$ ,  $x \in [0, 1]$ . However, the  $f_n$  occur very frequently as an example in other cases, so we have chosen to present it here.  $\diamond$

We have proved that  $T$  is singular of type (2), i.e.  $0 \in \sigma_c(T)$  lies in the continuous spectrum for  $T$ .

**Example 1.9** Let  $H = L^2(\mathbb{R})$ , and let  $g$  be a bounded continuous real function defined on  $\mathbb{R}$ . Prove that the operator  $T$  given by

$$Tf(x) = g(x)f(x), \quad f \in L^2(\mathbb{R}),$$

belongs to  $B(H)$ .

Find a necessary and sufficient condition on  $g$  that  $T$  is regular.

When  $g$  is bounded,  $\|g\|_\infty < +\infty$ , then

$$\|Tf\|_2^2 = \int_{-\infty}^{+\infty} g(x)^2 |f(x)|^2 dx \leq \|g\|_\infty^2 \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|g\|_\infty^2 \cdot \|f\|_2^2,$$

hence  $\|Tf\|_2 \leq \|g\|_\infty \cdot \|f\|_2$  for all  $f \in H$ , and we infer that  $T \in B(H)$  with  $\|T\| \leq \|g\|_\infty$ .

Then we shall find when  $T$  is regular, i.e. when  $T$  fulfils the following three conditions:

- 1) The equation  $Tf = 0$  has only the trivial solution  $f = 0$ , so  $T^{-1}$  exists.
- 2) The inverse operator  $T^{-1}$  is densely defined, i.e.

$$D(T^{-1}) = T(L^2(\mathbb{R}))$$

is dense in  $L^2(\mathbb{R})$ .

- 3) The inverse operator  $T^{-1}$  is bounded.

We now check each of these conditions:

- 1) It follows from  $Tf(x) = g(x) \cdot f(x)$  that  $Tf = 0$ , if and only if  $g(x) \cdot f(x) = 0$  for almost every  $x \in \mathbb{R}$ . Therefore, if we want always to conclude that  $f = 0$  (in  $L^2(\mathbb{R})$ ), then we must assume that  $g(x) \neq 0$  for almost every  $x \in \mathbb{R}$ .
- 2) Then we want that  $T^{-1}$  is bounded. It follows from  $Tf(x) = g(x)f(x) = h(x)$  that

$$f(x) = T^{-1}h(x) = \frac{1}{g(x)} h(x),$$

and then the same consideration as above shows that we must require that

$$\left\| \frac{1}{g} \right\|_\infty < +\infty.$$

- 3) Based on the conditions above, assume that

$$0 < b \leq |g(x)| < a < +\infty, \quad \text{for all } x \in \mathbb{R}.$$

Then clearly all three conditions are fulfilled, so these conditions are sufficient that both  $T$  and  $T^{-1} \in B(H)$ .



**Example 1.10** Let  $(e_k)$  denote an orthonormal basis in a Hilbert space  $H$ , and let the operator  $T$  be defined by

$$T \left( \sum_{k=1}^{+\infty} a_k e_k \right) = \sum_{k=2}^{+\infty} a_k e_{k-1}.$$

Prove that  $\lambda$  is an eigenvalue for  $T$ , if and only if  $|\lambda| < 1$ .  
Find  $\sigma(T)$  and  $\rho(T)$ .

Assume that  $\lambda \in \sigma_p(T)$ , thus there exists

$$x = \sum_{k=1}^{+\infty} x_k e_k, \quad \text{where} \quad 0 < \sum_{k=1}^{+\infty} |x_k|^2 < +\infty,$$

such that  $Tx = \lambda x$ , i.e.

$$\sum_{k=2}^{+\infty} x_k e_{k-1} = \sum_{k=1}^{+\infty} x_{k+1} e_k = \lambda \sum_{k=1}^{+\infty} x_k e_k.$$

When we identify the coordinates we get

$$x_{k+1} = \lambda x_k, \quad k \in \mathbb{N}.$$

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Choosing  $x_1 = 1$ , we get by either induction or by recursion – both methods can be applied – that  $x_k = \lambda^{k-1}$ , and an eigenvector corresponding to the eigenvalue  $\lambda$  must *necessarily* be of the form

$$x = x_1 \sum_{k=1}^{+\infty} \lambda^{k-1} e_k.$$

This *candidate* belongs to the Hilbert space, if and only if

$$\sum_{k=1}^{+\infty} |\lambda^{k-1}|^2 = \sum_{k=0}^{+\infty} |\lambda|^{2k} < +\infty,$$

i.e. if and only if  $|\lambda| < 1$ . We infer that

$$\sigma_p(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

If on the other hand  $\lambda \in \mathbb{C}$  satisfies  $|\lambda| < 1$ , then we get by insertion that  $x = \sum_{k=1}^{+\infty} \lambda^{k-1} e_k$  is an eigenvector, so  $\lambda \in \sigma_p(T)$ , and we have proved that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

Then assume that  $\lambda \in \mathbb{C}$  satisfies  $|\lambda| > 1$ . We shall prove that  $\lambda \in \varrho(T)$ .

**Remark 1.4** If here one tries directly to find the inverse operator  $T_\lambda^{-1}$ , thus try to solve the equation  $T_\lambda x = y$  with respect to  $x \in H$  for given  $yn \in H$ , then we end up with an unpleasant infinite system of equations of the form

$$(1) \quad x_{k+1} - \lambda x_k = y_k, \quad k \in \mathbb{N},$$

where the solution also must satisfy

$$\sum_{k=1}^{+\infty} |x_k|^2 < +\infty.$$

Even this is possible, it is very difficult to solve this system of equations. Hence we search an alternative method of solution.  $\diamond$

We note that

$$\|Tx\| = \left\| \sum_{k=1}^{+\infty} x_{k+1} e_k \right\| \leq \|x\|,$$

where we get equality, when  $x_1 = 0$ . This shows that  $\|T\| = 1$ .

It follows from

$$T_\lambda = T - \lambda I = -\lambda I \left( I - \frac{1}{\lambda} T \right), \quad |\lambda| > 1,$$

and

$$\left\| \frac{1}{\lambda} T \right\| = \frac{1}{|\lambda|} < 1,$$

by using the Neumann series that

$$T_\lambda^{-1} = -\frac{1}{\lambda} \left( I - \frac{1}{\lambda} T \right)^{-1} \in B(H).$$

**Remark 1.5** The explicit solution is given by the Neumann series

$$x = T_\lambda^{-1} y = -\frac{1}{\lambda} \sum_{j=0}^{+\infty} \frac{1}{\lambda^j} T^j y,$$

which can also be found directly, if we work on (1). However, the precise solution is not so interesting in this connection.  $\diamond$

We infer that

$$\{\lambda \in \mathbb{C} \mid |\lambda| > 1\} \subseteq \varrho(T).$$

Now,  $\sigma(T)$  is *closed* and disjoint from  $\varrho(T)$ , and

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma(T),$$

hence

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} \quad \text{og} \quad \varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

**Example 1.11** Consider the Banach space  $(C([0, 1]), \|\cdot\|_\infty)$ . Let  $v \in C([0, 1])$  be real, and let the operator  $T$  be defined by

$$Tf(x) = v(x)f(x).$$

Find  $\sigma(T)$  and  $\varrho(T)$ .

We conclude from

$$\|Tf\|_\infty = \|v(x)f(x)\|_\infty \leq \|v\|_\infty \|f\|_\infty,$$

where we get equality by choosing  $f = v$ , that  $\|T\| = \|v\|_\infty$ . Then it follows that

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|v\|_\infty\}.$$

Now,  $v$  is continuous, and  $[0, 1]$  is compact, hence  $v([0, 1])$  is also compact. Let  $\lambda \notin v([0, 1])$ . Then there exists a  $b_\lambda > 0$ , such that

$$|v(x) - \lambda| \geq b_\lambda \quad \text{for all } x \in [0, 1].$$

Then

$$T_\lambda f(x) = \{v(x) - \lambda\}f(x) = g(x) \in C([0, 1])$$

for

$$f(x) = T_\lambda^{-1}g(x) = \frac{g(x)}{v(x) - \lambda} \in C([0, 1]).$$

It follows that  $\|T_\lambda\| \leq \frac{1}{b_\lambda}$ , hence  $T_\lambda \in B(C([0, 1]))$ , and

$$\varrho(T) \supseteq \mathbb{C} \setminus v([0, 1]) \quad \text{and} \quad \sigma(T) \subseteq v([0, 1]).$$

If conversely  $\lambda \in v([0, 1])$ , then there exists an  $x_0 \in [0, 1]$ , such that  $v(x_0) = \lambda$ . Then the equation  $T_\lambda f = g$  cannot be solved for any  $g$ , for which  $f(x_0) \neq 0$ , because then the candidate  $f$  then will not be continuous at  $x_0$ . Hence we finally get

$$\sigma(T) = v([0, 1]) \quad \text{and} \quad \varrho(T) = \mathbb{C} \setminus v([0, 1]).$$

**Example 1.12** Consider in the Banach space  $\ell^\infty$  the operator  $T$  given by

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Find  $\varrho(T)$ ,  $\sigma_p(T)$ ,  $\sigma_c(T)$  and  $\sigma_r(T)$ .

We get from  $\|Tx\|_\infty \leq \|x\|_\infty$  with equality for

$$|x_1| \leq \sup_{i \geq 2} |x_i|,$$

that  $\|T\| = 1$ , hence  $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ .

Therefore, if  $\lambda \in \sigma_p(T)$ , then  $|\lambda| \leq 1$ , and there exists an  $x \neq 0$ , such that  $Tx = \lambda x$ , i.e.

$$x_{k+1} = \lambda x_k = \dots = \lambda^k x_1.$$

We can therefore put  $x_1 = 1$  for an eigenvector, and thus any eigenvector has the form of a constant times

$$(1, \lambda, \lambda^2, \dots, \lambda^{n-1}, \dots).$$

It follows by insertion that this candidate indeed is an eigenvector, if it belongs to  $\ell^\infty$ , i.e. if  $|\lambda| \leq 1$ . We conclude that

$$\sigma_p(T) = \sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and

$$\varrho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\},$$

and  $\sigma_c(T) = \sigma_r(T) = \emptyset$ .

**Example 1.13** Let  $T : \ell^2 \rightarrow \ell^2$  denote the operator

$$T(x_1, x_2, \dots, x_n, \dots) = (x_2, x_4, \dots, x_{2n}, \dots).$$

Find  $\|T\|$ .

Find all eigenvalues for  $T$ .

Show that the eigenspace corresponding to any eigenvalue is infinite dimensional.

Determine the operators  $T^*$ ,  $TT^*$  and  $T^*T$ .

Determine  $\sigma(T)$  and  $\rho(T)$ .

1) We infer from

$$\|Tx\|^2 = \sum_{n=1}^{+\infty} |x_{2n}|^2 \leq \sum_{n=1}^{+\infty} |x_n|^2 = \|x\|^2$$

for every  $x \in \ell^2$  that  $\|T\| \leq 1$ .

For  $x = (0, x_2, 0, x_4, 0, x_6, 0, \dots)$  we get in particular that

$$\|Tx\|^2 = \|T(0, x_2, 0, x_4, 0, x_6, \dots)\|^2 = \sum_{n=1}^{+\infty} |x_{2n}|^2 = \sum_{n=1}^{+\infty} |x_n|^2 = \|x\|^2,$$

and we conclude that  $\|T\| = 1$ .

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- 2) Assume that  $\lambda \in \sigma_p(T)$ . Then there exists an  $x \in \ell^2 \setminus \{0\}$ , such that  $Tx = \lambda x$ . We get for the  $n$ -th coordinate of this equation that

$$(2) \quad x_{2n} = \lambda x_n, \quad n \in \mathbb{N}.$$

If  $\lambda = 0$ , then we get the conditions  $x_{2n} = 0$ ,  $n \in \mathbb{N}$ . It follows that if

$$\sum_{n=0}^{+\infty} |x_{2n+1}|^2 < +\infty,$$

then  $(x_1, 0, x_3, 0, x_5, 0, \dots)$  is an eigenvector corresponding to the eigenvalue  $\lambda = 0$ , hence  $0 \in \sigma_p(T)$ , and the eigenspace corresponding to  $\lambda = 0$  is spanned by  $\{e_{2n-1} \mid n \in \mathbb{N}\}$ , hence it is infinite dimensional, cf. the third question.

Assume that  $\lambda \in \sigma_p(T) \setminus \{0\}$ . Then it follows from (2) with  $n = 2^{m-1}q$  that

$$x_{2^m q} = \lambda x_{x^{n-1}q} = \lambda^2 x_{2^{m-2}q} = \dots = \lambda^m x_q, \quad m \in \mathbb{N}.$$

We get in particular for  $q = 1$ ,

$$x_{2^m} = \lambda^m x_1.$$

If we put  $x_1 = 1$  and  $x^r = 0$ , when  $r$  is not of the form  $2^n$ , we get an eigenvector, if and only if

$$\sum_{n=0}^{+\infty} |\lambda^n|^2 < +\infty.$$

This condition is fulfilled if and only if  $|\lambda| < 1$ . Hence we conclude that the point spectrum is given by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

- 3) Assume that  $\lambda \in \sigma_p(T)$ , so  $|\lambda| < 1$ . Then we get by a simple computation that every odd index  $2q + 1$ ,  $q \in \mathbb{N}_0$ , determines an eigenvector  $x$  by

$$x_{(2q+1) \cdot 2^n} = \lambda^n, \quad n \in \mathbb{N}_0, \quad \text{og} \quad x_r = 0 \text{ otherwise.}$$

All these eigenvectors are linearly independent, so we conclude that the eigenspace corresponding to an eigenvalue  $\lambda \in \sigma_p(T)$  is infinite dimensional.

- 4) Now,  $T \in B(\ell^2)$ , and  $\|T\| = 1$ , so  $T^* \in B(\ell^2)$  and  $\|T^*\| = 1$ .

We have for every  $x \in \ell^2$  and every  $y \in \ell^2$  that

$$\begin{aligned} (Tx, y) &= ((x_2, x_4, x_6, \dots), (y_1, y_2, y_3, \dots)) = \sum_{n=1}^{+\infty} x_{2n} \overline{y_n} \\ &= ((0, x_2, 0, x_4, 0, \dots), (0, y_1, 0, y_2, 0, \dots)) = (x, T^*y), \end{aligned}$$

so we infer that

$$T^*y = T^*(y_1, y_2, y_3, \dots) = (0, y_1, 0, y_2, 0, y_3, \dots), \quad y \in \ell^2.$$

Furthermore,

$$TT^*x = T(T^*(x_1, x_2, x_3, \dots)) = T(0, x_1, 0, x_2, 0, x_3, \dots) = (x_1, x_2, x_3, \dots) = x,$$

i.e.  $TT^* = I$ , and

$$T^*Tx = T^*(T(x_1, x_2, x_3, \dots)) = T^*(x_2, x_4, x_6, \dots) = (0, x_2, 0, x_4, 0, x_6, \dots),$$

proving that  $T^*T = P$  is the projection onto the subspace of  $\ell^2$  which is spanned by  $\{e_{2n} \mid n \in \mathbb{N}\}$ .

5) It follows from  $\|T\| = 1$  that

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\} = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

Furthermore,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and the spectrum is closed, hence

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} \quad \text{og} \quad \rho(T) = \mathbb{C} \setminus \sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

**Remark 1.6** It is also easy to prove that

$$\sigma_p(T^*) = \emptyset.$$

In fact, we get from  $T^*y = \lambda y$  that

$$(0, y_1, 0, y_2, 0, y_3, \dots) = \lambda(y_1, y_2, y_3, y_4, y_5, y_6, \dots).$$

If  $\lambda = 0$ , then the right hand side is 0, and this implies that  $y_n = 0$ , thus  $y = 0$ , and  $0 \notin \sigma_p(T^*)$ . If  $\lambda \neq 0$ , then

$$0 = \lambda y_{2n+1}, \quad n \in \mathbb{N}_0, \quad \text{and} \quad y_n = \lambda y_{2n}, \quad n \in \mathbb{N}.$$

The former equation gives  $y_{2n+1} = 0$ , which is then inserted into the latter (follows by an iteration, when  $n$  is even) to give  $y_{2n} = 0$ , hence  $y = 0$ , and we have proved that  $\sigma_p(T^*) = \emptyset$ .

Now,  $\sigma_p(T^*) = \emptyset$ , hence also  $\sigma_r(T) = \emptyset$ . Since  $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$  is a disjoint splitting of the spectrum, we conclude that

$$\begin{aligned} \rho(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}, \\ \sigma(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}, \\ \sigma_p(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}, \\ \sigma_c(T) &= \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}, \\ \sigma_r(T) &= \emptyset. \quad \diamond \end{aligned}$$



**Example 1.14** Let  $X$  denote the Banach space of  $C([-1, 1])$ -functions equipped with the usual sup-norm  $\|\cdot\|_\infty$ , and let  $T \in B(X)$  be given by

$$Tf = f(0) + f.$$

- 1) Find the norm of  $T$ .
- 2) Determine the resolvent set  $\rho(T)$  for  $T$  and find

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

for all  $\lambda \in \rho(T)$ .

- 3) Show that the spectrum for  $T$  is a pure point spectrum and find all eigenvalues and corresponding eigenfunctions.
- 4) Show that all  $f \in X$  can be written as a sum of eigenfunctions belonging to different eigenspaces, and show that this decomposition is unique.

- 1) Clearly,

$$\|Tf\|_\infty \leq |f(0)| + \|f\|_\infty \leq \|f\|_\infty + \|f\|_\infty = 2\|f\|_\infty,$$

where we obtain equality if e.g.  $f$  is a real function with maximum at 0, i.e.  $\|T\| = 2$ .

- 2) Then we shall check when it is possible for all  $g \in X$  to solve the equation

$$(T - \lambda I)g = g, \quad f \in X.$$

We get

$$(3) \quad g(x) = Tf(x) - \lambda f(x) = f(0) + f(x) - \lambda f(x).$$

In particular for  $x = 0$ ,

$$g(0) = f(0) + f(0) - \lambda f(0) = (2 - \lambda)f(0).$$

Now, the solution  $f$  must be continuous, so this equation *cannot* be solved for arbitrary  $g \in X$ , when  $\lambda = 2$ , hence  $2 \in \sigma(T)$ .

If  $\lambda \neq 2$ , then

$$f(0) = \frac{1}{2 - \lambda} g(0),$$

which gives by insertion into (3),

$$g(x) - \frac{1}{2 - \lambda} g(0) = (1 - \lambda)f(x).$$

Hence, if  $\lambda = 1$ , then this equation *cannot* be solved for an arbitrary  $g \in X$ , so  $1 \in \sigma(T)$ . If we assume that  $\lambda \neq 1$ , then we get the candidate of the solution

$$f(x) = T_\lambda^{-1}g(x) = \frac{1}{1 - \lambda}g(x) - \frac{1}{(1 - \lambda)(2 - \lambda)}g(0),$$

which is clearly continuous, when  $g$  is continuous. Finally,

$$\|T_\lambda^{-1}g\|_\infty \leq \left\{ \frac{1}{|1-\lambda|} + \frac{1}{|1-\lambda|\cdot|2-\lambda|} \right\} \|g\|_\infty = C(\lambda) \|g\|_\infty.$$

This implies that  $\rho(T) \supseteq \mathbb{C} \setminus \{1, 2\}$ , and because we have proved above that  $\{1, 2\} \subseteq \sigma(T)$ , it follows that

$$\rho(T) = \mathbb{C} \setminus \{1, 2\} \quad \text{and} \quad \sigma(T) = \{1, 2\}.$$

3) Here we shall prove that  $\lambda = 1$  and  $\lambda = 2$  are eigenvalues, i.e. we shall prove that the equation

$$Tf = f(0) + f(x) = \lambda f(x)$$

has non-trivial solutions for  $\lambda = 1$  and  $\lambda = 2$ .

If  $\lambda = 1$ , then a check gives

$$f(0) + f(x) = f(x),$$

and the condition becomes  $f(0) = 0$ . Any function  $f \in C([-1, 1])$ , for which  $f(0) = 0$ , is therefore an eigenfunction corresponding to the eigenvalue  $\lambda = 1$ .

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If  $\lambda = 2$ , then

$$f(0) + f(x) = 2f(x),$$

and we get the condition  $f(x) = f(0)$  for all  $x \in [-1, 1]$ . This shows that every constant function  $f(x) = c$  is an eigenfunction corresponding to the eigenvalue  $\lambda = 2$ , and we have proved that

$$\sigma(T) = \{1, 2\} = \sigma_p(T).$$

4) Let  $f \in C([-1, 1])$ . Then we have the following splitting of  $f$ ,

$$f(x) = \{f(x) - f(0)\} + f(0) = g(x) + h(x),$$

where  $g(x) = f(x) - f(0)$  satisfies  $g(0) = 0$ , so  $g$  belongs to the eigenspace corresponding to  $\lambda = 1$ , and where  $h(x) = f(0)$  is constant, hence  $h(x)$  belongs to the eigenspace of the eigenvalue. This proves the existence.

If conversely

$$f(x) = g(x) + h(x)$$

is such a splitting, then

$$Tf(x) = f(x) + f(0) = Tg(x) + Th(x) = g(x) + 2h(x),$$

and we get the two equations

$$\begin{cases} g(x) + 2h(x) = f(x) + f(0), \\ g(x) + h(x) = f(x), \end{cases}$$

from which we get  $h(x) = f(0)$  by subtraction, and then

$$g(x) = f(x) - h(x) = f(x) - f(0),$$

and we have proved the uniqueness.

**Example 1.15** Let  $H$  denote a Hilbert space and let  $T \in B(H)$ . Assume that we have for some  $m \in \mathbb{N}$  that  $T^m = 0$ .

Show that

$$(I - \lambda T)^{-1} = \sum_{n=0}^{m-1} \lambda^n T^n \in B(H),$$

and deduce that  $\mathbb{C} \setminus \{0\} \subset \rho(T)$ .

Show next that  $\sigma(T) = \sigma_p(T) = \{0\}$ .

We have  $T^m = 0$ , and

$$\begin{aligned} (I - \lambda T) \sum_{n=0}^{m-1} \lambda^n T^n &= \sum_{n=0}^{m-1} \lambda^n T^n - \sum_{n=0}^{m-1} \lambda^{n+1} T^{n+1} = I + \sum_{n=1}^{m-1} \lambda^n T^n - \sum_{n=1}^m \lambda^n T^n \\ &= I - \lambda^m T^m = I, \end{aligned}$$

and analogously because  $T$  is defined everywhere,

$$\sum_{n=0}^{m-1} \lambda^n T^n (I - \lambda T) = I.$$

We therefore conclude that

$$\sum_{n=0}^{m-1} \lambda^n T^n = I + \sum_{n_1}^{m-1} \lambda^{n_1} T^{n_1} = (I - \lambda T)^{-1} \quad \text{for every } \lambda \in \mathbb{C}.$$

If  $\mu \neq 0$ , then

$$(T - \mu I)^{-1} = -\frac{1}{\mu} \left( I - \frac{1}{\mu} T \right)^{-1} = -\frac{1}{\mu} \sum_{n=0}^{m-1} \frac{1}{\mu^n} T^n \in B(H),$$

proving that  $\varrho(T) \supseteq \mathbb{C} \setminus \{0\}$ .

Clearly,  $T^m = 0$  implies that  $T^m f = T(T^{m-1}f) = 0$  for every  $f \in H$ . Hence if  $T^{m-1}f \neq 0$  for some  $f \in H$ , then  $T^{m-1}f$  is an eigenvector for  $T$ , corresponding to  $\lambda = 0$ .

First find the smallest  $m \in \mathbb{N}$ , such that  $T^m = 0$  and  $T^{m-1} \neq 0$ . It follows from this that

$$\sigma(T) = \sigma_p(T) = \{0\},$$

and hence

$$\varrho(T) = \mathbb{C} \setminus \{0\}.$$

**Example 1.16** Let  $E$  be a Banach space and let  $P \in B(E)$  satisfy  $P^2 = P$ .

- 1) Show that  $P - \lambda I$  is injective for  $\lambda \in \mathbb{C} \setminus \{0, 1\}$ .
- 2) Show that  $P - \lambda I$  is surjective for  $\lambda \in \mathbb{C} \setminus \{0, 1\}$ , and find  $(P - \lambda I)^{-1}$ .
- 3) Show that  $\sigma(P) = \sigma_p(P) = \{0, 1\}$ .

**Remark 1.7** The latter claim of the example is not true, if  $P = 0$  or  $I$ . In fact, it is well-known that

$$\sigma(0) = \sigma_p(0) = \{0\} \quad \text{and} \quad \sigma(I) = \sigma_p(I) = \{1\},$$

and it is obvious that both  $0^2 = 0$  and  $I^2 = I$ . Of a similar reason we must assume in (2) that  $\lambda \notin \{0, 1\}$ , while (1) also holds for 0 and  $I$ .  $\diamond$

- 1) Let  $\lambda \in \mathbb{C} \setminus \{0, 1\}$ , and assume that

$$(P - \lambda I)x = Px - \lambda x = 0,$$

i.e.  $Px = \lambda x$ . Then also

$$Px = P^2x = \lambda Px.$$

Because  $\lambda \neq 1$ , we must have  $Px = 0$ , and since also  $\lambda \neq 0$ , we get

$$x = \frac{1}{\lambda} Px = 0,$$

and we have proved that  $P - \lambda I$  is injective.

- 2) Let again  $\lambda \in \mathbb{C} \setminus \{0, 1\}$ . Because  $P^2 = P$ , the formal Neumann series for  $(P - \lambda I)^{-1}$  can in principle be reduced to  $\mu P - \frac{1}{\lambda} I$ , where we shall find  $\mu$  and then prove that this is indeed the inverse operator. A check gives

$$\begin{aligned} \left(\mu P - \frac{1}{\lambda} I\right) (P - \lambda I) &= (P - \lambda I) \left(\mu P - \frac{1}{\lambda} I\right) = I + \mu P^2 - \lambda \mu P - \frac{1}{\lambda} P \\ &= I + \left\{ \mu - \lambda \mu - \frac{1}{\lambda} \right\} P = I + \left\{ \mu(1 - \lambda) - \frac{1}{\lambda} \right\} P. \end{aligned}$$

Choosing  $\mu = \frac{1}{\lambda(1 - \lambda)}$  we get that the inverse operator is given by

$$(P - \lambda I)^{-1} = \frac{1}{\lambda(1 - \lambda)} P - \frac{1}{\lambda} I \in B(E)$$

and that in particular,  $P - \lambda I$  is surjective.

- 3) It follows from (2) that  $\rho(P) \supseteq \mathbb{C} \setminus \{0, 1\}$ , hence  $\sigma(P) \subseteq \{0, 1\}$ . We have also assumed that  $P \neq 0$  and  $P \neq I$ , hence

$$\{x \in M \mid Px = 0\} \neq \{0\}, M,$$

and

$$\{x \in M \mid Px = x\} \neq \{0\}, M,$$

are the eigenspaces corresponding to  $\lambda = 0$  and  $\lambda = 1$ , respectively, hence

$$\sigma(P) = \sigma_p(P) = \{0, 1\}.$$

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## 2 The adjoint of a bounded operator

**Example 2.1** Let  $T \in B(H)$  where  $H$  is a Hilbert (or just Banach) space. Show that  $\|R_\lambda(T)\| \rightarrow 0$  for  $|\lambda| \rightarrow \infty$ .

Since  $T \in B(H)$ , we see that  $R_\lambda(T) = (T - \lambda I)^{-1}$  exists for every  $\lambda \in \mathbb{C}$ , for which  $|\lambda| > \|T\|$ . Then by the Neumann series,

$$R_\lambda(T) = (T - \lambda I)^{-1} = -\frac{1}{\lambda} \left( I - \frac{1}{\lambda} T \right)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{+\infty} \frac{1}{\lambda^n} T^n.$$

We get the estimate

$$\|R_\lambda(T)\| \leq \frac{1}{|\lambda|} \sum_{n=0}^{+\infty} \left\{ \frac{\|T\|}{|\lambda|} \right\}^n = \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{\|T\|}{|\lambda|}} \rightarrow 0 \quad \text{for } |\lambda| \rightarrow +\infty,$$

and the claim is proved.

**Example 2.2** Let  $T$  be a self adjoint operator in a Hilbert space  $H$ . Show that if  $D(T) = H$ , then  $T$  is bounded.

When  $T$  is self adjoint, then  $T$  is closed, and since  $D(T) = H$  is closed, it follows from the *Closed Graph Theorem* that  $T$  is bounded.

**Example 2.3** Let  $T$  be a bounded operator on a Hilbert space  $H$  and assume that  $N$  and  $M$  are closed subspaces of  $H$ . Show that

$$T(M) \subset N \quad \text{if and only if} \quad T^*(N^\perp) \subset M^\perp.$$

Show moreover that

$$\ker(T) = T^*(H)^\perp \quad \text{and} \quad \ker(T)^\perp = \overline{T^*(H)}.$$

We assume that  $T(M) \subseteq N$ , and we shall prove that  $T^*(N^\perp) \subseteq M^\perp$ .

Let  $x \in M$  and  $y \in N^\perp$ . By the assumption,  $Tx \in N$ , thus

$$0 = (Tx, y) = (x, T^*y).$$

Now,  $x \in M$  was arbitrary, so it follows that  $T^*y \in M^\perp$ . This holds for every  $y \in N^\perp$ , hence

$$T^*(N^\perp) \subseteq M^\perp.$$

Then by iteration,  $T^{**}(M^{\perp\perp}) \subseteq N^{\perp\perp}$ . However,  $T^{**} = T$  and  $M^{\perp\perp} = M$ , and  $N^{\perp\perp} = N$ , so we conclude that

$$T(M) \subseteq N \quad \text{if and only if} \quad T^*(N^\perp) \subseteq M^\perp.$$

If  $x \in \ker(T)$ , then  $Tx = 0$ , and  $\ker(T)$  is a closed subspace. Then put  $M = \ker(T)$  and  $N = \{0\}$ , and it follows from the above that

$$T^*(N^\perp) = T^*(H) \subseteq \ker(T)^\perp, \quad \text{thus} \quad \{T^*(H)\}^\perp \supseteq \ker(T).$$

If conversely  $x \in \{T^*(H)\}^\perp$ , then for every  $y \in H$ ,

$$0 = (x, T^*y) = (Tx, y),$$

so  $Tx = 0$ , and we have  $x \in \ker(T)$ . We have proved that

$$\ker(T) = \{T^*(H)\}^\perp.$$

Finally, it follows from this equation that

$$\ker(T)^\perp = \{T^*(H)\}^{\perp\perp} = \overline{T^*(H)},$$

where the bar means the closure of the set.

**Example 2.4** Let  $T$  be a bounded operator on a Hilbert space  $H$  with  $\|T\| = 1$ , and assume that we can find  $x_0 \in H$  such that  $Tx_0 = x_0$ . Show that also  $T^*x_0 = x_0$ .

First we get

$$\begin{aligned} 0 &\leq \|T^*x_0 - x_0\|^2 = (T^*x_0 - x_0, T^*x_0 - x_0) \\ &= (T^*x_0, T^*x_0) - (x_0, T^*x_0) - (T^*x_0, x_0) + (x_0, x_0) \\ &= \|T^*x_0\|^2 - (Tx_0, x_0) - (x_0, Tx_0) + \|x_0\|^2 \\ &= \|T^*x_0\|^2 - (x_0, x_0) - (x_0, x_0) + \|x_0\|^2 \\ &= \|T^*x_0\|^2 - \|x_0\|^2, \end{aligned}$$



from which  $\|T^*x_0\|^2 \geq \|x_0\|^2$ , or

$$\|x_0\| \leq \|T^*x_0\| \leq \|T^*\| \cdot \|x_0\| = \|T\| \cdot \|x_0\| = \|x_0\|.$$

Thus we must have equality everywhere, and therefore in particular,  $\|x_0\| = \|T^*x_0\|$ , hence by insertion,

$$\|T^*x_0 - x_0\|^2 = \|T^*x_0\|^2 - \|x_0\|^2 = \|x_0\|^2 - \|x_0\|^2 = 0.$$

This shows that  $T^*x_0 - x_0$ , or after a rearrangement,  $T^*x_0 = x_0$ .

**Example 2.5** Let  $(e_n)$  denote an orthonormal basis in a Hilbert space  $H$ , and consider the operator

$$T \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} a_k e_{k+1}.$$

Find the adjoint  $T^*$  and show that  $T^*$  is an extension of  $T^{-1}$ .

Put

$$x = \sum_{k=1}^{+\infty} x_k e_k \in H \quad \text{and} \quad y = \sum_{k=1}^{+\infty} y_k e_k \in D(T^*) = H.$$

then

$$\begin{aligned} (Tx, y) &= \left( \sum_{k=1}^{+\infty} x_k e_{k+1}, \sum_{k=1}^{+\infty} y_k e_k \right) = \left( \sum_{k=2}^{+\infty} x_{k-1} e_k, \sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=2}^{+\infty} x_{k-1} \overline{y_k} = \sum_{k=1}^{+\infty} x_k \overline{y_{k+1}} \\ &= \left( \sum_{k=1}^{+\infty} x_k e_k, \sum_{k=1}^{+\infty} y_{k+1} e_k \right) = (x, T^*y), \end{aligned}$$

from which

$$T^*y = T^* \left( \sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=1}^{+\infty} y_{k+1} e_k.$$

It follows from  $D(T^{-1}) = \{e_1\}^\perp$  and

$$T^{-1}y = T^{-1} \left( \sum_{k=2}^{+\infty} y_k e_k \right) = \sum_{k=1}^{+\infty} y_{k+1} e_k \quad \text{for } y \in D(T^{-1}),$$

that  $T^{-1}y = T^*y$  for all  $y \in D(T^{-1}) \subset H$ , hence  $T^{-1} \subset T^*$ .

Finally, we notice that  $T^*e_1 = 0$ , thus  $T^{-1} \neq T^*$ .

**Example 2.6** Let  $(e_n)$  denote an orthonormal basis in a Hilbert space  $H$ , and consider the operator

$$T \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=2}^{\infty} \sqrt{k-1} a_k e_{k-1}.$$

Show that  $T$  is a densely defined unbounded operator, and find  $T^*$ .

It follows from  $\|e_n\| = 1$  and

$$\|Te_n\| = \sqrt{n-1} \rightarrow +\infty \quad \text{for } n \rightarrow +\infty,$$

that  $T$  is unbounded.

Put

$$x = \sum_{k=1}^{+\infty} x_k e_k \quad \text{and} \quad y = \sum_{n=1}^{+\infty} y_n e_n.$$

Then

$$\begin{aligned} (Tx, y) &= \left( \sum_{k=1}^{+\infty} \sqrt{k} x_{k+1} e_k, \sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{n=1}^{+\infty} \sqrt{n} \cdot x_{n+1} \overline{y_n} \\ &= (x, T^*y) = \left( \sum_{n=1}^{+\infty} x_{n+1} e_{n+1}, \sum_{k=1}^{+\infty} \sqrt{k} \cdot y_k e_{k+1} \right) = \left( x, \sum_{k=2}^{+\infty} \sqrt{k-1} \cdot y_{k-1} e_k \right), \end{aligned}$$

and we infer that

$$T^*y = T^* \left( \sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=2}^{+\infty} \sqrt{k-1} \cdot y_{k-1} e_k.$$

Then we shall explain that the formal computations above are legal. Thus, we shall prove that

$$D(T) = \left\{ x \in H \mid \sum_{k=2}^{+\infty} k |a_k|^2 < +\infty \right\}$$

is dens in  $H$ . Let  $x \in H$  be arbitrary. To any  $\varepsilon > 0$  there exists an  $N$ , such that

$$\sum_{k=N+1}^{+\infty} |a_k|^2 < \varepsilon^2.$$

Choose  $x_N = (a_1, a_2, \dots, a_N, 0, 0, \dots) \in D(T)$ . Then  $\|x - x_N\| < \varepsilon$ . This proves that  $D(T)$  is dense in  $H$ , thus  $T^*$  exists and the formal computations above are correct, when  $x \in D(T)$  and  $y \in D(T^*)$ .

We infer from

$$\|T^*y\|^2 = \sum_{k=2}^{+\infty} (k-1) |y_{k-1}|^2 = \sum_{k=1}^{+\infty} k |y_k|^2 \quad (= \|Ty\|^2),$$

that  $D(T^*) = D(T)$ .

**Example 2.7** Consider the operator  $T : \ell^2 \rightarrow \ell^2$  given by

$$T(x_1, x_2, \dots, x_n, \dots) = \left( \frac{1}{2} x_2, \frac{2}{3} x_3, \dots, \frac{n}{n+1} x_n, \dots \right).$$

- 1) Determine  $\|T\|$ .
- 2) Find all eigenvalues  $\sigma_p(T)$  and corresponding eigenvectors.
- 3) Determine the adjoint  $T^*$  and  $\sigma_p(T^*)$  and the resolvent  $\rho(T)$ .

1) It is obvious that  $\|Tx\| \leq \|x\|$ . Then it follows from

$$\|T(e_n)\| = \frac{n}{n+1} \rightarrow 1 \quad \text{for } n \rightarrow +\infty,$$

that  $\|T\| = 1$ .

2) Assume that  $\lambda \in \sigma_p(T)$  is an eigenvalue, and let  $x \in \ell^2$  be a corresponding eigenvector. Then we get for the coordinates,

$$\lambda x_n = \frac{n}{n+1} x_{n+1}, \quad n \in \mathbb{N},$$

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hence by a rearrangement and recursion,

$$x_{n+1} = \lambda \cdot \frac{n+1}{n} x_n = \dots = \lambda^n \cdot \frac{n+1}{n} \frac{n}{n-1} \dots \frac{2}{1} \cdot x_1 = \lambda^n (n+1)x_1,$$

hence

$$x_n = n \cdot \lambda^{n-1} x_1, \quad n \in \mathbb{N}.$$

It follows that

$$\sum_{n=1}^{+\infty} |x_n|^2 = \sum_{n=1}^{+\infty} n^2 |\lambda|^{2(n-1)} |x_1|^2 = |x_1|^2 \sum_{n=1}^{+\infty} n^2 |\lambda|^{2(n-1)},$$

where the series is convergent, if and only if  $|\lambda| < 1$ , thus

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\},$$

and a corresponding eigenvector is

$$x_\lambda = (1, 2\lambda, 3\lambda^2, \dots, n\lambda^{n-1}, \dots).$$

3) We see that  $T^*$  exists in  $B(\ell^2)$ , so

$$\begin{aligned} (x, T^*y) &= (Tx, y) = \sum_{n=1}^{+\infty} \frac{n}{n+1} x_{n+1} \overline{y_n} = \sum_{n=2}^{+\infty} x_n \cdot \frac{n-1}{n} \overline{y_{n-1}} \\ &= \left( (x_1, x_2, \dots, x_n, \dots), \left( 0, \frac{1}{2} y_1, \frac{2}{3} y_2, \dots, \frac{n-1}{n} y_{n-1}, \dots \right) \right), \end{aligned}$$

and we get

$$T^*y = \left( 0, \frac{1}{2} y_1, \frac{2}{3} y_2, \dots, \frac{n-1}{n} y_{n-1}, \dots \right), \quad y \in \ell^2.$$

Assume that  $\lambda \in \sigma_p(T^*)$  is an eigenvalue for  $T^*$ . Then

$$\lambda y_1 = 0, \quad \lambda y_n = \frac{n-1}{n} y_{n-1}, \quad n \in \mathbb{N} \setminus \{1\}.$$

We have two possibilities: Either  $\lambda = 0$ , or  $y_1 = 0$ .

(a)  $\lambda = 0$ . It follows from the latter equation that  $y_{n-1} = 0$  for  $n \in \mathbb{N} \setminus \{1\}$ , meaning that  $y = 0$ , and we conclude that  $0 \notin \sigma_p(T^*)$ .

(b)  $\lambda \neq 0$ . In this case,  $y_1 = 0$ , and then it follows by induction on

$$y_n = \frac{n-1}{n\lambda} y_{n-1}, \quad n \in \mathbb{N} \setminus \{1\},$$

that  $y_n = 0$ , and hence  $y = 0$ . We conclude that  $\lambda \notin \sigma_p(T^*)$ .

Summing up,

$$\sigma_p(T^*) = \emptyset.$$

Hence  $\sigma_r(T) = \emptyset$ . Furthermore,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\| = 1\},$$

and because  $\sigma(T)$  is closed, we must have

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

Utilizing that

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup_r(T) = \sigma_p(T) \cup \sigma_c(T)$$

is a disjoint splitting, we finally find the continuous spectrum

$$\sigma_c(T) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\},$$

and the resolvent set

$$\rho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

**Example 2.8** Let  $T : \ell^2 \rightarrow \ell^2$  be the linear operator given by

$$T(x_1, x_2, \dots, x_n, \dots) = (x_1 + x_2, x_2 + x_3, \dots, x_n + x_{n+1}, \dots).$$

- 1) Find the point spectrum  $\sigma_p(T)$  and determine all eigenvectors associated to  $\lambda \in \sigma_p(T)$ .
- 2) Determine  $\|T\|$ .
- 3) Determine the adjoint  $T^*$  and find also the point spectrum  $\sigma_p(T^*)$ .
- 4) Let  $S = T - I$ . Determine  $\|S\|$ .
- 5) Find  $\sigma_c(T)$  and  $\sigma_r(T)$  with the help of  $S$  above.

- 1) We shall find the non-trivial solutions of the equation

$$Tx = \lambda x.$$

The coordinate equation of this equation becomes

$$x_n + x_{n+1} = \lambda x_n, \quad n \in \mathbb{N},$$

thus

$$(4) \quad x_{n+1} = (\lambda - 1)x_n, \quad n \in \mathbb{N}.$$

If  $\lambda = 1$ , then  $x_{n+1} = 0$ , so we can only choose  $x_1 \neq 0$ . On the other hand,  $e_1$  is clearly an eigenvector and  $1 \in \sigma_p(T)$ .

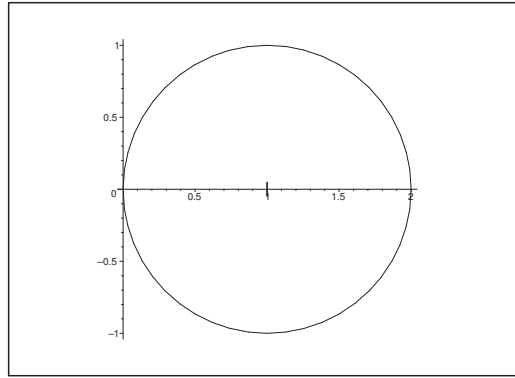


Figure 1: The point spectrum  $\sigma_p(T)$  is the open set inside the circle.

If  $\lambda \neq 1$ , then we can divide (4) by  $(\lambda - 1)^{n+1} \neq 0$ . Then it follows by recursion that

$$\frac{x_{n+1}}{(\lambda - 1)^{n+1}} = \frac{x_n}{(\lambda - 1)^n} = \cdots = \frac{x_1}{\lambda - 1},$$

so  $x_n = (\lambda)^{n-1}x_1$ . Choosing  $x_1 = 1$  we see that one *candidate* of an eigenvector is given by its coordinates  $x_n = (\lambda - 1)^{n-1}$ . Because

$$\sum_{n=1}^{+\infty} |x_n|^2 = \sum_{n=1}^{+\infty} |\lambda - 1|^{2(n-1)} = \sum_{n=0}^{+\infty} |\lambda - 1|^{2n}$$

is convergent, if and only if  $|\lambda - 1| < 1$ , it follows that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$$

with the eigenvectors

$$(1, \lambda - 1, (\lambda - 1)^2, \dots, (\lambda - 1)^{n-1}, \dots), \quad \text{for } |\lambda - 1| < 1.$$

We notice for  $\lambda = 1$  that we get precisely  $(1, 0, 0, \dots)$ .

2) From

$$2 \in \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\},$$

and a consideration of the figure, it follows that  $\|T\| \geq 2$ .

On the other hand, an application of Minkowski's inequality gives

$$\|Tx\| = \|x + (0, x_1, x_2, \dots)\| \leq \|x\| + \|x\| = 2\|x\|,$$

proving that  $\|T\| \leq 2$ .

Summing up,  $\|T\| = 2$ .

3) It follows from

$$\begin{aligned}(Tx, y) &= \sum_{n=1}^{+\infty} (x_n + x_{n+1}) \overline{y_n} = \sum_{n=1}^{+\infty} x_n \overline{y_n} + \sum_{n=2}^{+\infty} x_n \overline{y_{n-1}} \\ &= x_1 \overline{y_1} + \sum_{n=2}^{+\infty} x_n (\overline{y_{n-1}} + \overline{y_n}) = (x, T^*y) = \sum_{n=1}^{+\infty} x_n \overline{(T^*y)_n},\end{aligned}$$

that

$$T^*y = (y_1, y_1 + y_2, y_2 + y_3, \dots, y_{n-1} + y_n, \dots),$$

or written in coordinates,

$$(T^*y)_1 = y_1, \quad (T^*y)_n = y_{n-1} + y_n \quad \text{for } n \geq 2.$$

The equation  $T^*y = \lambda y$  is written in coordinates as

$$y_1 = \lambda y_1 \quad \text{and} \quad y_{n-1} + y_n = \lambda y_n \quad \text{for } n \geq 2,$$

thus

$$(\lambda - 1)y_1 = 0 \quad \text{and} \quad (\lambda - 1)y_n = y_{n-1} \quad \text{for } n \geq 2.$$

We get from the first equation that either  $\lambda = 1$  or  $y_1 = 0$ . If  $\lambda = 1$ , then it follows from the last equations that  $y_{n-1} = 0$  for all  $n \geq 2$ , hence  $y = 0$ , and  $\lambda = 1$  is not an eigenvalue for  $T^*$ .

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If  $\lambda \neq 1$  and  $y_1 = 0$ , then we see by recursion on

$$y_n = \frac{1}{\lambda - 1} y_{n-1}$$

that the only solution is  $y = 0$ .

Summing up,  $\sigma_p(T^*) = \emptyset$ .

Then of course,  $\sigma_r(T) = \emptyset$ .

4) Because

$$(Sx)_n = (Tx)_n - x_n = x_{n+1},$$

and  $\|Sx\| \leq \|x\|$  with equality for  $x_1 = 0$ , it follows immediately that  $\|S\| = 1$ .

5) We get from  $T = S + I$  that  $T - \lambda I = S - (\lambda - 1)I$ , so

$$\begin{aligned} \lambda \in \sigma_p(T) & \quad \text{if and only if} \quad \lambda - 1 \in \sigma_p(S), & \quad \text{thus.} \quad \sigma_p(T) = 1 + \sigma_p(S), \\ \lambda \in \sigma_c(T) & \quad \text{if and only if} \quad \lambda - 1 \in \sigma_c(S), & \quad \text{thus} \quad \sigma_c(T) = 1 + \sigma_c(S), \\ \lambda \in \sigma_r(T) & \quad \text{if and only if} \quad \lambda - 1 \in \sigma_r(S), & \quad \text{thus} \quad \sigma_r(T) = 1 + \sigma_r(S). \end{aligned}$$

It is not surprising that the various parts of the spectrum for  $T$  is obtained by translating the corresponding parts of the spectrum for  $S$ . We now conclude from

$$\sigma_p(S) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\},$$

and

$$\sigma_r(S) = \emptyset, \quad (\text{because } \sigma_r(T) = \emptyset),$$

and from  $\sigma(S)$  being closed, and

$$\sigma_p(S) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma(S) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|S\|\} = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

that

$$\sigma(S) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and hence that

$$\sigma_c(S) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Finally, by utilizing the translation, we get

$$\begin{aligned} \sigma(T) &= \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq 1\}, \\ \sigma_p(T) &= \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}, \\ \sigma_c(T) &= \{\lambda \in \mathbb{C} \mid |\lambda - 1| = 1\} \\ \sigma_r(T) &= \emptyset. \end{aligned}$$



**Example 2.9** We consider in  $\ell^2$  the operator

$$T(x_1, x_2, \dots, x_n, \dots) = \left( 2x_2, \frac{3}{2}x_3, \dots, \frac{n+1}{n}x_{n+1}, \dots \right).$$

- 1) Find  $\|T\|$ .
- 2) Find  $\sigma_p(T)$  and find the eigenspace associated to all  $\lambda \in \sigma_p(T)$ .
- 3) Determine the adjoint  $T^*$ .
- 4) Determine  $\sigma_r(T)$ .
- 5) Let  $\lambda \notin \sigma_p(T) \cup \sigma_r(T)$ . For  $k \in \mathbb{N}$  we define an operator  $I_k$  on  $\ell^2$  by

$$I_k((x_1, x_2, \dots, x_k, x_{k+1}, \dots)) = (0, 0, \dots, 0, x_k, x_{k+1}, \dots),$$

and we define  $T_k = I_k T$ . Show that there is a  $k \in \mathbb{N}$  such that

$$\|T_k\| < \lambda.$$

Use this to solve the equation

$$(T_k - \lambda I_k)x = y$$

for a given  $y \in \ell^2$ . Finally, show that the equation

$$(T - \lambda I)x = y$$

has a solution  $x = (T - \lambda T)^{-1}y$  for all  $y \in \ell^2$ .

- 6) Find  $\sigma(T)$  and  $\rho(T)$  (e.g. by use of the Closed Graph Theorem).

- 1) From  $1 + \frac{1}{n} \leq 2$  for all  $n \in \mathbb{N}$ , follows for every  $x \in \ell^2$  that

$$\|Tx\|^2 = \sum_{n=1}^{+\infty} \left(1 + \frac{1}{n}\right)^2 |x_{n+1}|^2 \leq 2^2 \sum_{n=1}^{\infty} |x_{n+1}|^2 \leq \{2\|x\|\}^2,$$

proving that  $\|T\| \leq 2$ .

On the other hand,

$$T(0, 1, 0, 0, \dots) = (2, 0, 0, 0, \dots),$$

and we infer that  $\|T\| = 2$ .

- 2) Assume that  $Tx = \lambda x$ , thus

$$\frac{n+1}{n}x_{n+1} = \lambda x_n, \quad n \in \mathbb{N}.$$

For  $\lambda = 0$  we get  $x = (1, 0, 0, \dots)$  as an eigenvector, and 0 is an eigenvalue,  $0 \in \sigma_p(T)$ .

If  $\lambda \neq 0$ , then a multiplication by  $n \lambda^{-(n+1)}$  follows by a recursion gives that

$$(n+1) \lambda^{-(n+1)} x_{n+1} = n \lambda^{-n} x_n = \dots = 1 \cdot \lambda^{-1} x_1,$$

and we get the coordinates of the candidate

$$x_n = \frac{1}{n} \lambda^{n-1} x_1, \quad n \in \mathbb{N}.$$

the corresponding sequence lies in  $\ell^2$  for  $x_1 \neq 0$ , if and only if

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} |\lambda|^{2(n-1)} < +\infty.$$

It is well-known that  $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$ , so this condition is equivalent to  $|\lambda| \leq 1$ , and we conclude that

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

and an eigenvector corresponding to  $\lambda \in \sigma_p(T)$  is given by

$$x_1 \left( 1, \frac{\lambda}{2}, \frac{\lambda^2}{3}, \dots, \frac{\lambda^{n-1}}{n}, \dots \right).$$

3) If  $x, y \in \ell^2$ , then

$$(Tx, y) = \sum_{n=1}^{+\infty} (Tx)_n \overline{y_n} = \sum_{n=1}^{+\infty} \frac{n+1}{n} x_{n+1} \overline{y_n} = \sum_{n=2}^{+\infty} x_n \cdot \overline{\frac{n}{n-1} y_{n-1}} = (x, T^*y),$$

hence

$$T^*(y_1, y_2, \dots, y_n, \dots) = \left( 0, 2y_1, \frac{3}{2}y_2, \dots, \frac{n}{n-1}y_{n-1}, \dots \right),$$

or written in coordinates,

$$\begin{cases} (T^*y)_1 = 0, & \text{for } n = 1, \\ (T^*y)_n = \frac{n}{n-1} y_{n-1}, & \text{for } n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

4) We prove that  $\sigma_p(T^*) = \emptyset$ , because this will imply that  $\sigma_r(T) = \emptyset$ .

Assume that  $\lambda \in \sigma_p(T^*)$ . It follows from the equation  $T^*y = \lambda y$  that

$$\begin{cases} 0 = \lambda y_1, & \text{for } n = 1, \\ \frac{n}{n-1} y_{n-1} = \lambda y_n, & \text{for } n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

If  $\lambda = 0$ , then clearly  $y = 0$ , so  $0 \notin \sigma_p(T^*)$ .

If  $\lambda \neq 0$ , then  $y_1 = 0$ . Multiply the last coordinate equation by  $\frac{1}{n} \lambda^{n-1}$ . Then it follows by recursion that

$$\frac{\lambda^n}{n} y_n = \frac{\lambda^{n-1}}{n-1} y_{n-1} = \cdots = \frac{\lambda}{1} y_1 = 0,$$

from which  $y_n = 0$  for all  $n \in \mathbb{N}$ , and there is no eigenvectors. Hence,  $\sigma_p(T^*) = \emptyset$ , and therefore  $\sigma_r(T) = \emptyset$ .

5) If

$$\lambda \notin \sigma_p(T) \cup \sigma_r(T) = \sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\},$$

then  $|\lambda| > 1$ . It follows from

$$\|T_k x\|^2 = \sum_{n=k}^{+\infty} \left(\frac{n+1}{n}\right)^2 |x_{n+1}|^2 \leq \left(\frac{k+1}{k}\right)^2 \|x\|^2,$$

that

$$\|T_k\| \leq \frac{k+1}{k} = 1 + \frac{1}{k},$$

where we can obtain equality, so

$$\|T_k\| = 1 + \frac{1}{k}.$$


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Because  $|\lambda| > 1$ , we can choose  $k$  so big that

$$\|T_k\| = 1 + \frac{1}{k} < |\lambda|.$$

Now,  $\lambda \notin \sigma_p(T) \cup \sigma_r(T)$ , so  $(T - \lambda I)^{-1}$  exists and is densely defined.

It follows from  $\|T_k\| < |\lambda|$ , that

$$(T_k - \lambda I_k)^{-1} \in B(I_k \ell^2, \ell^2),$$

where  $I_k \ell^2$  is a Hilbert space which is isomorphic to  $\ell^2$ .

The equation

$$Tx - \lambda x = y, \quad y \in \ell^2,$$

has the coordinate form

$$\frac{n+1}{n} x_{n+1} - \lambda x_n = y_n, \quad n \in \mathbb{N}.$$

Thus it follows from  $\lambda \neq 0$  that

$$\begin{cases} x_n = \frac{1}{\lambda} \left( \frac{n+1}{n} x_{n+1} - y_n \right), & n \in \{1, \dots, k-1\}, \\ T_k x - \lambda I_k x = I_k y. \end{cases}$$

It follows from the above that the latter equation can be solved,

$$I_k x = (T_k - \lambda I_k)^{-1} I_k y \quad \text{for all } y \in \ell^2.$$

Hence for a given  $y \in \ell^2$ ,

$$I_k x = (0, \dots, 0, x_k, x_{k+1}, \dots) = (T_k - \lambda I_k)^{-1} I_k y$$

is uniquely determined. The recursion formula

$$x_n = \frac{1}{\lambda} \left\{ \frac{n+1}{n} x_{n+1} - y_n \right\}, \quad \text{for } n \in \{1, \dots, k-1\},$$

determines the remaining elements of  $x$ , so  $(T - \lambda I)^{-1}$  is defined everywhere.

- 6) If  $|\lambda| > 1$ , then it follows from the above that  $(T - \lambda I)^{-1}$  is defined everywhere. Now,  $T - \lambda I$  is closed, so  $(T - \lambda I)^{-1}$  is also closed. Then it follows from the Closed Graph Theorem that  $\lambda \in \rho(T)$  for every  $\lambda \in \mathbb{C}$ , for which  $|\lambda| > 1$ . Hence

$$\sigma(T) = \sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}, \quad \sigma_r(T) = \sigma_c(T) = \emptyset,$$

and

$$\rho(T) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}.$$

**Remark 2.1** This example shows that it is possible that every  $\lambda$  for which  $|\lambda| = \|T\|$  belongs to the resolvent set,  $\rho \in \rho(T)$ . So far we have only seen examples, in which there is always at least one  $\lambda \in \sigma(T)$ , such that  $|\lambda| = \|T\|$ . This is not the case in the present example.  $\diamond$

### 3 Self adjoint operators

**Example 3.1** Let  $T \in B(H)$ . Show that we can write  $T$  as

$$T = A + iB,$$

where  $A$  and  $B$  are uniquely determined, bounded self adjoint operators.

First assume that  $T$  can be written in the form  $T = A + iB$ , where  $A$  and  $B$  are self adjoint. Then

$$\begin{aligned} (Tx, y) &= (Ax + iBx, y) = (Ax, y) + i(Bx, y) \\ &= (x, Ay) + i(x, By) = (x, Ay - iBy) = (x, (A - iB)y) = (x, T^*y), \end{aligned}$$

and it follows that if

$$T = A + iB \quad \text{then} \quad T^* = A - iB.$$

We get by simple addition or subtraction,

$$A = \frac{1}{2} (T + T^*) \quad \text{and} \quad B = \frac{1}{2i} (T - T^*).$$

Conversely, if

$$A = \frac{1}{2} (T + T^*) \quad \text{and} \quad B = \frac{1}{2i} (T - T^*),$$

then clearly,  $T = A + iB$ . Furthermore,  $A$  and  $B$  are obviously linear and

$$\|A\| \leq \frac{1}{2} \{\|T\| + \|T^*\|\} = \|T\|, \quad \|B\| \leq \frac{1}{2} \{\|T\| + \|T^*\|\} = \|T\|,$$

so  $A$  and  $B$  are bounded. Finally,

$$(Ax, y) = \left( \frac{1}{2} \{T + T^*\}x, y \right) = \left( x, \frac{1}{2} \{T^* + T\}y \right) = \left( x, \frac{1}{2} \{T + T^*\}y \right) = (x, Ay),$$

and

$$(Bx, y) = \left( \frac{1}{2i} \{T - T^*\}x, y \right) = \left( x, -\frac{1}{2i} \{T^* - T\}y \right) = \left( x, \frac{1}{2i} \{T - T^*\}y \right) = (x, By),$$

shows that both  $A$  and  $B$  are self adjoint.

**Example 3.2** Show that  $T \in B(H)$  is self adjoint if and only if one of the following conditions is satisfied:

$$(Tx, x) = (x, Tx) \quad \text{for all } x \in H,$$

and

$$(Tx, x) \in \mathbb{R} \quad \text{for all } x \in H.$$

We assume implicitly that  $H$  is a complex Hilbert space.

We have  $T \in B(H)$ , thus  $T$  is self adjoint if and only if  $T^* = T$ , thus if and only if

$$(5) \quad (Tx, y) = (x, Ty) \quad \text{for all } x, y \in H.$$

Choosing  $y = x$  in (5) we get in particular the first condition above, thus

$$(6) \quad (Tx, x) = (x, Tx) \quad \text{for all } x \in H.$$

This condition is equivalent with

$$(Tx, x) = (x, Tx) = \overline{(Tx, x)} \quad (\in \mathbb{R}),$$

and it follows that the two conditions are equivalent. It only remains to prove that (6) implies that  $T$  is self adjoint.

Assume (6). We shall prove (5). We get by replacing  $x$  in (6) by  $x + y$  that

$$\begin{array}{rcccc} (T(x+y), x+y) & = & (Tx, x) & + (Tx, y) + (Ty, x) + & (Ty, y), \\ (x+y, T(x+y)) & = & (x, Tx) & + (x, Ty) + (y, Tx) + & (y, Ty), \\ \star & & \star & & \star \end{array}$$

It follows from the assumption (6) that the three columns marked with a  $\star$  inside each column are mutually equal. Hence by a subtraction and a rearrangement,

$$(7) \quad (Tx, y) + (Ty, x) = (x, Ty) + (y, Tx).$$

If we write  $x + iy$  in (6) instead of  $x$ , then we get analogously,

$$\begin{array}{rcccc} (T(x+iy), x+iy) & = & (Tx, x) & -i(Tx, y) + i(Ty, x) + & (Ty, y), \\ (x+iy, T(x+iy)) & = & (x, Tx) & -i(x, Ty) + i(y, Tx) + & (y, Ty), \\ \star & & \star & & \star \end{array}$$

We conclude as before by utilizing that the columns marked with a  $\star$  by the assumption (6) are identical that

$$(8) \quad (Tx, y) - (Ty, x) = (x, Ty) - (y, Tx).$$

We get by adding (7) and (8), followed by a division by 2,

$$(Tx, y) = (x, Ty).$$

This is true for all  $x, y \in H$ , so we have proved (5), thus  $T$  is self adjoint.

**Example 3.3** Let  $S$  and  $T$  be bounded, self adjoint operators on a Hilbert space. Show that  $ST + TS$  and  $i(ST - TS)$  are self adjoint.

The proof is simple, because  $S, T \in B(H)$  and

$$(ST + TS)^* = (ST)^* + (TS)^* = T^*S^* + S^*T^* = ST + TS,$$

and

$$\{i(ST - TS)\}^* = -i\{(ST)^* - (TS)^*\} = -i\{T^*S^* - S^*T^*\} = i(ST - TS).$$

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**Example 3.4** Let  $T$  be a bounded self adjoint operator. Define the numbers

$$m = \inf\{(Tx, x) \mid \|x\| = 1\},$$

and

$$M = \sup\{(Tx, x) \mid \|x\| = 1\}.$$

Show that  $\sigma(T) \subset [m, M]$ , and show that both  $m$  and  $M$  belong to  $\sigma(T)$ .

Show that  $\|T\| = \max\{|m|, |M|\}$ .

We deduce from the definitions of  $m$  and  $M$  that

$$m\|x\|^2 \leq (Tx, x) \leq M\|x\|^2 \quad \text{for all } x \in H.$$

Now,  $T \in B(H)$  is self adjoint, so  $\sigma(T) \subseteq \mathbb{R}$ . Choose  $\lambda \in \mathbb{R} \setminus [m, M]$ . We shall prove that  $\lambda \in \varrho(T)$ . First assume that  $\lambda < m$ . Then

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= (Tx - \lambda x, Tx - \lambda x) \\ &= (Tx - mx + (m - \lambda)x, Tx - mx + (m - \lambda)x) \\ &= \|Tx - mx\|^2 + (m - \lambda)^2\|x\|^2 + 2\{m - \lambda\}(Tx - mx, x). \end{aligned}$$

It follows from  $m - \lambda > 0$  and  $(Tx - mx, x) = (Tx, x) - m(x, x) \leq 0$  that we have the estimate,

$$\|(T - \lambda I)x\|^2 \geq 0 + (m - \lambda)^2\|x\|^2 + 0 = (m - \lambda)^2\|x\|^2,$$

which implies that  $T - \lambda I$  is injective, and  $(T - \lambda I)^{-1}$  exists and is bounded. Then

$$\lambda \in \varrho(T) \cup \sigma_r(T).$$

Because  $T$  is self adjoint, the residual spectrum is  $\sigma_r(T) = \emptyset$ , hence  $\lambda \in \varrho(T)$ .

If instead  $\lambda > M$ , then we get analogously

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= (Tx - \lambda x, Tx - \lambda x) \\ &= (Mx - Tx + (\lambda - M)x, Mx - Tx + (\lambda - M)x) \\ &= \|Mx - Tx\|^2 + (\lambda - M)^2\|x\|^2 + 2\{\lambda - M\}(Mx - Tx, x) \\ &\geq (\lambda - M)^2\|x\|^2, \end{aligned}$$

because  $\lambda - M > 0$  and  $(Mx - Tx, x) = M\|x\|^2 - (Tx, x) \geq 0$ . As before we infer that  $(T - \lambda I)^{-1}$  exists and is bounded. We have proved that  $\mathbb{C} \setminus [m, M] \subseteq \varrho(T)$ , and it follows that  $\sigma(T) \subseteq [m, M]$ .

Using a well-known formula we get

$$\|T\| = \sup\{|(Tx, x)| \mid \|x\| = 1\} = \max\{|m|, |M|\}.$$

Assume e.g. that  $\|T\| = |M| = M \geq 0$ , and let  $\lambda = M$ . Then

$$M \in \sigma_p(T) \cup \sigma_c(T) \cup \varrho(T).$$

We shall prove that  $M \notin \varrho(T)$ . This is done INDIRECTLY.



Assume that  $M \in \varrho(T)$ , thus  $(T - MI)^{-1} \in B(H)$ . Then there exists a  $c > 0$ , such that

$$\|(T - MI)^{-1}x\| \leq \frac{1}{c} \|x\| \quad \text{for all } x \in H.$$

If we put  $y = (T - MI)^{-1}x$ , then  $x = (T - MI)y$ , hence

$$\|(T - MI)y\| \geq c\|y\| \quad \text{for all } y \in H.$$

This implies that  $\|T - MI\| \geq c > 0$ .

From  $M = \sup\{(Tx, x) \mid \|x\| = 1\}$  follows the existence of a sequence  $x_n$ ,  $\|x_n\| = 1$ , of unit vectors, such that

$$(Tx_n, x_n) \rightarrow M = \|T\| \quad \text{for } n \rightarrow +\infty,$$

and we conclude from

$$(Tx_n, x_n) \leq \|Tx_n\| \cdot \|x_n\| = \|Tx_n\| \leq \|T\| = M,$$

that also  $\|Tx_n\| \rightarrow M$ . Then for every such sequence,

$$\begin{aligned} 0 &\leq \|(T - MI)x_n\|^2 = (Tx_n - Mx_n, Tx_n - Mx_n) \\ &= \|Tx_n\|^2 + M^2\|x_n\|^2 - 2M(Tx_n, x_n) \\ &\rightarrow M^2 + M^2 - 2M^2 = 0, \end{aligned}$$

which shows that the estimate  $\|(T - MI)x_n\| \geq c\|x_n\| = c > 0$  is not true, and we have derived a contradiction. Therefore,  $M \notin \varrho(T)$ , i.e.  $M \in \sigma(T)$ .

An analogous argument shows that if  $\|T\| = |m| = -m$ , then  $m \in \sigma(T)$ .

Finally, assume that  $|m| = -m < M$ . It follows from the above that  $M \in \sigma(T)$ . We shall prove that also  $m \in \sigma(T)$ . First notice that  $T - MI$  of course is self adjoint. Then it follows from

$$((T - MI)x, x) = (Tx, x) - M\|x\|^2,$$

and

$$m\|x\|^2 \leq (Tx, x) \leq M\|x\|^2,$$

that

$$(m - M)\|x\|^2 \leq ((T - MI)x, x) \leq (M - M)\|x\|^2 = 0,$$

and

$$\inf\{((T - MI)x, x) \mid \|x\| = 1\} = m - M < 0.$$

Then from the above,  $m - M \in \sigma(T - MI)$ , which means that

$$(T - MI) - (m - M)I = T - mI$$

is not regular, so  $m \in \sigma(T)$ .

**Example 3.5** Consider in  $L^2(\mathbb{R})$  the operator  $Q$  defined by

$$Qf(x) = x f(x),$$

with

$$D(Q) = \{f \in L^2(\mathbb{R}) \mid Qf \in L^2(\mathbb{R})\}.$$

Show that  $Q$  is self adjoint.

Let  $f, g \in D(Q)$ , thus  $f, g \in L^2(\mathbb{R})$  and  $x \cdot f(x), x \cdot g(x) \in L^2(\mathbb{R})$ . Because  $Q$  is densely defined, we get

$$(Qf, g) = \int_{-\infty}^{+\infty} x f(x) \overline{g(x)} dx = \int_{-\infty}^{+\infty} f(x) \cdot \overline{x g(x)} dx = (f, Qg),$$

proving that  $Q$  is symmetric,  $Q \subseteq Q^*$ . It remains to prove that  $D(Q) = D(Q^*)$ . To do this it suffices to prove that  $Q$  is a closed operator.

Assume that  $(f_n) \subset D(Q)$  and  $f_n \rightarrow f \in L^2(\mathbb{R})$ , and  $x f_n \rightarrow g \in L^2(\mathbb{R})$ . We shall prove that  $g(x) = x \cdot f(x)$  almost everywhere. We find

$$\|g - x f\|_2^2 = \int_{-1}^1 |g(x) - x f(x)|^2 dx + \left\{ \int_{-\infty}^{-1} + \int_1^{+\infty} |g(x) - x f(x)|^2 dx \right\}.$$

Here,  $\int_{-1}^1 |g(x) - x f(x)|^2 dx = 0$ , because  $f \in L^2([-1, 1])$  implies that also  $x \cdot f \in L^2([-1, 1])$ , noting that the interval is bounded. This means that  $g(x) = x \cdot f(x)$  for almost every  $x \in [-1, 1]$ . If  $|x| \geq 1$ , then we get  $f_n \rightarrow f$  and  $f_n \rightarrow \frac{g(x)}{x}$ , both in the sense of  $L^2$ , because

$$\int_{|x| \geq 1} \left| \frac{g(x)}{x} \right|^2 dx \leq \int_{|x| \geq 1} |g(x)|^2 dx < +\infty.$$

The limit value is unique, hence  $f(x) = \frac{g(x)}{x}$  almost everywhere for  $|x| \geq 1$ . Hence we conclude that  $g(x) = x f(x)$  for almost every  $x \in \mathbb{R}$ .

This proves that  $Q$  is closed, which again implies by the above that  $Q = Q^*$ , and we have proved that  $Q$  is self adjoint.

**Example 3.6** Show that the set of self adjoint operators is closed in  $B(H)$ .

We shall only prove that if  $(T_n) \subset B(H)$  is a sequence of self adjoint operators, converging towards  $T \in B(H)$ , then  $T$  is also self adjoint. The condition  $T_n \rightarrow T$  for  $n \rightarrow +\infty$  means that

$$Tx = \lim_{n \rightarrow +\infty} T_n x \quad \text{for all } x \in H.$$

Therefore, if  $x, y \in H$ , then

$$(Tx, y) = \lim_{n \rightarrow +\infty} (T_n, y) = \lim_{n \rightarrow +\infty} (x, T_n y) = (x, Ty),$$

proving that  $T \subseteq T^*$ . Because  $D(T) = H$ , we have  $T = T^*$ , hence  $T$  is self adjoint.

**Example 3.7** Let  $(e_n)$  denote an orthonormal basis in a Hilbert space  $H$ , and let  $(r_k)$  be all the rational numbers in  $]0, 1[$ , arranged as a sequence. Consider the operator

$$T \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} r_k a_k e_k.$$

Show that  $T$  is self adjoint and that  $\|T\| = 1$ . Find  $\rho(T)$  and determine the point spectrum and the continuous spectrum for  $T$ .

First note that

$$\|Tx\|^2 = \sum_{k=1}^{+\infty} r_k^2 |x_k|^2 \leq \sum_{k=1}^{+\infty} |x_k|^2 = \|x\|^2,$$

thus  $T \in B(H)$  and  $\|T\| \leq 1$ . Furthermore,

$$(Tx, y) = \sum_{k=1}^{+\infty} r_k x_k \overline{y_k} = \sum_{k=1}^{+\infty} x_k \overline{r_k y_k} = (x, Ty),$$

proving that  $T$  is self adjoint. This implies that the residual spectrum is empty,  $\sigma_r(T) = \emptyset$ .

From  $Te_k = r_k e_k$  follows that every  $r_k \in \sigma_p(T)$ , and we conclude further from  $0 < r_k \leq \|T\|$  that

$$\|T\| \geq \sup_{k \in \mathbb{N}} r_k = 1,$$

hence  $\|T\| = 1$ .

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Conversely, if  $Tx = \lambda x$ , then

$$Tx - \lambda x = \sum_{k=1}^{+\infty} (r_k - \lambda) x_k e_k = 0,$$

so either  $\lambda = r_k$  or  $x_k = 0$ . This shows that

$$\sigma_p(T) = \mathbb{Q} \cap ]0, 1[ = \{r_k \mid k \in \mathbb{N}\}.$$

Assume that  $\lambda < 0$ . Then

$$\|Tx - \lambda x\|^2 = \left\| \sum_{k=1}^{+\infty} (r_k + |\lambda|) x_k e_k \right\|^2 = \sum_{k=1}^{+\infty} (r_k + |\lambda|)^2 |x_k|^2 \geq |\lambda|^2 \sum_{k=1}^{+\infty} |x_k|^2 = |\lambda|^2 \cdot \|x\|^2,$$

from which we infer that  $\|Tx - \lambda x\| \geq |\lambda| \cdot \|x\|$ , hence  $\lambda \in \varrho(T)$ . It follows that

$$\varrho(T) \supseteq \mathbb{C} \setminus [0, 1].$$

On the other hand,  $\sigma(T)$  is closed, so it follows from

$$\sigma(T) \supseteq \sigma_p(T) = \mathbb{Q} \cap ]0, 1[,$$

that  $\sigma(T) \supseteq [0, 1]$ . From  $\varrho(T)$  and  $\sigma(T)$  being disjoint we conclude that

$$\varrho(T) = \mathbb{C} \setminus [0, 1] \quad \text{and} \quad \sigma(T) = [0, 1].$$

Now,  $\sigma_r(T) = \emptyset$  for self adjoint operators, and  $\sigma_p(T) = \mathbb{Q} \cap ]0, 1[$ , hence the continuous spectrum is

$$\sigma_c(T) = \sigma(T) \setminus \sigma_p(T) = ([0, 1] \setminus \mathbb{Q}) \cup \{0, 1\}.$$

**Example 3.8** Let  $(e_k)$  be an orthonormal basis in a Hilbert space  $H$ , and assume that  $T \in B(H)$  has the matrix representation  $\mathbf{T} = (t_{jk})$  with respect to the orthonormal basis  $(e_k)$  (see VENTUS, HILBERT SPACES, EXAMPLE 2.7). Derive a necessary and sufficient condition on the  $t_{jk}$  that  $T$  is self adjoint.

In VENTUS, HILBERT SPACES, EXAMPLE 2.7 we derived that  $t_{jk} = (Te_j, e_k)$ , and

$$T \left( \sum_{j=1}^{+\infty} x_j e_j \right) = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} x_j t_{jk} e_k = \sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} x_j t_{jk} \right\} e_k.$$

If

$$x = \sum_{j=1}^{+\infty} x_j e_j \quad \text{and} \quad y = \sum_{k=1}^{+\infty} y_k e_k,$$

then

$$\begin{aligned} (Tx, y) &= \left( \sum_{k=1}^{+\infty} \left\{ \sum_{j=1}^{+\infty} x_j t_{jk} \right\} e_k, \sum_{k=1}^{+\infty} y_k e_k \right) = \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} x_j t_{jk} \overline{y_k} = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} \overline{y_k} t_{jk} x_j \\ &= \left( \sum_{j=1}^{+\infty} x_j e_j, \sum_{j=1}^{+\infty} \left\{ \sum_{k=1}^{+\infty} \overline{t_{jk}} y_k \right\} e_j \right) = (x, T^* y). \end{aligned}$$

Hence

$$T^*y = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} y_k t_{jk}^* e_j = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} y_k \overline{t_{kj}} e_j,$$

so  $\mathbf{T}^* = (t_{jk}^*) = (\overline{t_{kj}})$ . This means that  $\mathbf{T}^*$  is obtained from  $\mathbf{T}$  by taking the transpose and apply complex conjugating.

It follows from the above that  $T$  is self adjoint if and only if  $\mathbf{T}^* = \mathbf{T}$ , i.e. if and only if

$$\overline{t_{kj}} = t_{jk} \quad \text{for all } j, k = 1, 2, 3, \dots$$

Note that

$$\overline{t_{kj}} = \overline{(Te_k, e_j)} = (e_j, Te_k),$$

so the example shows that in this case  $T$  is self adjoint, if

$$(Te_j, e_k) = (e_j, Te_k) \quad \text{for all } j, k \in \mathbb{N},$$

and there is nothing new in that statement.

**Example 3.9** Let  $H = L^2(\mathbb{R})$ , and let  $V$  denote a bounded real continuous function. We define the operator  $T$  by

$$Tf(x) = V(x) \cdot f(x), \quad f \in L^2(\mathbb{R}).$$

Prove that  $T$  is a bounded self adjoint operator.

In Quantum Mechanics the operator  $T$  is called a potential operator.

It follows from  $\|V\|_\infty < +\infty$  that

$$\begin{aligned} \|Tf\|_2^2 &= \int_{-\infty}^{+\infty} V(x)f(x) \cdot \overline{V(x)f(x)} dx = \int_{-\infty}^{+\infty} V(x)^2 |f(x)|^2 dx \\ &\leq \|V\|_\infty^2 \int_{-\infty}^{+\infty} |f(x)|^2 dx = \|V\|_\infty^2 \cdot \|f\|_2^2, \end{aligned}$$

hence

$$\|Tf\|_2 \leq \|V\|_\infty \cdot \|f\|_2 \quad \text{for every } f \in L^2(\mathbb{R}).$$

We conclude that  $T \in B(V)$  and  $\|T\| \leq \|V\|_\infty$ .

Utilizing that  $V(x)$  is real we see that

$$(Tf, g) = \int_{-\infty}^{+\infty} V(x)f(x) \cdot \overline{g(x)} dx = \int_{-\infty}^{+\infty} f(x) \cdot \overline{V(x)g(x)} dx = (f, Tg),$$

which shows that  $T$  is self adjoint.

**Example 3.10** Let  $H$  denote a Hilbert space. Introduce in the set of all self adjoint operators from  $B(H)$  a relation  $\leq$  by

$$S \leq T, \quad \text{if } T - S \geq 0,$$

cf. EXAMPLE 6.1. Prove that  $\leq$  is a partial relation.

It follows from  $S - S = 0 \geq 0$  that  $S \leq S$ .

Assume that  $S \leq T$  and  $T \leq U$ , thus  $T - S \geq 0$  and  $U - T \geq 0$ .

We shall prove that  $S \leq U$ , i.e. that  $U - S \geq 0$ .

We have

$$\begin{aligned} ((U - S)x, x) &= ((U - T) + (T - S)x, x) \\ &= ((U - T)x, x) + ((T - S)x, x) \geq 0. \end{aligned}$$

This holds for every  $x \in H$ , hence the claim is proved.

**Example 3.11** Let  $H$  be a Hilbert space and let  $T \in B(H)$  be positive and self adjoint.

Show that

$$\|(Tx, y)\|^2 \leq (Tx, x)(Ty, y),$$

for all  $x, y \in H$ .

We shall here be aware of two possible obstacles. First,  $(Tx, y)$  could be a complex number, and secondly  $(Tx, x)$  could be 0, so we must never divide by  $(Tx, x)$ .

Let  $x, y \in H$  be given, and choose  $\alpha \in \mathbb{R}$  such that

$$(Tx, y) = |(Tx, y)| e^{i\alpha}.$$

Using the assumption it follows for any  $\lambda \in \mathbb{C}$  that

$$\begin{aligned} 0 &\leq (T(\lambda x + y), \lambda x + y) \\ &= |\lambda|^2 (Tx, x) + \lambda (Tx, y) + \bar{\lambda} (Ty, x) + (Ty, y) \\ &= |\lambda|^2 (Tx, x) + \lambda (Tx, y) + \bar{\lambda} (y, Tx) + (Ty, y) \\ &= |\lambda|^2 (Tx, x) + 2 \operatorname{Re}\{\lambda (Tx, y)\} + (Ty, y), \end{aligned}$$

where we have used that  $T$  is self adjoint, hence

$$(Ty, x) = (y, Tx) = \overline{(Tx, y)}.$$

Choosing in particular  $\lambda = \mu e^{-i\alpha}$ ,  $\mu \in \mathbb{R}$ , then

$$\mu^2 (Tx, x) + 2\mu |(Tx, y)| + (Ty, y) \geq 0 \quad \text{for all } \mu \in \mathbb{R}.$$

All coefficients are real, so the condition of the discriminant  $B^2 - AC \leq 0$  holds, thus

$$|(Tx, y)|^2 \leq (Tx, x)(Ty, y) \quad \text{for all } x, y \in H,$$

and the claim is proved.

**Example 3.12** 1) Let  $V$  denote a normed space. Show that

$$\|x - y\| \geq | \|x\| - \|y\| | \quad \text{for all } x, y \in V.$$

- 2) Let  $T$  be a bounded, linear and self adjoint operator on a Hilbert space. Assume that  $T$  is surjective and show that  $T$  is then injective.
- 3) Assume that  $T$  is a closed linear operator on a normed space  $X$ . Show that  $\ker(T)$  is closed in  $X$ .
- 4) Let  $H$  denote a Hilbert space and assume that  $(x_n)$  and  $(y_n)$  are two sequences in the closed unit ball of  $H$  such that  $(x_n, y_n) \rightarrow 1$ . Show that  $\|x_n - y_n\| \rightarrow 0$ .
- 5) Let  $(x_n)$  and  $(y_n)$  denote two orthonormal sequences in a Hilbert space  $H$ , and assume that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1.$$

Show that if  $(x_n)$  is an orthonormal basis, then so is  $(y_n)$ .

- 1) It follows from the triangle inequality that

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|,$$

and analogously (or just by interchanging letters)

$$\|y\| \leq \|x - y\| + \|x\|.$$

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By a rearrangement,

$$\left. \begin{array}{l} \|x\| - \|y\| \\ \|y\| - \|x\| \end{array} \right\} \leq \|x - y\|,$$

hence

$$\|x - y\| \geq |\|x\| - \|y\||.$$

2) We shall prove that if  $Tx = 0$ , then  $x = 0$ . We get for every  $y \in H$  that

$$0 = (0, y) = (Tx, y) = (x, Ty).$$

From  $T$  being surjective follows that the image of  $T$  is all of  $H$ , so  $x$  is perpendicular to  $H$ , thus  $x = 0$ , and  $T$  is injective.

3) Let  $T$  be closed, thus the graph  $\mathcal{G}(T)$  is closed as a subset of  $X \times X$ . Let  $(x_n) \subset \ker(T)$  denote a convergent sequence in  $X$ , i.e.  $x_n \rightarrow x$ . Then  $((x_n, 0)) \subset \mathcal{G}(T)$ , and

$$(x_n, 0) \rightarrow (x, 0) \in \overline{\mathcal{G}(T)} = \mathcal{G}(T),$$

which shows that  $x \in \ker(T)$ .

4) Here,

$$\begin{aligned} \|x_n - y_n\|^2 &= (x_n - y_n, x_n y_n) = (x_n, x_n) - (y_n, x_n) - (x_n, y_n) + (y_n, y_n) \\ &= \|x_n\|^2 + \|y_n\|^2 - 2 \operatorname{Re} \{(x_n, y_n)\}, \end{aligned}$$

and since all  $x_n$  and  $y_n$  belong to the unit ball, we have

$$0 \leq \|x_n - y_n\|^2 \leq 1 + 1 - 2 \operatorname{Re} \{(x_n, y_n)\} \rightarrow 2 - 2 = 0 \quad \text{for } n \rightarrow \infty,$$

proving that

$$\|x_n - y_n\| \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

5) Let  $x \in H$  be perpendicular to all  $y_n$ . From  $(x_n)$  being an orthonormal basis and  $(x, y_n) = 0$  we get

$$x = \sum_{n=1}^{\infty} (x, x_n) x_n = \sum_{n=1}^{\infty} \{(x, y_n) + (x, x_n - y_n)\} x_n = \sum_{n=1}^{\infty} (x, x_n - y_n) x_n.$$

This implies the estimate, when we apply that  $(x_n)$  is orthonormal and the Cauchy-Schwarz inequality,

$$\|x\| = \sum_{n=1}^{\infty} |(x, x_n - y_n)| \leq \sum_{n=1}^{\infty} \|x\|^2 \cdot \|x_n - y_n\|^2 = \|x\|^2 \sum_{n=1}^{\infty} \|x_n - y_n\|^2.$$

It follows from the assumption that  $\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1$ , so the only possibility for this inequality is when  $x = 0$ , hence  $x = 0$  is the only vector in  $H$ , which is perpendicular on all  $y_n$ . This shows that  $(y_n)$  is an orthonormal basis.



**Example 3.13** Let  $(x_n) \subset \ell^2$  and define the sequence  $y = (y_n)$  by

$$y_n = x_{n+1} + n x_n + x_{n-1},$$

where we put  $x_0 = 0$  whenever it is necessary.

1. Show that  $y \in \ell^2$  if and only if  $(n x_n) \in \ell^2$ .

Let

$$D = \{x \in \ell^2 \mid (n x_n) \in \ell^2\},$$

and define a linear operator  $T : D \rightarrow \ell^2$  by  $Tx = y$ , where  $y$  is given above.

2. Show that  $D$  is dense in  $\ell^2$ .

3. Show that  $T$  is self adjoint.

1) It follows from

$$(y_n) = (x_{n+1}) + (n x_n) + (x_{n-1}),$$

and that  $\ell^2$  is a vector space that if  $(x_n)$  and  $(n x_n) \in \ell^2$ , then  $(y_n) \in \ell^2$ .

If conversely  $(x_n)$  and  $(y_n) \in \ell^2$ , then it follows from

$$(n x_n) = (y_n) - (x_{n+1}) - (x_{n-1}),$$

that  $(n x_n) \in \ell^2$ .

ALTERNATIVELY, we have the following possible, though not very brilliant variant,

$$\begin{aligned} \sum_{n=1}^{+\infty} y_n^2 &= \sum_{n=1}^{+\infty} (x_{n+1} + n x_n + x_{n-1})^2 \\ &= \sum_{n=1}^{+\infty} x_{n+1}^2 + \sum_{n=1}^{+\infty} (n x_n)^2 + \sum_{n=1}^{+\infty} x_{n-1}^2 + 2 \sum_{n=1}^{+\infty} x_{n+1} n x_n + 2 \sum_{n=1}^{+\infty} n x_n x_{n-1} + 2 \sum_{n=1}^{+\infty} x_{n+1} x_{n-1} \\ &\leq \|x\|_2^2 + \sum_{n=1}^{+\infty} (n x_n)^2 + 2\|x\|_2 \left\{ \sum_{n=1}^{+\infty} (n x_n)^2 \right\}^{\frac{1}{2}} + 2 \left\{ \sum_{n=1}^{+\infty} (n x_n)^2 \right\} \|x\|_2 + 2\|x\|_2 \|x\|_2 \\ &= 4\|x\|_2^2 + 4\|x\| \left\{ \sum_{n=1}^{+\infty} (n x_n)^2 \right\}^{\frac{1}{2}} + \sum_{n=1}^{+\infty} (n x_n)^2 \left( \left\{ \sum_{n=1}^{+\infty} (n x_n)^2 \right\}^{\frac{1}{2}} + 2\|x\|_2 \right)^2. \end{aligned}$$

Hence, if  $\sum_{n=1}^{+\infty} (n x_n)^2 < +\infty$ , then  $\sum_{n=1}^{+\infty} y_n^2 < +\infty$ , so  $y \in \ell^2$ .

Conversely, if  $y \in \ell^2$ , then by a rearrangement,

$$n x_n = y_n - x_{n+1} - x_{n-1},$$

hence

$$\begin{aligned}
\sum_{n=1}^{+\infty} (n x_n)^2 &= \sum_{n=1}^{+\infty} (y_n - x_{n+1} - x_{n-1})^2 \\
&= \sum_{n=1}^{+\infty} y_n^2 + \sum_{n=1}^{+\infty} x_{n+1}^2 + \sum_{n=1}^{+\infty} x_{n-1}^2 - 2 \sum_{n=1}^{+\infty} y_n x_{n+1} - 2 \sum_{n=1}^{+\infty} y_n x_{n-1} + 2 \sum_{n=1}^{+\infty} x_{n+1} x_{n-1} \\
&\leq \|y\|_2^2 + \|x\|_2^2 + \|x\|_2^2 + 2\|y\|_2 \|x\|_2 + 2\|y\|_2 \|x\|_2 + 2\|x\|_2 \|x\|_2 \\
&= \|y\|_2^2 + 4\|y\|_2 \|x\|_2 + 4\|x\|_2^2 = \{\|y\|_2 + 2\|x\|_2\}^2 < +\infty.
\end{aligned}$$

We conclude that  $(n x_n) \in \ell^2$ .

- 2) Let  $D = \{x \in \ell^2 \mid (n x_n) \in \ell^2\}$ , and let  $z \in \ell^2$  be arbitrary, i.e.  $\sum_{n=1}^{+\infty} z_n^2 < +\infty$ . To any  $\varepsilon > 0$  there exists an  $N \in \mathbb{N}$ , such that

$$\sum_{n=N+1}^{+\infty} z_n^2 < \varepsilon^2.$$

Define  $x = (x_n)$  by

$$x_n = \begin{cases} z_n & \text{for } n = 1, 2, \dots, N, \\ 0 & \text{for } n > N. \end{cases}$$

Then

$$\sum_{n=1}^{+\infty} (n x_n)^2 = \sum_{n=1}^N n^2 x_n^2 < +\infty,$$

because the sum is finite, so  $x \in D$ , and

$$\|z - x\|_2 = \left\{ \sum_{n=1}^{+\infty} (z_n - x_n)^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{n=N+1}^{+\infty} z_n^2 \right\}^{\frac{1}{2}} < (\varepsilon^2)^{\frac{1}{2}} = \varepsilon,$$

which shows that  $x$  approximates  $z$ , and we get that  $D$  is dense in  $\ell^2$ . Clearly,  $D$  is a subspace, because  $(x_n), (y_n), (n x_n), (n y_n) \in \ell^2$  for every  $\lambda \in \mathbb{R}$  imply that  $(x_n + \lambda y_n)$  and  $(n(x_n + \lambda y_n)) = (n x_n + \lambda n y_n) \in \ell^2$ . Finally, it is obvious that  $T$  is linear.

- 3) Because  $T$  is densely defined, the adjoint  $T^*$  exists. Let  $x \in D$ , and let  $y \in \mathcal{D}(T^*)$ . Then

$$(Tx, y) = (x, T^*y),$$

thus

$$\begin{aligned}
 (Tx, y) &= \sum_{n=1}^{+\infty} (x_{n+1} + n x_n + x_{n-1}) y_n \\
 &= \sum_{n=1}^{+\infty} x_{n+1} y_n + \sum_{n=1}^{+\infty} n x_n y_n + \sum_{n=1}^{+\infty} x_{n-1} y_n \\
 &= \sum_{n=2}^{+\infty} x_n y_{n-1} + \sum_{n=1}^{+\infty} x_n n y_n + \sum_{n=0}^{+\infty} x_n y_{n+1} \\
 &= \sum_{n=1}^{+\infty} x_n y_{n-1} + \sum_{n=1}^{+\infty} x_n n y_n + \sum_{n=1}^{+\infty} x_n y_{n+1} \\
 &= \sum_{n=1}^{+\infty} x_n (y_{n+1} + n y_n + y_{n-1}) = (x, T^* y).
 \end{aligned}$$

The splitting of the sums in the second equality is legal, because each of the three series on the right hand side is absolutely convergent by the Cauchy-Schwarz inequality. Hence we conclude that

$$T^* y = (y_{n+1} + n y_n + y_{n-1}),$$

thus  $D \subseteq \mathcal{D}(T^*)$ , and  $T \subseteq T^*$ , so  $T$  is at least symmetric.

It follows from the result of (1) that  $(y_{n+1} + n y_n + y_{n-1}) \in \ell^2$ , when  $(y_n) \in \ell^2$ , if and only if  $(n y_n) \in \ell^2$ . Hence  $\mathcal{D}(T^*) = D$ , and  $T = T^*$ , and we have proved that  $T$  is self adjoint.

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## 4 Isometric operators

**Example 4.1** Let  $T \in B(H)$ . An operator is called isometric if  $\|Tx\| = \|x\|$  for all  $x \in H$ . Show that the following conditions are equivalent for  $T \in B(H)$ .

- 1)  $T$  is isometric.
- 2)  $T^*T = I$ .
- 3)  $(Tx, Ty) = (x, y)$  for all  $x, y \in H$ .

(3)  $\Rightarrow$  (2). This is almost trivial:

$$(x, y) = (Tx, Ty) = (T^*Tx, y) \quad \text{for all } x, y \in H,$$

thus  $T^*Tx = x$  for all  $x \in H$ , and hence  $T^*T = I$ .

(2)  $\Rightarrow$  (1). If  $T^*T = I$ , then

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = (x, x) = \|x\|^2,$$

proving that  $T$  is isometric.

(1)  $\Rightarrow$  (3). If  $T$  is isometric, we get as above,

$$(T^*Tx, x) = (Ix, x), \quad \text{thus} \quad ((T^*T - I)x, x) = 0,$$

for all  $x \in H$ . Then it follows from EXAMPLE 1.8 in VENTUS, FUNCTIONAL ANALYSIS, HILBERT SPACES that  $T^*T - I = 0$ , if  $H$  is a complex Hilbert space, hence  $T^*T = I$ .

**Example 4.2** Let  $T \in B(H)$  be an isometric operator. Show that  $T(H)$  is a closed subspace. Show that  $T(H) = H$  if  $H$  is finite dimensional.

Give an example of an isometric operator with  $T(H) \neq H$ .

- 1) When  $T \in B(H)$  is isometric, i.e.  $\|Tx\| = \|x\|$  for all  $x \in H$ , then in particular  $T$  is injective, thus  $T^{-1} : T(H) \rightarrow H$  exists.

Put  $y = Tx$ . Then it follows from the above that  $\|T^{-1}y\| = \|x\|$ , and  $T^{-1}$  is continuous (though not necessarily defined in all of  $H$ ).

Now,  $H$  is closed, so  $T(H) = (T^{-1})^{-1}(H)$  is also closed.

- 2) Let  $H$  be finite dimensional,  $\dim H = n$ , and denote by  $\{e_1, \dots, e_n\}$  a basis of  $H$ .

When  $T$  is isometric, then  $T$  is injective. In fact,  $0 = \|Tx\| = \|x\|$  implies trivially that  $x = 0$ .

We claim that the images  $\{Te_1, \dots, Te_n\}$  of the basis vectors are linearly independent. Assume that

$$0 = \lambda_1 Te_1 + \dots + \lambda_n Te_n \quad (= T(\lambda_1 e_1 + \dots + \lambda_n e_n)).$$

The operator  $T$  is injective, so also  $\lambda_1 e_1 + \dots + \lambda_n e_n = 0$ . Here  $\{e_1, \dots, e_n\}$  is a basis, so  $\lambda_1 = \dots = \lambda_n = 0$ . It follows that  $Te_1, \dots, Te_n$  are linearly independent, so  $n \leq \dim T(H) \leq n$ , thus  $\dim T(H) = n$ . This is only possible, if  $T(H) = H$ , because  $T : H \rightarrow H$ .

3) Let  $(e_k)_{k \in \mathbb{N}}$  denote an orthonormal basis in an infinite dimensional Hilbert space. Define  $T \in B(H)$  by

$$Tx = T \left( \sum_{k=1}^{\infty} x_k e_k \right) = \sum_{k=1}^{\infty} x_k e_{k+1}.$$

Then clearly  $T$  is isometric,  $\|Tx\| = \|x\|$  for all  $x \in H$ , and

$$T(H) = \{e_1\}^{\perp} \neq H.$$

**Example 4.3** Let  $T \in B(H)$  be an isometric operator and let  $M$  and  $N$  denote closed subspaces of the Hilbert space  $H$ . Show that

$$T(M) = N \implies T(M^{\perp}) \subset N^{\perp}.$$

Show that  $T$  is isometric if and only if for any orthonormal basis  $(e_k)$ ,  $(Te_k)$  is an orthonormal sequence.

Assume that  $T \in B(H)$  is isometric, and let  $M$  and  $N \subseteq H$  be closed subspaces, and assume that  $T(M) = N$ . We shall prove that for every  $x \in M^{\perp}$  and for every  $y \in N$  we have that  $(Tx, y) = 0$ .

From  $y \in N = T(M)$  follows that there exists a  $z \in M$ , such that  $y = Tz$ , and then we get from EXAMPLE 4.1, (3) that

$$(Tx, y) = (Tx, Tz) = (x, z) = 0,$$

because  $x \in M^{\perp}$  and  $z \in M$ . It follows that  $T(M^{\perp}) \subseteq N^{\perp}$ .

Let  $(e_k)$  denote an orthonormal basis, and assume that  $T$  is isometric. We get again from EXAMPLE 4.1, (3) that

$$(Te_j, Te_k) = (e_j, e_k) = \delta_{jk},$$

(Kronecker symbol), which shows that  $(Te_k)$  is an orthonormal *sequence*. Of course  $(Te_k)$  needs *not* be a basis. An example is given in EXAMPLE 4.2.

If conversely there exists an orthonormal basis  $(e_k)$ , such that  $(Te_k)$  is an orthonormal sequence, then

$$Tx = \sum_{k=1}^{+\infty} x_k Te_k, \quad \text{thus} \quad \|Tx\|^2 = \sum_{k=1}^{+\infty} |x_k|^2 = \|x\|^2,$$

and  $T$  is isometric.

**Remark 4.1** The answer of the latter question above shows that if there is just one orthonormal basis  $(e_k)$ , such that  $(Te_k)$  is an orthonormal sequence, then every orthonormal basis has this property.  $\diamond$

**Example 4.4** Let  $T \in B(H)$  be an isometric operator. Show that  $TT^*$  is a projection and determine its range.

Assume that  $T \in B(H)$  is isometric. We shall prove that  $TT^*$  is a projection, i.e.  $TT^*$  must satisfy the two conditions,

$$(TT^*x, y) = (x, TT^*y) \quad \text{for all } x, y \in H,$$

and

$$(TT^*)^2 = TT^*.$$

We get

$$(TT^*x, y) = (T^*x, T^*y) = (x, TT^*y),$$

and the first condition is fulfilled. Then apply the result  $T^*T = I$  from EXAMPLE 4.1, (2),

$$(TT^*)^2 = TT^*TT^* = T(T^*T)T^* = TIT^* = TT^*,$$

and it follows that  $P = TT^*$  is a projection.

The range of the projection  $P = TT^*$  is given by  $Px = TT^*x = x$ , i.e.  $TT^*H$ . Now,

$$T^*(H) = \overline{T^*(H)} = \ker(T)^\perp,$$

thus  $TT^*(H) = T(\ker(T)^\perp)$ . It follows from

$$H = \ker(T) \oplus \ker(T)^\perp,$$

that

$$TT^*(H) = T(\ker(T)^\perp) = T(\ker(T) \oplus \ker(T)^\perp) = T(H),$$

and the range is  $TH$ .

**Example 4.5** Consider the Hilbert space  $L^2([0, \infty))$ . Let  $h > 0$  and define the operator  $T$  by

$$\begin{aligned} Tf(x) &= 0 && \text{for } 0 \leq x < h, \\ Tf(x) &= f(x-h) && \text{for } h \leq x. \end{aligned}$$

Show that  $T$  is isometric and determine  $T^*$ . Find  $TT^*$  and  $T^*T$ .

First notice that

$$\|Tf\|_2^2 = \int_0^{+\infty} |Tf(x)|^2 dx = \int_h^{+\infty} |f(x-h)|^2 dx = \int_0^{+\infty} |f(x)|^2 dx = \|f\|_2^2,$$

which shows that  $T$  is isometric. Then it follows from EXAMPLE 4.1, (2) that  $T^*T = I$ .

Let  $f, g \in H$ . Then

$$\begin{aligned}(Tf, g) &= \int_0^{+\infty} Tf(x)\overline{g(x)} dx = \int_h^{+\infty} f(x-h)\overline{g(x)} dx \\ &= \int_0^{+\infty} f(x)\overline{g(x+h)} dx = (f, T^*g),\end{aligned}$$

and we conclude that

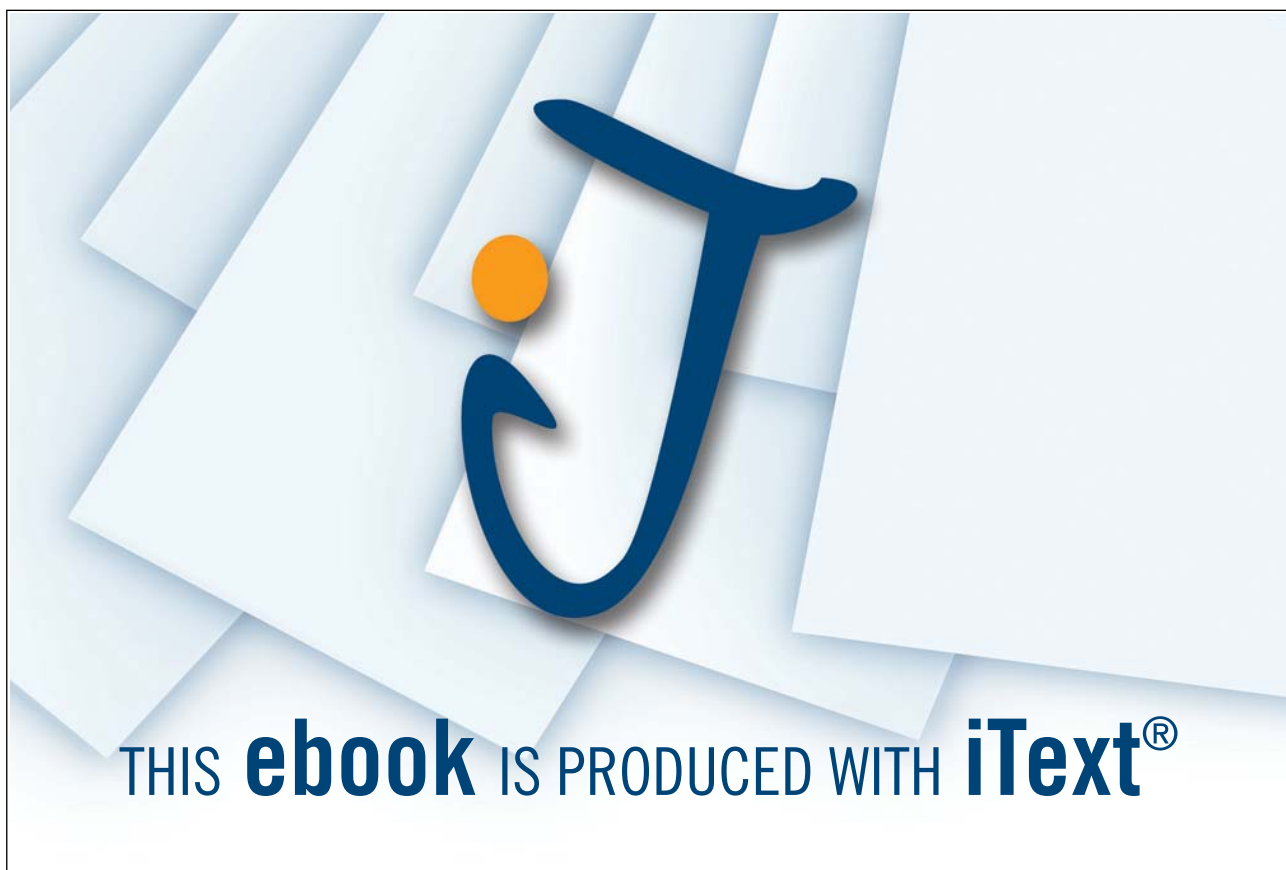
$$T^*g(x) = g(x+h) \quad \text{for } x \in [0, +\infty[.$$

Then finally we get

$$TT^*g(x) = Tg(x+h) = \begin{cases} g(x+h-h) = g(x) & \text{for } x \in [h, +\infty[, \\ 0 & \text{for } x \in [0, h[, \end{cases}$$

thus  $TT^*g = 1_{[h, +\infty[} \cdot g$ .

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## 5 Unitary and normal operators

**Example 5.1** An operator  $T \in B(H)$  is called unitary if it is isometric and surjective. Show that the following conditions are equivalent for an operator  $T \in B(H)$ ,

- (a)  $T$  is unitary.
- (b)  $T$  is bijective and  $T^{-1} = T^*$ .
- (c)  $T^* = TT^* = I$ .
- (d)  $T$  and  $T^*$  are isometric.
- (e)  $T$  is isometric and  $T^*$  is injective.
- (f)  $T^*$  is unitary.

(a)  $\Rightarrow$  (b). Assume that  $T$  is unitary, thus  $T(H) = H$ , and  $\|Tx\| = \|x\|$  for  $x \in H$ . Clearly,  $Tx = 0$  implies that  $x = 0$ , so  $T$  is injective, and  $T^{-1}$  exists and is continuous with  $\|T\|^{-1} = 1$ . (Sketch of proof: Put  $y = Tx$ , etc.) From  $D(T^{-1}) = T(H) = H$ , we even get that  $T^{-1} \in B(H)$ , and we conclude that  $T$  is bijective.

Then it follows from EXAMPLE 4.1, (2) that  $T^*T = I$ , and from the definition of  $T^{-1}$  we get  $T^{-1}T = I$ . Hence,

$$0 = (T^* - T^{-1})T, \quad \text{thus} \quad (T^* - T^{-1})T(H) = \{0\}.$$

From  $T(H) = H$  follows that  $T^* - T^{-1}$  is identically 0 on all of  $H$ , thus  $T^* = T^{-1}$ .

(b)  $\Rightarrow$  (c). Assume that  $T$  is bijective and that  $T^{-1} = T^*$ . Then

$$T^*T = T^{-1}T = I \quad \text{and} \quad TT^* = TT^{-1} = I.$$

(c)  $\Rightarrow$  (d). Let  $T^*T = TT^* = I$ . It follows from EXAMPLE 4.1, (2) that  $T$  is isometric. Then we conclude from

$$(TT^*)^* = (T^*)^*T^* = I^* = I,$$

that  $T^*$  is also isometric by EXAMPLE 4.1, (2).

(d)  $\Rightarrow$  (e). If  $T$  and  $T^*$  are isometric, then  $T^*$  is in particular injective.

(e)  $\Rightarrow$  (a). Assume that  $T$  is isometric and that  $T^*$  is injective. We shall prove (a), so it only remains to prove that  $T(H) = H$ .

Because  $T(H)$  is closed, it suffices to prove that if

$$(Ty, x) = 0 \quad \text{for all } y \in H,$$

then  $x = 0$ . We have

$$0 = (Ty, x) = (y, T^*x) \quad \text{for all } y \in H.$$

When we in particular choose  $y = T^*x$ , then

$$(T^*x, T^*x) = \|T^*x\|^2 = 0, \quad \text{thus} \quad T^*x = 0.$$

Now,  $T^*$  is injective, so  $x = 0$ .



Summing up we have proved that (a)–(e) are equivalent. We shall only prove that we can add (f) to this family of equivalent conditions.

(a)  $\wedge$  (d)  $\Rightarrow$  (f). If  $T$  is unitary, then  $T^*$  and  $T^{**} = T$  are isometric, so  $T^*$  is unitary by (d).

(f)  $\wedge$  (d)  $\Rightarrow$  (a). If  $T^*$  is unitary, then  $T^*$  and  $T^{**} = T$  are isometric, and  $T$  is unitary by (d).

**Example 5.2** Let  $(e_k)$  denote an orthonormal basis in a Hilbert space  $H$  and let  $T \in B(H)$  be given by

$$T \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} \lambda_k a_k e_k.$$

Show that  $T$  is unitary if and only if  $|\lambda_k| = 1$  for all  $k$ .

We conclude from

$$\|Tx\|^2 = \left\| \sum_{k=1}^{\infty} \lambda_k x_k e_k \right\|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 |x_k|^2,$$

that if  $|\lambda_k| = 1$  for all  $k$ , then  $\|Tx\| = \|x\|$ , hence  $T$  is isometric.

If there exists a  $k$ , such that  $|\lambda_k| \neq 1$ , then  $\|Te_k\| = |\lambda_k| \neq 1 = \|e_k\|$ , and  $T$  is not isometric.

We have proved that  $T$  is isometric, if and only if  $|\lambda_k| = 1$  for all  $k \in \mathbb{N}$ . We shall only prove that if  $|\lambda_k| = 1$  for all  $k \in \mathbb{N}$ , then  $T(H) = H$ , because this implies by EXAMPLE 5.1 that  $T$  is unitary.

Let  $y \in H$ , i.e.

$$y = \sum_{k=1}^{\infty} y_k e_k \quad \text{and} \quad \sum_{k=1}^{\infty} |y_k|^2 < \infty.$$

If there exists an  $x \in H$ , such that  $Tx = y$ , then

$$\sum_{k=1}^{\infty} \lambda_k x_k e_k = \sum_{k=1}^{\infty} y_k e_k \quad \text{and} \quad \sum_{k=1}^{\infty} |x_k|^2 < \infty.$$

It is seen by the identification that since  $\lambda_k \cdot \overline{\lambda_k} = |\lambda_k|^2 = 1$ , we have only the possibility that  $\lambda_k x_k = y_k$ , thus

$$x_k = \frac{y_k}{\lambda_k} = \overline{\lambda_k} y_k.$$

We shall only prove that the *candidate*

$$x = \sum_{k=1}^{\infty} \overline{\lambda_k} y_k e_k$$

belongs to  $H$ . This is trivial, because

$$\sum_{k=1}^{\infty} |x_k|^2 = \sum_{k=1}^{\infty} |\overline{\lambda_k}|^2 |y_k|^2 = \sum_{k=1}^{\infty} |y_k|^2 = \|y\|^2 < \infty,$$

so  $x \in H$ , and  $Tx = y$ . This proves that  $T(H) = H$ , and it then follows from EXAMPLE 5.1 that  $T$  is unitary.

**Example 5.3** Let  $T \in B(H)$  be unitary. Show that

$$\sigma(T) \subset \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let  $|\lambda| \neq 1$ . Because  $T$  is unitary, we get in particular that  $\|T\| = \|x\|$ , hence

$$\|Tx - \lambda x\| \geq |\|Tx\| - \|\lambda x\|| = |1 - |\lambda|| \cdot \|x\|.$$

It follows that  $(T - \lambda I)^{-1}$  exists for every  $\lambda \in \mathbb{C}$ , for which  $|\lambda| \neq 1$ . We shall finish the proof by showing that  $(T - \lambda I)^{-1}$  is densely defined in  $H$ , because then

$$\varrho(T) \supseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid |z| = 1\} \quad \text{and} \quad \sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}.$$

Assume that  $(T - \lambda I)^{-1}$  is not densely defined for some  $\lambda \in \mathbb{C}$ . Then there exists an  $y \neq 0$ , such that

$$y \perp (T - \lambda I)D(T - \lambda I) = (T - \lambda I)(H),$$

thus

$$0 = ((T - \lambda I)x, y) = (x, (T^* - \bar{\lambda}I)y) = (x, 0) \quad \text{for all } x \in H.$$

We conclude that  $T^*y - \bar{\lambda}y = 0$ , hence  $\bar{\lambda}$  is even an eigenvalue for  $T^* = T^{-1}$ .

By EXAMPLE 5.1,  $T^*$  is also unitary, thus  $|\bar{\lambda}| = 1$ , and hence also  $|\lambda| = 1$ . Then it follows by contraposition that if  $|\lambda| \neq 1$ , then  $(T - \lambda I)^{-1}$  is densely defined. Then

$$\varrho(T) \supseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid |z| = 1\} \quad \text{and} \quad \sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}.$$

**Example 5.4** An operator  $T \in B(H)$  is normal if

$$TT^* = T^*T.$$

Show that  $T$  is normal if and only if  $\|T^*x\| = \|Tx\|$  for all  $x \in H$ .

If  $T \in B(H)$  is normal, i.e.  $T^*T = TT^*$ , then

$$\|T^*x\|^2 = (T^*x, T^*x) = (TT^*x, x) = (T^*Tx, x) = (Tx, Tx) = \|Tx\|^2,$$

and we conclude that  $\|T^*x\| = \|Tx\|$  for all  $x \in H$ .

Assume conversely that  $\|T^*x\| = \|Tx\|$  for all  $x \in H$ . Then

$$0 = \|T^*x\|^2 - \|Tx\|^2 = (T^*x, T^*x) - (Tx, Tx) = (TT^*x, x) - (T^*Tx, x) = ((TT^* - T^*T)x, x).$$

The space  $H$  is complex. so it follows that  $TT^* - T^*T = 0$ , hence  $T^*T = TT^*$  as required.

**Example 5.5** Let  $T \in B(H)$  be normal. Show that

$$\|(T - \lambda I)x\| = \|(T^* - \bar{\lambda} I)x\|$$

for all  $x \in H$ . Show that  $\sigma_r(T)$  is empty.

If  $T$  is normal, then  $T^*T = TT^*$ , and we get

$$\begin{aligned} \|(T - \lambda I)x\|^2 &= ((T - \lambda I)x, (T - \lambda I)x) \\ &= (Tx, Tx) - \lambda(x, Tx) - \bar{\lambda}(Tx, x) + |\lambda|^2(x, x) \\ &= (T^*Tx, x) - \lambda(T^*x, x) - \bar{\lambda}(x, T^*x) + |\lambda|^2(x, x) \\ &= (TT^*x, x) - (T^*x, \bar{\lambda}x) - (\bar{\lambda}x, T^*x) + (\bar{\lambda}x, \bar{\lambda}x) \\ &= (T^*x, T^*x) - (T^*x, \bar{\lambda}x) - (\bar{\lambda}x, T^*x) + (\bar{\lambda}x, \bar{\lambda}x) \\ &= ((T^* - \bar{\lambda}I)x, (T^* - \bar{\lambda}I)x) = \|(T^* - \bar{\lambda}I)x\|^2, \end{aligned}$$

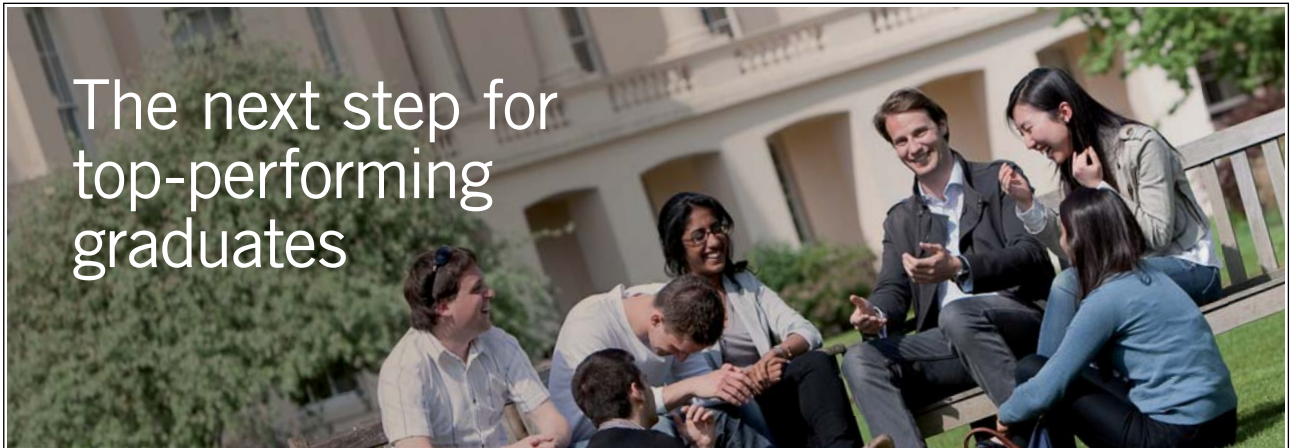
and the first claim is proved.

It follows that  $\lambda$  is an eigenvalue for  $T$  (of eigenvector  $x$ ), if and only if  $\bar{\lambda}$  is an eigenvalue for  $T^*$  (the same eigenvector  $x$ ), thus

$$\sigma_p(T^*) = \overline{\sigma_p(T)}.$$

On the other hand,  $\sigma_r(T) \subseteq \overline{\sigma_p(T^*)} = \sigma_p(T)$ , and because  $\sigma_r(T)$  and  $\sigma_p(T)$  are disjoint, we must have  $\sigma_r(T) = \emptyset$ .

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**Example 5.6** Let  $H = L^2([0, 1])$  and consider the operator

$$Tf(x) = \sqrt{3}x f(x^3).$$

- 1) Show that  $T \in B(H)$  and find  $\|T\|$ .
- 2) Show that  $T^{-1}$  exists and that  $T^{-1} \in B(H)$ . Determine  $T^{-1}g(y)$  for  $g \in H$ , and find  $\|T^{-1}\|$ .
- 3) Show that  $\sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = \|T\|\}$ .

- 1) The operator  $T$  is obviously linear.

Then by the change of variable  $y = x^3$ ,

$$\|Tf\|_2^2 = \int_0^1 |Tf(x)|^2 dx = \int_0^1 3x^2 |f(x^3)|^2 dx = \int_0^1 |f(y)|^2 dy = \|f\|_2^2,$$

hence  $T$  is isometric ( $\|Tf\|_2 = \|f\|_2$ ), thus  $T \in B(H)$  and  $\|T\| = 1$ .

- 2) We shall prove that the equation

$$Tf(x) = g(x), \quad g \in L^2([0, 1]),$$

always has a uniquely determined solution, thus  $T^{-1} : H \rightarrow H$ . It follows by the definition that we shall solve

$$Tf(x) = \sqrt{3}x f(x^3) = g(x).$$

Utilizing the monotone change of variable  $x = \sqrt[3]{y}$ , we get

$$f(y) = \frac{1}{\sqrt{3}} \cdot \frac{1}{x} g(x) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt[3]{y}} \cdot g(\sqrt[3]{y}) = T^{-1}g(y),$$

hence

$$T^{-1}g(x) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt[3]{x}} g(\sqrt[3]{x}), \quad g \in H \in L^2([0, 1]).$$

We get from the computation in (1) that  $Tf = g$  and  $f = T^{-1}g$  that

$$\|Tf\|_2 = \|g\|_2 = \|f\|_2 = \|T^{-1}g\|_2, \quad T^{-1} \in B(H),$$

and  $T^{-1}$  is also isometric,  $\|T^{-1}g\|_2 = \|g\|_2$ , and  $\|T^{-1}\| = 1$ .

We say that  $T$  is *unitary*, cf. EXAMPLE 5.1.

- 3) This has already been proved in EXAMPLE 5.3. However, let us do it again. If  $|\lambda| > 1$ , then

$$T - \lambda I = -\lambda \left( I - \frac{1}{\lambda} T \right), \quad \text{where } \left\| \frac{1}{\lambda} T \right\| = \frac{1}{|\lambda|} < 1,$$

thus  $(T - \lambda I)^{-1} \in B(H)$ , and  $(T - \lambda I)^{-1}$  is given by the Neumann series

$$(T - \lambda I)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{+\infty} \frac{1}{\lambda^n} T^n.$$

Then let  $|\lambda| < 1$ . From  $T^{-1} \in B(H)$  follows that  $T - \lambda I = T(I - \lambda T^{-1})$ . From  $\|\lambda T^{-1}\| = |\lambda| < 1$  follows by a Neumann series that

$$(T - \lambda I)^{-1} = (I - \lambda T^{-1})^{-1} T^{-1} = \left( \sum_{n=0}^{+\infty} \lambda^n (T^{-1})^n \right) T^{-1} = \sum_{n=0}^{+\infty} \lambda^n (T^{-1})^{n+1},$$

hence  $(T - \lambda I)^{-1} \in B(H)$ , and we conclude that

$$\rho(T) \supseteq \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\} \quad \text{and} \quad \sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

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## 6 Positive operators and projections

**Example 6.1** An operator  $T \in B(H)$  is positive if

$$(Tx, x) \geq 0 \quad \text{for all } x \in H,$$

and we write  $T \geq 0$ .

Prove the following:

- 1)  $T \geq 0$  implies that  $T$  is self adjoint.
- 2) If  $S, T \geq 0$ ,  $\alpha \geq 0$ , then  $S + \alpha T \geq 0$ .
- 3) If  $T \geq 0$  and  $S \in B(H)$ , then  $S^*TS \geq 0$ .
- 4) If  $T \in B(H)$  then  $T^*T \geq 0$ ,
- 5) If  $T$  is an orthogonal projection then  $T \geq 0$ .

- 1) Assume that  $T \in B(H)$  is positive, i.e.  $(Tx, x) \geq 0$  for every  $x \in H$ . Then

$$(T^*x, x) = (x, Tx) = \overline{(Tx, x)} = (Tx, x) \geq 0,$$

and  $T^*$  is also positive, and

$$((T^* - T)x, x) = 0 \quad \text{for every } x \in H.$$

Then assume that the vector space is complex. Then it follows that  $T^* - T = 0$ , i.e.  $T^* = T$ , and we have proved that  $T$  is self adjoint.

- 2) This is trivial: For every  $x \in H$ ,

$$((S + \alpha T)x, x) = (Sx, x) + \alpha(Tx, x) \geq 0 + \alpha \cdot 0 = 0.$$

- 3) It follows from  $Sx \in H$  for every  $x \in H$  that

$$(S^*TSx, x) = (T(Sx), Sx) \geq 0.$$

- 4) This is again trivial. In fact, for every  $x \in H$ ,

$$(T^*Tx, x) = (Tx, Tx) = \|Tx\|^2 \geq 0.$$

- 5) Let  $T$  denote an orthogonal projection. Then

$$T^* = T \quad \text{and} \quad T^2 = T.$$

It follows from (4) that

$$T^*T = TT = T^2 = T$$

is positive, hence  $T \geq 0$ .

**Example 6.2** Let  $P_M$  and  $P_N$  denote the orthogonal projections of the closed subspaces  $M$  and  $N$  of a Hilbert space  $H$ . Show that  $M \subset N$  implies that  $P_M \leq P_N$ .

If  $M \subseteq N$ , then

$$H = N \oplus N^\perp = M \oplus (M^\perp \cap N) \oplus N^\perp,$$

which means that every element  $x \in H$  has a unique decomposition

$$x = x_M + x_N + x^\perp, \quad \text{where } x_M \in M, \quad x_N \in M^\perp \cap N, \quad x^\perp \in N^\perp.$$

Then

$$P_M x = P_M(x_M + x_N + x^\perp) = x_M \quad \text{and} \quad P_N x = P_N(x_M + x_N + x^\perp) = x_M + x_N.$$

It follows that

$$\begin{aligned} ((P_N - P_M)x, x) &= (x_M + x_N - x_M, x_M + x_N + x^\perp) = (x_N, x_M + x_N + x^\perp) \\ &= (x_N, x_M) + (x_N, x_N) + (x_N, x^\perp) \\ &= 0 + \|x_N\|^2 + 0 = \|x_N\|^2 \geq 0, \end{aligned}$$

hence  $P_N - P_M \geq 0$ , and whence  $P_M \leq P_N$ .

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**Example 6.3** An operator  $T \in B(H)$  is called a contraction

$$\|Tx\| \leq \|x\| \quad \text{for all } x \in H.$$

Show that the following conditions are equivalent for an operator  $T \in B(H)$ :

- 1)  $T$  is a contraction,
- 2)  $\|T\| \leq 1$ ,
- 3)  $T^*T \leq I$ ,
- 4)  $TT^* \leq I$ ,
- 5)  $T^*$  is a contraction,
- 6)  $T^*T$  is a contraction.

(1)  $\Rightarrow$  (2). Let  $T \in B(H)$  denote a contraction, thus  $\|Tx\| \leq \|x\|$  for all  $x \in H$ . Then

$$\|T\| = \sup\{\|Tx\| \mid \|x\| \leq 1\} \leq \sup\{\|x\| \mid \|x\| \leq 1\} = 1,$$

and we have proved (2).

(2)  $\Rightarrow$  (3). Assume that  $\|T\| \leq 1$ . Then

$$\begin{aligned} (9) \quad ((I - T^*T)x, x) &= (x, x) - (T^*Tx, x) = \|x\|^2 - (Tx, Tx) \\ &= \|x\|^2 - \|Tx\|^2 \geq \|x\|^2 - 1 \cdot \|x\|^2 = 0, \end{aligned}$$

and we have proved that  $I = T^*T \geq 0$ , hence  $T^*T \leq I$ , and we have proved that (3).

(3)  $\Rightarrow$  (1). Assume that  $T^*T \leq I$ . By repeating (9) we see that  $\|x\|^2 - \|Tx\|^2 \geq 0$ , thus  $\|Tx\| \leq \|x\|$ , and we have proved (1).

It follows from the above that the former three conditions (1)–(3) are equivalent.

(1)  $\Leftrightarrow$  (5). If  $T$  is a contraction, then by (2),  $\|T^*\| = \|T\| \leq 1$ , and we infer that  $T^*$  is a contraction.

If conversely  $T^*$  is a contraction, then  $T^{**} = T$  is contraction.

We have proved that the conditions (1)–(3) and (5) are equivalent.

(1)  $\Leftrightarrow$  (4). If (1) is fulfilled, then also (3) and (5), and it follows that (5) is equivalent with

$$(T^*)^* T^* = TT^* \leq I,$$

and (1)–(5) are all equivalent.

(1)  $\Rightarrow$  (6). If  $T$  is a contraction, then we have proved that  $\|T^*\| = \|T\| \leq 1$ , and it follows that

$$\|TT^*\| \leq \|T^*\| \cdot \|T\| \leq 1^1 = 1,$$

thus  $T^*T$  is a contraction by (2), and we have proved (6).



(6)  $\Rightarrow$  (1). If  $T^*T$  is a contraction, then

$$\|T^*Tx\| \leq \|x\| \quad \text{for all } x \in H,$$

hence by the Cauchy-Schwarz inequality

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*Tx\| \cdot \|x\| \leq \|x\|^2.$$

We infer that  $\|Tx\| \leq \|x\|$  for every  $x \in H$ , and  $T$  is by the definition a contraction.

We have proved that the six conditions (1)–(6) are equivalent.

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## 7 Compact operators

**Example 7.1** Let  $S$  and  $T$  be linear and bounded operators and assume that  $S$  is compact. Show that  $ST$  and  $TS$  are compact.

According to the definition,  $S \in B(H)$  is compact, if  $\overline{S(X)}$  is compact for every bounded set  $X \subset H$ .

Consider  $S, T \in B(H)$ , and let  $S$  be compact. If  $X$  is bounded, then  $T(X)$  is also bounded. In fact, if

$$M = \sup\{\|x\| \mid \|x\| \in X\},$$

then

$$\|Tx\| \leq \|T\| \cdot \|x\| \leq \|T\| \cdot M \quad \text{for all } x \in X.$$

It follows that  $\overline{ST(X)} = \overline{S(T(X))}$  is compact, hence the composite operator  $ST$  is compact.

Since  $T$  is continuous, it follows that  $\overline{TS(X)} \subseteq \overline{T(\overline{S(X)})}$ . Now,  $\overline{S(X)}$  is compact for every bounded set  $X$ , and  $T$  is continuous, hence  $\overline{T(\overline{S(X)})}$  is also compact. Now every closed subset of a compact set is compact, hence  $\overline{TS(X)}$  is compact, and the composite operator  $TS$  is compact.

**Example 7.2** Let  $S$  and  $T$  be compact operators in  $B(H)$ , and let  $\alpha \in \mathbb{C}$ . Show that  $S + \alpha T$  is compact.

Denote by  $X$  a bounded set. Then  $\overline{S(X)}$  and  $\overline{T(X)}$  are both compact sets, because  $S$  and  $T$  are compact operators. Choose any sequence  $(x_n) \subseteq (S + \alpha T)(X)$ . Then we can find other sequences  $(y_n) \subseteq X$  and  $(z_n) \subseteq X$ , such that

$$x_n = Sy_n + \alpha Tz_n.$$

The set  $\overline{S(X)}$  is compact, hence there exists a subsequence  $(y_{n_j})$ , such that  $Sy_{n_j} \rightarrow y$ , and we obtain the subsequence  $(x_{n_j})$  by

$$x_{n_j} = Sy_{n_j} + \alpha Tz_{n_j}.$$

If  $\alpha = 0$ , there is nothing to prove. If  $\alpha \neq 0$ , it follows by a rearrangement that

$$Tz_{n_j} = \frac{1}{\alpha} x_{n_j} - \frac{1}{\alpha} Sy_{n_j} \in T(X).$$

The set  $\overline{T(X)}$  is compact, so there is a subsequence  $(n_{j_k})$ , such that  $Tz_{n_{j_k}} \rightarrow z$ . This implies that the subsequence  $(x_{n_{j_k}})$  is convergent,

$$x_{n_{j_k}} = Sy_{n_{j_k}} + \alpha Tz_{n_{j_k}} \rightarrow y + \alpha z.$$

We have proved that any sequence  $(x_n)$  from  $(S + \alpha T)(X)$  has a convergent subsequence, hence  $\overline{(S + \alpha T)(X)}$  is compact. Furthermore,  $X$  is any bounded set in  $H$ , so we infer that  $S + \alpha T$  is compact.

**Remark 7.1** This result shows that the set of compact operators in  $B(H)$  is a subspace of  $B(H)$ . Then it follows from the result of EXAMPLE 7.1 that the subspace of compact operators is even a so-called two-sided ideal in  $B(H)$  with the composition of operators as multiplication.  $\diamond$

**Example 7.3** Let  $(e_k)$  denote an orthonormal basis in a Hilbert space  $H$ , and define the operator  $T$  by

$$T \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=2}^{\infty} \frac{1}{k} a_k e_{k-1}.$$

Show that  $T$  is compact and find  $T^*$ . Find  $\sigma_p(T)$  and  $\sigma_p(T^*)$ .

Define  $T_n$ ,  $n \geq 2$ , by

$$T_n \left( \sum_{k=1}^{+\infty} a_k e_k \right) = \sum_{k=2}^n \frac{1}{k} a_k e_{k-1}.$$

Then  $T_n$  is of finite rank, thus also compact. It follows from

$$(T - T_n) \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=n+1}^{+\infty} \frac{1}{k} a_k e_{k-1},$$

that

$$\left\| (T - T_n) \left( \sum_{k=1}^{+\infty} a_k e_k \right) \right\|^2 = \sum_{k=n+1}^{+\infty} \frac{1}{k^2} |a_k|^2 \leq \frac{1}{(n+1)^2} \sum_{k=n+1}^{+\infty} |a_k|^2 \leq \frac{1}{(n+1)^2} \left\| \sum_{k=1}^{+\infty} a_k e_k \right\|^2.$$

thus  $\|(T - T_n)x\| \leq \frac{1}{n+1} \|x\|$  for all  $x \in H$ , and we have proved that  $\|T - T_n\| \leq \frac{1}{n+1}$ , hence  $\|T - T_n\| \rightarrow 0$  for  $n \rightarrow +\infty$ . It follows that  $T$  is compact.

Then we check when  $T_\lambda = T - \lambda I$  is injective. It follows by recursion from

$$T_\lambda \left( \sum_{k=1}^{+\infty} a_k e_k \right) = \sum_{k=1}^{+\infty} \left\{ \frac{1}{k+1} a_{k+1} - \lambda a_k \right\} e_k = 0,$$

that

$$a_{k+1} = (k+1)\lambda a_k = \cdots = (k+1)!\lambda^k a_1, \quad k \in \mathbb{N}.$$

If  $\lambda \neq 0$ , then

$$\sum_{n=1}^{+\infty} |a_n|^2 = \sum_{k=1}^{+\infty} |a_1|^2 (k!|\lambda|^{k-1})^2.$$

Now,  $(k!|\lambda|^{k-1})^2 \rightarrow +\infty$  for  $k \rightarrow +\infty$ , thus this series is only convergent, if  $a_1 = 0$ , and hence all  $a_k = 0$ . Therefore, when  $\lambda \neq 0$ , then  $T_\lambda x = 0$  implies that  $x = 0$ , thus  $T_\lambda$  is injective for  $\lambda \neq 0$ . In

particular we get for the point spectrum  $\sigma_p(T) \subseteq \{0\}$ . On the other hand  $Te_1 = 0 = 0 \cdot e_1$ , thus 0 is an eigenvalue, and  $\sigma_p(T) = \{0\}$ .

Then we search the adjoint operator  $T^*$ . Let

$$x = \sum_{k=1}^{+\infty} x_k e_k \quad \text{og} \quad y = \sum_{k=1}^{+\infty} y_k e_k.$$

Then

$$(Tx, y) = \left( \sum_{k=2}^{+\infty} \frac{1}{k} x_k e_{k-1}, \sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{k=2}^{+\infty} \frac{1}{k} x_k \cdot \overline{y_{k-1}} = \left( \sum_{k=2(1)}^{+\infty} x_k e_k, \sum_{n=2}^{+\infty} \frac{1}{n} y_{n-1} e_n \right) = (x, T^*y),$$

from which

$$T^* \left( \sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{n=2}^{+\infty} \frac{1}{n} y_{n-1} e_n = \sum_{n=1}^{+\infty} \frac{1}{n+1} y_n e_{n+1}.$$

Assume that  $\mu \in \sigma_p(T^*)$  is an eigenvalue for  $T^*$ . Then there is a  $y = \sum_{n=1}^{+\infty} y_n e_n \neq 0$ , for which

$$(10) \quad (T^* - \mu I) \left( \sum_{n=1}^{+\infty} y_n e_n \right) = -\mu y_1 e_1 + \sum_{n=2}^{+\infty} \left\{ \frac{1}{n} y_{n-1} - \mu y_n \right\} e_n = 0.$$

Here we derive the conditions

$$\mu y_1 = 0 \quad \text{and} \quad \frac{1}{n} y_{n-1} = \mu y_n, \quad n \geq 2.$$

If  $\mu = 0$ , then it follows immediately from (10) that  $y = 0$ , thus  $0 \notin \sigma_p(T^*)$ .

If  $\mu \neq 0$ , then

$$y_1 = 0 \quad \text{and} \quad y_n = \frac{1}{n\mu} y_{n-1}, \quad n \geq 2,$$

and it follows by either induction or by recursion that  $y = 0$ , contradiction the assumption. We therefore conclude that  $\sigma_p(T^*) = \emptyset$ . This implies that the residual spectrum for  $T$  is empty,  $\sigma_r(T) = \emptyset$ .

**Remark 7.2** It is also possible here to find  $\sigma(T)$  and  $\sigma(T^*)$ , though this is not an easy task. For completeness the derivations are given in the following.

It follows immediately from the expressions of  $T$  and  $T^*$  that

$$\|T\| = \|T^*\| = \frac{1}{2},$$

hence

$$\sigma(T) \subseteq \left\{ z \in \mathbb{C} \mid |z| \leq \frac{1}{2} \right\} \quad \text{and} \quad \sigma(T^*) \subseteq \left\{ z \in \mathbb{C} \mid |z| \leq \frac{1}{2} \right\}.$$

It follows from the expression of  $T^*$ ,

$$T^* \left( \sum_{n=1}^{+\infty} y_n e_n \right) = \sum_{n=1}^{+\infty} \frac{1}{n+1} y_n e_{n+1},$$

that  $T^*$  is injective, so  $(T^*)^{-1}$  exists. Then from  $e_1 \perp T^*D(T^*)$  follows that  $(T^*)^{-1}$  is not densely defined. This means that  $0 \in \sigma_r(T^*)$ .

It follows from  $T^* \in B(H)$  and  $T \in B(H)$ , that  $T^{**} = \overline{T} = T$ . We have already proved that

$$\sigma_p(T) = \sigma_p(T^{**}) = \{0\}.$$

so it follows by contraposition that  $\sigma_r(T^*) = \{0\}$ . We have proved

$$\sigma_p(T) = \{0\}, \quad \sigma_r(T) = \emptyset, \quad \sigma_p(T^*) = \emptyset, \quad \sigma_r(T^*) = \{0\}.$$

Then we *claim* that

$$(11) \quad \sigma_c(T) = \sigma_c(T^*) = \emptyset.$$

First notice that if (11) holds, then it easily follows that

$$\sigma(T) = \sigma(T^*) = \{0\} \quad \text{and} \quad \varrho(T) = \varrho(T^*) = \mathbb{C} \setminus \{0\}.$$

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In order to prove (11) we shall need the following theorem:

**Theorem 7.1** *Assume that  $T \in B(H)$  is compact, and choose  $\lambda \neq 0$ . If  $T_\lambda = T - \lambda I$  is injective, then the range  $(T - \lambda I)(H)$  is closed.*

First assume that Theorem 7.1 holds. Let  $\lambda \in \sigma_c(T)$ . Then  $\sigma_p(T) = \{0\}$ , and because  $\sigma_p(T)$  and  $\sigma_c(T)$  are disjoint, we must have  $\lambda \neq 0$ . Then it follows from the definition of  $\sigma_c(T)$  that  $T - \lambda I$  is injective and that  $(T - \lambda I)(H)$  is dense in  $H$ . Theorem 7.1 shows that  $(T - \lambda I)(H)$  is closed, hence  $(T - \lambda I)(H) = H$ , and whence  $(T - \lambda I)^{-1}$  is bounded by the theorem of bounded inverse. This means that  $\lambda \in \rho(T)$ , contradicting the assumption that  $\lambda \in \sigma_c(T)$ . We conclude that  $\sigma_c(T) = \emptyset$ .

The proof of  $\sigma_c(T^*) = \emptyset$  is apart from a very small modification exactly the same as that above. This modification is that we this time shall use that because  $\sigma_r(T^*) = \{0\}$ , we must have  $\lambda \neq 0$  for any possible  $\lambda \in \sigma_c(T)$ .  $\diamond$

PROOF OF THEOREM 7.1. Let  $y = \lim_{n \rightarrow +\infty} y_n$ , where  $y_n = (T - \lambda I)x_n$ .

- 1) Assume that  $(x_n)$  has a bounded subsequence. Because  $T$  is compact, there must exist another subsequence  $(x_{n_i})$  such that the image sequence  $(Tx_{n_i})$  is convergent. From follows

$$x_{n_i} = \frac{1}{\lambda} (Tx_{n_i} - y_{n_i}),$$

that  $x_{n_i} \rightarrow x$  and  $y = (T - \lambda I)x$ , hence  $y \in (T - \lambda I)(H)$ , and we have proved that  $(T - \lambda I)(H)$  is closed in this case.

- 2) Then assume that  $(x_n)$  does *not* have any bounded subsequence. Then  $\|x_n\| \rightarrow +\infty$ . We define

$$z_n = \frac{x_n}{\|x_n\|}, \quad \|z_n\| = 1,$$

thus  $(T - \lambda I)z_n \rightarrow 0$ . There is a subsequence  $(z_{n_i})$ , such that  $(Tz_{n_i})$  is convergent. However,  $\left(z_{n_i} - \frac{1}{\lambda} Tz_{n_i}\right)$  is convergent, so  $z_{n_i} \rightarrow z$ , where  $\|z\| = 1$  and  $(T - \lambda I)z = 0$ , contradicting that  $T - \lambda I$  is injective. Hence the sequence  $(x_n)$  must have a bounded subsequence, and we are back in case (1) above, and the claim is proved.  $\square$

**Example 7.4** *Let  $T$  be a bounded operator on a Hilbert space  $H$ . Show that:*

- 1) *If  $T$  is compact, then  $T^*$  is also compact.*
- 2) *If  $T^*T$  is compact, then  $T$  is compact.*
- 3) *If  $T$  is self adjoint and  $T^n$  is compact for some  $n$ , then  $T$  is compact.*

- 1) Assume that  $T$  is compact. Let  $X$  be a bounded set, and let  $(y_n) \subseteq T^*(X)$  be any sequence, thus there exists a sequence  $(x_n) \subseteq X$ , such that  $y_n = T^*x_n$ .

We shall prove that there exists a subsequence  $(x_{n_j})$ , such that  $(T^*x_{n_j})$  is convergent. This is done INDIRECTLY. Assume that  $T^*$  is not compact. Then there exists a bounded sequence  $(\varphi_n)$ ,

which converges weakly towards  $\varphi$ , such that  $(T^* \varphi_n)$  does not converge strongly towards  $T^* \varphi$ , thus there exist a subsequence  $(f_n)$  and an  $\eta > 0$ , such that

$$\|T^* f_n - T^* \varphi\| > \eta \quad \text{for all } n \in \mathbb{N},$$

hence

$$\eta \leq \|T^* f_n - T^* \varphi\| \leq \|T^*\| \cdot \|f_n - \varphi\| \quad (< M),$$

and whence

$$\|f_n - \varphi_n\| \geq \frac{\eta}{\|T^*\|}.$$

Now,  $(T^* f_n - T^* \varphi)$  is bounded and it converges weakly towards 0, hence  $TT^* f_n$  converges strongly towards  $TT^* \varphi$ , i.e.

$$\eta^2 \leq \|T^* (f_n - \varphi)\|^2 = (TT^* (f_n - \varphi), f_n - \varphi) \leq \|TT^* (f_n - \varphi)\| \cdot \|f_n - \varphi\| \rightarrow 0$$

for  $n \rightarrow +\infty$ . This gives a contradiction,  $\eta > 0$  being fixed, and our assumption that  $T^*$  is not compact, must be wrong. We therefore conclude that  $T^*$  is compact as claimed above.

2) It follows trivially from EXAMPLE 7.1 that if  $T$  is compact, then  $T^*T$  is also compact.

Assume that  $T^*T$  is compact, and also *assume* (thus an INDIRECT proof) that  $T$  is *not* compact. Then there exists a bounded sequence  $(\varphi_n)$ , which converges weakly towards  $\varphi$ , such that (cf. (1))

$$\|T\varphi_n - T\varphi\| \geq \eta \quad \text{for all } n \in \mathbb{N}.$$

Because  $(\varphi_n - \varphi)$  is bounded and weakly convergent, it follows that  $(T^*T\varphi_n - T^*T\varphi)$  is strongly convergent, and we get

$$\begin{aligned} \eta^2 &\leq \|T(\varphi_n - \varphi)\|^2 = (T(\varphi_n - \varphi), T(\varphi_n - \varphi)) \\ &= (T^*T(\varphi_n - \varphi), \varphi_n - \varphi) \leq \|T^*T(\varphi_n - \varphi)\| \cdot \|\varphi_n - \varphi\| \\ &\leq \|T^*T(\varphi_n - \varphi)\| \cdot M \rightarrow 0 \quad \text{for } n \rightarrow +\infty, \end{aligned}$$

which is a contradiction, because  $\eta > 0$  is a given constant. We therefore conclude that  $T$  is compact.

3) Finally, assume that  $T$  is self adjoint,  $T^* = T$ , and that  $T^n$  is compact for some given  $n \in \mathbb{N}$ .

If  $n = 2m$  is even, then it follows from  $T$  being self adjoint that

$$T^n = T^{2m} = (T^m)^* (T^m)$$

is compact. Then we infer from (2) that  $T^m$  is compact, where  $m = \frac{n}{2} < n$ .

If instead  $n = 2m - 1$  is odd, then

$$T^{n+1}T^nT = T^{2m} = (T^m)^* (T^m)$$

is compact, cf. EXAMPLE 7.1, and we infer as above that  $T^m$  is compact, where  $m = \frac{n+1}{2} < n$ , when  $n > 1$ .

By recursion we get after a finite number of steps that  $T^3$  is compact, and hence that  $T^2 = T \star T$  is also compact, which by (2) implies that  $T$  is compact.

**Example 7.5** Let  $T : \ell^2 \rightarrow \ell^2$  be the linear operator given by

$$T(x_1, x_2, \dots, x_{2n-1}, x_{2n}, \dots) = \left( x_2, x_1, \frac{1}{2}x_4, \frac{1}{2}x_3, \dots, \frac{1}{n}x_{2n}, \frac{1}{n}x_{2n-1}, \dots \right).$$

- 1) Find  $\|T\|$ .
- 2) Find  $T^*$ .
- 3) Prove that  $T$  is compact.
- 4) Find the spectrum and resolvent set for  $T$ , and determine a set of basis vectors for the eigenspace associated to  $\lambda \in \sigma_p(T)$ .

1) In general,

$$\|Tx\|^2 = \sum_{n=1}^{+\infty} \frac{1}{n^2} \left\{ |x_{2n}|^2 + |x_{2n-1}|^2 \right\} \leq \sum_{n=1}^{+\infty} |x_n|^2 = \|x\|^2,$$

thus  $\|T\| \leq 1$ .

On the other hand,

$$\|Te_1\| = \|e_2\| = 1 = \|e_1\| \quad \text{and} \quad \|Te_2\| = \|e_1\| = 1 = \|e_2\|,$$

so  $\|T\| = 1$ , and  $T \in B(\ell^2)$ .

2) Because  $T \in B(\ell^2)$ , we also have  $T^* \in B(\ell^2)$ , and  $\|T^*\| = \|T\|$ . Then

$$\begin{aligned} (Tx, y) &= \sum_{n=1}^{+\infty} \left\{ \frac{1}{n} x_{2n} \overline{y_{2n-1}} + \frac{1}{n} x_{2n-1} \overline{y_{2n}} \right\} \\ &= \sum_{n=1}^{+\infty} \left\{ x_{2n-1} \overline{\frac{1}{n} y_{2n}} + x_{2n} \overline{\frac{1}{n} y_{2n-1}} \right\} = (x, T^*y) = (x, Ty), \end{aligned}$$

hence  $T = T^*$ , and  $T$  is self adjoint.

3) We get that  $T$  is compact from  $T_n \rightarrow T$ , where

$$T_n(x_1, x_2, \dots) = \left( x_2, x_1, \frac{1}{2}x_4, \frac{1}{2}x_3, \dots, \frac{1}{n}x_{2n}, \frac{1}{n}x_{2n-1}, 0, 0, \dots \right)$$

is of finite rank, thus compact, and where

$$\|(T - T_n)x\|^2 = \sum_{k=n+1}^{+\infty} \frac{1}{k^2} \left\{ |x_{2k}|^2 + |x_{2k-1}|^2 \right\} \leq \frac{1}{(n+1)!} \|x\|^2,$$

i.e.

$$\|T - T_n\| \leq \frac{1}{n+1} \rightarrow 0 \quad \text{for } n \rightarrow +\infty.$$



4) Because  $T$  is self adjoint and compact, we can apply the main theorem, thus

$$\sigma_p(T) = \{\lambda_n \mid n \in \mathbb{N}\}.$$

Now  $Tx = 0$  implies that  $x = 0$ , hence  $0 \notin \sigma_p(T)$ , which means that  $\sigma_c(T) = \{0\}$  and  $\sigma_r(T) = \emptyset$ , because  $T$  is self adjoint.

The eigenvalue problem  $Tx = \lambda x$ ,  $\lambda \neq 0$ , is now written in coordinates

$$\begin{cases} \frac{1}{n} x_{2n} &= \lambda x_{2n-1}, \\ \frac{1}{n} x_{2n-1} &= \lambda x_{2n}, \end{cases} \quad \text{i.e.} \quad \begin{cases} -\lambda x_{2n-1} + \frac{1}{n} x_{2n} &= 0, \\ \frac{1}{n} x_{2n-1} - \lambda x_{2n} &= 0, \end{cases} \quad n \in \mathbb{N},$$

which has non-trivial solutions, if and only if there exists an  $n \in \mathbb{N}$ , such that

$$\begin{vmatrix} -\lambda & \frac{1}{n} \\ \frac{1}{n} & -\lambda \end{vmatrix} = 0, \quad \text{i.e.} \quad \lambda^2 = \frac{1}{n^2}.$$

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We get the eigenvalues  $\lambda = \pm \frac{1}{n}$ ,  $n \in \mathbb{N}$ , corresponding to e.g. the eigenvectors

$$\begin{cases} e_{2n-1} + e_{2n}, & \lambda_n = \frac{1}{n}, \\ e_{2n-1} - e_{2n}, & \lambda_{-n} = -\frac{1}{n}, \end{cases} \quad n \in \mathbb{N}.$$

We finally get

$$\sigma_p(T) = \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\} \right\}, \quad \sigma_c(T) = \{0\}, \quad \sigma_r(T) = \emptyset,$$

and

$$\varrho(T) = \mathbb{C} \setminus \left( \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\} \right\} \right).$$

## Index

- adjoint operator, 28
- Cauchy-Schwarz inequality, 53, 56, 70
- Closed Graph Theorem, 28, 38
- compact operator, 71
- complement, 29
- contraction, 69
- double sided shift operator, 10
- isometric operator, 57
- kernel, 29, 52
- Minkowski's inequality, 35
- Neumann series, 18, 27, 28, 65
- normal operator, 61, 63
- positive operator, 67
- potential operator, 50
- projection, 22, 59, 67
- regular operator, 5
- resolvent, 5
- self adjoint operator, 42, 75
- shift operator, 7
- spectrum, 5
- translation operator, 5
- unitary operator, 61, 65
- Weierstraß's Approximation Theorem, 14