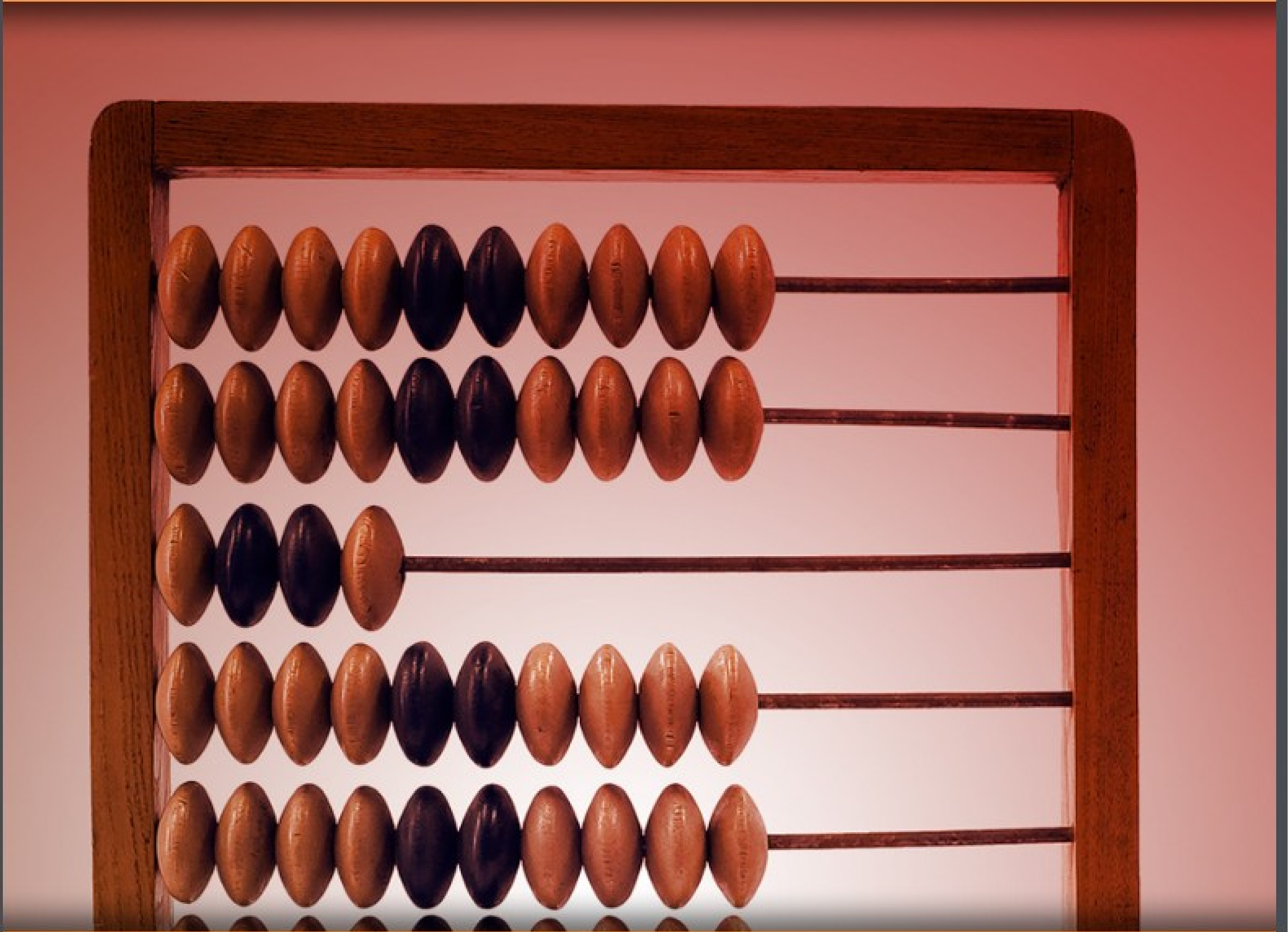
Real Functions in One Variable - Taylor's...

Leif Mejlbro



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Real Functions in One Variable

Examples of Taylor's Formula and Limit Processes

Calculus 1c-6

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Contents

Preface

In this volume I present some examples of *Taylor's formula and Limit Processes*, cf. also *Ventus: Calculus 1a, Functions of One Variable*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
5th August 2007

1 Taylor's formula for simple functions

Example 1.1 *Find the two first derivatives of the function*

$$f(x)=\sqrt{1+x},\qquad x>-1.$$

A. Simple differentiations.

D. Just differentiate.

I. If $f(x)=\sqrt{1+x}$, $x>-1$, then

$$f'(x)=\frac{1}{2}\,\frac{1}{\sqrt{1+x}}\quad\text{and}\quad f''(x)=-\frac{1}{4}\,\frac{1}{(1+x)\sqrt{1+x}},\qquad x>-1.$$

Example 1.2 *Set up Taylor's formula for $n=2$ with the point of expansion $x_0=0$ for the function*

$$f(x)\sqrt{1+x}.$$

A. Taylor's formula for $n=2$.

D. Perform the differentiations, or use the results from Example 1.1.

I. From

$$f(x)=\sqrt{1+x},\quad f'(x)=\frac{1}{2}\,\frac{1}{\sqrt{1+x}},\quad f''(x)=-\frac{1}{4}\,\frac{1}{(x+1)\sqrt{1+x}},$$

we get for $x_0=0$,

$$f(0)=1,\qquad f'(0)=\frac{1}{2},\qquad f''(0)=-\frac{1}{4}.$$

Then by insertion into Taylor's formula for $n=2$,

$$\begin{aligned}\sqrt{1+x} &= f(0)+f'(0)\,(x-0)+\frac{1}{2}\,f''(\xi)\,(x-0)^2\\ &= 1+\frac{1}{2}\,x-\frac{1}{8}\,\frac{1}{(1+\xi)^{3/2}}\,x^2,\end{aligned}$$

where ξ lies somewhere between 0 and x.

Example 1.3 *Find the first two derivatives of the function*

$$f(x)=\text{ Arctan }2x.$$

A. Simple differentiations.

D. Just differentiate.

I. When $f(x)=\text{Arctan }2x$, then

$$f'(x)=\frac{2}{1+4x^2}\quad\text{and}\quad f''(x)=-\frac{16x}{(1+4x^2)^2}.$$

Example 1.4 *Set up Taylor's formula for $n = 2$ with the point of expansion $x_0 = 0$ for the function*

$$f(x) = \text{Arctan } 2x.$$

A. Taylor's formula for $n = 2$.

D. Differentiate or use the results from Example 1.3.

I. When

$$f(x) = \text{Arctan } 2x, \qquad f'(x) = \frac{2}{1+4x^2}, \qquad f''(x) = -\frac{16x}{(1+4x^2)^2},$$

we get at the point of expansion $x_0 = 0$,

$$f(0) = 0, \qquad f'(0) = 2, \qquad f''(0) = 0.$$

Then by Taylor's formula,

$$\begin{aligned} \text{Arctan } 2x &= 0 + 2(x-0) - \frac{1}{2}\,\frac{16\xi}{(1+4\xi^2)^2}\,(x-0)^2 \\ &= 2x - \frac{8\xi}{(1+4\xi^2)^2}\,x^2, \end{aligned}$$

where $\xi = \xi(x)$ lies somewhere between 0 and x.

Example 1.5 *The mean value theorem (or Taylor's formula for $n = 1$) states that for any continuous function $f(x)$ there exists a point ξ between x_0 and x such that*

$$f(x) - f(x_0) = (x - x_0)f'(\xi).$$

Find such a point ξ for

1) $f(x) = \sin x$, $x_0 = 0$, $x = \pi$,

2) $f(x) = x^n$ $(n > 1)$, $x_0 = 0$, $x = 1$.

A. Applications of the mean value theorem.

D. Set up the mean value theorem in the two given cases, and then find ξ.

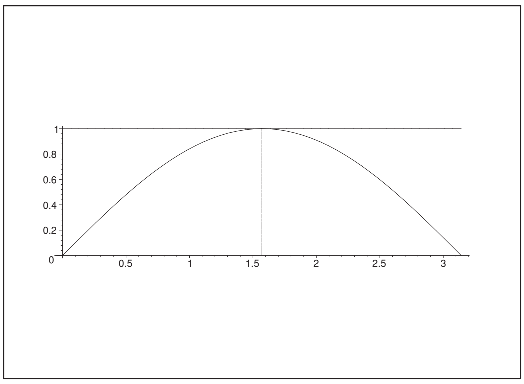

Figure 1: The graph of $f(x) = \sin x$, and the tangent parallel to the x-axis, corresponding to $\xi = \dfrac{\pi}{2}$.

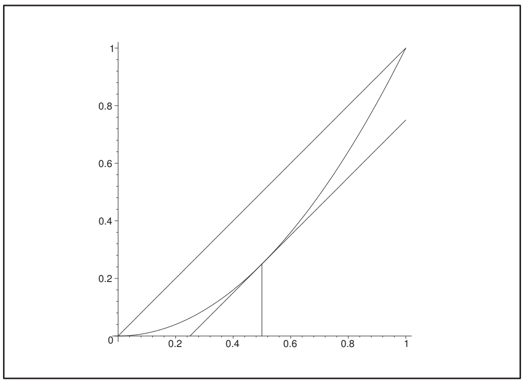

Figure 2: The graph of $f(x) = x^2$, and the tangent parallel to the line through the end point, corresponding to $\xi = \dfrac{1}{2}$.

I. 1) From $f'(x) = \cos x$ we get

$$\sin x - \sin x_0 = (x - x_0) \cdot \cos \xi.$$

When $x_0 = 0$ or $x = \pi$ we get the equation

$$0 - 0 = 0 = \pi \cdot \cos \xi,$$

thus $\xi = \dfrac{\pi}{2}$.

2) From $f'(x) = nx^{n-1}$ we get

$$x^n - x_0^n = n\xi^{n-1} \cdot (x - x_0).$$

For $x_0 = 0$ and $x = 1$ we get the equation

$$1 - 0 = 1 = n\xi^{n-1} \cdot (1 - 0) = n\xi^{n-1},$$

thus

$$\xi_n = \sqrt[n-1]{\frac{1}{n}} \qquad (\to 1 \text{ for } n \to +\infty).$$

Example 1.6 *Assume that the function $f(x)$ is three times continuously differentiable, which means that the third derivative exists and is continuous, in a neighbourhood of the point $x_0 \in \mathbb{R}$, and assume that $f'(x_0) = 0$.*

1) Prove that if $f''(x_0) < 0$, then $f(x)$ has a maximum at the point x_0.

2) Now, we further assume that $f''(x_0) = 0$ and $f'''(x_0) \neq 0$. Apply Taylor's formula to decide whether $f(x)$ has a maximum or a minimum or none of the kind at the point x_0.

A. Maximum/minimum.

D. Taylor expansion of order 2.

I. 1) It follows immediately from

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0) \cdot (x - x_0)^2 + (x - x_0)^2 \varepsilon(x - x_0),$$

that if $f''(x_0) < 0$, then $f(x) < f(x_0)$ in a neighbourhood of x_0 (excl. x_0 itself), such that $f(x)$ has a maximum at the point x_0.

2) If we assume that $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then

$$f(x) = f(x_0) + \frac{1}{6} f''(x_0) \cdot (x - x_0)^3 + (x - x_0)^3 \varepsilon(x - x_0),$$

and $f(x) - f(x_0)$ is of the same sign as $f'''(x_0)$ for $x > 0$ and of the opposite sign of $f'''(x_0)$ for $x < 0$. Hence $f(x)$ has neither a maximum nor a minimum at x_0.

Example 1.7 *Assume that the function $f(x)$ is four times continuously differentiable, which means that the fourth derivative exists and is continuous, in a neighbourhood of the point $x_0 \in \mathbb{R}$, and assume that*

$$f'(x_0) = f''(x_0) = f'''(x_0) = 0,$$

while $f^{(4)}(x_0) \neq 0$.

1) *Prove by means of Taylor's formula that if $f^{(4)}(x_0) > 0$, then $f(x)$ has a minimum at the point x_0.*

2) *What is the conclusion, if instead $f^{(4)}(x_0) < 0$?*

A. Maximum/minimum.

D. Use Taylor's formula.

I. 1) It follows from Taylor's formula, that in a neighbourhood of x_0,

$$f(x) = f(x_0) + \frac{1}{4!}\, f^{(4)}(x_0) \cdot (x - x_0)^4 + (x - x_0)^4\, \varepsilon(x - x_0),$$

because the first three derivatives of $f(x)$ are 0 at x_0.

It follows immediately, when $f^{(4)}(x_0) > 0$ that $f(x) > f(x_0)$ in a neighbourhood of x_0, $x \neq x_0$, so $f(x)$ must have a (local) minimum at x_0.

2) If instead $f^{(4)}(x_0) < 0$, then $f(x) < f(x_0)$ in a neighbourhood of x_0, $x \neq x_0$, hence $f(x)$ has a local maximum at x_0.

Example 1.8 *Assume that the function $f(x)$ is of class C^∞ in a neighbourhood of the point $x_0 \in \mathbb{R}$, and that $f'(x_0) = 0$. Let p denote the first number of 2, 3, 4, ..., for which $f^{(p)}(x_0) \neq 0$. This means that*

$$f'(x_0) = f''(x_0) = \cdots = f^{(p-1)}(x_0) = 0, \qquad f^{(p)}(x_0) \neq 0.$$

1) *Set up Taylor's formula for $f(x)$ i x_0 with p as point of expansion.*

2) *Formulate and prove a theorem which states that $f(x)$ has a maximum or a minimum or none of the kind at the point x_0.*

A. Maximum/minimum.

D. Apply Taylor's formula.

I. 1) This is trivial,

$$f(x) = f(x_0) + \frac{1}{p!}\, f^{(p)}(x_0) \cdot (x - x_0)^p + (x - x_0)^p\, \varepsilon(x - x_0).$$

2) Here we shall split into the cases, whether p is odd or even.

   a) If p is odd, then $(x - x_0)^p$ changes its sign in a neighbourhood of x_0, so we have neither a maximum nor a minimum.

   b) If $p = 2n$ is even, we must split according to whether $f^{(2n)}(x_0) > 0$ or $f^{(2n)}(x_0) < 0$.

i) If $f^{(2nm)}(x_0) > 0$, then $f(x) > f(x_0)$ in a neighbourhood of x_0, $x \neq x_0$, so $f(x)$ has a local minimum at x_0.

ii) If $f^{(2n)}(x_0) < 0$, then $f(x) < f(x_0)$ in a neighbourhood of x_0, $x \neq x_0$, so $f(x)$ has a local maximum at x_0.

Example 1.9 *Find all values of the constant a, for which there exists a $\delta > 0$, such that the parabola $y = 1 + a\,x^2$ lies above the chain curve $y = \cosh x$ for $0 < |x| < \delta$.*

A. Local comparison of graphs.

D. Find the Taylor expansion of $y = \cosh x$ with the expansion point $x_0 = 0$ and then analyze.

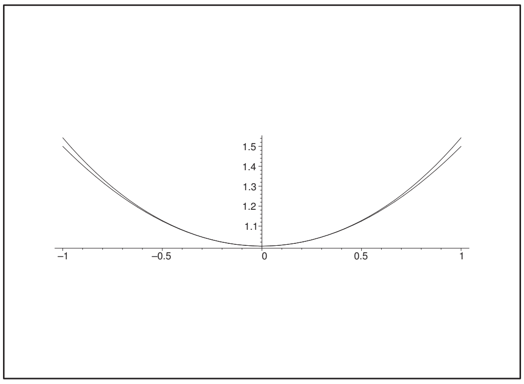

Figure 3: The graphs of $y = \cosh x$ and the limit case $y = 1 + \frac{1}{2}\,x^2$. When $x \neq 0$, then the graph of the polynomial is always lying below the graph of the chain curve.

I. We conclude from

$$y = \cosh x = 1 + \frac{1}{2}\,x^2 + x^2\varepsilon(x),$$

that if $a > \frac{1}{2}$, then there always exists a $\delta > 0$, such that the parabola $y = 1 + a\,x^2$ lies above the chain curve $y = \cosh x$ for $0 < |x| < \delta$. The figure indicates for $a = \frac{1}{2}$ the biggest value of a, for which this is not possible.

Example 1.10 *Find the Taylor expansion of degree $n = 6$ for the functions*

(1) $f(x) = \sin x^2$, (2) $f(x) = e^{2x}$, (3) $f(x) = \ln\left(1 + x^3\right)$.

A. Taylor expansions.

D. Substitute in known Taylor expansions.

I. 1) From

$$\sin y = y - \frac{1}{3!}\,y^3 + y^3\varepsilon(y^3),$$

we get by the substitution $y = x^2$,

$$f(x) = \sin x^2 = x^2 - \frac{1}{6}\,x^6 + x^6\varepsilon(x).$$

2) From

$$e^y = 1 + \frac{1}{1!}\,y + \frac{1}{2!}\,y^2 + \frac{1}{3!}\,y^3 + \frac{1}{4!}\,y^4 + \frac{1}{5!}\,y^5 + \frac{1}{6!}\,y^6 + y^6\varepsilon(y),$$

we get by the substitution $y = 2x$,

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}\,x^3 + \frac{2}{3}\,x^4 + \frac{4}{15}\,x^5 + \frac{4}{45}\,x^6 + x^6\varepsilon(x).$$

3) From

$$\ln(1+y) = y - \frac{1}{2}\,y^2 + y^2\varepsilon(y),$$

we get by the substitution $y = x^3$,

$$f(x) = \ln\left(1+x^3\right) = x^3 - \frac{1}{2}\,x^6 + x^6\varepsilon(x).$$

Example 1.11 *Find the Taylor expansion for $n = 4$ and $x_0 = 0$ for the function*

$$f(x) = \frac{1}{(1+x)^2}, \qquad x > -1.$$

A. Taylor expansion.

D. Differentiate five times.

I. When $f(x) = (1+x)^{-2}$, we get by differentiation

$$f'(x) = -2(1+x)^{-3}, \quad f''(x) = 3!(1+x)^{-4}, \quad f^{(3)}(x) = -4!(1+x)^{-5},$$
$$f^{(4)}(x) = 5!(1+x)^{-6}, \qquad f^{(5)}(x) = -6!(1+x)^{-7}.$$

Thus,

$$\begin{aligned} f(x) &= f(0) + \frac{1}{1!}\,f'(0)\,x + \frac{1}{2!}\,f''(0)\,x^2 + \frac{1}{3!}\,f^{(3)}(0)\,x^3 \\ &\qquad + \frac{1}{4!}\,f^{(4)}(0)\,x^4 + \frac{1}{5!}\,f^{(5)}(\xi)\,x^5 \\ &= 1 - \frac{2!}{1!}\,x + \frac{3!}{2!}\,x^2 - \frac{4!}{3!}\,x^3 + \frac{5!}{4!}\,x^4 - \frac{6!}{5!}\,\frac{1}{(1+\xi)^7}\,x^6 \\ &= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \frac{6}{(1+\xi)^7}\,x^6, \end{aligned}$$

where $x > -1$, and ξ is some number between 0 and x.

Example 1.12 *Find the Taylor polynomial $P_2(x)$ of second order at the point $x_0 = 0$ for the function*

$$f(x) = \ln(1+e^x), \qquad x \in \mathbb{R}.$$

A. Taylor expansion, cf. Example 2.15.

D. Differentiate two times and find the coefficients.

I. From

$$\begin{aligned} f(x) &= \ln(1+e^x), & f(0) &= \ln 2, \\ f'(x) &= \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}, & f'(0) &= \frac{1}{2}, \\ f''(x) &= \frac{e^x}{(1+e^x)^2}, & f''(0) &= \frac{1}{4}, \end{aligned}$$

we obtain the Taylor polynomial expanded from $x_0 = 0$,

$$P_2(x) = \ln 2 + \frac{1}{2}\cdot\frac{1}{4}\,x^2 = \ln 2 + \frac{1}{2}\,x + \frac{1}{8}\,x^2.$$

Example 1.13 *Indicate on a figure the domain of the function*

$$f(x,y) = \ln(4+x-y^2).$$

Find the approximating Taylor polynomial of first order, when $(x,y) = (1,-2)$ *is used as the point of expansion.*

A. Domain of a function at an approximating polynomial (in two variables).

D. Apply the usual procedure of solution.

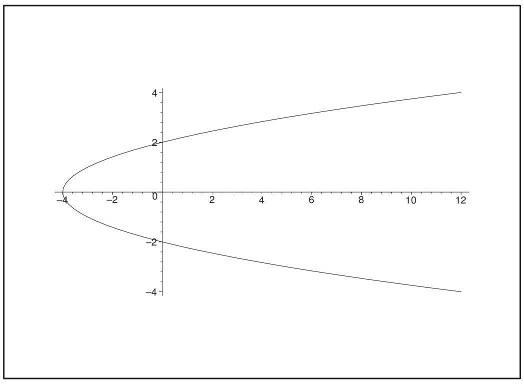

Figure 4: The domain is the open set inside the parabola of the equation $x = y^2 - 4$.

I. The domain is the open set inside the parabola on the figure. By differentiation we get

$$\begin{aligned}
f(x,y) &= \ln(4+x-y^2), & f(1,-2) &= 0,\\
f_x'(x,y) &= \frac{1}{4+x-y^2}, & f_x'(1,2) &= 1,\\
f_y'(x,y) &= -\frac{2y}{4+x-y^2}, & f_y'(1,-2) &= 4,
\end{aligned}$$

hence

$$P_1(x,y) = 0 + 1\cdot(x-1) + 4\cdot(y+2) = x-1+2(y+2).$$

2 Estimates of remainder terms

Example 2.1 *Give an estimate of the expression*

$$U=\left|\frac{1}{(1+\xi)^{\frac{3}{2}}}\right|x^2,$$

where ξ lies between 0 and x, in the cases of

$$(1)\;\; |x|\le\frac{1}{10},\qquad (2)\;\; |x|\le\frac{1}{2}.$$

A. A latent estimate of a remainder term.

D. Estimate by making the (positive) denominator as small as possible, and the (positive) numerator as big as possible.

I. 1) When $x\in\left[-\frac{1}{10},-\frac{1}{10}\right]$, and ξ lies between 0 and x, then the expression becomes largest when $-\frac{1}{10}=x=\xi$, thus

$$\left|\frac{1}{(1+\xi)^{\frac{3}{2}}}\right|x^2\le\frac{1}{\left(1-\frac{1}{10}\right)^{\frac{3}{2}}}\cdot\left(\frac{1}{10}\right)^2=\left(\frac{10}{9}\right)^{\frac{3}{2}}\cdot\left(\frac{1}{10}\right)^2=\frac{1}{\sqrt{10}}\cdot\frac{1}{27}\approx 0.0117.$$

2) We use the same method for $|x|\le\frac{1}{2}$. Here, the expression is largest when $-\frac{1}{2}=x=\xi$, thus

$$\left|\frac{1}{(1+\xi)^{\frac{3}{2}2}}\right|x^2\le\frac{1}{\left(1-\frac{1}{2}\right)^{\frac{3}{2}}}\cdot\left(\frac{1}{2}\right)^2=2^{\frac{3}{2}}\cdot\frac{1}{2^2}=\frac{1}{\sqrt{2}}\approx 0.7071.$$

Example 2.2 *Give an estimate of the expression*

$$\left|\frac{\xi}{1+\xi^2}\right|x^2,$$

where ξ lies between 0 and x, and when

$$(1)\;\; |x|\le\frac{1}{2},\qquad (2)\;\; |x|\le 2.$$

A. A latent estimate of a remainder term.

D. Find the maximum of $\left|\frac{\xi}{1+\xi^2}\right|$ in the two intervals and estimate.

I. The function $\varphi(\xi)=\frac{\xi}{1+\xi^2}$ is odd, hence it is sufficient only to consider $x>0$ and $0\le\xi\le x$. We conclude from

$$\varphi'(\xi)=\frac{1-\xi^2}{(1+\xi^2)^2},$$

that $\varphi(\xi)$ is increasing for $\xi\in[0,1[$, and decreasing for $\xi\in]1,+\infty[$. Maximum is $\varphi(1)=\frac{1}{2}$.

1) When $0 \le \xi \le x \le \frac{1}{2}$, the maximum is obtained for $\xi = x = \frac{1}{2}$. Therefore, if $|x| \le \frac{1}{2}$, then

$$\left|\frac{\xi}{1+\xi^2}\right| x^2 \le \frac{\frac{1}{2}}{1+\left(\frac{1}{2}\right)^2} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{10}.$$

2) If $0 \le \xi \le x \le 2$, then $\varphi(\xi)$ is largest for $\xi = 1$, and x^2 is largest for $x = 2$. Since $|\varphi(\xi)|$ is even, we get for general $|x| \le 2$ the estimate

$$\left|\frac{\xi}{1+\xi^2}\right| x^2 \le \frac{1}{1+1^2} \cdot 2^2 = 2.$$

Example 2.3 *Let $P_1(x)$ denote the Taylor polynomial of order 1 with the point of expansion $x_0 = 0$ for the function $f(x) = \text{Arctan}\, 2x$. Give an estimate of the remainder term $R_1(x)$, when*

$$(1)\ \ |x| \leq \frac{1}{10}, \qquad (2)\ \ |x| \leq \frac{1}{2}.$$

A. Taylor expansion and an estimate of the remainder term.

D. Differentiate two times or apply Example 1.3 and Example 1.4, and apply Taylor's formula. Estimate the remainder term.

I. Let $f(x) = \text{Arctan}\, 2x$. Then

$$f'(x) = \frac{2}{1+4x^2} \quad \text{and} \quad f''(x) = -\frac{16x}{(1+4x^2)^2}.$$

When $x_0 = 0$ is the point of expansion, we get

$$\text{Arctan}\, 2x = f(x_0)0f'(x_0)\cdot(x-x_0) + \frac{1}{2}\, f''(\xi)\,(x-x_0)^2 = 2x - \frac{8\xi}{(1+4\xi^2)^2}\, x^2,$$

where ξ lies somewhere between 0 and x.
It follows that $P_1(x) = 2x$ and that

$$|f(x) - P_1(x)| = \left|\frac{8\xi}{(1+4\xi^2)^2}\right| x^2 = |R_1(x)|.$$

Now Arctan $2x$ is an odd function, so we can assume in the estimation of the remainder term that $x > 0$, thus $0 \leq \xi \leq x$.

The function

$$\varphi(\xi) = \frac{8\xi}{(1+4\xi^2)^2}, \qquad \xi \in [0, x],$$

has the derivative

$$\varphi'(\xi) = \frac{8}{(1+4\xi^2)^3}\,(1-12\xi^2).$$

Hence $\varphi(\xi)$ is increasing for $\xi \in \left[0, \frac{1}{2\sqrt{3}}\right[$ and decreasing for $\xi > \frac{1}{2\sqrt{3}}$. In particular, $\varphi(\xi)$, $\xi > 0$, has its maximum for $\xi = \frac{1}{2\sqrt{3}}$.

1) If $|x| \leq \frac{1}{10} < \frac{1}{2\sqrt{3}}$, then $\varphi(\xi)$, $\xi \geq 0$, is maximum for $\xi = \frac{1}{10}$. This is also the case of x^2, hence we get the estimate of the remainder term

$$|R_1(x)| = \left|\frac{8\xi}{(1+4\xi^2)^2}\right| x^2 \leq \frac{8\cdot\frac{1}{10}}{\left(1+\frac{4}{100}\right)^2}\cdot\left(\frac{1}{10}\right)^2 \approx 0,0074.$$

2) If $|x| \leq \frac{1}{2}$, then $\frac{1}{2\sqrt{3}} < \frac{1}{2}$, so $\varphi(\xi)$, $\xi > 0$, attains its maximum at $\xi = \frac{1}{2\sqrt{3}}$, and x^2 its maximum at $x = \frac{1}{2}$. Thus we get the estimate of the remainder term

$$|R_1(x)| = \left|\frac{8\xi}{(1+4\xi^2)^2}\right| x^2 \leq \frac{8 \cdot \frac{1}{2\sqrt{3}}}{\left(1+\frac{4}{12}\right)^2} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{\sqrt{3}} \cdot \frac{9}{16} \approx 0,3248.$$

Example 2.4 *Given the function*

$$f(x) = x\,\cos x, \qquad x \in \mathbb{R}.$$

1) Set up Taylor's formula for $n = 2$ with the point of expansion $x_0 = 0$ for $f(x)$.

2) Estimate the remainder term $R_1(x)$, when $|x| \leq 1$.

3) Prove that

$$|x\,\cos x - x| \leq x^2 \qquad \textit{for } |x| \leq 1.$$

A. Taylor expansion and estimate of a remainder term.

D. Differentiate two times and apply Taylor's formula. Estimate the remainder term. We get some problems in (3).

I. 1) When $f(x) = x\,\cos x$ we get

$$f'(x) = \cos x - x\,\sin x, \qquad f''(x) = -2\sin x - x\,\cos x,$$

thus with the point of extension $x_0 = 0$,

$$\begin{aligned} f(x) &= x\,\cos x \\ &= f(x_0) + f'(x_0)\,(x-x_0) + \frac{1}{2}\,f''(\xi)\cdot(x-x_0)^2 \\ &= x - \frac{1}{2}\,\{2\,\sin\xi + \xi\,\cos\xi\}\cdot x^2, \end{aligned}$$

for some ξ between 0 and x.

2) The function $2\sin\xi + \xi\cos\xi$ is odd with the derivative

$$3\cos\xi - \xi\sin\xi \geq 3\,\cos\frac{\pi}{3} - \frac{\pi}{3}\cdot\sin\frac{\pi}{3} = \frac{3}{2} - \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2} > 0 \qquad \text{for } \xi \in [0,1].$$

The maximum is attained for $|2\sin\xi + \xi\cos\xi|$ in the interval $[-1,1]$ for $\xi = 1$. Hence

$$|2\sin\xi + \xi\cos\xi| \leq 2\cdot\sin 1 + \cos 1 \approx 2\cdot 1,1116,$$

and we get the estimate of the remainder term

$$|R_1(x)| \leq \frac{1}{2}\,\{2\sin 1 + \cos 1\}x^2 \approx 1,1116\cdot x^2 \leq 1,1116.$$

3) By the estimate from (2) we get

$$|x\,\cos x - x| \le \frac{1}{2}\,|2\sin 1 + 1\cdot\cos 1|\cdot x^2 \approx 1,1116x^2,$$

which is not sufficient.

Instead we estimate directly, where we use that $\sin^2 x \le x^2$ for every x,

$$\begin{aligned}|x\,\cos x - x| &= |x|(1-\cos x) = 2\,\sin^2\left(\frac{x}{2}\right)\cdot|x| \\ &\le 2\cdot\left(\frac{x}{2}\right)^2\cdot|x| = \frac{1}{2}\,|x|^3 \le \frac{1}{2}\,|x|^2,\end{aligned}$$

for $|x|\le 1$.

Example 2.5 *Given the function*

$$f(x) = \ln(1+\sin x), \qquad x \in \left]-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right[.$$

1) Set up Taylor's formula for $n = 2$ with the point of expansion $x_0 = 0$ for $f(x)$.

2) Prove that

$$|f(x) - x| \le \frac{1}{2}\cdot 10^{-2} \qquad \textit{for } x \in \left[0\,,\,\frac{1}{10}\right].$$

A. Taylor expansion of second order from $x_0 = 0$. Cf. also Example 2.6.

D. Differentiate two times: then estimate the remainder term.

I. 1) First calculate

$$f'(x) = \frac{\cos x}{1+\sin x}, \qquad f''(x) = \frac{-\sin x\cdot(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = -\frac{1}{1+\sin x},$$

hence

$$\ln(1+\sin x) = f(0) + f'(0)\,x + \frac{1}{2}\,f''(\xi)\cdot x^2 = x - \frac{1}{2}\cdot\frac{1}{1+\sin\xi}\cdot x^2$$

for some ξ between 0 and x.

2) If $0 \le \xi \le x \le \dfrac{1}{10}$, then

$$|f(x) - x| = \frac{1}{2}\left|\frac{1}{1+\sin\xi}\right| x^2 \le \frac{1}{2}\cdot\frac{1}{1+0}\,x^2 \le \frac{1}{2}\cdot 10^{-2}.$$

Example 2.6 *Given the function*

$$f(x) = \ln(1+\sin x), \qquad x \in \left]-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right[.$$

1) Find the Taylor polynomial $P_2(x)$ of order 2 with the point of expansion $x_0 = 0$ for $f(x)$.

2) Prove that

$$|f(x) - P_2(x)| \le 2\cdot 10^{-4} \qquad \textit{for } x \in \left[0, \frac{1}{10}\right].$$

A. Same function as in Example 2.5; we shall only develop one further step. Taylor polynomial, estimate of remainder term.

D. Differentiate three times. Set up the Taylor polynomial and then continue with the estimate of the remainder term.

I. When $f(x) = \ln(1+\sin x)$ we get

$$f'(x) = \frac{\cos x}{1+\sin x}, \quad f''(x) = -\frac{1}{1+\sin x}, \quad f^{(3)}(x) = \frac{\cos x}{(1+\sin x)^2},$$

where $f^{(3)}(x) > 0$ for $x \in \left[0, \dfrac{1}{10}\right]$.

1) The Taylor polynomial $P_2(x)$ with point of expansion $x_0 = 0$ is

$$P_2(x) = f(0) + f'(0) \cdot x + \frac{1}{2} f''(0)x^2 = x - \frac{1}{2} x^2.$$

2) Then by Taylor's formula,

$$f(x) = P_2(x) + \frac{1}{3!} f^{(3)}(\xi) \cdot x^3$$

for some ξ between 0 and x. If $x \in \left[0, \frac{1}{10}\right]$ then $f^{(3)}(\xi)\, x^3 > 0$, hence

$$|f(x) - P_2(x)| = f(x) - P_2(x) = \frac{1}{6} f^{(3)}(\xi)\, x^3 = \frac{1}{6} \frac{\cos\xi}{(1+\sin\xi)^2} x^3.$$

The numerator $\cos\xi$ decreases and the denominator $(1+\sin\xi)^2$ increases when ξ runs through $\left[0, \frac{1}{10}\right]$, hence $f^{(3)}(\xi)$ is largest for $\xi = 0$. Then we get the estimate

$$|f(x) - P_2(x)| \le \frac{1}{6} \cdot \frac{\cos 0}{(1+0)^2} \cdot 10^{-3} = \frac{2}{12} \cdot 10^{-3} < 2 \cdot 10^{-4}$$

for $x \in \left[0, \frac{1}{10}\right]$.

Example 2.7 *Find The Taylor polynomial $P_n(x)$ for each of the following functions with the given point of expansion x_0 and for the given n. Give an estimate of the remainder term for $|x| < 0.2$:*

1) $f(x) = \tan x$, $x_0 = 0$, $n = 2$.

2) $f(x) = \ln\cos x$, $x_0 = 0$, $n = 3$.

3) $f(x) = \sinh x$, $x_0 = 0$, $n = 4$.

A. Taylor polynomials and estimates of remainder terms. In all three cases the point of expansion is $x_0 = 0$.

D. Differentiate in each case $n+1$ times with due respect to following the estimate of the remainder term. Find the polynomials.

I. 1) If $f(x) = \tan x$ and $n = 2$, then

$$f'(x) = \frac{1}{\cos^2 x}, \quad f''(x) = \frac{2\sin x}{\cos^3 x}, \quad f^{(3)}(x) = \frac{6\sin^2 x}{\cos^4 x} + \frac{2}{\cos^2 x}.$$

Thus

$$P_2(x) = f(0) + f'(0) \cdot x + \frac{1}{2} f''(0) \cdot x^2 = x,$$

and

$$\begin{aligned} R_2(x) &= \frac{1}{3!}\,f^{(3)}(\xi)\cdot x^3 = \frac{1}{3}\cdot\frac{3\sin^2\xi+\cos^2\xi}{\cos^4\xi}\cdot x^3 \\ &= \left\{\frac{1}{\cos^4\xi}-\frac{\frac{2}{3}}{\cos^2\xi}\right\}x^3. \end{aligned}$$

When $|x|<0.2$ and ξ lies somewhere between 0 and x we get the estimate of the remainder term

$$\begin{aligned} |R_2(x)| &\leq \left|\frac{1}{\cos^2\xi}-\frac{2}{3}\right|\cdot\frac{1}{\cos^2\xi}\cdot x^3 \\ &\leq \left\{\frac{1}{\cos^2\frac{1}{5}}-\frac{2}{3}\right\}\cdot\frac{1}{\cos^2\frac{1}{5}}\cdot(0,2)^3\approx 0,003118. \end{aligned}$$

2) If $f(x)=\ln\cos x$ and $n=3$, then

$$f'(x)=-\tan x,\qquad f''(x)=-\frac{1}{\cos^2 x},$$

$$f^{(3)}(x)=-\frac{2\sin x}{\cos^3 x},\qquad f^{(4)}(x)=-\left\{\frac{6}{\cos^2 x}-4\right\}\frac{1}{\cos^2 x},$$

Thus,

$$P_3(x)=f(0)+f'(0)\,x+\frac{1}{2}\,f''(0)\,x^2+\frac{1}{6}\,f^{(3)}(0)\,x^3=-\frac{1}{2}\,x^2,$$

and

$$R_3(x)=\frac{1}{4!}\,f^{(4)}(\xi)\cdot x^4=\frac{1}{4}\left\{\frac{1}{\cos^2\xi}-\frac{2}{3}\right\}\frac{1}{\cos^2\xi}\cdot x^4.$$

If we notice that this remainder term is $\frac{x}{4}$ times the remainder term in (1), we end up with the estimate of the remainder term

$$|R_3(x(|\leq\frac{1}{4}\left\{\frac{1}{\cos^2\xi}-\frac{2}{3}\right\}\frac{1}{\cos^2\xi}\,x^4\leq 0,003118\cdot 0,05=0.000156.$$

3) If $f(x)=\sinh x$ and $n=4$, then

$$f(x)=f^{(2)}(x)=f^{(4)}(x)=\sinh x,$$

and

$$f'(x)=f^{(3)}(x)=f^{(5)}(x)=\cosh x,$$

thus

$$P_4(x)=x+\frac{1}{3!}\,x^3=x+\frac{1}{6}\,x^3,$$

and

$$R_4(x) = \frac{1}{5!}\cosh\xi \cdot x^5.$$

For $|x| \le 0,2 = \dfrac{1}{5}$ we get the estimate of the remainder term

$$|R_4(x)| \le \frac{1}{120}\cosh\left(\frac{1}{5}\right)\cdot\left(\frac{1}{5}\right)^5 \approx 2,72\cdot 10^{-6}.$$

Example 2.8 *1) Find the Taylor polynomial $P_2(x)$ with the point of expansion $x_0 = 0$ of the function*

$$f(x) = \sqrt[3]{1+x}.$$

2) Give an estimate of the remainder term $R_2(x)$ when $x \in [0\,;\,0.02]$.

3) What are the bounds for this estimate for $\sqrt[3]{1.02}$?

A. Taylor expansion.

D. Differentiate three times. Find $P_2(x)$ and estimate $R_2(x)$.

I. 1) We obtain from $f(x) = \sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$ that

$$f'(x) = \frac{1}{3}\,(1+x)^{-\frac{2}{3}}, \qquad f''(x) = -\frac{2}{9}\,(1+x)^{-\frac{5}{3}},$$

$$f^{(3)}(x) = \frac{10}{27}\,(1+x)^{-\frac{8}{3}}.$$

Hence,

$$f(x) = f(0) + f'(0)\cdot x + \frac{1}{2}\,f''(0)\cdot x^2 + \frac{1}{3!}\,f^{(3)}(\xi)\cdot x^3$$

for some ξ lying between 0 and x, i.e.

$$P_2(x) = 1 + \frac{1}{3}\,x - \frac{1}{9}\,x^2$$

with the remainder term

$$R_2(x) = \frac{5}{81}\cdot\frac{1}{(1+\xi)^{\frac{8}{3}}}\,x^3.$$

2) When $0 \le \xi \le x \le 0,02$, we get $(1+\xi)^{\frac{8}{3}} \ge 1$, so

$$|R_2(x)| \le \frac{5}{81}\cdot x^3.$$

3) Putting $x = 0,02$ we get

$$|R_2(0,02)| \le \frac{5}{81}\cdot\left(\frac{2}{100}\right)^3 = \frac{40}{81}\cdot 10^{-6} < \frac{1}{2}\cdot 10^{-6},$$

corresponding to

$$\sqrt[3]{1,02} - P_2(0,02)| < \frac{1}{2}\cdot 10^{-6}.$$

Here,

$$P_2(0,02) = 1 + \frac{1}{3}\cdot 0,02 - \frac{1}{9}\cdot 0.02^2 \approx 1,006622.$$

Example 2.9 *What is the smallest order of the Taylor expansion of the function*

$$f(x) = \sqrt[3]{1+x}, \qquad x \geq -1,$$

if we want the Taylor polynomial only to deviate from $f(x)$ by at most 10^{-2} on the interval $[0,1]$?

A. Estimate of a remainder term.

D. Differentiate $f(x)$ n times, until

$$\left|\frac{1}{(n+1)!}\, f^{(n+1)}(\xi)\right|$$

is $< 10^{-2}$, whenever $0 \leq \xi \leq x \leq 1$.

I. If $f(x) = (1+x)^{\frac{1}{3}}$, then

$$f^{(n)}(x) = k_n\,(1+x)^{\frac{1}{3}-n},$$

where k_n is some constant, which is calculated below.
Let $n \geq 1$. Then $(1+\xi)^{\frac{1}{3}-n}$ is largest for $\xi = 0$, corresponding to the value 1, so we shall "only" find n such that

$$\frac{1}{(n+1)!}\,|k_{n+1}| < 10^{-2}.$$

First note that

$$\frac{1}{n!}\,|k_n| = \frac{1}{n!}\left|\left(\frac{1}{3}-n+1\right)k_{n-1}\right| = \frac{|3n-4|}{3n}\cdot\frac{1}{(n-1)!}\,|k_{n-1}|\,.$$

If we put

$$a_n = \frac{1}{n!}\,|k_n|\,, \quad n \geq 1,$$

then

$$a_1 = \frac{1}{3} \qquad \text{og} \qquad a_n = \frac{3n-4}{3n}\cdot a_{n-1} \quad \text{for } n \geq 2,$$

and we continue successively

$$a_2 = \frac{2}{6}\,a_1 = \frac{1}{9}, \qquad a_3 = \frac{5}{9}\cdot a_2 = \frac{5}{81}, \qquad a_4 = \frac{8}{12}\cdot a_3 = \frac{10}{243},$$

$$a_5 = \frac{11}{15}\cdot a_4 = \frac{22}{729}, \qquad a_6 = \frac{14}{18}\cdot a_5 = \frac{154}{6561}, \qquad a_7 = \frac{17}{21}\cdot a_6 = \frac{374}{19683},$$

$$a_8\frac{20}{24}\cdot a_7 = \frac{935}{59049}, \quad a_9 = \frac{23}{27}\cdot a_8 = \frac{21505}{1594329}, \quad a_{10} = \frac{26}{30}\cdot a_9 = \frac{55913}{4782969},$$

$$a_{11} = \frac{29}{33}\cdot a_{10} = \frac{1\,621\,577}{157\,837\,977}, \qquad a_{12} = \frac{32}{36}\,a_{11} = \frac{12\,972\,616}{1\,420\,541\,793} < \frac{1}{100}.$$

Since we first obtain $|R^{(n)}(x)| < \dfrac{1}{100}$ for $n+1 = 12$, we must approximate $f(x)$ by $P_{11}(x)$ in order to obtain the given estimate over the interval $[0,1]$.

Example 2.10 *Calculate* $\sin 1$ *by using the first five terms different from 0 of the Taylor polynomial. Estimate the remainder term.*

A. Taylor expansion and estimate of the remainder term.

D. Differentiate $f(x) = \sin x$ 10–11 times. Then set up the Taylor polynomial and find the value for $x = 1$. Finally, estimate the remainder term.

I. It may seem insurmountable to differentiate 10–11 times. However, the periodicity of the functions reduces this task to

$$\begin{aligned}
f(x) &= f^{(4)}(x) = f^{(8)}(x) = \sin x, \\
f'(x) &= f^{(5)}(x) = f^{(9)}(x) = \cos x, \\
f''(x) &= f^{(6)}(x) = f^{(10)}(x) = -\sin x, \\
f^3(x) &= f^{(7)}(x) = f^{(11)}(x) = -\cos x.
\end{aligned}$$

By an expansion from $x_0 = 0$ we get

$$P_9(x) = x - \frac{1}{3!}\,x^3 + \frac{1}{5!}\,x^5 - \frac{1}{7!}\,x^7 + \frac{1}{9!}\,x^9,$$

thus $\sin 1$ is approximated by

$$P(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{71}{9!} = \frac{5}{6} + \frac{2953}{9!} \approx 0,84146.$$

We get for the remainder term,

$$|R_9(x)| = \frac{1}{10!}\,|\sin\xi|\cdot x^{10}\frac{1}{10!} = \frac{1}{3\,628\,800} \le \frac{1}{3}\cdot 10^{-6};$$

but since $f^{10}(0) = 0$, we even get the better estimate

$$|R_{10}(x)| = \frac{1}{11!}\,|\cos\xi|\cdot x^{11} \le \frac{1}{11!} = \frac{1}{39\,916\,800} \le \frac{1}{3}\cdot 10^{-7}.$$

Example 2.11 *Consider the function*

$$f(x) = \cos\left(\frac{x}{2}\right)$$

and the corresponding approximating polynomials $P_n(x)$ with the same point of expansion $x_0 = 0$ of this function.

1) Find $P_n(x)$, such that

$$|f(x) - P_n(x)| < 10^{-4} \qquad \textit{for all } x \in \left[-\frac{1}{10}, \frac{1}{10}\right].$$

2) Find $P_n(x)$, such that

$$|f(x) - P_n(x)| < 10^{-2} \qquad \textit{for all } x \in [-\pi, \pi].$$

A. Taylor polynomial and estimate of the remainder term.

D. Find n, such that $R_n(x)$ satisfies the given estimates.

I. We have

$$R_n(x) = \frac{1}{(n+1)!}\,f^{(n+1)}(\xi)\cdot x^{n+1},$$

where ξ lies somewhere between 0 and x, and

$$f^{(n+1)}(\xi) = \frac{1}{2^{n+1}}\cdot\cos\left(\frac{\xi}{2} + (n+1)\cdot\frac{\pi}{2}\right),$$

where $\left|\cos\left(\frac{\xi}{2} + (n+1)\cdot\frac{\pi}{2}\right)\right| \le 1$ for every ξ. In general,

$$|R_n(x)| \le \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+1}.$$

1) We shall first find n, such that

$$|R_n(x)| \le \frac{1}{(n+1)!} \cdot \left(\frac{1}{20}\right) < 10^{-4}$$

for every $|x| \le \dfrac{1}{10}$. Use the method of trial and error:
For $n=1$ we get $\dfrac{1}{2} \cdot \left(\dfrac{1}{20}\right)^2 = \dfrac{1}{800} > 10^{-4}$.
For $n=2$ we get $\dfrac{1}{6} \cdot \left(\dfrac{1}{20}\right)^3 = \dfrac{1}{48\,000} < 10^{-4}$.
This shows that if $x \in \left[-\dfrac{1}{10}, \dfrac{1}{10}\right]$ then we can use $n=2$, and we find that

$$P_2(x) = 1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 = 1 - \frac{x^2}{8}$$

is a good approximation of $\cos\left(\dfrac{x}{2}\right)$ in the interval $\left[-\dfrac{1}{10}, \dfrac{1}{10}\right]$. If we use MAPLE to sketch the graphs, it is not possible on the figure to distinguish between the graphs.

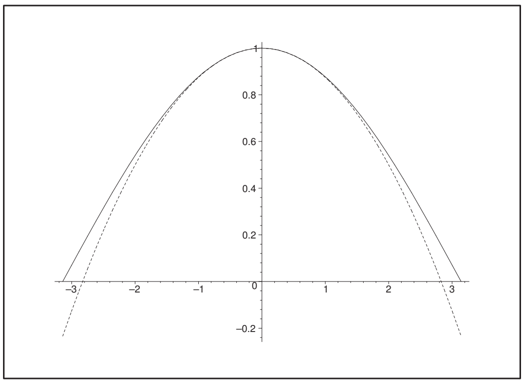

Figure 5: The graphs of $\cos\left(\dfrac{x}{2}\right)$ and $1 - \dfrac{1}{8}x^2$ for $x \in [-\pi, \pi]$. We cannot distinguish between the two graphs in the interval $\left[-\dfrac{1}{10}, \dfrac{1}{10}\right]$.

2) Then we shall find n, such that

$$|R_n(x)| \le \frac{1}{(n+1)!}\left(\frac{\pi}{2}\right)^{n+1} < 10^{-2}$$

for every $x \in [-\pi, \pi]$. Since $\dfrac{\pi}{2} > \dfrac{3}{2} > 0$, we must at least require that $(n+1)! > 100$, i.e. $n \ge 4$. Now, $\left(\dfrac{3}{2}\right)^4 > 2^2 = 4$, so even $(n+1)! > 400$, i.e. $n \ge 5$. Since $\cos\left(\dfrac{x}{2}\right)$ is an even function, $n=6$ is the first realistic candidate. When $n=6$ we get the following estimate of the remainder term,

$$\frac{1}{7!}\left(\frac{\pi}{2}\right)^7 \approx 0,004682 < 10^{-2},$$

hence $n = 6$ can indeed be chosen, where

$$P_6(x) = 1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 - \frac{1}{6!}\left(\frac{x}{2}\right)^6 = 1 - \frac{1}{8}\,x^2 + \frac{1}{384}\,x^4 - \frac{1}{46\,080}\,x^6.$$

Example 2.12 *Let $x \in [0,1[$. Use Taylor's formula to prove that*

(1) $(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}\,x + \frac{3}{8}\,(1-\xi)^{-\frac{5}{2}}\,x^2$

for some ξ lying between 0 and x.
According to Albert Einstein, the kinetic energy of a particle is given by

$$E_{\text{kin}}(v) = m_0 c^2 \left\{ \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} - 1 \right\}, \qquad 0 \le v < c,$$

where m_0 is the mass of the particle at rest, c is the speed of light ($= 3 \cdot 10^5$ km/s), and v is the speed of the particle. It is well-known that the classical kinetic energy is

$$T(v) = \frac{1}{2}\,m_0 v^2.$$

The relative error by replacing $E_{\text{kin}}(v)$ by $T(v)$ is defined by

$$F = \frac{E_{\text{kin}}(v) - T(v)}{E_{\text{kin}}(v)}.$$

2. *Prove by means of (1) that*

$$F < \frac{3\left(\frac{v}{c}\right)^2}{4\left\{1-\left(\frac{v}{c}\right)^2\right\}^{\frac{5}{2}}}.$$

3. *Prove by means of the result in (2) and a pocket calculator that if $v \le 3 \cdot 10^4$ km/s, then $F < 10^{-2}$. Hence, up to these velocities the relative error is at most 1 %.*

A. Applications of Taylor expansions.

D. Differentiate $(1-x)^{-\frac{1}{2}}$ two times.

I. 1) When $f(x) = (1-x)^{-\frac{1}{2}}$, $x \in [0,1[$, then $f(0) = 1$, and

$$f'(x) = \frac{1}{2}\,(1-x)^{-\frac{3}{2}}, \qquad f''(x) = \frac{3}{4}\,(1-x)^{-\frac{5}{2}}.$$

By Taylor's formula there exists a $\xi \in [0,x]$, such that

$$\begin{aligned} f(x) &= (1-x)^{-\frac{1}{2}} = f(0) + \frac{1}{1!}\,f'(0)\,x + \frac{1}{2!}\,f''(\xi)\,x^2 \\ &= 1 + \frac{1}{2}\,x + \frac{3}{8}\,(1-\xi)^{-\frac{5}{2}}\,x^2. \end{aligned}$$

2) Then by insertion we get for some $\xi \in \left[0, \left(\frac{v}{c}\right)^2\right]$ that

$$\begin{aligned}
F &= \frac{E_{kin}(v) - T(v)}{E_{kin}(v)} \\
&= \frac{m_0c^2\left\{\left(1-\left[\frac{v}{c}\right]^2\right)^{-\frac{1}{2}} - 1\right\} - \frac{1}{2}\,m_0v^2}{m_0c^2\left\{\left(1-\left[\frac{v}{c}\right]^2\right)^{-\frac{1}{2}} - 1\right\}} \\
&= \frac{\left(1-\left[\frac{v}{c}\right]^2\right)^{-\frac{1}{2}} - 1 - \frac{1}{2}\left[\frac{v}{c}\right]}{\left(1-\left[\frac{v}{c}\right]^2\right)^{-\frac{1}{2}} - 1} \\
&= \frac{\left\{1+\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}(1-\xi)^{-\frac{5}{2}}\cdot\left(\frac{v}{c}\right)^4\right\} - 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2}{\left\{1+\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}(1-\xi)^{-\frac{5}{2}}\left(\frac{v}{c}\right)^4\right\} - 1} \\
&= \frac{\frac{3}{8}(1-\xi)^{-\frac{5}{2}}\left(\frac{v}{c}\right)^2}{\frac{1}{2}\left\{1+\frac{3}{4}(1-\xi)^{-\frac{5}{2}}\left(\frac{v}{c}\right)^2\right\}},
\end{aligned}$$

which is clearly positive. Then

$$\begin{aligned}
F &= \frac{3}{4}\cdot\frac{(1-\xi)^{-\frac{5}{2}}\left(\frac{v}{c}\right)^2}{1+\frac{3}{4}(1-\xi)^{-\frac{5}{2}}\left(\frac{v}{c}\right)^2} = \frac{3}{4}\cdot\frac{\left(\frac{v}{c}\right)^2}{(1-\xi)^{\frac{5}{2}}+\frac{3}{4}\left(\frac{v}{c}\right)^2} \\
&\le \frac{3}{4}\cdot\frac{\left(\frac{v}{c}\right)^2}{(1-\xi)^{\frac{5}{2}}} \le \frac{3}{4}\cdot\frac{\left(\frac{v}{c}\right)^2}{\left\{1-\left(\frac{v}{c}\right)^2\right\}^{\frac{5}{2}}},
\end{aligned}$$

because we increase a positive fraction by decreasing the denominator. In fact, we first delete $\frac{3}{4}$, and then replace ξ by its maximum $\left(\frac{v}{c}\right)^2$.

3) If $\frac{v}{c} \le \frac{1}{10}$, then

$$\begin{aligned}
F &< \frac{3\cdot\left(\frac{1}{10}\right)^2}{4\left\{1-\frac{1}{100}\right\}^{\frac{5}{2}}} = \frac{3}{4}\cdot\left(\frac{100}{99}\right)^{\frac{5}{2}}\cdot 10^{-2} \\
&< \frac{3}{4}\cdot\left(\frac{100}{99}\right)^3\cdot 10^{-2} = \frac{3}{4}\cdot\left(1+\frac{3}{99}+\frac{3}{99^2}+\frac{1}{99^3}\right)\cdot 10^{-2} \\
&< \frac{3}{4}\cdot\frac{104}{100}\cdot 10^{-2} < 10^{-2}.
\end{aligned}$$

Example 2.13 *Consider the function $f(x) = e^x$ and Taylor's formula with the point of expansion $x_0 = 0$,*

(2) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_{n+1}(x),$

where $R_{n+1}(x)$ denotes the remainder term. We shall in the following only consider the case $x > 0$.

1. *Prove that $R_{n+1}(x) > 0$, and then show that*

(3) $e^x > \frac{x^p}{p!}, \qquad x \geq 0, \quad$ *every* $p \in \mathbb{N}$.

2. *Prove from (1) that for every $\alpha \in \mathbb{R}_+$,*

$$\frac{x^\alpha}{e^x} \to 0 \qquad \textit{for } x \to +\infty.$$

In the final question we shall prove how we from (2) can prove that e is an irrational number. We shall take for granted that $e \in]2, 3[$. Then we apply a proof by contraposition, so we assume that

(4) $e = \frac{m}{n}, \qquad m \in \mathbb{N}, \quad n \in \mathbb{N}, \quad n \geq 2.$

we shall then prove that this assumption will lead to a contradiction, hence that the assumption is false.

3. *Assume that (4). Then by (2) for $x = 1$,*

$$\frac{m}{n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_{n+1}(1).$$

First prove that $0 < n!\,R_{n+1}(1) < 1$. Multiply the equation by $n!$ and then derive the contradiction.

A. Taylor expansion. There are given some guidelines.

D. Follow the guidelines.

I. 1) Since $f^{(n)}(x) = e^x > 0$, we get

$$R_{n+1}(x) = \frac{e^\xi}{(n+1)!}\,x^{n+1} > 0, \qquad \text{for } x > 0,$$

hence

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_{n+1}(x) > \frac{x^n}{n!},$$

and we derive (3), i.e.

$$e^x > \frac{x^p}{p!}, \qquad x \geq 0, \quad \text{every } p \in \mathbb{N}.$$

2) When $p > \alpha$ it follows from (3) for $x > 0$ that

$$0 < \frac{x^\alpha}{e^x} = \frac{x^p}{e^x} \cdot \frac{1}{x^{p-\alpha}} < \frac{p!}{x^{p-\alpha}} \to 0 \qquad \text{for } x \to +\infty.$$

3) Assume that

$$e = \frac{m}{n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^{\xi}}{(n+1)!}, \qquad \xi \in [0,1].$$

Then

$$0 < n!\, R_{n+1}(1) = \frac{e^{\xi}}{n+1} \le \frac{e}{n+1} < 1 \qquad \text{for } n \ge 2,$$

because $e < 3$.

When we multiply the equation by $n!$ we get

$$m(n-1)! = n! \left\{ 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right\} + \frac{e^{\xi}}{n+1},$$

where

$$m(n-1)! \qquad \text{and} \qquad n! \left\{ 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right\}$$

are both integers. Since $\dfrac{e^{\xi}}{n+1}$ is not an integer, we have reached our contradiction.

Example 2.14 *(Cf. Example 4.23) Given the function*

$$f(x) = \cos\left(\frac{1}{2}\,x^2 + x\right), \qquad x \in \mathbb{R}.$$

1) Find the Taylor polynomial $P_2(x)$ with the point of expansion $x_0 = 0$ for $f(x)$.

2) Prove by Taylor's formula that

$$|f(x) - P_2(x)| < 8 \cdot 10^{-3} \qquad \textit{for } |x| < \frac{1}{5}.$$

A. Taylor expansion and estimates of remainder term. The example is the same as the first two bullets in Example 4.23.

D. Differentiate and then find the coefficients.

I. 1) We get successively by differentiation

$$\begin{aligned}
f'(x) &= -(x+1)\sin\left(\frac{1}{2}\,x^2+x\right),\\
f''(x) &= -(x+1)^2\cos\left(\frac{1}{2}\,x^2+x\right) - \sin\left(\frac{1}{2}\,x^2+x\right),\\
f^{(3)}(x) &= (x+1)^3\sin\left(\frac{1}{2}\,x^2+x\right) - 3(x+1)\cos\left(\frac{1}{2}\,x^2+x\right).
\end{aligned}$$

This gives

$$P_2(x) = f(0) + f'(0)\cdot x + \frac{1}{2!}\,f''(0)\cdot x^2 = 1 + 0 - \frac{1}{2}\,x^2 = 1 - \frac{1}{2}\,x^2.$$

2) According to Taylor's formula there exists a ξ lying between 0 and x, such that

$$f(x) = P_2(x) = \frac{1}{3!}\,f^{(3)}(\xi)\cdot x^3.$$

From this we derive the estimate

$$\begin{aligned}
|f(x) - P_2(x)| &= \frac{1}{6}\left|f^{(3)}(\xi)\right|\cdot |x|^3\\
&= \frac{|x|^3}{6}\left|(\xi+1)^3\sin\left(\frac{1}{2}\,\xi^2+\xi\right) - 3(\xi+1)\cos\left(\frac{1}{2}\,\xi^2+\xi\right)\right|.
\end{aligned}$$

Now

$$\left|\frac{1}{2}\,x^2 + x\right| < \frac{1}{50} + \frac{1}{5} = \frac{11}{50} \qquad \text{for } |x| < \frac{1}{5},$$

and $|\xi| \le |x| < \frac{1}{5}$, so we get the estimate

$$\begin{aligned}
|f(x) - P_2(x)| &< \frac{1}{6}\cdot\frac{1}{5^3}\left\{\left(\frac{1}{5}+1\right)^3\sin\frac{11}{50} + 3\left(\frac{1}{5}+1\right)\cos 0\right\}\\
&< \frac{8}{1000}\cdot\frac{1}{6}\left\{\frac{6^3}{5^3}\cdot\frac{11}{50} + 3\cdot\frac{6}{5}\cdot 1\right\}\\
&= \frac{8}{1000}\left\{\frac{6^2\cdot 11}{125\cdot 50} + \frac{3}{5}\right\} < 8\cdot 10^{-3}\cdot\frac{2}{3}\\
&< 8\cdot 10^{-3}.
\end{aligned}$$

Example 2.15 *Find the Taylor polynomial $P_1(x)$ of first order at the point $x_0 = 0$ for the function*

$$f(x) = \ln(1+e^x), \qquad x \in \mathbb{R}.$$

Then prove that

$$|f(x) - P_1(x)| \le \frac{1}{16} \qquad \textit{for } x \in \left[0, \frac{1}{2}\right].$$

A. Taylor polynomial and estimate of the remainder term. Cf. Example 1.12.

D. Since we later shall estimate the remainder term, we differentiate twice.

I. From

$$\begin{aligned} f(x) &= \ln(1+e^x), & f(0) &= \ln 2, \\ f'(x) &= \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}, & f'(0) &= \frac{1}{2}, \end{aligned}$$

we get

$$P_1(x) = \ln 2 + \frac{1}{2}\,x.$$

Since

$$\begin{aligned} f''(x) &= \frac{e^x}{(1+e^x)^2} = \frac{1}{(1+e^x)(1+e^{-x})} \\ &= \frac{1}{1+e^x+e^{-x}+1} = \frac{1}{2(1+\cosh x)}, \end{aligned}$$

it follows for $x \in \left[0, \frac{1}{2}\right]$ $\left(\text{and even for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right)$, that

$$\begin{aligned} |f(x) - P_1(x)| &\le \frac{1}{2!} \cdot \frac{1}{2^2} \cdot \max_{|x| \le \frac{1}{2}} \frac{1}{2(1+\cosh x)} \\ &= \frac{1}{2} \cdot \frac{1}{2^2} \cdot \frac{1}{2 \cdot 2} = \frac{1}{32}, \end{aligned}$$

hence we get a better estimate than wanted.

Example 2.16 *1) Find the Taylor polynomial $P_2(x)$ of order 2 at the point $x_0 = 0$ for the function $e^x \sin x$, $x \in \mathbb{R}$.*

2) Prove that

$$|e^x \sin x - P_2(x)| < 0,02, \qquad \text{for } x \in \left[0, \frac{1}{3}\right].$$

A. Taylor polynomial. Cf. Example 2.17.

D. The Taylor coefficients are found by differentiation.

I. 1) **First variant.** By successive differentiation we get

$$\begin{array}{rclcrcl}
f(x) & = & e^x \sin x, & \qquad & f(0) & = & 0, \\
f'(x) & = & e^x\{\sin x + \cos x\}, & & f'(0) & = & 1, \\
f''(x) & = & 2e^x \cos x, & & f''(0) & = & 2, \\
f^{(3)}(x) & = & 2e^x\{\cos x - \sin x\}, & & & &
\end{array}$$

where we shall use the third derivative in the estimate of the remainder term.

It follows that

$$P_2(x) = x + \frac{1}{2} \cdot 2x^2 = x + x^2.$$

Second variant. From

$$e^x = 1 + x + \frac{1}{2}\,x^2 + \cdots \quad \text{og} \quad \sin x = x - \frac{1}{6}\,x^3 + \cdots,$$

it follows by simple multiplying the two expressions that

$$f(x) = x + x^2 + \frac{1}{3}\,x^3 + \cdots,$$

hence

$$P_2(x) = x + x^2.$$

2) If $x \in \left[0, \frac{1}{3}\right]$, then

$$\begin{aligned} |e^x \sin x - P_2(x)| &\leq \frac{1}{3!} \sum_{x \in [0,\frac{1}{3}]} \left|f^{(3)}(x)\right| \cdot \left(\frac{1}{3}\right)^3 \\ &\leq \frac{1}{6} \cdot \frac{1}{27} \cdot 2\sqrt[3]{e} = \frac{\sqrt[3]{e}}{81} < \frac{1}{50} = 0,02. \end{aligned}$$

Example 2.17 *1) Find the Taylor polynomial $P_2(x)$ of order 2 at the point $x_0 = 0$ for the function $e^x \sin x$, $x \in \mathbb{R}$.*

2) Prove that

$$|e^x \sin x - P_2(x)| < 0,02, \qquad \text{if } x \in \left[0, \frac{1}{3}\right].$$

3) Prove also that we even have

$$|e^x \sin x - P_2(x)| < 0,0125, \qquad \text{if } x \in \left[0, \frac{1}{3}\right].$$

A. Taylor polynomial. The first two bullets are the same as the first two bullets in Example 2.16.

D. The Taylor coefficients are found by differentiation. We can reuse Example 2.16 in the first two questions.

I. 1) **First variant.** By successive differentiation we get

$$\begin{array}{lclclcl} f(x) & = & e^x \sin x, & \qquad & f(0) & = & 0, \\ f'(x) & = & e^x\{\sin x + \cos x\}, & & f'(0) & = & 1, \\ f''(x) & = & 2e^x \cos x, & & f''(0) & = & 2, \\ f^{(3)}(x) & = & 2e^x\{\cos x - \sin x\}, & & & & \end{array}$$

where we save the third derivative for the estimate of the remainder term.

We see that

$$P_2(x) = x + \frac{1}{2} \cdot 2x^2 = x + x^2.$$

Second variant. From

$$e^x = 1 + x + \frac{1}{2}\,x^2 + \cdots \quad \text{and} \quad \sin x = x - \frac{1}{6}\,x^3 + \cdots ,$$

we get by simply multiplying the two expressions that

$$f(x) = x + x^2 + \frac{1}{3}\,x^3 + \cdots ,$$

hence

$$P_2(x) = x + x^2.$$

2) If $x \in \left[0, \frac{1}{3}\right]$, then

$$\begin{aligned} |e^x \sin x - P_2(x)| &\leq \frac{1}{3!} \sum_{x \in [0, \frac{1}{3}]} \left| f^{(3)}(x) \right| \cdot \left(\frac{1}{3}\right)^3 \\ &\leq \frac{1}{6} \cdot \frac{1}{27} \cdot 2\sqrt[3]{e} = \frac{\sqrt[3]{e}}{81} < \frac{1}{50} = 0,02. \end{aligned}$$

3) From

$$f^{(4)}(x) = -4e^x \sin x,$$

follows that $f^{(3)}(x)$ is decreasing in $\left[0, \frac{1}{3}\right]$, and since we already know that $f^{(3)}(x) > 0$, we get

$$\sup_{x \in [0, \frac{1}{3}]} \left| f^{(3)}(x) \right| = f^{(3)}(0) = 2.$$

Hence we obtained the improved estimate

$$\begin{aligned} |e^x \sin x - P_2(x)| &\leq \frac{1}{3!} \sup_{x \in [0, \frac{1}{3}]} \left| f^{(3)}(x) \right| \cdot \left(\frac{1}{3}\right)^3 \\ &\leq \frac{1}{6} \cdot 2 \cdot \frac{1}{27} = \frac{1}{81} < \frac{1}{80} = 0,0125. \end{aligned}$$

Example 2.18 *1) Let $x_0 = 0$ be the chosen expansion point. Find the Taylor polynomial $P_3(x)$ of third order for the function*

$$f(x) = (1+x)^2 \ln(1+x).$$

2) Prove by an estimate of the remainder term that we have the inequality

$$-10^{-5} \leq f(x) - P_3(x) \leq 0$$

for $0 \leq x \leq \dfrac{1}{10}$.

A. Taylor expansion and estimate of the remainder term.

D. Differentiate and find the Taylor coefficients.

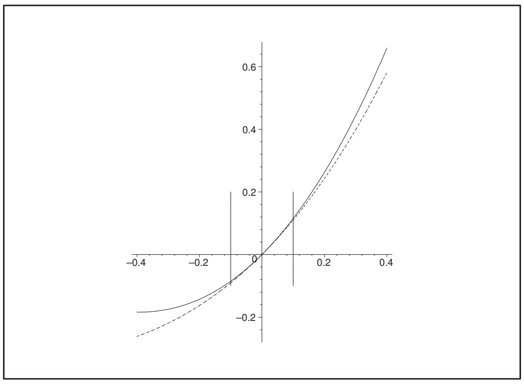

Figure 6: The graphs of $f(x) = (1+x)^2 \ln(1+x)$ and $P_3(x)$, $-0,4 \leq x \leq 0,4$ with an indication of the interval $[-0,1;0,1]$.

I. 1) We obtain by differentiation,

$$\begin{array}{ll}
f(x) = (1+x)^2 \ln(1+x), & f(0) = 0, \\
f'(x) = 2(1+x)\ln(1+x) + (1+x), & f'(0) = 1, \\
f''(x) = 2\ln(1+x) + 3, & f''(0) = 3, \\
f^{(3)}(x) = \dfrac{2}{1+x}, & f^{(3)}(0) = 2, \\
f^{(4)}(x) = -\dfrac{2}{(1+x)^2}. &
\end{array}$$

Thus the Taylor polynomial at $x_0 = 0$ is given by

$$P_3(x) = x + \frac{3}{2}\,x^2 + \frac{1}{3}\,x^3.$$

2) If $x \in \left[0, \dfrac{1}{10}\right]$, then $f^{(4)} < 0$, hence $f(x) - P_3(x) \leq 0$ in the same interval, and we get the estimate

$$|f(x) - P_3(x)| \leq \frac{1}{4!} \cdot \left(\frac{1}{10}\right)^4 \cdot \sup_{x \in [0, \frac{1}{10}]} \left|f^{(4)}(x)\right| = 10^{-5} \cdot \frac{5}{12} \cdot \frac{2}{(1+0)^2} < 10^{-5},$$

thus

$$-10^{-5} \le f(x) - P_3(x) \le 0 \qquad \text{for } x \in \left[0, \frac{1}{10}\right].$$

Example 2.19 *Given the function*

$$f(x) = \frac{1}{\cos x}, \qquad x \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right].$$

1) Prove that the Taylor polynomial of order 2 with the point of expansion $x_0 = 0$ is $P_2(x) = 1 + \frac{1}{2}x^2$.

2) Prove that the remainder term $R_2(x)$ satisfies the following estimate in the given interval

$$|R_2(x)| \le 0,3.$$

A. Taylor polynomial and estimate of the remainder term

D. Differentiate three times

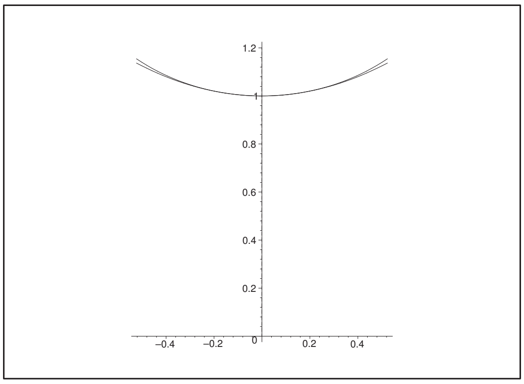

Figure 7: The graphs of $f(x) = \frac{1}{\cos x}$ (above) and the approximation $P_2(x) = 1 + \frac{1}{2}x^2$ (below) in the interval $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

I. By differentiation we get for $x \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right)$,

$$\begin{aligned}
f(x) &= \frac{1}{\cos x}, && f(0) = 1,\\
f'(x) &= \frac{\sin x}{\cos^2 x}, && f'(0) = 0,\\
f''(x) &= \frac{1}{\cos x} + 2 \cdot \frac{\sin^2 x}{\cos^3 x} = \frac{1 + \sin^2 x}{\cos^3 x}, && f''(0) = 1,
\end{aligned}$$

and

$$\begin{aligned}
f^{(3)}(x) &= \frac{2\sin x}{\cos^2 x} + 3 \cdot \sin x \cdot \frac{1 + \sin^2 x}{\cos^4 x}\\
&= \frac{\sin x}{\cos^4 x}\left\{2\cos^2 x + 3 + 3\sin^2 x\right\} = \frac{\sin x}{\cos^4 x}\{5 + \sin^2 x\}.
\end{aligned}$$

1) It follows from the above that

$$P_2(x) = f(0) + \frac{1}{1!}\,f'(0)\cdot x + \frac{1}{2!}\,f''(0)\cdot x^2 = 1 + \frac{1}{2}\,x^2.$$

2) Also, we get the remainder term from the above

$$R_2(x) = \frac{1}{3!}\,f^{(3)}(\xi)\cdot x^3 = \frac{1}{6}\cdot\frac{\sin\xi}{\cos^4\xi}\cdot(5+\sin^2\xi)\cdot x^3,$$

where ξ is some point lying between 0 and x. Since $\sin\xi$ is increasing and $\cos\xi$ is decreasing for $\xi \in \left[0,\frac{\pi}{6}\right]$, and since $R_2(x)$ is even because $f(x)$ is even, we find that $|R_2(x)|$ is largest for $(\xi, x) = \pm\left(\frac{\pi}{6},\frac{\pi}{6}\right)$, thus

$$\begin{aligned}
|R_2(x)| &\le \frac{1}{6}\cdot\frac{\sin\frac{\pi}{6}}{\cos^4\frac{\pi}{6}}\cdot\left\{5+\sin^2\frac{\pi}{6}\right\}\cdot\left(\frac{\pi}{6}\right)^3\\
&= \frac{1}{6}\cdot\frac{\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)^4}\cdot\left\{5+\left(\frac{1}{2}\right)^2\right\}\cdot\left(\frac{\pi}{6}\right)^3\\
&= \frac{1}{12}\cdot\frac{16}{9}\cdot\frac{21}{4}\cdot\left(\frac{\pi}{6}\right)^3 = \frac{7}{9}\cdot\left(\frac{\pi}{6}\right)^3 \approx 0,112 < 0,3.
\end{aligned}$$

3 Approximating polynomials

Example 3.1 *Find the Taylor polynomial $P_3(x)$ for the function $f(x) = \dfrac{x}{x^2+1}$ with the point of expansion $x_0 = 0$. Sketch the graphs of P_3 and f.*

A. Comparison between a function and one of its Taylor polynomials.

D. Find the Taylor polynomial and sketch the graphs.

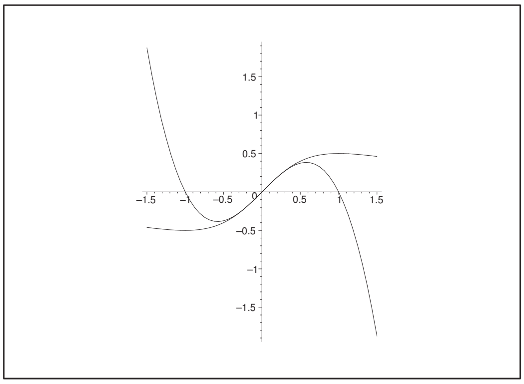

Figure 8: The graphs of $P_3(x)$ and $f(x) \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

I. We get by differentiation

$$\begin{aligned}
f(x) &= \frac{x}{x^2+1} \\
f'(x) &= \frac{1-x^2}{(1+x^2)^2} = \frac{2}{(1+x^2)^2} - \frac{1}{1+x^2}, \\
f''(x) &= -\frac{8x}{(1+x^2)^3} + \frac{2x}{(1+x^2)^2}, \\
f^{(3)} &= \frac{48x^2}{(1+x^2)^4} - \frac{8}{(1+x^2)^3} - \frac{8x^2}{(1+x^2)^3} + \frac{2}{(1+x^2)^2}.
\end{aligned}$$

Here it is no need to further reduce these expressions because we shall only put $x_0 = 0$. This gives

$$P_3(x) = x + \frac{1}{3!}(-8+2)x^3 = x - x^3.$$

For $P_3(x) = x - x^3$ we get $P_3'(x) = 1 - 3x^2$, corresponding to $P_3'(x) = 0$ for $x = \pm\dfrac{1}{\sqrt{3}} \approx \pm 0,57735$, where

$$P_3\left(\pm\frac{1}{\sqrt{3}}\right) = \pm\frac{2}{3}\,\frac{1}{\sqrt{3}} \approx \pm 0,3849.$$

Example 3.2 *1) Find the Taylor polynomial $P_8(x)$ with the point of expansion $x_0 = 0$ for the polynomial*

$$Q(x) = (x+a)^8,$$

where a is a real number.

2) Prove that we for all x get $P_8(x) = Q(x)$.

3) Use the method to se up a formula for $(x+a)^n$.

A. Taylor polynomial for the *polynomial* $(x+a)^n$.

D. The Taylor polynomial is found in the usual way.

I. 1) We get

$$\begin{aligned} P_8(x) &= Q(0) + \frac{Q'(0)}{1!}\,x + \frac{Q''(0)}{2!}\,x^2 + \frac{Q^{(3)}(0)}{3!}\,x^3 + \frac{Q^{(4)}(0)}{4!}\,x^4 \\ &\quad + \frac{Q^{(5)}(0)}{5!}\,x^5 + \frac{Q^{(6)}(0)}{6!}\,x^6 + \frac{Q^{(7)}(0)}{7!}\,x^7 + \frac{Q^{(8)}(0)}{8!}\,x^8. \end{aligned}$$

We see that we by differentiation get

$$Q^{(j)}(x) = 8 \cdot 7 \cdots (8-j+1)\,(x+a)^{8-j}, \qquad j = 1, \dots, 8,$$

thus

$$Q^{(j)}(0) = 8 \cdot 7 \cdots (9-j)a^{8-j} = \frac{8!}{(8-j)!}\,a^{8-j}, \qquad j = 1, \dots, 8,$$

which is also true for $j = 0$,

$$Q(0) = Q^{(0)}(0) = a^8.$$

From these results follows that a general term of the Taylor expansion is

$$\frac{Q^{(j)}(0)}{j!}\,x^j = \frac{8!}{j!(8-j)!}\,a^{8-j}\,x^j = \binom{8}{j}\,a^{8-j}\,x^j, \qquad j = 0, 1, \dots, 8.$$

Then we find the Taylor polynomial

$$P_8(x) = \sum_{j=0}^{8} \binom{8}{j}\,a^{8-j}\,x^j.$$

2) Since $Q^{(9)}(x) \equiv 0$, it follows from Taylor's formula that

$$Q(x) = (x+a)^8 = P_8(x) + \frac{1}{9!}\,Q^{(9)}(\xi)\,x^9 = P_8(x).$$

3) When we use the same method as above we obtain the general binomial formula

$$(x+a)^n = \sum_{j=0}^{n} \binom{n}{j}\,a^{n-j}\,x^j.$$

Example 3.3 *Prove that for positive x,*

$$1+\frac{1}{2}\,x^3-\frac{1}{8}\,x^6<\sqrt{1+x^3}<1+\frac{1}{2}\,x^3-\frac{1}{8}\,x^6+\frac{1}{16}\,x^9,$$

and find the corresponding bounds for

$$\int_0^{\frac{1}{2}}\sqrt{1+x^3}\,dx.$$

A. Taylor expansions with hidden estimate of the remainder term.

D. Put $y=x^3$, and then find the Taylor expansion with respect to y.

I. If we put $f(y)=\sqrt{1+y}=(1+y)^{\frac{1}{2}}$, then

$$f'(y)=\frac{1}{2}\,(1+y)^{-\frac{1}{2}},\qquad f''(y)=-\frac{1}{4}\,(1+y)^{-\frac{3}{2}},$$

$$f^{(3)}(y)=\frac{3}{8}\,(1+y)^{-\frac{5}{2}},\qquad f^{(4)}(y)=-\frac{15}{16}\,(1+y)^{-\frac{7}{2}}.$$

Thus $f(0)=1$ and

$$f^{(2n-1)}(y)>0 \quad \text{and} \quad f^{(2n)}(y)<0, \qquad \text{for } n\in\mathbb{N},$$

so

$$\begin{aligned}
P_2(y) &= 1+\frac{1}{2}\,y-\frac{1}{2!}\cdot\frac{1}{4}\,y^2, && R_2(y)>0,\\
P_3(y) &= 1+\frac{1}{2}\,y-\frac{1}{2!}\cdot\frac{1}{4}\,y^2+\frac{1}{3!}\cdot\frac{3}{8}\,y^3, && R_3(y)<0,
\end{aligned}$$

and we conclude that

$$P_2(y)=1+\frac{y}{2}-\frac{y^2}{8}<f(y)=\sqrt{1+y}<1+\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}, \qquad y>0.$$

If we put $y=x^3$, then

$$1+\frac{x^3}{2}-\frac{x^6}{8}<\sqrt{1+x^3}<1+\frac{x^3}{2}-\frac{x^6}{8}+\frac{x^9}{16}.$$

From

$$\int_0^{\frac{1}{2}}\left\{1+\frac{x^3}{2}-\frac{x^6}{8}\right\}dx=\left[x+\frac{x^4}{8}-\frac{x^7}{56}\right]_0^{\frac{1}{2}}=\frac{1}{2}+\frac{1}{128}-\frac{1}{56}\cdot\frac{1}{128}\approx 0,507673,$$

and

$$\int_0^{\frac{1}{2}}\left\{1+\frac{x^3}{2}-\frac{x^6}{8}+\frac{x^9}{x^9}16\right\}dx=\frac{1}{2}+\frac{55}{56}\cdot\frac{1}{128}+\frac{1}{160}\cdot\frac{1}{2^{10}}\approx 0,507679,$$

we conclude that

$$\begin{aligned}
0,507673 \quad &\approx \quad \int_0^{\frac{1}{2}}\left\{1+\frac{x^3}{2}-\frac{x^6}{8}\right\}dx<\int_0^{\frac{1}{2}}\sqrt{1+x^3}\,dx\\
&< \quad \int_0^{\frac{1}{2}}\left\{1+\frac{x^3}{2}-\frac{x^6}{8}9+\frac{x^9}{16}\right\}dx\approx 0,507679,
\end{aligned}$$

and we have found a good approximation of the integral of $\sqrt{1+x^3}$,

$$0,507673<\int_0^{\frac{1}{2}}\sqrt{1+x^3}\,dx<0,507679.$$

Example 3.4 *Find the Taylor polynomial of order 3 with any point of expansion x_0 for the function*

$$f(x) = (1+x)^3.$$

A. Taylor polynomial; any point of expansion. Obviously, the function is *always* equal to its Taylor polynomial of order 3, because $f(x)$ itself is a polynomial of third degree.

D. We have here two possibilities:

1) the binomial formula,
2) the method of differentiation.

I. 1) We get by the binomial formula

$$\begin{aligned} f(x) &= (1+x)^3 = (\{1+x_0\}+\{x-x_0\})^3 \\ &= (1+x_0)^3 + 3(1+x_0)^2(x-x_0) + 3(1+x_0)(x-x_0)^2 + (x-x_0)^3. \end{aligned}$$

2) By successive differentiation we get

$$\begin{array}{ll} f(x) = (1+x)^3, & f(x_0) = (1+x_0)^3, \\ f'(x) = 3(1+x)^2, & f'(x_0) = 3(1+x_0)^2, \\ f''(x_0) = 6(1+x), & f''(x_0) = 6(1+x_0), \\ f^{(3)}(x) = 6, & f^{(3)}(x_0) = 6. \end{array}$$

The Taylor polynomial is

$$\begin{aligned} P_3(x) &= f(x) = f(x_0) + f'(x_0)\,(x-x_0) + \frac{1}{2!}\,f''(x_0)\,(x-x_0)^2 + \frac{1}{3!}\,f^{(3)}(x_0)\,(x-x_0)^3 \\ &= (1+x_0)^3 + 3(1+x_0)^2(x-x_0) + 3(1+x_0)(x-x_0)^2 + (x-x_0)^3. \end{aligned}$$

Example 3.5 *Find the Taylor polynomial $P_8(x)$ with the point of expansion $x_0 = 0$ for the functions*

(1) $f(x) = \sin 2x$, (2) $f(x) = \cos 2x$, (3) $f(x) = (1+x^2)^2$.

A. Taylor expansions.

D. Perform a simple calculation in (3). In (1) and (2) we differentiate.

I. 1) If we put $g(y) = \sin y$, then

$$\begin{array}{ll} g(y) = g^{(4)}(y) = g^{(8)}(y) = \sin y, & g'(y) = g^{(5)}(y) = \cos y, \\ g''(y) = g^{(6)}(y) = -\sin y, & g^{(3)}(y) = g^{(7)}(y) = -\cos y, \end{array}$$

hence

$$P_{g,8}(y) = y - \frac{1}{3!}\,y^3 + \frac{1}{5!}\,y^5 - \frac{1}{7!}\,y^7.$$

Now, $f(x) = g(2x)$, so we get

$$P_8(x) = P_{g,8}(2x) = 2x - \frac{4}{3}\,x^3 + \frac{4}{15}\,x^5 - \frac{8}{315}\,x^7.$$

2) Analogously for $g(y) = \cos y$,

$$g(y) = g^{(4)}(y) = g^{(8)}(y) = \cos y, \qquad g'(y) = g^{(5)}(y) = -\sin y,$$
$$g''(y) = g^{(6)}(y) = -\cos y, \qquad g^{(3)}(y) = g^{(7)}(y) = \sin y,$$

thus

$$P_{g,8}(y) = 1 - \frac{1}{2!}\,y^2 + \frac{1}{4!}\,y^4 - \frac{1}{6!}\,y^6 + \frac{1}{8!}\,y^8.$$

Then by putting $f(x) = g(2x)$,

$$P_8(x) = P_{g,8}(2x) = 1 - 2x^2 + \frac{2}{3}\,x^4 - \frac{4}{45}\,x^6 + \frac{2}{315}\,x^8.$$

3) We simply get by a squaring

$$P_8(x) = (1+x^2)^2 = 1 + 2x^2 + x^4.$$

Example 3.6 *Prove that a Taylor polynomial with the point of expansion $x_0 = 0$ of an odd (an even, resp.) function only contains odd (even, resp.) powers of x.*

A function $f(x)$ is called *odd*, if $f/-x) = -f(x)$, and it is called *even*, if $f(-x) = f(x)$.

A. Taylor expansion of an odd (even, resp.) function.

D. Differentiate the definitions of an odd, (an even, resp.)function.

I. When $f(-x) = -f(x)$ is odd we get by differentiation,

$$\frac{d^n}{dx^n}\,f(-x) = (-1)^n f^{(n)}(x) = -f^{(n)}(x).$$

By a rearrangement followed by putting $x = 0$ we get

$$\{(-1)^n + 1\}f^{(n)}(0) = 0.$$

If $n = 2m$ is even, then $(-1)^{2m} + 1 = 2 \neq 0$, and $f^{(2m)}(0) = 0$. We conclude that the Taylor polynomial only contains powers of x of odd exponents.

If instead $f(-x) = f(x)$ is even, then we get by differentiation,

$$\frac{d^n}{dx^n}\,f(-x) = (-1)^n f^{(n)}(-x) = f^{(n)}(x).$$

If we put $x = 0$, we get by a rearrangement,

$$\{(-1)^n - 1\}f^{(n)}(0) = 0.$$

When $n = 2m+1$ is odd, then $(-1)^{2m+1} - 1 = -2 \neq 0$, so $f^{(2m+1)}(0) = 0$. We conclude that the Taylor polynomial only contain powers of x of even exponents.

Example 3.7 *Find the Taylor polynomial different from 0 of lowest degree for the functions*

(1) $f(x) = 6\sin x - 6x + x^3$, (2) $f(x) = \ln(1+x) - x$.

A. The meaning is that one shall find the smallest n, for which

$$\frac{1}{n!} f^{(n)}(0) \neq 0.$$

The simplest method is of course to insert known series, but this is not the purpose, so we shall here choose the most difficult method, which also will indicate the order of the zero at $x = 0$ for the function.

D. Differentiate and put $x = 0$.

I. 1) We get by successive differentiation

$$\begin{array}{ll}
f(x) = 6\sin x - 6x + x^3, & f(0) = 0, \\
f'(x) = 6\cos x - 6 + 3x^2, & f'(0) = 6 - 6 + 0 = 0, \\
f''(x) = -6\sin x + 6x, & f''(0) = 0, \\
f^{(3)}(x) = -6\cos x + 6, & f^{(3)}(0) = 0, \\
f^{(4)}(x) = 6\sin x, & f^{(4)}(0) = 0, \\
f^{(5)}(x) = 6\cos x, & f^{(5)}(0) = 6.
\end{array}$$

The searched Taylor polynomial is

$$P_5(x) = \frac{1}{5!}\, f^{(5)}(0)\, x^5 = \frac{6}{120}\, x^5 = \frac{1}{20}\, x^5.$$

2) We get by successive differentiation

$$\begin{array}{ll}
f(x) = \ln(1+x) - x, & f(0) = 0, \\
f'(x) = \dfrac{1}{1+x} - 1, & f'(0) = 0, \\
f''(x) = -\dfrac{1}{(1+x)^2}, & f''(0) = -1.
\end{array}$$

The searched Taylor polynomial is

$$P_2(x) = \frac{1}{2!}\, f^{(2)}(0)\, x^2 = -\frac{1}{2}\, x^2.$$

Example 3.8 *Find the Taylor expansion of order $n = 3$ for the functions*

(1) $f(x) = 2^{2x} - \ln(1+x^2)$, (2) $f(x) = \sin 2x + \sqrt{1+x^2}$.

A. Taylor expansions.

D. Differentiate three times.

I. 1) We get by successive differentiation

$$\begin{array}{ll}
f(x) = e^{2x} - \ln(1+x^2), & f(0) = 1, \\
f'(x) = 2e^{2x} - \dfrac{2x}{1+x^2}, & f'(0) = 2, \\
f''(x) = 4e^{2x} - \dfrac{2}{1+x^2} + \dfrac{4x^2}{(1+x^2)^2}, & f''(0) = 4 - 2 = 2, \\
f^{(3)}(x) = 8e^{2x} + x\{\cdots\}, & f^{(3)}(0) = 8,
\end{array}$$

hence

$$\begin{aligned}
e^{2x} - \ln(1+x^2) &= 1 + 2x + \frac{2}{2!}\, x^2 + \frac{8}{3!}\, x^3 + x^3\varepsilon(x) \\
&= 1 + 2x + x^2 + \frac{4}{3}\, x^3 + x^3\, \varepsilon(x).
\end{aligned}$$

2) We get by successive differentiation

$$\begin{aligned}
&f(x)=\sin 2x+\sqrt{1+x^2}, && f(0)=1,\\
&f'(x)=2\cos 2x+\frac{x}{\sqrt{1-x^2}}, && f'(0)=2,\\
&f''(x)=-4\sin 2x+\frac{1}{\sqrt{1+x^2}}-\frac{x^2}{(1+x^2)^{\frac{3}{2}}}, && f^{(3)}(0)=-8,
\end{aligned}$$

hence

$$f(x)=1+2x+\frac{1}{2}\,x^2-\frac{4}{3}\,x^3+x^3\varepsilon(x).$$

Remark. Since we later always instead ought to use the method of direct insertion of known series, we add this variant, though this was not the purpose of the example.

a) In the first case we get

$$\begin{aligned}
f(x) &= e^{2x}-\ln(1+x^2)\\
&= \left\{1+2x+\frac{4x^2}{2}+\frac{8x^3}{6}+x^3\varepsilon(x)\right\}-\left\{x^2-\frac{x^4}{2}+x^4\varepsilon(x)\right\}\\
&= 1+2x+x^2+\frac{4}{3}\,x^3+x^3\varepsilon(x).
\end{aligned}$$

b) In the second case we get

$$\begin{aligned}
f(x) &= \sin 2x+\sqrt{1+x^2}\\
&= \left\{2x-\frac{8x^3}{6}+x^3\varepsilon(x)\right\}+\left\{10\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}x^2+x^3\varepsilon(x)\right\}\\
&= 1+2x+\frac{1}{2}\,x^2-\frac{4}{3}\,x^3+x^3\varepsilon(x).
\end{aligned}$$

Example 3.9 *Find the Taylor expansion of order $n=6$ for the functions*

(1) $f(x)=\cos 3x-\ln(1-x^2)$, (2) $f(x)=\sqrt{1-x}+\sin(x^2)$.

A. Taylor expansions.

D. Either differentiate six times (this is not done here), or insert known series. We shall here use the latter method.

I. 1) Since

$$\cos y=1-\frac{1}{2!}\,y^2+\frac{1}{4!}\,y^4-\frac{1}{6!}\,y^6+y^6\varepsilon(y),$$

and

$$-\ln(1-z)=z+\frac{1}{2}\,z^2+\frac{1}{3}\,x^3+z^3\varepsilon(z),$$

we get by the substitutions $y=3x$ and $z=x^2$ that

$$\begin{aligned}
f(x) &= \cos 3x-\ln(1-x^2)\\
&= 1-\frac{1}{2}\,3^2x^2+\frac{1}{4!}\,3^4x^4-\frac{1}{6!}\,3^6x^6+x^6\varepsilon(x)\\
&\qquad +x^2+\frac{1}{2}\,x^4+\frac{1}{3}\,x^6+x^6\varepsilon(x)\\
&= 1-\frac{7}{2}\,x^2+\frac{31}{8}\,x^4-\frac{163}{240}\,x^6+x^6\varepsilon(x).
\end{aligned}$$

2) Since

$$\begin{aligned}\sqrt{1-x} &= 1+\binom{\frac{1}{2}}{1}(-x)+\binom{\frac{1}{2}}{2}(-x)^2+\binom{\frac{1}{2}}{3}(-x)^3\\ &\quad +\binom{\frac{1}{2}}{4}(-x)^4+\binom{\frac{1}{2}}{5}(-x)^5+\binom{\frac{1}{2}}{6}(-x)^6+x^6\varepsilon(x)\\ &= 1-\frac{1}{2}\,x-\frac{1}{8}\,x^2-\frac{1}{16}\,x^3-\frac{5}{128}\,x^4-\frac{7}{256}\,x^5-\frac{21}{1024}\,x^6+x^6\varepsilon(x),\end{aligned}$$

and

$$\sin(x^2)=\frac{1}{1!}(x^2)-\frac{1}{3!}(x^2)^3+x^6\varepsilon(x)=x^2-\frac{1}{6}\,x^6+x^2\varepsilon(x),$$

we get

$$\begin{aligned}f(x) &= \sqrt{1-x}+\sin(x^2)\\ &= 1-\frac{1}{2}\,x+\frac{7}{8}\,x^2-\frac{1}{16}\,x^3-\frac{5}{128}\,x^4-\frac{7}{256}\,x^5-\frac{575}{3072}\,x^6+x^6\varepsilon(x).\end{aligned}$$

Example 3.10 *Find the Taylor expansion of order $n=8$ for the functions*

(1) $f(x)=e^{-x^2}-\cos x$, (2) $f(x)=\sin x-2xe^{-x^2}$.

A. Taylor expansions.

D. When the order is as big as $n=8$, one should probably avoid the method of successive differentiations. Instead we insert known series development.

I. 1) From the series of the exponential we get

$$e^{-x^2}=1-x^2+\frac{1}{2}\,x^4-\frac{1}{6}\,x^6+\frac{1}{24}\,x^8+x^8\varepsilon(x).$$

Furthermore,

$$\cos x=1-\frac{1}{2}\,x^2+\frac{1}{24}\,x^4-\frac{1}{720}\,x^6+\frac{1}{40\,320}\,x^8+x^8\varepsilon(x),$$

hence

$$\begin{aligned}f(x) &= e^{-x^2}-\cos x\\ &= -\frac{1}{2}\,x^2+\left(\frac{1}{2}-\frac{1}{24}\right)x^4-\left(\frac{1}{6}-\frac{1}{6!}\right)x^6+\left(\frac{1}{24}-\frac{1}{8!}\right)x^8+x^8\varepsilon(x)\\ &= -\frac{1}{2}\,x^2+\frac{11}{24}\,x^4-\frac{119}{720}\,x^6+\frac{1679}{40\,320}\,x^8+x^8\varepsilon(x).\end{aligned}$$

2) Since

$$f(x)=\sin x-2xe^{-x^2}=\frac{d}{dx}\left\{e^{-x^2}-\cos x\right\},$$

one may wrongly conclude that by differentiation of the result of (1) should obtain

$$\begin{aligned}f(x) &= \sin x-2xe^{-x^2}\\ &= -x+\frac{11}{6}\,x^3-\frac{119}{120}\,x^5+\frac{1679}{5040}\,x^7+x^7\varepsilon(x).\end{aligned}$$

THIS METHOD IS IN GENERAL WRONG, BECAUSE WE DO NOT KNOW HOW TO DIFFERENTIATE THE UNSPECIFIED $\varepsilon(x)$-TERM! These derivatives may in some cases be very large, indeed. The annoying thing of this example is that it can here be proved (with some more theory!) that in this present case this method is legal, but this theory is not in general known to to the students at this stage!

We use instead that

$$\sin x = \frac{1}{1!}\,x - \frac{1}{3!}\,x^3 + \frac{1}{5!}\,x^5 - \frac{1}{7!}\,x^{+}x^8\varepsilon(x),$$

and that if follows from (5) that

$$-2xe^{-x^2} = -2x + 2x^3 - x^5 + \frac{1}{3}\,x^7 + x^8\varepsilon(x),$$

hence

$$\begin{aligned} f(x) &= \sin x - 2xe^{-x^2} \\ &= -x + \left(2 - \frac{1}{6}\right)x^3 - \left(1 - \frac{1}{5!}\right)x^5 + \left(\frac{1}{3} - \frac{1}{7!}\right)x^7 + x^8\varepsilon(x) \\ &= -x + \frac{11}{6}\,x^3 - \frac{119}{120}\,x^5 + \frac{1179}{5040}\,x^7 + x^8\varepsilon(x). \end{aligned}$$

We see that apart from the order of the $\varepsilon(x)$-term we obtain the same result as if we formally had differentiated the result of (1).

Example 3.11 *Find the Taylors expansion with the point of expansion $x_0 = 0$ and any order n of the functions*

$$(1)\ \ f(x) = 2^x, \qquad (2)\ \ f(x) = \frac{1}{2+x}.$$

A. Taylor expansion.

D. Find a general expression for $f^{(n)}(x)$.

I. 1) If $f(x) = 2^x = e^{x \ln 2}$, then $f^{(k)}(x) = (\ln 2)^k \cdot 2^x$, so

$$\frac{f^{(k)}(0)}{k!} = \frac{(\ln 2)^k}{k!},$$

and the Taylor expansion is

$$f(x) = 2^x = \sum_{k=0}^{n} \frac{1}{k!}\,(\ln 2)^k\, x^k + x^n \varepsilon(x).$$

2) If

$$f(x) = \frac{1}{2+x} = (x+2)^{-1} \qquad \left[= \frac{1}{2} \cdot \frac{1}{1+\dfrac{x}{2}} \right],$$

then

$$f^{(k)}(x) = (-1)^k\, k!(x+2)^{-k-1},$$

so

$$\frac{f^{(k)}(0)}{k!} = \frac{(-1)^k}{2^{k+1}},$$

and the Taylor expansion is

$$f(x) = \frac{1}{2+x} = \frac{1}{2}\sum_{k=0}^{n}(-1)^k \left(\frac{x}{2}\right)^k + x^n \varepsilon(x),$$

which we of course also could have obtained directly by putting $y = \dfrac{x}{2}$ into the development of $\dfrac{1}{2} \cdot \dfrac{1}{1+y}$.

Example 3.12 *Find the Taylor expansion with the point of expansion $x_0 = 0$ and of any order n for the functions*

(1) $f(x) = \sin x + \cos x$, (2) $f(x) = \sqrt{1+x^2} - \sqrt{1-x^2}$.

A. Taylor expansion for any n.

D. Substitute into known developments.

I. 1) Strictly speaking we should here consider the two cases of even and odd order. We shall here lazily restrict ourselves to the case of odd order, so we develop up to order $2n+1$. From

$$\sin x = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} + x^{2n+1}\varepsilon(x),$$

and

$$\cos x = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}\, x^{2k} + x^{2n+1}\varepsilon(x),$$

we get

$$\begin{aligned} f(x) &= \sin x + \cos x \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}\, x^{2k} + \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} + x^{2n+1}\varepsilon(x). \end{aligned}$$

2) If we expand to order $2n$, then

$$\begin{aligned} f(x) &= (1+x^2)^{\frac{1}{2}} - (1-x^2)^{\frac{1}{2}} \\ &= \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} x^{2k} + \sum_{k=0}^{n} (-1)^{k+1} \binom{\frac{1}{2}}{k} x^{2k} + x^{2n} \\ &= \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} \left\{1 + (-1)^{k+1}\right\} x^{2k} + x^{2n}\varepsilon(x) \\ &= 2 \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{\frac{1}{2}}{2k+1} x^{4k+2} + x^{2n}\varepsilon(x), \end{aligned}$$

where $\left[\dfrac{n-1}{2}\right]$ denotes the integer part of $\dfrac{n-1}{2}$, i.e. the biggest integer $\le \dfrac{n-1}{2}$. We obtain the latter result by putting n odd or even into the second last equation.

Example 3.13 *1) Find the Taylor polynomial $P_n(x)$ of order n with the point of expansion $x_0 = 0$ for the function $f(x) = \cos^2 x$.*

2) Find the Taylor polynomial $Q_n(x)$ of order n with the point of expansion $x_0 = 0$ for the function $g(x) = \sin^2 x$.

3) Prove that $P_n(x) + Q_n(x) = 1$ for every $x \in \mathbb{R}$.

A. Taylor polynomials. The fundamental trigonometric relation.

D. Express $\cos^2 x$ and $\sin^2 x$ by $\cos 2x$ before the Taylor expansion.

I. 1) We get from $\cos^2 x = \dfrac{1}{2} + \dfrac{1}{2}\cos 2x$ that

$$\begin{aligned}\cos^2 x &= \frac{1}{2} + \frac{1}{2}\left\{1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 + \cdots + \frac{(-1)^n}{(2n)!}(2x)^{2n}\right\} \\ &\qquad\qquad + x^{2n}\varepsilon(x) \\ &= 1 - \frac{2}{2!}\,x^2 + \frac{2^3}{4!}\,x^4 + \cdots + (-1)^n\,\frac{2^{2n-1}}{(2n)!}\,x^{2n} + x^{2n}\varepsilon(x) \\ &= 1 - x^2 + \frac{1}{3}\,x^4 + \cdots + (-1)^n \cdot \frac{4^n}{2(2n)!}\,x^{2n} + x^{2n}\varepsilon(x).\end{aligned}$$

Notice that $P_{2n+1}(x) = P_{2n}(x)$.

2) Analogously, we get from $\sin^2 x = \dfrac{1}{2} - \dfrac{1}{2}\cos 2x$ that

$$\sin^2 = \frac{2}{2!}\,x^2 - \frac{2^3}{4!}\,x^4 + \cdots + (-1)^n \cdot \frac{2^{2n-1}}{(2n)!}\,x^{2n} + x^{2n}\varepsilon(x).$$

Here

$$\begin{aligned}Q_{2n+1}(x) &= Q_{2n}(x) \\ &= \frac{2}{2!}\,x^2 - \frac{2^3}{4!}\,x^4 + \cdots + (-1)\cdot\frac{2^{2n-1}}{(2n)!}\,x^{2n} \\ &= 1 - P_{2n}(x) = 1 - P_{2n+1}(x).\end{aligned}$$

3) It follows clearly from the remark in (2) that

$$P_n(x) + Q_n(x) = 1 \qquad \text{for every } n \in \mathbb{N}.$$

We could not expect this result even if we know that $\cos^2 x + \sin^2 x = 1$.

Example 3.14 *1) Find the approximating polynomial $P_3(x)$ of at most third degree with the point of expansion $x_0 = 0$ for the function*

$$f(x) = \ln(1+2x), \qquad x \in \left]-\frac{1}{2}, +\infty\right[.$$

2) Prove that

$$\ln(1+2x) < P_3(x) \qquad \textit{for every } x > -\frac{1}{2}, \quad x \neq 0.$$

A. Approximating polynomial.

D. Differentiate four times.

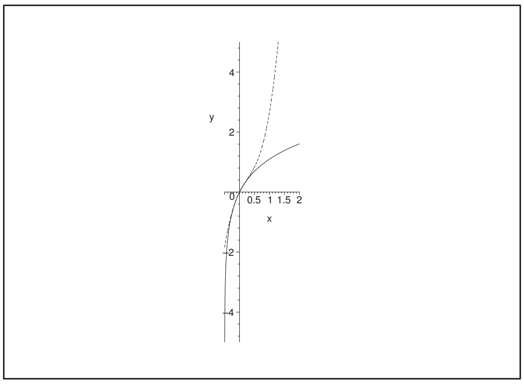

Figure 9: The graph of $y = \ln(1+2x)$ and its approximating polynomial $y = 2x - 2x^2 + \frac{8}{3}\,x^3$ (dotted).

I. 1) We get by successive differentiations

$$\begin{array}{lll}
f(x) & = \ln(1+2x), & f(0), \\
f'(x) & = \dfrac{2}{1+2x} = \dfrac{1}{x+\frac{1}{2}}, & f'(0) = 2, \\
f''(x) & = -\dfrac{1}{\left(x+\frac{1}{2}\right)^2}, & f''(0) = -4, \\
f^{(3)}(x) = \dfrac{2}{\left(x+\frac{1}{2}\right)^3}, & f''(0) = -4, & \\
f^{(3)}(x) = \dfrac{2}{\left(x+\frac{1}{2}\right)^3}, & f^{(3)}(0) = 16, & \\
f^{(4)}(x) & = -\dfrac{6}{\left(x+\frac{1}{2}\right)^4}. &
\end{array}$$

Hence

$$P_3(x) = 2x - 2x^2 + \frac{8}{3}\,x^3,$$

with the remainder term

$$R_3(x) = -\frac{1}{4!}\,\frac{6}{\left(\xi+\frac{1}{2}\right)^4}\,x^4 = -\frac{4}{(1+2\xi)^4}\,x^4,$$

for some ξ between 0 and x.

2) From

$$\ln(1+2x) = P_3(x) + R_3(x)$$

and

$$R_3(x) = -\frac{4x^4}{(1+2\xi)^4} < 0$$

for every $x > -\dfrac{1}{2}$ and every ξ between 0 and x, we conclude that

$$\ln(1+2x) < P_3(x) \qquad \text{for every } x > -\frac{1}{2}.$$

Example 3.15 *Write a MAPLE programme, which sketches the graph of the Taylor polynomial of order 6 with the point $x_0 = 0$ as extension point in the interval $[-3, 3]$ for the function*

$$f(x) = \ln(1 + \sqrt{1 + \sin x}), \qquad x \in \mathbb{R}.$$

A. MAPLE programme.

D. This example, found in some textbook, is rather strange because one would usually instead directly sketch the graph of the function itself. One will almost always lose some information by considering the Taylor polynomial instead. An exception is of course when the function is itself a polynomial and we want the Taylor polynomial of at least the same order as the degree of the polynomial.

D. The first command is

```
taylor(ln(1+sqrt(1+sin(x))),x=0,6);
```

We get the result

$$\ln(2) + \frac{1}{4}\,x - \frac{3}{32}\,x^2 + \frac{1}{96}\,x^3 - \frac{3}{1024}\,x^4 + \frac{1}{1536}\,x^5 + O\left(x^6\right)$$

Then we remove the term $O\left(x^6\right)$, and e.g. continue by

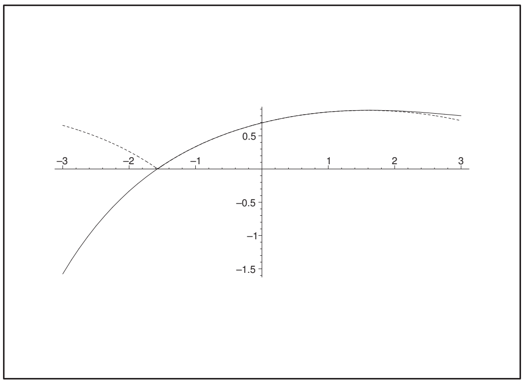

Figure 10: The graphs of $P_6(x)$ and $f(x) = \ln(1 + \sqrt{1 + \sin x})$, (dotted), $x \in [-3, 3]$.

```
plot([[t,ln(2)+t/4-3*t^2/32+t^3/96-3*t^4/1024+t^5/1536,t=-3..3],
  [t,ln(1+sqrt(1+sin(t))),t=-3..3]],linestyle=[1,2],color=black);
```

Hereby we get the figure where we for comparison also have sketched the graph of the function itself (here dotted line): It is seen that the graph of the function has a kink for $x = -\dfrac{\pi}{2}$, a phenomenon which is never possible for any Taylor polynomial.

Example 3.16 *Find for every one of the following functions $f(x)$ an expression for $f^{(n)}(0)$, and then find the approximating polynomial of at most degree n at the point $x_0 = 0$ for $f(x)$.*

$$(1)\ \ f(x) = a^x, \quad (2)\ \ f(x) = \frac{1}{1-x}, \quad (3)\ \ f(x) = \frac{1}{2+x}.$$

A. Approximating polynomials. Note that (1) is almost the same as Example 3.11 (1).

D. Just differentiate.

I. 1) It follows immediately from $f(x) = a^x = e^{x \ln a}$ that

$$f^{(n)}(x) = (\ln a)^n e^{x \ln a}, \qquad f^{(n)}(0) = (\ln a)^n,$$

hence

$$P_n(x) = 1 + \frac{1}{1!}\ln a \cdot x + \frac{1}{2!}(\ln a)^2 x^2 + \cdots + \frac{1}{n!}(\ln a)^2\, x^n.$$

2) When $f(x) = \dfrac{1}{1-x}$ we get

$$f'(x) = \frac{1!}{(1-x)^2} \qquad \text{and} \qquad f''(x) = \frac{2!}{(1-x)^3}.$$

Assume that

$$f^{(n)} = \frac{n!}{(1-x)^{n+1}}.$$

We see that this is true for $n = 0$, $n = 1$ and $n = 2$. Then by a differentiation of the assumption we get

$$f^{(n+1)}(x) = \frac{(n+1)!}{(1-x)^{n+2}} = \frac{(n+1)!}{(1-x)^{(n+1)+1}},$$

i.e. a formula of the same structure, only with n replaced by $n+1$. Hence we conclude by induction that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \quad \text{and} \quad f^{(n)}(0) = n!, \quad \frac{f^{(n)}(0)}{n!} = 1.$$

Hence

$$P_n(x) = 1 + x + x^2 + \cdots + x^n.$$

3) When

$$f(x) = \frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1 + \frac{x}{2}}$$

we get (cf. (2))

$$f^{(n)}(x) = \frac{1}{2}(-1)^n \cdot \frac{n!}{\left(1 + \frac{x}{2}\right)^{n+1}} \cdot \frac{1}{2^n},$$

so

$$f^{(n)}(0) = (-1)^n \cdot \frac{n!}{2^{n+1}},$$

and we have

$$P_n(x) = \frac{1}{2} - \frac{1}{2^2}\,x + \frac{1}{2^3}\,x^2 + \cdots + (-1)^n \cdot \frac{1}{2^{n+1}}\,x^n.$$

Example 3.17 *One often applies Taylor expansions in physics and technical sciences to approximate functions which cannot be calculated directly. The procedure is often that one expands up to some given order and neglect the remainder term. One example is*

$$(5)\quad T(A) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - A^2 \sin^2 t}}\,dt.$$

One cannot calculate the integral directly. The task is now to give a procedure which shows how $T(A)$ depends on A for small values of A (this integral occurs e.g. in the formula of the oscillation time for a mathematical pendulum).

1) Find the Taylor expansion of order $n = 4$ for the function

$$f(x) = \frac{1}{\sqrt{1 - x^2 k^2}}.$$

2) Find the Taylor expansion of order $n = 4$ for the integrand in (5), where A is the variable while t in this connection is kept fixed.

3) Replace the integrand in (5) by the found approximation and then find a corresponding approximation of $T(A)$.

[There will occur some integrals which the student must find in a table, because they are not known at this stage.]

A. Approximation of an elliptic integral.

D. Find the Taylor expansion of the integrand.

I. 1) We get by a Taylor expansion

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{1 - x^2 k^2}} = \left(1 - k^2 x^2\right)^{-\frac{1}{2}} \\ &= 1 + \binom{-\frac{1}{2}}{1}\left(-k^2 x^2\right) + \binom{-\frac{1}{2}}{2}\left(-k^2 x^2\right)^2 + x^4 \varepsilon(x) \\ &= 1 + \frac{k^2}{2}\,x^2 + \frac{3}{8}\,k^4 x^4 + x^4 \varepsilon(x). \end{aligned}$$

2) Replace kx by $A \sin t$. Then it follows directly from (1) that

$$\frac{1}{\sqrt{1 - A^2 \sin^2 t}} = 1 + \frac{1}{2}\,A^2 \sin^2 t + \frac{3}{8}\,A^4 \sin^4 t + A^4 \varepsilon(A).$$

3) Finally, we shall calculate

$$\int_0^{\frac{\pi}{2}} \left\{1 + \frac{1}{2}\,A^2 \sin^2 t + \frac{3}{8}\,A^4 \sin^4 t\right\}\, dt.$$

Now $\sin^2 t = \dfrac{1}{2}\{1 - \cos 2t\}$, so

$$\begin{aligned}
\sin^4 t &= \frac{1}{4}\,\{1 - 2\cos 2t + \cos^2 2t\}\\
&= \frac{1}{4}\left\{1 - 2\cos 2t + \frac{1+\cos 4t}{2}\right\}\\
&= \frac{1}{4} - \frac{1}{2}\,\cos 2t + \frac{1}{8} + \frac{1}{8}\,\cos 4t\\
&= \frac{3}{8} - \frac{1}{2}\,\cos 2t + \frac{1}{8}\,\cos 4t.
\end{aligned}$$

Then by insertion,

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \left\{1 + \frac{1}{2}\,A^2 \sin^2 t + \frac{3}{8}\,A^4 \sin^4 t\right\}\, dt\\
&= \frac{\pi}{2} + \frac{1}{2}\,A^2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2t)\, dt + \frac{3}{8}\,A^4 \cdot \frac{1}{8}\int_0^{\frac{\pi}{2}} \{3 - 4\cos 2t + \cos 4t\}\, dt\\
&= \frac{\pi}{2} + \frac{A^2}{4}\left[t - \frac{1}{2}\,\sin 2t\right]_0^{\frac{\pi}{2}} + \frac{3}{64}\,A^4\left[3t - 2\sin 2t + \frac{1}{4}\,\sin 4t\right]_0^{\frac{\pi}{2}}\\
&= \frac{\pi}{2} + \frac{A^2}{4}\cdot\frac{\pi}{2} + \frac{9}{64}\,A^4 \cdot \frac{\pi}{2}\\
&= \frac{\pi}{2}\left\{1 + \frac{1}{4}\,A^2 + \frac{9}{64}\,A^4\right\}.
\end{aligned}$$

Example 3.18 *In this example we shall derive a formula which approximates the length of a circular arc. This formula is due to Huygens (1629–1695). We shall use the figure and the notation given in the text. We shall assume that the angle φ satisfies $\varphi \in \left[0, \frac{\pi}{2}\right]$.*

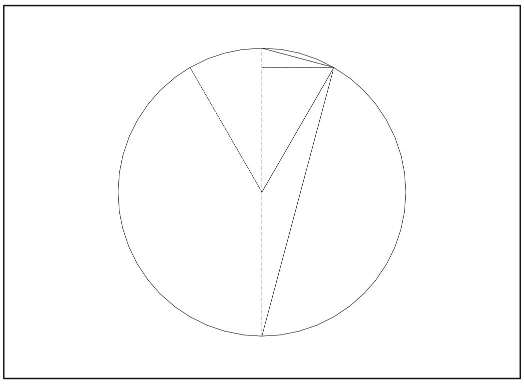

Figure 11: Circle of radius r, centre angle φ, thus periphery angle $\frac{\varphi}{2}$ below. The arc is denoted by ℓ, and the corresponding cord is denoted by d. Finally, we let s denote the height on the dotted vertical diagonal.

The approximating expression $\tilde{\ell}$ of the length ℓ is given in the form

$$\tilde{\ell} = ad + bs,$$

where a and b are constants which will be found below.

1) First prove that

$$\tilde{\ell} = 2ar\,\sin\frac{\varphi}{2} + br\,\sin\varphi.$$

2) We consider $\tilde{\ell}$ as a function of φ. Find the approximating polynomial $P_3(\varphi)$ and the corresponding remainder term $R_3(\varphi)$ with the point of expansion $\varphi = 0$.

3) Find the constants a and b, such that $P_3(\varphi) = \ell = 2r\varphi$, and set up the corresponding approximation $\tilde{\ell}$ expressed by d and s.

4) Prove that

$$|\ell - \tilde{\ell}| \le \frac{r}{180} \cdot \varphi^5.$$

A. An approximation with given guidelines.

D. Follow the guidelines.

I. 1) It follows from some simple geometric considerations (look at some rectangular triangles) that

$$d = 2r\,\sin\frac{\varphi}{2} \quad \text{and} \quad s = r\,\sin\varphi,$$

thus

$$\tilde{\ell} = ad + bs = 2ar\,\sin\frac{\varphi}{2} + br\,\sin\varphi.$$

2) Then by some differentiations,

$$\begin{aligned}
\frac{d\tilde{\ell}}{d\varphi} &= ar\,\cos\frac{\varphi}{2}+br\,\cos\varphi, & \tilde{\ell}'(0)&=ar+br,\\
\frac{d^2\tilde{\ell}}{d\varphi^2} &= -\frac{ar}{2}\,\sin\frac{\varphi}{2}-br\,\sin\varphi, & \tilde{\ell}''(0)&=0,\\
\frac{d^3\tilde{\ell}}{d\varphi^3} &= -\frac{ar}{4}\,\cos\frac{\varphi}{2}-br\,\cos\varphi, & \tilde{\ell}^{(3)}(0)&=-\frac{ar}{4}-br,\\
\frac{d^4\tilde{\ell}}{d\varphi^4} &= \frac{ar}{8}\,\sin\frac{\varphi}{2}+br\,\sin\varphi, & \tilde{\ell}^{(4)}(0)&=0,
\end{aligned}$$

$$\frac{d^5\tilde{\ell}}{d\varphi^5}=\frac{ar}{16}\,\cos\frac{\varphi}{2}+br\,\cos\varphi,$$

and we get by insertion into Taylor's formula that

$$\begin{aligned}\tilde{\ell}(\varphi) &= \frac{\ell'(0)}{1!}\,\varphi + \frac{\ell''(0)}{3!}\,\varphi^3 + \frac{\ell^{(5)}(\xi)}{5!}\,\varphi^5\\ &= (a+b)r\varphi - \frac{1}{24}\,(a+4b)r\varphi^3 + \frac{1}{120}\,\left\{\frac{a}{16}\,\cos\frac{\xi}{2} + b\,\cos\xi\right\} r\varphi^5,\end{aligned}$$

where ξ lies somewhere between 0 and φ. Then

$$P_3(\varphi) = P_4(\varphi) = (a+b)r\varphi - \frac{1}{24}\,(a+4b)r\varphi^3,$$

and the remainder term is

$$R_4(\varphi) = \frac{1}{120}\left\{\frac{a}{16}\,\cos\frac{\xi}{2} + b\,\cos\xi\right\} r\varphi^5.$$

3) Thus, if we put $P_3(\varphi) = \ell = 2r\varphi$, then

$$P_3(\varphi) = (a+b)r\varphi - \frac{1}{24}\,(a+4b)r\varphi^3 = 2r\varphi,$$

and we obtain the conditions

$$a+b=2 \qquad \text{og} \qquad a+4b=0.$$

Hence, $a=-4b$ and $2=-3b$, i.e. $b=-\dfrac{2}{3}$, so finally $a=\dfrac{8}{3}$. Then we get by insertion

$$\tilde{\ell} = ad + bs = \frac{8}{3}\,d - \frac{2}{3}\,s.$$

4) Putting $a=\dfrac{8}{3}$ and $b=-\dfrac{2}{3}$ the expression of the remainder term becomes

$$\begin{aligned}R_4(\varphi) &= \frac{1}{120}\left\{\frac{8}{3}\cdot\frac{1}{16}\,\cos\frac{\xi}{2} - \frac{2}{3}\,\cos\xi\right\} r\varphi^5\\ &= \frac{1}{120}\left\{\frac{1}{6}\,\cos\frac{\xi}{2} - \frac{2}{3}\,\cos\xi\right\} r\varphi^5.\end{aligned}$$

When $\varphi \in \left[0, \dfrac{\pi}{2}\right]$, then both $\cos\dfrac{\xi}{2}$ and $\cos\xi$ are positive, so

$$\begin{aligned}|R_4(\varphi)| &\le \frac{1}{120}\,\max\left\{\frac{1}{6}\,\cos\frac{\xi}{2},\,\frac{2}{3}\,\cos\xi\right\}\cdot r\varphi^5\\ &\le \frac{1}{120}\cdot\frac{2}{3}\cdot r\varphi^5 = \frac{r}{180}\,\varphi^5.\end{aligned}$$

Remark. Since the function $\dfrac{1}{6}\,\cos\dfrac{\xi}{2} - \dfrac{2}{3}\,\cos\xi$ is increasing in $\left[0, \dfrac{\pi}{2}\right]$, one can actually prove that

$$|R_4(\varphi)| \le \frac{r}{240}\,\varphi^5, \qquad \varphi \in \left[0, \frac{\pi}{2}\right]. \qquad \Diamond$$

4 Limit processes

Example 4.1 *Find the following limits by means of a Taylor expansion:*

$$(1)\quad \lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}, \qquad (2)\quad \lim_{x\to 0}\frac{6\sin x-6x+x^3}{x^3}.$$

A. A Taylor expansion.

D. Find the order of the zero in the denominator. Then expand the numerator to the same order, and then finally take the limit.

I. 1) Since x^2 is 0 of order 2, we expand the numerator $f(x)=\ln(1+x)-x$ to the order 2. Then $f(0)=0$ and

$$f'(x)=\frac{1}{1+x}-1, \qquad f''(x)=-\frac{1}{(1+x)^2},$$

so $f'(0)=0$ and $f''(0)=-1$. Then

$$f(x)=\ln(10x)-x=-\frac{1}{2}\,x^2+x^2\varepsilon(x), \qquad \text{where } \varepsilon(x)\to 0 \text{ for } x\to 0.$$

Finally, we get by insertion and a limit process that

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=\lim_{x\to 0}\frac{-\frac{1}{2}\,x^2+x^2\varepsilon(x)}{x^2}=-\frac{1}{2}.$$

2) Since the denominator x^3 is 0 of order 3, we expand the numerator

$$f(x)=6\sin x-6x+x^3, \qquad f(0)=0,$$

to the order 3. Then by differentiation,

$$\begin{array}{ll} f'(x)=6\cos x-6+3x^2, & f'(0)=0,\\ f''(x)=-6\sin x+6x, & f''(0)=0,\\ f^{(3)}(x)=-6\cos x+6, & f^{(3)}(0)=0, \end{array}$$

hence

$$f(x)=0+x^3\varepsilon(x), \qquad \text{hvor } \varepsilon(x)\to 0 \text{ for } x\to 0.$$

Finally, we get by insertion and a limit process,

$$\lim_{x\varepsilon 0}\frac{6\sin x-6x+x^3}{x^3}=\lim_{x\to 0}\frac{x^3\varepsilon(x)}{x^3}=0.$$

Remark. The method above is the one which should be used when one is learning the technique. Later on one should instead use that the Taylor expansions taken from $x_0=0$ are known for most of the important functions. I shall in the following also demonstrate how one uses such tables.

1) By using a table we get

$$\ln(1+x) - x = \left\{x - \frac{x^2}{2} + x^2\varepsilon(x)\right\} - x = -\frac{x^2}{2} + x^2\varepsilon(x),$$

thus

$$\frac{\ln(1+x)-x}{x^2} = \frac{-\dfrac{x^2}{2} + x^2\varepsilon(x)}{x^2} = -\frac{1}{2} + \varepsilon(x) \to -\frac{1}{2} \qquad \text{for } x \to 0.$$

2) By using a table we get

$$\begin{aligned} 6\sin x - 6x + x^3 &= 6\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\varepsilon(x)\right\} - 6x + x^3 \\ &= \frac{1}{20}\,x^5 + x^5\varepsilon(x), \end{aligned}$$

thus

$$\frac{6\sin x - 6x + x^3}{x^3} = \frac{\dfrac{1}{20}\,x^5 + x^5\varepsilon(x)}{x^3} = \frac{1}{20}\,x^2 + x^2\varepsilon(x) \to 0 \qquad \text{for } x \to 0.$$

Example 4.2 *Find the following limits by means of a Taylor expansion,*

$$(1)\quad \lim_{x\to 0}\frac{x(e^x+1)-2(e^x-1)}{x^3}, \qquad (1)\quad \lim_{x\to 0}\frac{\sin x - x}{x^3}.$$

A. Limits by a Taylor expansion.

D. Then denominator is in both cases 0 of order 3 at $x_0 = 0$. We therefore expand the numerator to the order 3. We shall here use the direct method, where we assume the series expansions known.

I. 1) From

$$e^x = 1 + x + \frac{1}{2}\,x^2 + \frac{1}{6}\,x^3 + x^3\varepsilon(x),$$

we get by insertion

$$\begin{aligned}
&\lim_{x\to 0}\frac{x(e^1+1)-2(e^x-1)}{x^3} \\
&\lim_{x\to 0}\frac{x\left\{2+x+\frac{1}{2}\,x^2+x^2\varepsilon(x)\right\}-2\left\{x+\frac{1}{2}\,x^2+\frac{1}{6}\,x^3+x^3\varepsilon(x)\right\}}{x^3} \\
= &\lim_{x\to 0}\frac{\frac{1}{2}\,x^3-\frac{1}{3}\,x^3+x^3\varepsilon(x)}{x^3} = \frac{1}{6}.
\end{aligned}$$

2) From $\sin x = x - \frac{1}{6}\,x^3 + x^3\varepsilon(x)$ we get by insertion,

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{-\frac{1}{6}\,x^3 + x^3\varepsilon(x)}{x^3} = -\frac{1}{6}.$$

Example 4.3 *Find by means of a Taylor expansion the following limits,*

$$(1)\quad \lim_{x\to 0}\frac{\ln(1+x)}{e^{2x}-1}, \qquad (2)\quad \lim_{x\to 0}\frac{1-\cos^2 x}{x\tan x}.$$

A. Limits found by means of a Taylor expansion.

D. The order of the zero at 0 of the denominator is 1 in (1), and 2 in (2), so the numerators should be expanded similarly.

I. 1) From $\ln(1+x) = x + x\varepsilon(x)$ and $e^{2x} - 1 = 2x + x\varepsilon(x)$, we get by insertion that

$$\lim_{x\to 0}\frac{\ln(1+x)}{e^{2x}-1} = \lim_{x\to 0}\frac{x+x\varepsilon(x)}{2x+x\varepsilon(x)} = \lim_{x\to 0}\frac{1+\varepsilon x}{2+\varepsilon(x)} = \frac{1}{2}.$$

2) Since $1 - \cos^2 x = \sin^2 x$, we can actually here make a shortcut, because

$$\frac{1-\cos^2 x}{x\tan x} = \frac{\sin^2 x}{x\cdot\dfrac{\sin x}{\cos x}} = \frac{\sin x}{x}\cdot\cos x.$$

Then we either get directly (known from high school) that

$$\lim_{x\to 0} \frac{1-\cos^2 x}{x \tan x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \cos x = 1 \cdot 1 = 1,$$

or more elaborated,

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos^2 x}{x \tan x} &= \lim_{x\to 0} \frac{\sin x}{x} \cdot \cos x \\ &= \lim_{x\to 0} \frac{x + x\varepsilon(x)}{x} \cdot \{1 + x\varepsilon(x)\} \\ &= \lim_{x\to 0} \{1+\varepsilon(x)\}\{1+\varepsilon(x)\} = 1.\end{aligned}$$

Example 4.4 *Prove that the limits*

$$(1)\quad \lim_{x\to 0+} \frac{\ln x}{x^2+x+1}, \qquad (2)\quad \lim_{x\to 0} \frac{\sin(x+x^3)-x}{x^5},$$

do not exist.

A. Both limits should be divergent.

D. Clearly, the denominator is $\neq 0$ for $x = 0$ in (1), thus since the numerator tends to $-\infty$, it follows by inspection that (1) is divergent. We shall use a little more in (2) to get to the same conclusion.

I. 1) Since the numerator tends to $-\infty$, and the denominator tends to 1 for $x \to 0$, we see that

$$\lim_{x\to 0+} \frac{\ln x}{x^2+x+1} = -\infty$$

is divergent.

2) Put $f(x) = \sin(x+x^3) - x$. Then $f(0) = 0$, and

$$\begin{array}{ll} f'(x) = (1+3x^2)\cos x - 1, & f'(0) = 0, \\ f''(x) = -(1+3x^2)\sin x + 6x\cos x, & f''(0) = 0, \\ f^{(3)}(x) = (5-3x^2)\cos x - 6x\sin x, & f^{(3)}(0) = 5, \end{array}$$

hence

$$f(x) = \sin(x+x^3) - x = \frac{5}{3!}\,x^3 + x^3\varepsilon(x).$$

Then by insertion,

$$\frac{\sin(x+x^3)-x}{x^5} = \frac{\frac{5}{6}\,x^3 + x^3\varepsilon(x)}{x^5} = \frac{\frac{5}{6} + \varepsilon x}{x^2} \to +\infty \qquad \text{for } x \to 0,$$

and

$$\lim_{x\to 0} \frac{\sin(x+x^3)-x}{x^5} = +\infty$$

is divergent.

Example 4.5 *Find the limits*

$$(1)\quad \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\sqrt{1-\sin x}}, \qquad (2)\quad \lim_{x\to 0} \frac{(\ln x)^2}{(x^2-1)^2}, \qquad (3)\quad \lim_{x\to\frac{\pi}{2}} \frac{1-\cos x}{x},$$

if they exist.

A. Limit processes. In all three cases one should use common sense instead of some standard method. The two first cases are divergent, and indeed it does not in (1) give sense to write lim without any specification of the limit process, because one may obtain different possible limit values according to whether one approaches $\frac{\pi}{2}$ from above or from below.

D. In all three cases we either inspect, or rearrange in a convenient way.

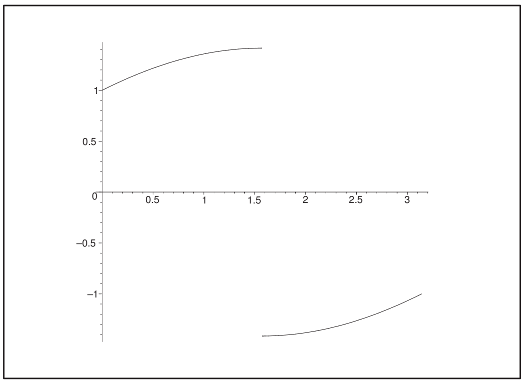

Figure 12: The graph of $f(x) = \dfrac{\cos x}{\sqrt{1-\sin x}}$, $x \in [0,\pi] \setminus \left\{\dfrac{pi}{2}\right\}$.

I. 1) If $x < \dfrac{\pi}{2}$ during the limit process, then

$$\cos x = +\sqrt{(1+\sin x)(1-\sin x)},$$

hence

$$\lim_{x\to\frac{pi}{2}-} \frac{\cos x}{\sqrt{1-\sin x}} = \lim_{x\to\frac{\pi}{2}-} \left(+\sqrt{1+\sin x}\right) = \sqrt{2}.$$

If $x > \dfrac{\pi}{2}$ during the limit process, then

$$\cos x = -\sqrt{(1+\sin x)(1-\sin x)},$$

hence

$$\lim_{x\to\frac{\pi}{2}+} \frac{\cos x}{\sqrt{1-\sin x}} = \lim_{x\to\frac{\pi}{2}+} \left(-\sqrt{1+\sin x}\right) = -\sqrt{2}.$$

We conclude that the limit does not exist.
It is illustrated on the figure what happens in a neighbourhood of $x = \dfrac{\pi}{2}$.

2) If $x \to 0+$, then the numerator tends to $+\infty$ while the denominator tends to 1, so

$$\lim_{x \to 0+} \frac{(\ln x)^2}{(x^2-1)^2} = +\infty.$$

Remark. In the case $x \to 1$, we put

$$g(x) = \ln x, \qquad g(1) = 0,$$
$$g'(x) = \frac{1}{x}, \qquad g'(1) = 1,$$

and $g(x) = \ln x = (x-1) + (x-1)\varepsilon(x-1)$. Then by insertion

$$\lim_{x \to 1} \frac{(\ln x)^2}{(x^2-1)^2} = \lim_{x \to 1} \left(\frac{\ln x}{x-1} \right)^2 \cdot \frac{1}{(x+1)^2}$$
$$= \frac{1}{4} \lim_{x \to 1} \left\{ \frac{x - 1 + (x-1)\varepsilon(x-1)}{x-1} \right\}^2 = \frac{1}{4}.$$

3) The denominator is $\neq 0$ for $x = \dfrac{\pi}{2}$, hence

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \cos x}{x} = \frac{2}{\pi}.$$

Example 4.6 *Consider the functions below for $x \to 0$ by first putting all the terms on the same fraction line:*

$$(1)\quad f(x) = \frac{1}{\ln(1+x)} - \frac{1}{x^2}, \qquad (2)\quad f(x) = \frac{1}{\sin x} - \frac{1}{e^x - 1}.$$

A. A limit process. A reasonable guess is that (1) is divergent, because $\ln(1+x) = x + x\varepsilon(x)$ is of lower degree than x^2.

D. Put all the terms on the same fraction line and the find the Taylor expansions of the numerator and the denominator.

I. 1) We get by some Taylor expansions,

$$\begin{aligned} f(x) &= \frac{1}{\ln(1+x)} - \frac{1}{x^2} = \frac{x^2 - \ln(1+x)}{x^2\ln(1+x)} = \frac{x^2 - x + x\varepsilon(x)}{x^2\{x + x\varepsilon(x)\}} \\ &= \frac{x}{x^3} \cdot \frac{-1 + x + \varepsilon(x)}{1 + \varepsilon(x)} = -\frac{1}{x^2} \cdot \frac{1 + \varepsilon(x)}{1 + \varepsilon(x)}. \end{aligned}$$

The last factor tends to 1, so $f(x) \to -\infty$ for $x \to 0$.

Remark. Suppose now that the example contains an error, so that we should have had $\frac{1}{x}$ instead of $\frac{1}{x^2}$. Then we get the following expansions

$$\begin{aligned} f(x) &= \frac{1}{\ln(1+x)} - \frac{1}{x} = \frac{x - \ln(1+x)}{x\ln(1+x)} = \frac{x - x + \frac{1}{2}x^2 + x^2\varepsilon(x)}{x\{x + x\varepsilon(x)\}} \\ &= \frac{\frac{1}{2} + \varepsilon(x)}{1 + \varepsilon} \to \frac{1}{2} \quad \text{for } x \to 0. \quad \diamond \end{aligned}$$

2) Analogously,

$$\begin{aligned} f(x) &= \frac{1}{\sin x} - \frac{1}{e^x - 1} = \frac{e^x - 1 - \sin x}{\{e^x - 1\}\sin x} \\ &= \frac{x + \frac{1}{2!}x^2 + x^2\varepsilon(x) - x + \frac{1}{3!}x^3 + x^3\varepsilon(x)}{\{x + x\varepsilon(x)\}\{x + x\varepsilon(x)\}} \\ &= \frac{\frac{1}{2}x^2 + x^2\varepsilon(x)}{x^2\{1 + \varepsilon(x)\}} \to \frac{1}{2} \qquad \text{for } x \to 0. \end{aligned}$$

Example 4.7 *1) What is the sign for a term of the form $x^2 + \varepsilon(x) \cdot x^2$, when x is close to 0?*

2) Find the sign of a term of the form $x^n + \varepsilon(x) \cdot x^n$, $n \in \mathbb{N}$, when x is close to 0.

3) Find

$$(a) \quad \lim_{x \to 0} \frac{\ln(1+x) - x}{x^4}, \qquad (b) \quad \lim_{x \to 0} \left(\frac{-1}{\sin x} + \frac{1}{x^2} \right).$$

A. Limit processes.

D. Use common sense.

I. 1) Since $x^2 > 0$ is dominating for $x \neq 0$ close to zero, the sign must be positive.

2) If n is even, then $x^n + \varepsilon(x) \cdot x^n$ is positive in a neighbourhood of zero.
If n is odd, then $x^n + \varepsilon(x) \cdot x^n$ is positive for x positive, and negative for x negative.

3) a) We get by a Taylor expansion,

$$\begin{aligned} \frac{\ln(1+x) - x}{x^4} &= \frac{x - \frac{1}{2}x^2 + x^2\varepsilon(x) - x}{x^4} \\ &= \frac{1}{x^2}\left\{ -\frac{1}{2} + \varepsilon(x) \right\} \to -\infty \qquad \text{for } x \to 0. \end{aligned}$$

b) We get by a Taylor expansion,

$$\begin{aligned} -\frac{1}{\sin x} + \frac{1}{x^2} &= \frac{-x^2 + \sin x}{x^2 \sin x} = \frac{x + x\varepsilon(x)}{x^2\{x + x\varepsilon(x)\}} \\ &= \frac{1}{x^2} \cdot \frac{1 + \varepsilon(x)}{1 + \varepsilon(x)} \to +\infty \qquad \text{for } x \to 0. \end{aligned}$$

Example 4.8 *Let*

$$f(x) = \frac{\sqrt[3]{\ln x}}{x - 1}, \qquad x \in]1, +\infty[.$$

What happens to $f(x)$ under each of the limit processes $x \to 1+$ and $x \to +\infty$?
Consider each of the following functions under the given limit process by putting the dominating term outside as a factor:

1) $f(x) = e^x - x^2$ for $x \to +\infty$.

2) $f(x) = \ln(1 + x^2) + x$ for $x \to -\infty$.

3) $f(x) = \ln x + \dfrac{1}{x}$ for $x \to 0+$.

A. Limit processes.

D. Apply the rules of magnitude.

I. We get from $\ln x = \ln\{1 + (x - 1)\} = (x - 1) + (x - 1)\varepsilon(x - 1)$,

$$\begin{aligned} f(x) &= \frac{\sqrt[3]{\ln x}}{x-1} = \frac{\sqrt[3]{(x-1)+(x-1)\varepsilon(x-1)}}{x-1} \\ &= \frac{1}{(x-1)^{\frac{2}{3}}}\,\sqrt[3]{1+\varepsilon(x)} \to +\infty \qquad \text{for } x \to 1+. \end{aligned}$$

Since $\ln x < x - 1$, we also have

$$0 < f(x) = \frac{\sqrt[3]{\ln x}}{x-1} < \frac{\sqrt[3]{x-1}}{x-1} = \frac{1}{(x-1)^{\frac{2}{3}}} \to 0 \qquad \text{for } x \to +\infty,$$

thus

$$\lim_{x \to +\infty} \frac{\sqrt[3]{\ln x}}{x-1} = 0.$$

1) From

$$f(x) = e^x - x^2 = e^x \left\{1 - \frac{x^2}{e^x}\right\}$$

and the rules of magnitude follows that $f(x) \to +\infty$ for $x \to +\infty$.

2) From

$$f(x) = \ln(1+x^2) + x = x\left\{1 + \frac{\ln(1+x^2)}{x}\right\},$$

and the rules of magnitude follows that $f(x) \to -\infty$ for $x \to -\infty$.

3) It follows from

$$f(x) = \ln x + \frac{1}{x} = \frac{1 + x \cdot x}{x},$$

where $x \cdot \ln x \to 0$ for $x \to 0+$ that at $f(x) \to +\infty$ for $x \to 0+$.

Example 4.9 *Let f, $g :]0,\infty[\to \mathbb{R}$ be two differentiable functions where $g'(x) \neq 0$ for every $x \in]0,\infty[$. Check in each of the following cases whether the claim is correct or wrong:*

1) Assume that $f(x) \to \infty$ for $x \to 0$, and $g(x) \to -\infty$ for $x \to 0$.
If $\dfrac{f'(x)}{g'(x)} \to c$ for $x \to 0$, where $c \in \mathbb{R}$, then $\dfrac{f(x)}{g(x)} \to c$ for $x \to 0$.

2) Assume that $f(x) \to 0$ for $x \to 0$, and $g(x) \to 0$ for $x \to 0$.
If $\dfrac{f(x)}{g(x)} \to c$ for $x \to 0$, where $c \in \mathbb{R}$, then $\dfrac{f'(x)}{g'(x)} \to c$ for $x \to 0$.

3) Assume that $f(x) \to 0$ for $x \to \infty$, and $g(x) \to \infty$ for $x \to \infty$.
If $f'(x)g'(x) \to c$ $(\in \mathbb{R})$ for $x \to \infty$, then $f(x)g(x) \to c$ for $x \to \infty$.

A. General limit processes and l'Hospital's rules.

D. Analyze each of the cases. Give counterexamples, if possible.

I. 1) If we put $h(x) = -g(x)$, then

$$f(x) \to +\infty \quad \text{and} \quad h(x) \to +\infty \qquad \text{for } x \to 0+,$$

and $h'(x) = -g'(x) \neq 0$. It follows from *l'Hospital's second rule* that

$$\frac{f'(x)}{h'(x)} = \frac{f'(x)}{-g'(x)} \to -c \qquad \text{for } x \to 0+,$$

implies

$$\frac{f(x)}{g(x)} = -\frac{f(x)}{h(x)} \to -(-c) = c \qquad \text{for } x \to 0+.$$

2) This claim is wrong. Let e.g.

$$f(x) = x \sin \frac{1}{x} \qquad \text{og} \qquad g(x) = \sqrt{x}.$$

Then $f(x) \to 0$ and $g(x) \to 0$ for $x \to 0+$, because e.g.

$$|f(x)| = \left| x \cdot \sin \frac{1}{x} \right| \le x \to x \to 0 \qquad \text{for } x \to 0+.$$

Furthermore,

$$\left| \frac{f(x)}{g(x)} \right| = \sqrt{x} \left| \sin \frac{1}{x} \right| \le \sqrt{x} \to 0 \qquad \text{for } x \to 0+,$$

so $\dfrac{f(x)}{g(x)} \to 0$ for $x \to 0+$.

Finally,

$$f'(x) = \sin \frac{1}{x} + x \left\{ -\cos \frac{1}{x} \right\} \cdot \left\{ -\frac{1}{x^2} \right\} = \sin \frac{1}{x} + \frac{1}{x} \cos \frac{1}{x},$$

and $g'(x) = \dfrac{1}{2} \dfrac{1}{\sqrt{x}}$, thus

$$\frac{f'(x)}{g'(x)} = 2\sqrt{x} \sin \frac{1}{x} + \frac{2}{\sqrt{x}} \cos \frac{1}{x},$$

where

$$\left| 2\sqrt{x} \sin \frac{1}{x} \right| \le 2\sqrt{x} \to 0 \qquad \text{for } x \to 0+,$$

while $\dfrac{2}{\sqrt{x}} \to +\infty$ and $\cos \dfrac{1}{x}$ oscillates "wildly" between -1 and 1.

Hence the limit does not exist.

3) This claim is also wrong.

Let e.g. $f(x) = \dfrac{1}{x}$ and $g(x) = x \ln x$. Then $f(x) \to 0$ and $g(x) \to +\infty$ for $x \to +\infty$.

Furthermore, $f'(x) = -\dfrac{1}{x^2}$ and $g'(x) = 1 + \ln x$, thus

$$f'(x)g'(x) = -\frac{1 + \ln x}{x^2} \to 0 \qquad \text{for } x \to +\infty$$

due the the rules of magnitude. Clearly,

$$f(x)g(x) = \ln x \to +\infty \qquad \text{for } x \to +\infty.$$

Example 4.10 *It is well-known that the function h given by*

$$h(x) = \begin{cases} e, & \text{for } x = 0, \\ (1+x)^{\frac{1}{x}}, & \text{for } x > 0, \end{cases}$$

is continuous.

1) Find $h'(x)$ for $x > 0$.

2) Then find the limit $\lim_{x \to 0+} h'(x)$.

A. Differentiation and a limit process.

D. Differentiate.

I. 1) For $x > 0$ we get

$$h(x) = (1+x)^{\frac{1}{x}} = \exp\left(\frac{\ln(1+x)}{x}\right).$$

Hence,

$$\begin{aligned}
h'(x) &= \exp\left(\frac{\ln(1+x)}{x}\right)\cdot\left\{\frac{1}{x(1+x)}-\frac{\ln(1+x)}{x^2}\right\}\\
&= (1+x)^{\frac{1}{x}}\cdot\frac{1}{1+x}\cdot\frac{x-(1+x)\ln(1+x)}{x^2}\qquad\text{for } x>0.
\end{aligned}$$

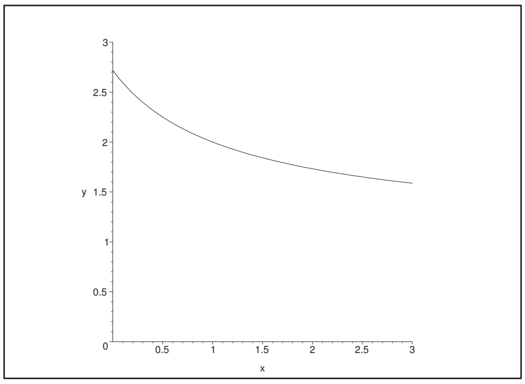

Figure 13: The graph of $y=(1+x)^{\frac{1}{x}}$, $x>0$.

2) For $x\to 0+$ it is well-known that

$$(1+x)^{\frac{1}{x}}\to e \qquad \text{and} \qquad \frac{1}{1+x}\to 1,$$

and

$$\begin{aligned}
\frac{x-(1+x)\ln(1+x)}{x^2} &= \frac{x-(1+x)\left\{x-\frac{1}{2}\,x^2+x^2\varepsilon(x)\right\}}{x^2}\\
&= \frac{1}{x^2}\left\{x-x+\frac{1}{2}\,x^2+x^2\varepsilon(x)-x^2+x^2\varepsilon(x)\right\}\\
&= \frac{1}{x^2}\left\{-\frac{1}{2}\,x^2+x^2\varepsilon(x)\right\}\to-\frac{1}{2}.
\end{aligned}$$

Finally we get

$$\lim_{x\to 0+} h'(x) = e\cdot 1\cdot\left(-\frac{1}{2}\right) = -\frac{e}{2}.$$

Example 4.11 *1) Find the approximating polynomial of at most second degree with the point of expansion $x_0 = 0$ for the function*

$$f(x)\sqrt{\cos x}, \qquad x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

2) Find

$$\lim_{x\to 0}\left\{\frac{\sqrt{\cos x}}{x^2} - \frac{\sin x}{x^3}\right\}.$$

3) Prove that

$$\sqrt{\cos x} \geq 1 - \frac{3}{10}\,x^2 \qquad \textit{for alle } x \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right].$$

A. Taylor expansion; limit process; estimate of remainder term.

D. Differentiate $f(x)$.

I. 1) We get by successive differentiation

$$\begin{aligned}
f(x) &= \sqrt{\cos x}, & f(0) &= 1,\\
f'(x) &= -\frac{1}{2}\,\frac{\sin x}{\sqrt{\cos x}} = -\frac{1}{2}\,\tan x \cdot f(x), & f'(0) &= 0,\\
f''(x) &= -\frac{1}{2}\,(1+\tan^2 x)f(x)0\frac{1}{4}\,\tan^2 x \cdot f(x) & &\\
&= -\frac{1}{4}\,(2+\tan^2 x)f(x), & f''(0) &= -\frac{1}{2},\\
f^{(3)}(x) &= -\frac{1}{4}\cdot 1\tan x \cdot (1+\tan^2 x)f(x) & &\\
&\qquad +\frac{1}{8}\,\tan x \cdot (2+\tan^2 x)f(x) & &\\
&= \frac{1}{8}\,\tan x \cdot f(x) \cdot \left\{-4 - 4\tan^2 x + 2 + \tan^2 x\right\} & &\\
&= -\frac{1}{8}\,(2+3\tan^2 x)\cdot \tan x \cdot \sqrt{\cos x}. & &
\end{aligned}$$

Hence

$$f(x) = 1 - \frac{1}{4}\,x^2 - \frac{1}{48}\,(2+3\tan^2 \xi)\tan\xi\,\sqrt{\cos\xi}\cdot x^3$$

for some ξ between 0 and x.

2) We get from

$$f(x) = 1 - \frac{1}{4}\,x^2 + x^2\varepsilon\left(x^2\right) \quad \text{og} \quad \sin x = x - \frac{1}{6}\,x^6 + x^3\varepsilon(x)$$

that

$$\begin{aligned}
&\lim_{x\to 0}\left\{\frac{\sqrt{\cos x}}{x^2} - \frac{\sin x}{x^3}\right\}\\
&\qquad = \lim_{x\to 0}\frac{x\sqrt{\cos x} - \sin x}{x^3}\\
&\qquad = \lim_{x\to 0}\frac{1}{x^3}\left\{x - \frac{1}{4}\,x^3 + x^3\,\varepsilon(x) - x + \frac{1}{6}\,x^3 + x^3\varepsilon(x)\right\}\\
&\qquad = \lim_{x\to 0}\frac{-\frac{1}{12}\,x^3 + x^3\varepsilon(x)}{x^3} = -\frac{1}{12}.
\end{aligned}$$

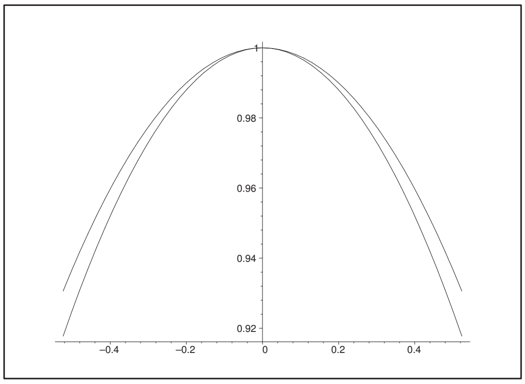

Figure 14: The graphs of $\sqrt{\cos x}$ and $1-\dfrac{3}{10}\,x^2$ for $x \in \left[-\dfrac{\pi}{6},\dfrac{\pi}{6}\right)$.

3) We get from the expansion of $f(x)$ with a remainder term,

$$\begin{aligned}\sqrt{\cos x} &= 1-\frac{1}{4}\,x^2-\frac{1}{48}\,x^2-\frac{1}{48}\,(2+3\tan^2\xi)\tan\xi\sqrt{\cos\xi}\cdot x^3\\ &= \left\{1-\frac{3}{10}\,x^2\right\}+\left\{\frac{1}{20}\,x^2-\frac{1}{48}\,(2+3\tan^2\xi)\tan\xi\sqrt{\cos\xi}\cdot x^3\right\}.\end{aligned}$$

Thus, the claim will be proved, if we can prove that

$$\frac{1}{20}\,x^2-\frac{1}{48}\,(2+3\tan^2\xi)\tan\xi\sqrt{\cos\xi}\cdot x^3 \geq 0 \qquad \text{for } |x| \leq \frac{\pi}{6},$$

where ξ lies somewhere between 0 and x. When we divide by x^2 (for $x \neq 0$) and rearrange, we see that it is sufficient to prove that

$$(2+3\tan^2\xi)\tan\xi\sqrt{\cos\xi}\cdot x \leq \frac{48}{20}=\frac{12}{5} \qquad \text{for } |\xi| \leq |x| \leq \frac{\pi}{6}.$$

Here, we can estimate upwards by replacing both ξ and x by $\dfrac{\pi}{6}$, because $\tan\xi\sqrt{\cos\xi}=\dfrac{\sin\xi}{\sqrt{\cos\xi}}$ is increasing in the given interval. Hence

$$\begin{aligned}(2+3\tan^2\xi)\tan\xi\sqrt{\cos\xi}\cdot x &\leq \left(2+\frac{3}{3}\right)\cdot\frac{1}{\sqrt{3}}\cdot\sqrt{\frac{\sqrt{3}}{2}}\cdot\frac{\pi}{6}\\ &= 3\cdot\frac{1}{\sqrt{2\sqrt{3}}}\cdot\frac{\pi}{6}<2<\frac{12}{5},\end{aligned}$$

and the claim is proved.

Example 4.12 *Let the function f be given by*

$$f(x) = \ln(1 + \sinh 2x).$$

1) Find the approximating polynomial of at most second degree for f with the point of expansion $x_0 = 0$.

2) Find the limit for $x \to 0$ of

$$\frac{f(x) - 2\sin x}{1 - \cos x}.$$

A. Approximating polynomial; limit process. The example is very similar to Example 5.6.

D. Differentiate $f(x)$.

I. 1) We get by successive differentiations,

$$\begin{array}{rclr}
f(x) & = & \ln(1+\sinh 2x), & f(0) = 0, \\
f'(x) & = & \dfrac{2\cosh 2x}{1+\sinh 2x}, & f'(x) = 2, \\
f''(x) & = & \dfrac{-4\cosh^2 2x}{(1+\sinh 2x)^2} + \sinh 2x \cdot \{\cdots\}, & f''(0) = -4,
\end{array}$$

hence,

$$P_2(x) = f(0) + \frac{f'(0)}{1!}\,x + \frac{f''(0)}{2!}\,x^2 = 2x - 2x^2.$$

2) From

$$f(x) = 2x - 2x^2 + x^2\varepsilon, \qquad \sin x = x + x^2\varepsilon(x),$$
$$\cos x = 1 - \frac{1}{2}\,x^2 + x^2\varepsilon(x),$$

follows that

$$\begin{aligned}\frac{f(x) - 1\sin x}{1-\cos x} &= \frac{2x - 2x^2 + x^2\varepsilon(x) - 2x + x^2\varepsilon(x)}{1 - \left(1 - \frac{1}{2}\,x^2\right) + x^2\varepsilon(x)} \\ &= \frac{-2x^2 + x^2\varepsilon(x)}{\frac{1}{2}\,x^2 + x^2\varepsilon(x)} = -4 \cdot \frac{1+\varepsilon(x)}{1+\varepsilon(x)},\end{aligned}$$

hence

$$\lim_{x \to 0} \frac{f(x) - 2\sin x}{1 - \cos x} = -4.$$

Example 4.13 *Find every solution $x(t)$ of the differential equation*

$$\frac{d^x x}{dt^2} - x = 0,$$

which also satisfies

$$\frac{d}{dt}\left(\frac{x(t)}{t}\right) \to 0 \qquad \textit{for } t \to 0.$$

A. Differential equation of second order and of constant coefficients. Limit process.

D. First find the complete solution of the differential equation, and then apply the condition.

I. The complete solution of the differential equation is

$$x(t) = c_1 \cosh t + c_2 \sinh t, \qquad t \in \mathbb{R}, \quad c_1,\, c_2 \in \mathbb{R} \text{ arbitrære.}$$

Clearly, $\left|\dfrac{\cosh t}{t}\right| \to +\infty$ for $t \to 0$, so the only possibility of a solution is when

$$\frac{x(t)}{t} = c \cdot \frac{\sinh t}{t}.$$

When this equation is differentiated we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{x(t)}{t}\right) &= c \cdot \frac{t\,\cosh t - \sinh t}{t^2} \\ &= c \cdot \frac{t\left\{1 + \frac{1}{2}\,t^2 + t^2\varepsilon(t)\right\} - t - \frac{1}{6}\,t^3 + t^3\varepsilon(t)}{t^2} \\ &= c \cdot \frac{\frac{1}{3}\,t^3 + t^3\varepsilon(t)}{t^2} = c\left\{\frac{1}{3}\,t + t\varepsilon(t)\right\} \to 0 \qquad \text{for } t \to 0.\end{aligned}$$

Example 3.18 *In this example we shall derive a formula which approximates the length of a circular arc. This formula is due to Huygens (1629–1695). We shall use the figure and the notation given in the text. We shall assume that the angle* φ *satisfies* $\varphi \in \left[0, \frac{\pi}{2}\right]$.

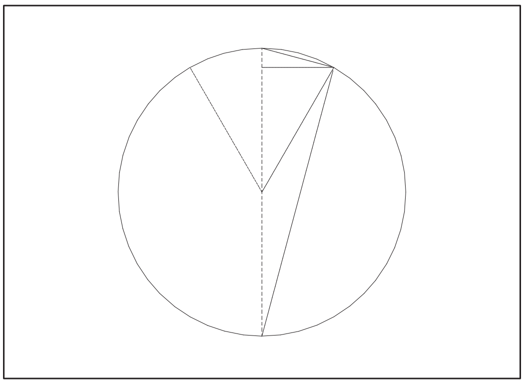

Figure 11: Circle of radius r, centre angle φ, thus periphery angle $\frac{\varphi}{2}$ below. The arc is denoted by ℓ, and the corresponding cord is denoted by d. Finally, we let s denote the height on the dotted vertical diagonal.

The approximating expression $\tilde{\ell}$ *of the length* ℓ *is given in the form*

$$\tilde{\ell} = ad + bs,$$

where a *and* b *are constants which will be found below.*

1) First prove that

$$\tilde{\ell} = 2ar\,\sin\frac{\varphi}{2} + br\,\sin\varphi.$$

2) We consider $\tilde{\ell}$ *as a function of* φ. *Find the approximating polynomial* $P_3(\varphi)$ *and the corresponding remainder term* $R_3(\varphi)$ *with the point of expansion* $\varphi = 0$.

3) Find the constants a *and* b, *such that* $P_3(\varphi) = \ell = 2r\varphi$, *and set up the corresponding approximation* $\tilde{\ell}$ *expressed by* d *and* s.

4) Prove that

$$|\ell - \tilde{\ell}| \le \frac{r}{180} \cdot \varphi^5.$$

A. An approximation with given guidelines.

D. Follow the guidelines.

I. 1) It follows from some simple geometric considerations (look at some rectangular triangles) that

$$d = 2r\,\sin\frac{\varphi}{2} \quad \text{and} \quad s = r\,\sin\varphi,$$

thus

$$\tilde{\ell} = ad + bs = 2ar\,\sin\frac{\varphi}{2} + br\,\sin\varphi.$$

Example 4.15 *Apply l'Hospital's rule to find the following limits*

$$(1)\quad \lim_{x\to 0}\frac{\ln(1+x)}{e^{2x}-1}, \qquad (2)\quad \lim_{x\to 0}\frac{1-\cos^2 x}{x\tan x}.$$

A. Limits found by an application of l'Hospital's rule.

D. Always start by checking the numerator and the denominator separately.

I. 1) Let $T(x)=\ln(1+x)$ and $N(x)=e^{2x}-1$. Then $T(0)=0$ and $N(0)=0$, thus

$$\lim_{x\to 0}\frac{\ln(1+x)}{e^{2x}-1}=\lim_{x\to 0}\frac{T'(x)}{N'(x)}=\lim_{x\to 0}\frac{\frac{1}{1+x}}{2e^{2x}}=\frac{1}{2}.$$

ALTERNATIVELY we get by using the Taylor expansions,

$$\frac{\ln(1+x)}{e^{2x}-1}=\frac{x+x\varepsilon(x)}{1+2x+x\varepsilon(x)-1}=\frac{1+\varepsilon(x)}{2+\varepsilon(x)}\to\frac{1}{2}\qquad \text{for } x\to 0.$$

2) We get by a reduction

$$\begin{aligned}\frac{1-\cos^2 x}{x\tan x} &= \frac{\sin^2 x}{x\cdot\frac{\sin x}{\cos x}}=\frac{\cos x\cdot\sin x}{x}\\ &= \cos x\cdot\frac{\sin x}{x}\to 1\cdot 1\qquad \text{for } x\to 0,\end{aligned}$$

because we know from the textbook that

$$\lim x\to 0\frac{\sin x}{x}=1.$$

ALTERNATIVELY we put $T(x)=1-\cos^2 x$ and $N(x)=x\cdot\tan x$. Then $T(0)=0$ and $N(0)=0$, hence

$$\lim_{x\to 0}\frac{1-\cos^2 x}{x\cdot\tan x}=\lim_{x\to 0}\frac{T'(x)}{N'(x)}=\lim_{x\to 0}\frac{2\sin x\cdot\cos x}{x(1+\tan^2 x)+\tan x}.$$

Now $T'(0)=0$ and $N'(0)=0$, so we proceed by

$$=\lim_{x\to 0}\frac{T''(x)}{N''(x)}=\lim_{x\to 0}\frac{2\cos^2 x-2\sin^2 x}{2(1+\tan^2 x)+x\cdot 2\tan x\cdot(1+\tan^2 x)}=\frac{2}{2}=1.$$

Example 4.16 *Calculate the integral*

$$I(\alpha)=\int_0^1 \frac{1}{\sqrt{1+\alpha x^2}}\,dx$$

for every $\alpha>0$*, and then find* $\lim_{\alpha\to\infty} I(\alpha)$.

A. Integral containing a parameter. Limit with respect to the parameter.

D. First find an indefinite integral.

I. An indefinite integral is e.g. given by

$$\begin{aligned}\int \frac{1}{\sqrt{1+\alpha x^2}}\,dx &= \frac{1}{\sqrt{\alpha}}\int_{y=\sqrt{\alpha}\,x} \frac{1}{\sqrt{1+y^2}}\,dy\\ &= \frac{1}{\sqrt{\alpha}}\,\mathrm{Arsinh}(\sqrt{\alpha}\cdot x)\\ &= \frac{1}{\sqrt{\alpha}}\ln\left(\sqrt{\alpha}\cdot x+\sqrt{1+\alpha x^2}\right).\end{aligned}$$

Then we get for the definite integral,

$$I(\alpha) = \int_0^1 \frac{1}{\sqrt{1+\alpha x^2}}\,dx = \frac{1}{\sqrt{\alpha}}\,\ln\left(\sqrt{\alpha}+\sqrt{1+\alpha}\right).$$

Now, a power function dominates a logarithm, so

$$\lim_{\alpha\to+\infty} I(\alpha) = \lim_{\alpha\to+\infty} \frac{\ln(\sqrt{\alpha}+\sqrt{1+\alpha})}{\sqrt{\alpha}} = 0.$$

Example 4.17 *1) Find the Taylor expansion of degree $n=2$ for the function*

$$f(x) = \sqrt{1+x+x^2}.$$

2) Find the Taylor expansion of degree $n=2$ for the function

$$f(x) = \sqrt{1+x+x^2} - 1 - \frac{1}{2}\,x.$$

3) Finally, find the limit

$$\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1-\frac{1}{2}\,x}{x^2}.$$

A. A limit found by means of Taylor expansions.

D. Find the Taylor expansions.

I. 1) We have

$$\begin{aligned} f(x) &= \sqrt{1+x+x^2} \\ &= 1 + \binom{\frac{1}{2}}{1}(x+x^2) + \binom{\frac{1}{2}}{2}(x+x^2)^2 + x^2\varepsilon(x) \\ &= 1 + \frac{1}{2}\left(x+x^2\right) - \frac{1}{8}\left(x^2+2x^3+x^4\right) + x^2\varepsilon(x) \\ &= 1 + \frac{1}{2}\,x + \frac{3}{8}\,x^2 + x^2\varepsilon(x). \end{aligned}$$

2) We now get from (1),

$$f(x) = \sqrt{1+x+x^2} - 1 - \frac{1}{2}\,x = \frac{3}{8}\,x^2 + x^2\varepsilon(x).$$

3) Finally, if follows from (2) that

$$\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1-\frac{1}{2}\,x}{x^2} = \lim_{x\to 0} \frac{\frac{3}{8}\,x^2 + x^2\varepsilon(x)}{x^2} = \frac{3}{8}.$$

Example 4.18 *Find the following limits*

$$(1)\quad \lim_{x\to 0}\frac{\ln(1+x)}{e^{2x}-1},\qquad (2)\quad \lim_{x\to 0}\frac{1-\cos^2 x}{x\cdot\tan x},$$

$$(3)\quad \lim_{x\to+\infty}\frac{\ln x}{e^{\sqrt{\ln x}}},\qquad (4)\quad \lim_{x\to+\infty}\left\{2x-x^2\ln\left[\left(1+\frac{1}{x}\right)^2\right]\right\}.$$

A. Limits.

D. We use a Taylor expansion in (1) and (4). In (2) and (3) other methods are easier to apply.

I. 1) We immediately get

$$\lim_{x\to 0}\frac{\ln(1+x)}{e^{2x}-1}=\lim_{x\to 0}\frac{x+x\varepsilon(x)}{1+2x+x\varepsilon(x)-1}=\lim_{x\to 0}\frac{x+x\varepsilon(x)}{2x+x\varepsilon(x)}=\frac{1}{2}.$$

2) By a small rearrangement we get

$$\lim_{x\to 0}\frac{1-\cos^2 x}{x\cdot\tan x}=\lim_{x\to 0}\frac{\sin^2 x}{x\cdot\dfrac{\sin x}{\cos x}}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\cos x=1.$$

3) If we put $y=\sqrt{\ln x}$, it follows that $y\to+\infty$ for $x\to+\infty$, and then it follows from the rules of magnitude that

$$\lim_{x\to+\infty}\frac{\ln x}{\exp(\sqrt{\ln x})}=\lim_{y\to+\infty}\frac{y^2}{e^y}=0.$$

4) Since $\dfrac{1}{x}\to 0$ for $x\to+\infty$, we get

$$\begin{aligned}
&\lim_{x\to+\infty}\left\{2x-x^2\ln\left[\left(1+\frac{1}{x}\right)^2\right]\right\}\\
&\quad=\lim_{x\to+\infty}\left\{2x-2x^2\ln\left(1+\frac{1}{x}\right)\right\}\\
&\quad=2\lim_{x\to+\infty}\left\{x-x^2\left(\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{x^2}+\frac{1}{x^2}\,\varepsilon\left(\frac{1}{x}\right)\right)\right\}\\
&\quad=\lim_{x\to+\infty}2\left\{x-x+\frac{1}{2}+\varepsilon(x)\right\}=1.
\end{aligned}$$

Example 4.19 *When one shall calculate the bending of a beam one often applies the so-called Berry functions $B_\beta(\lambda)$, which are defined by*

$$B_\beta(\lambda) = \frac{6\{\sqrt{\lambda}\,\cosh(\beta\sqrt{\lambda}) - \sinh(\sqrt{\lambda})\}}{\lambda\,\sinh(\sqrt{\lambda})}, \qquad \textit{for } \lambda > 0.$$

Here, β is a fixed real constant. We shall find a value of the function $B_\beta(\lambda)$ for $\lambda = 0$, such that the function becomes continuous at $\lambda = 0$.

Which value should one choose?

A. A limit process in λ.

D. Use Taylor expansions.

I. From

$$\cosh x = 1 + \frac{1}{2}\,x^2 + x^2\varepsilon(x) \quad \text{and} \quad \sinh x = x + \frac{1}{6}\,x^3 + x^3\varepsilon(x),$$

we get by insertion that

$$\begin{aligned}
B_\beta(\lambda) &= 6 \cdot \frac{\sqrt{\lambda}\cdot\cosh(\beta\sqrt{\lambda}) - \sinh(\sqrt{\lambda})}{\lambda\cdot\sinh(\sqrt{\lambda})} \\
&= 6 \cdot \frac{\sqrt{\lambda}\left\{1 + \beta^2\lambda + \lambda\varepsilon(\lambda)\right\} - \sqrt{\lambda} - \frac{1}{6}\,\lambda\sqrt{\lambda} + \lambda\sqrt{\lambda}\varepsilon(\lambda)}{\lambda\{\sqrt{\lambda} + \lambda\varepsilon(\lambda)\}} \\
&= 6 \cdot \frac{1 + \beta^2\lambda + \lambda\varepsilon(\lambda) - 1 - \frac{1}{6}\,\lambda + \lambda\varepsilon(\lambda)}{\lambda\{1 + \varepsilon(\lambda)\}} \\
&= 6 \cdot \left(\beta^2 - \frac{1}{6}\right) \cdot \frac{1 + \varepsilon(\lambda)}{1 + \varepsilon(\lambda)} \to 6\beta^2 - 1 \qquad \text{for } \lambda \to 0+\,.
\end{aligned}$$

Hence we shall put

$$B_\beta(0) = 6\beta^2 - 1.$$

Example 4.20 *(Cf. Example 4.21). Let $\varphi(t) = t - te^{-t^2}$, $t \in \mathbb{R}$.*

1) Find the Taylor expansion of order 3 (i.e. the approximating polynomial of at most third degree) for the function $\varphi(t)$ with the point of expansion $t_0 = 0$.

2) Find the limit

$$\lim_{t \to \infty} \frac{\varphi(t)}{\sinh t - \sin t \, \cos t}.$$

A. Approximating polynomial and limit.

D. We shall give two solutions of the first bullet.

I. Let $\varphi(t) = t - te^{-t^2}$. Then $\varphi(t)$ is of class C^∞. We consider the two variants:

First variant. From $e^u = 1 + u + u\,\varepsilon(u)$ for $u \to 0$, we get by the substitution $u = -t^2$ that

$$\varphi(t) = t\left\{1 - e^{-t^2}\right\} = t\left\{1 - \left(1 - t^2 + t^2\,\varepsilon(t)\right)\right\} = t^3 + t^3\,\varepsilon(t),$$

thus $P_3(t) = t^3$.

Second variant. By successive differentiations we get with the point of expansion $t_0 = 0$,

$$\begin{array}{rclrcl}
\varphi(t) & = & t - te^{-t^2}, & \varphi(0) & = & 0, \\
\varphi'(t) & = & 1 - e^{-t^2} + 2t^2e^{-t^2}, & \varphi'(0) & = & 0, \\
\varphi''(t) & = & 6\,t\,e^{-t^2} - 4\,t^3\,e^{-t^2}, & \varphi''(t) & = & 0, \\
\varphi^{(3)}(t) & = & 6\,e^{-t^2} - 24\,t^2\,e^{-t^2} + 8\,t^4\,e^{-t^2}, & \varphi^{(3)}(0) & = & 6.
\end{array}$$

Hence,

$$P_3(t) = \varphi(0) + \varphi'(0)\,t + \frac{\varphi''(t)}{2!}\,t^2 + \frac{\varphi^{(3)}(0)}{3!}\,t^3 = \frac{6}{3!}\,t^3 = t^3.$$

This bullet can be solved in at least four ways:

First variant. Using a simple rearrangement we get by some ε-functions that

$$\begin{aligned}
\sinh t - \sin t \cdot \cos t &= \sinh t - \frac{1}{2}\,\sin 2t \\
&= \left\{t + \frac{1}{3!}\,t^3 + t^3\,\varepsilon(t)\right\} - \frac{1}{2}\left\{2t - \frac{1}{3!}\,(2t)^3 + t^3\,\varepsilon(t)\right\} \\
&= \frac{1+4}{6}\,t^3 + t^3\,\varepsilon(t) = \frac{5}{6}\,t^3 + t^3\,\varepsilon(t).
\end{aligned}$$

Then apply the result from (1),

$$\frac{\varphi(t)}{\sinh t - \sin t \cdot \cos t} = \frac{t^3 + t^3\,\varepsilon(t)}{\frac{5}{6}\,t^3 + t^3\,\varepsilon(t)} = \frac{1+\varepsilon(t)}{\frac{5}{6} + \varepsilon(t)} \to \frac{6}{5} \qquad \text{for } t \to 0.$$

Second variant. Even if one does not use the trick of applying the result of the First variant, it is still possible to solve the problem by using ε-functions:

$$\begin{aligned}
&\sinh t - \sin t \cdot \cos t \\
&= \left\{t + \frac{1}{3!}\,t^3 + t^3\,\varepsilon(t)\right\} - \left\{t - \frac{1}{3!}\,t^3 + t^3\,\varepsilon(t)\right\}\left\{1 - \frac{1}{2!}\,t^2 + t^3\,\varepsilon(t)\right\} \\
&= t + \frac{1}{6}\,t^3 + t^3\,\varepsilon(t) - t + \frac{1}{2}\,t^3 + t^3\,\varepsilon(t) + \frac{1}{6}\,t^3 + t^3\varepsilon(t) + t^3\varepsilon(t) \\
&= \frac{5}{6}\,t^3 + t^3\,\varepsilon(t),
\end{aligned}$$

and then continue as in the **First variant**.

3. variant. The approximating polynomial by the method of differentiation. If we put

$$\psi(t) = \sinh t - \sin t \cdot \cos t,$$

then

$$\begin{array}{rclrcl}
\psi(t) & = & \sinh t - \sin t \cdot \cos t, & \psi(0) & = & 0, \\
\psi'(t) & = & \cosh t - \cos^2 t + \sin^2 t, & \psi'(0) & = & 0, \\
\psi''(t) & = & \sinh t + 4 \sin t \cdot \cos t, & \psi''(0) & = & 0, \\
\psi^{(3)}(t) & = & \cosh t + 4\cos^2 t - 4\sin^2 t, & \psi^{(3)}(0) & = & 5,
\end{array}$$

hence

$$\psi = \psi(0) + \psi'(0)\,t + \frac{\psi''(0)}{2!}\,t^2 + \frac{\psi^{(3)}(0)}{3!}\,t^3 + t^3\,\varepsilon(t) = \frac{5}{6}\,t^3 + t^3\,\varepsilon(t),$$

and then proceed like in the **First variant**.

4. variant. Application of l'Hospital's rule where we forget that some of the calculations already have been made in (1):

$$\begin{array}{ll} T(t)=\varphi(t)=t-t\,e^{-t^2}, & T(0)=0,\\ N(t)=\sinh t-\sin t\cdot\cos t, & N(0)=0.\end{array}$$

$$\begin{array}{ll} T'(t)=1-e^{-t^2}+2t^2\,e^{-t^2}, & T'(0)=0,\\ N'(t)=\cosh t-\cos^2 t+\sin^2 t, & N'(0)=0,\end{array}$$

$$\begin{array}{ll} T''(t)=6\,t\,e^{-t^2}-4\,t^3\,e^{-t^2}, & T''(0)=0,\\ N''(t)=\sinh t+4\,\sin t\cdot\cos t, & N''(0)=0,\end{array}$$

$$\begin{array}{ll} T^{(3)}(t)=6\,e^{-t^2}-24\,t^2\,e^{-t^2}+8\,t^4\,e^{-t^2}, & T^{(3)}(0)=6,\\ N^{(3)}(t)=\cosh t+4\cos^2 t-4\sin^2 t, & N^{(3)}(0)=5.\end{array}$$

From

$$T(0)=T'(0)=T''(0)=0, \qquad N(0)=N'(0)=N''(0)=0,$$

we conclude by successively applying l'Hospital's rule that

$$\begin{aligned}\lim_{t\to 0}\frac{\varphi(t)}{\sinh t-\sin t\cdot\cos t} &= \lim_{t\to 0}\frac{T(t)}{N(t)}=\lim_{t\to 0}\frac{T'(t)}{N'(t)}=\lim_{t\to 0}\frac{T''(t)}{N''(t)}\\ &= \lim_{t\to 0}\frac{T^{(3)}(t)}{N^{(3)}(t)}=\frac{T^{(3)}(t)}{N^{(3)}(0)}=\frac{6}{5}.\end{aligned}$$

Example 4.21 *(Cf. Example 4.20). Let the function $\varphi(x)$ be given by*

$$\varphi(x)=x-x\,e^{-x^2}, \qquad x\in\mathbb{R}.$$

1) Find the Taylor polynomial of order 3 for $\varphi(x)$ with the expansion point $x_0=0$.

2) Let $\psi(x)$ denote the indefinite integral of $\varphi(x)$, for which $\psi(0)=0$. Find by applying the result of (1) the Taylor polynomial of order 4 for $\psi(x)$ with the point of expansion $x_0=0$, and then find the limit

$$\lim_{x\to 0}\frac{\cos x-\exp\left(-\dfrac{1}{2}\,x^2\right)}{\psi(x)}.$$

A. The first bullet is the same as the first bullet of Example 4.20. The purpose of the present example is partly to find the Taylor expansion, and partly to apply this in a limit process-

D. 1) Use the exponential series.

2) Integrate this series and find the Taylor expansion of

$$\cos x-\exp\left(-\frac{1}{2}\,x^2\right).$$

Finally, go to the limit.

I. 1) From

$$\varphi(x)=x-x\,e^{-x^2}=x-x\left\{1-\frac{1}{1!}\,x^2+\frac{1}{2!}\,x^4+\cdots\right\}=x^3-\frac{1}{2}\,x^5+\cdots\,,$$

we get the Taylor polynomial

$$P_3(x)=x^3.$$

2) Clearly, $\psi(x)=\dfrac{1}{4}\,x^4+x^4\,\varepsilon(x)$. Since

$$\begin{aligned}&\cos x-\exp\left(-\frac{1}{2}\,x^2\right)\\&=1-\frac{1}{2!}\,x^2+\frac{1}{4!}\,x^4+\cdots-\left\{1-\frac{1}{1!}\,\frac{x^2}{2}+\frac{1}{2!}\left(\frac{x^2}{2}\right)^2+\cdots\right\}\\&=\frac{1}{24}\,x^4-\frac{1}{8}\,x^4+\cdots=-\frac{1}{12}\,x^4+\cdots\,,\end{aligned}$$

we get by insertion,

$$\lim_{x\to 0}\frac{\cos x-\exp\left(-\dfrac{1}{2}\,x^2\right)}{\psi(x)}=\lim_{x\to 0}\frac{-\dfrac{1}{12}\,x^4+x^4\,\varepsilon(x)}{\dfrac{1}{4}\,x^4+x^4\,\varepsilon(x)}=\frac{-\dfrac{1}{12}}{\dfrac{1}{4}}=-\frac{1}{3}.$$

Example 4.22 *Find the limit*

$$\lim_{x\to 0}\frac{\sin x^2-2\sqrt{1+x^2}+2}{x^4}.$$

A. Limit process.

D. Find the Taylor expansion of fourth order for the numerator.

I. From the expansions

$$\sin\left(x^2\right)=\frac{1}{1!}\,x^2-\frac{1}{3!}\,\left(x^2\right)^3+\cdots=x^2+x^4\,\varepsilon(x),$$

and

$$\begin{aligned}\sqrt{1+x^2}&=\left(1+x^2\right)^{\frac{1}{2}}=1+\binom{\frac{1}{2}}{1}x^2+\binom{\frac{1}{2}}{2}\left(x^2\right)^2+x^4\,\varepsilon(x)\\&=1+\frac{1}{2}\,x^2-\frac{1}{8}\,x^4+x^4\,\varepsilon(x),\end{aligned}$$

we get by insertion,

$$\lim_{x\to 0}\frac{\sin\left(x^2\right)-2\sqrt{1+x^2}+2}{x^4}=\lim_{x\to 0}\frac{x^2-2-x^2+\dfrac{1}{4}\,x^4+2+x^4\varepsilon(x)}{x^4}=\lim_{x\to 0}\frac{\dfrac{1}{4}\,x^4+x^4\,\varepsilon(x)}{x^4}=\frac{1}{4}.$$

Remark. The expression can also be considered as a function in $u = x^2 \to 0+$ for $x \to 0$. One may therefore instead change the variable to

$$\lim_{x\to 0} \frac{\sin\left(x^2\right) - 2\sqrt{1+x^2}+2}{x^4} = \lim_{u\to 0+} \frac{\sin u - 2\sqrt{1+u}+2}{u^2}$$

$$= \lim_{u\to 0+} \frac{u+u^2\,\varepsilon(u) - 2\left\{1+\frac{1}{2}\,u - \frac{1}{8}\,u^2+u^2\,\varepsilon(u)\right\}-2}{u^2} = \frac{1}{4}. \qquad \Diamond$$

Example 4.23 *(Cf. Example 2.14). Given the function*

$$f(x)=\cos\left(\frac{1}{2}\,x^2+x\right),\qquad x\in\mathbb{R}.$$

1) Find the Taylor polynomial $P_2(x)$ with the point of expansion $x_0=0$ for $f(x)$.

2) Prove that

$$|f(x)-P-2(x)|<8\cdot 10^{-3}\qquad \textit{for } |x|<\frac{1}{5}.$$

3) Find the limit

$$\lim_{x\to 0}\frac{\cos\left(\frac{1}{2}\,x^2+x\right)-1+\frac{1}{2}\,x^2}{x^3}.$$

A. Taylor expansion, error estimate, limit.

D. 1) Use one of the standard methods by the calculation of the Taylor expansion.

2) Estimate the remainder term of the Taylor expansion.

3) Use (1) and (2) in the limit process.

I. 1) The simplest method is to use the series expansion. (An alternative solution is given in the next bullet.) We find

$$\begin{aligned}f(x)&=1-\frac{1}{2!}\left(\frac{1}{2}\,x^2+x\right)^2+\frac{1}{4!}\left(\frac{1}{2}\,x^2+x\right)^4+\cdots\\&=1-\frac{1}{2}\left\{\frac{1}{4}\,x^4+x^3+x^2\right\}+\frac{1}{24}\,x^4\left\{1+\frac{1}{2}\,x\right\}^4+\cdots\\&=1-\frac{1}{2}\,x^2-\frac{1}{2}\,x^3+x^3\varepsilon(x).\end{aligned}$$

Hence,

$$P_2(x)=1-\frac{1}{2}\,x^2\quad\text{og}\quad P_3(x)=1-\frac{1}{2}\,x^2-\frac{1}{2}\,x^3,$$

where we shall use $P_3(x)$ in (3).

2) By successive differentiations we get

$$\begin{aligned}f'(x)&=-(x+1)\sin\left(\frac{1}{2}\,x^2+x\right),\\f''(x)&=-(x+1)^2\cos\left(\frac{1}{2}\,x^2+x\right)-\sin\left(\frac{1}{2}\,x^2+x\right),\\f^{(3)}(x)&=(x+1)^3\sin\left(\frac{1}{2}\,x^2+x\right)-3(x+1)\cos\left(\frac{1}{2}\,x^2+x\right).\end{aligned}$$

Thus again

$$P_2(x)=f(0)+f'(0)\cdot x+\frac{1}{2!}\,f''(0)\cdot x^2=1+0-\frac{1}{2}\,x^2=1-\frac{1}{2}\,x^2.$$

By Taylor's formula there exists a ξ somewhere between 0 and x, such that

$$f(x) = P_2(x) = \frac{1}{3!}\, f^{(3)}(\xi) \cdot x^3.$$

Then we have the estimate

$$\begin{aligned}|f(x) - P_2(x)| &= \frac{1}{6}\left|f^{(3)}(\xi)\right| \cdot |x|^3 \\ &= \frac{|x|^3}{6}\left|(\xi+1)^3 \sin\left(\frac{1}{2}\,\xi^2 + \xi\right) - 3(\xi+1)\cos\left(\frac{1}{2}\,\xi^2+\xi\right)\right|.\end{aligned}$$

Now,

$$\left|\frac{1}{2}\,x^2 + x\right| < \frac{1}{50} + \frac{1}{5} = \frac{11}{50} \qquad \text{for } |x| < \frac{1}{5},$$

and $|\xi| \le |x| < \dfrac{1}{5}$, so we get the estimate

$$\begin{aligned}|f(x) - P_2(x)| &< \frac{1}{6}\cdot\frac{1}{5^3}\left\{\left(\frac{1}{5}+1\right)^3 \sin\frac{11}{50} + 3\left(\frac{1}{5}+1\right)\cos 0\right\} \\ &< \frac{8}{1000}\cdot\frac{1}{6}\left\{\frac{6^3}{5^3}\cdot\frac{11}{50} + 3\cdot\frac{6}{5}\cdot 1\right\} \\ &= \frac{8}{1000}\left\{\frac{6^2\cdot 11}{125\cdot 50} + \frac{3}{5}\right\} < 8\cdot 10^{-3}\cdot\frac{2}{3} \\ &< 8\cdot 10^{-3}.\end{aligned}$$

3) From

$$\cos\left(\frac{1}{2}\,x^2 + x\right) = P_3(x) + x^3\varepsilon(x),$$

we get

$$\lim_{x\to 0}\frac{\cos\left(\frac{1}{2}\,x^2+x\right) - 1 + \frac{1}{2}\,x^2}{x^3} = \lim_{x\to 0}\frac{1 - \frac{1}{2}\,x^2 - \frac{1}{2}\,x^3 + x^3\,\varepsilon(x) - 1 + \frac{1}{2}\,x^2}{x^3}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{2}\,x^3 + x^3\,\varepsilon(x)}{x^3} = \lim_{x\to 0}\left\{-\frac{1}{2} + \varepsilon(x)\right\} = -\frac{1}{2}.$$

Example 4.24 *Find the limit*

$$\lim_{x\to 0}\frac{\ln\cosh x-\frac{1}{2}\,x^2}{x^4}.$$

A. Limit by an application of a Taylor expansion.

D. Set up Taylor's formula for $n=4$ and point of expansion $x_0=0$, and then insert.

I. When $f(x)=\ln\cosh x$, we have

$$f'(x)=\frac{\sin x}{\cosh x}=\tanh x,\qquad f''(x)=1-\tanh^2 x,$$

$$f^{(3)}(x)=-2\tanh x(1-\tanh^2 x)=2\tanh^3 x-2\tanh x,$$

$$f^{(4)}(x)=(6\tanh^2 x-2)(1-\tanh^2 x),$$

thus

$$\begin{aligned}\ln\cosh x &= f(0)+\frac{1}{1!}\,f'(0)\,x+\frac{1}{2!}\,f''(0)\,x^2+\frac{1}{3!}\,f^{(3)}(0)\,x^3\\ &\qquad +\frac{1}{4!}\,f^{(4)}(0)\,x^4+x^4\,\varepsilon(x)\\ &= 0+0+\frac{1}{2!}\cdot 1\cdot x^2+\frac{1}{3!}\cdot 0\cdot x^3+\frac{1}{4!}\,(-2)x^4+x^4\varepsilon(x)\\ &= \frac{1}{2}\,x^2-\frac{1}{12}\,x^4+x^4\varepsilon(x),\end{aligned}$$

hence by insertion,

$$\lim_{x\to 0}\frac{\ln\cosh x-\frac{1}{2}\,x^2}{x^4}=\lim_{x\to 0}\frac{-\frac{1}{12}\,x^4+x^4\varepsilon(x)}{x^4}=-\frac{1}{12}.$$

Example 4.25 *1) Set up Taylor's formula for $x_0=0$ and $n=3$ for the function*

$$f(x)=\ln\cos x,\qquad -\frac{\pi}{2}<x<\frac{\pi}{2}.$$

2) Find the limit

$$\lim_{x\to 0}\frac{\ln\cos x+\frac{1}{2}\,x^2}{x^2}.$$

A. Taylor's formula and a limit.

D. Differentiate three times.

I. Clearly, $f(x)$ is defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, because $\cos x > 0$ in this interval. Then by differentiation,

$$\begin{aligned} f(x) &= \ln\cos x, \\ f'(x) &= -\frac{\sin x}{\cos x} = -\tan x, \\ f''(x) &= -\tan^2 x - 1, \\ f^{(3)}(x) &= -2\tan x(1+\tan^2 x) = -\frac{2\sin x}{\cos^3 x}. \end{aligned}$$

If $x_0 = 0$, then

$$\begin{aligned} \ln\cos x &= f(0) + \frac{1}{1!}\,f'(0)\,x + \frac{1}{2!}\,f''(0)\,x^2 + \frac{1}{3!}\,f^{(3)}(\xi)\,x^3 \\ &= 0 + 0 + \frac{1}{2!}\,(-1)\,x^2 + \frac{1}{3!}\,(-2)\cdot\frac{\sin\xi}{\cos^3\xi}\,x^3 \\ &= -\frac{1}{2}\,x^2 - \frac{1}{3}\,\frac{\sin\xi}{\cos^3\xi}\,x^3. \end{aligned}$$

Finally we get for the limit process,

$$\lim_{x\to 0}\frac{\ln\cos x + \frac{1}{2}\,x^2}{x^2} = \lim_{x\to 0}\frac{-\frac{1}{2}\,x^2 + x^2\varepsilon(x) + \frac{1}{2}\,x^2}{x^2} = 0.$$

Example 4.26 *1) Find the Taylor expansion of order 4 at the point $x_0 = 0$ for the function*

$$f(x) = \sqrt{1+2x^2}, \qquad x \in \mathbb{R}.$$

2) Find

$$\lim_{x\to 0} \frac{\sqrt{1+2x^2}-(1+x^2)}{x^4}.$$

A. Taylor expansion and a limit.

D. Put $u = x^2$, and consider instead the Taylor expansion of $\sqrt{1+u}$ of second order.

I. 1) From

$$\begin{aligned}\sqrt{1+u} &= 1 + \binom{\frac{1}{2}}{1} u + \binom{\frac{1}{2}}{2} u^2 + \cdots \\ &= 1 + \frac{1}{2}\,u - \frac{1}{8}\,u^2 + \cdots,\end{aligned}$$

we get by the substitution $u = 2x^2$,

$$\begin{aligned}f(x) &= \sqrt{1+2x^2} = 1 + \frac{1}{2}\cdot 2x^2 - \frac{1}{8}\cdot\cdot\left(2x^2\right)^2 + \cdots \\ &= 1 + x^2 - \frac{1}{2}\,x^4 + \cdots,\end{aligned}$$

hence,

$$P_4(x) = 1 + x^2 - \frac{1}{2}\,x^4 \quad \text{og} \quad \sqrt{1+2x^2} = P_4(x) + x^4\,\varepsilon(x).$$

2) When we insert the result above we get

$$\frac{\sqrt{1+2x^2}-(1+x^2)}{x^4} = \frac{1+x^2-\frac{1}{2}\,x^4+x^4\varepsilon(x)-(1+x^2)}{x^4} = -\frac{1}{2}+\varepsilon(x),$$

and we conclude that

$$\lim_{x\to 0} \frac{\sqrt{1+2x^2}-(1+x^2)}{x^4} = -\frac{1}{2}.$$

Example 4.27 *1) Find the Taylor expansion of order 2 at the point $x_0 = 0$ for the function*

$$f(x) = \sqrt{1 + \frac{1}{2}\sin x}, \qquad x \in \mathbb{R}.$$

2) Then find the limit (e.g. by applying (1))

$$\lim_{x \to 0} \frac{\sqrt{1 + \frac{1}{2}\sin x} - \exp\left(\frac{x}{4}\right)}{\sin^2 x}.$$

A. Taylor expansion and a limit.

D. Even if I can find an alternative method of solution, I shall only apply the method of finding the coefficients by calculating the first two derivatives of $f(x)$.

I. 1) We have

$$\begin{array}{lclclcl}
f(x) & = & \sqrt{1 + \frac{1}{2}\sin x}, & & f(0) & = & 1, \\
f'(x) & = & \frac{1}{4}\,\frac{\cos x}{\sqrt{1 + \frac{1}{2}\sin x}}, & & f'(0) & = & \frac{1}{4}, \\
f''(x) & = & \sin x \cdot \{\cdots\} & & & & \\
 & & + \left(\frac{\cos x}{4}\right)^2 \left(-\frac{1}{2}\right) \frac{1}{\left(\sqrt{1 + \frac{1}{2}\sin x}\right)^3}, & & f''(0) & = & -\frac{1}{32},
\end{array}$$

thus

$$P_2(x) = 1 + \frac{1}{4}\,x - \frac{1}{64}\,x^2.$$

2) Since

$$\exp\left(\frac{x}{4}\right) = 1 + \frac{x}{4} + \frac{x^2}{32} + \cdots,$$

and

$$\sin^2 x = x^2 + \cdots,$$

we have

$$\begin{aligned}
\frac{\sqrt{1 + \frac{1}{2}\sin x} - \exp\left(\frac{x}{4}\right)}{\sin^2 x} &= \frac{1 + \frac{x}{4} - \frac{1}{64}\,x^2 - 1 - \frac{x}{4} - \frac{1}{32}\,x^2 + x^2\varepsilon(x)}{x^2 + x^2\varepsilon(x)} \\
&= -\left(\frac{1}{64} + \frac{1}{32}\right) + \varepsilon(x),
\end{aligned}$$

hence

$$\lim_{x \to 0} \frac{\sqrt{1 + \frac{1}{2}\sin x} - \exp\left(\frac{x}{4}\right)}{\sin^2 x} = -\frac{3}{64}.$$

Example 4.28 *1) Find the Taylor polynomial of order 3 at the point $x_0 = 0$ for the function*

$$f(x) = -\ln\cos x, \qquad x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

2) Find the limit

$$\lim_{x\to 0} \frac{\cos x - \ln\cos x - 1}{x^3}.$$

A. Taylor polynomial and a limit.

Either insert known Taylor series, or differentiate. Clearly, (1) must be applied in (2).

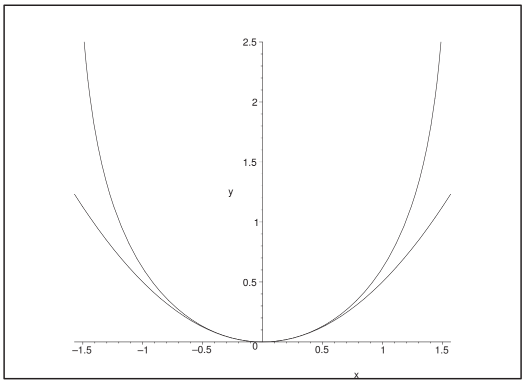

Figure 15: The graphs of $y = -\ln\cos x$ (above) and its approximating polynomial $P_2(x) = \dfrac{1}{2}\,x^2$.

I. 1) **First variant.** We see that

$$\begin{aligned} f(x) &= -\ln\cos x = -\ln\{1-(1-\cos x)\} \\ &= 1-\cos x + \cdots = \frac{1}{2}\,x^2 + \cdots, \end{aligned}$$

where the dots indicate terms of degree ≥ 4, thus

$$P_3(x) = \frac{1}{2}\,x^2.$$

Second variant. From

$$\begin{array}{lcllcl} f(x) &=& -\ln\cos x, & f(0) &=& 0, \\ f'(x) &=& \tan x, & f'(0) &=& 0, \\ f''(x) &=& 1+\tan^2 x, & f''(0) &=& 1, \\ f^{(3)}(x) &=& 2\tan x(1+\tan^2 x), & f^{(3)}(0) &=& 0, \end{array}$$

we get

$$P_3(x) = \frac{1}{2}\,x^2.$$

2) **First variant.** If we use the setup of (1) First variant, we get

$$\begin{aligned}\frac{\cos x-\ln\cos x-1}{x^3} &= \frac{1-\cos x+x^3\varepsilon(x)-(1-\cos x)}{x^3}\\ &= \varepsilon(x)\to 0 \qquad \text{for } x\to 0,\end{aligned}$$

thus

$$\lim_{x\to 0}\frac{\cos x-\ln\cos x-1}{x^3}=0.$$

Second variant. A more traditional procedure is the following,

$$\lim_{x\to 0}\frac{\cos x-\ln\cos x-1}{x^3}=\lim_{x\to 0}\frac{1-\frac{1}{2}\,x^2+\frac{1}{2}\,x^2-1+x^3\varepsilon(x)}{x^3}=\lim_{x\to 0}\varepsilon(x)=0.$$

Example 4.29 *1) Find the Taylor polynomial $P_3(x)$ of order 2 at the point $x_0 = 0$ for the function*

$$f(x) = \ln(1-x) + e^x, \qquad x < 1.$$

2) Prove that if $|x| < \frac{1}{4}$, then

$$|R_3(x)| = |f(x) - P_3(x)| < \frac{1}{250}.$$

3) Find the limit

$$\lim_{x \to 0} \frac{f(x) - 1}{x^3}.$$

A. Taylor polynomial, estimate of a remainder term and a limit process.

D. Either insert known series expansions, or differentiate.

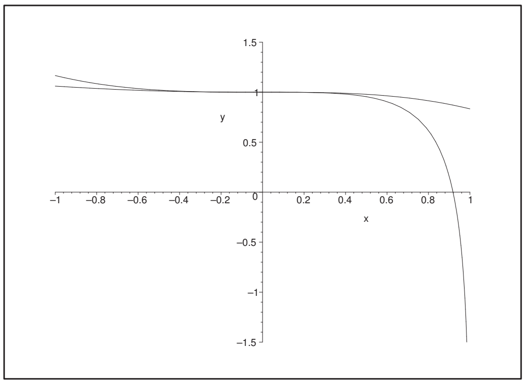

Figure 16: The graphs of $y = \ln(1-x) + e^x$ and its approximating polynomial $P_3(x)$ from $x_0 = 0$.

I. 1) **First variant.** By using well-known Taylor series we get for $x \in]-1,1[$,

$$\begin{aligned} f(x) &= \ln(1-x) + e^x \\ &= -x - \frac{1}{2}\,x^2 - \frac{1}{3}\,x^3 - \frac{1}{4}\,x^4 + \cdots \\ &\qquad +1 + x + \frac{1}{2}\,x^2 + \frac{1}{6}\,x^3 + \frac{1}{24}\,x^4 + \cdots \\ &= 1 - \frac{1}{6}\,x^3 - \frac{5}{24}\,x^4 + \cdots, \end{aligned}$$

hence,

$$P_3(x) = 1 - \frac{1}{6}\,x^3.$$

Second variant. It follows from

$$
\begin{array}{rclrcr}
f(x) & = & \ln(1-x)+e^x, & f(0) & = & 1,\\
f'(x) & = & -\dfrac{1}{1-x}+e^x, & f'(0) & = & 0,\\
f''(x) & = & -\dfrac{1}{(1-x)^2}+e^x, & f''(0) & = & 0,\\
f^{(3)}(x) & = & -\dfrac{2}{(1-x)^3}+e^x, & f^{(3)}(0) & = & -1,\\
f^{(4)}(x) & = & -\dfrac{6}{(1-x)^4}+e^x, & & &
\end{array}
$$

that

$$P_3(x)=1-\frac{1}{3!}\,x^3=1-\frac{1}{6}\,x^3.$$

2) If $|x|<\dfrac{1}{4}$, then

$$
\begin{aligned}
|f(x)-P_3(x)| &= |R_3(x)|<\frac{1}{4!}\cdot\left(\frac{1}{4}\right)^4\cdot\sup_{|x|\le\frac{1}{4}}\left|\frac{-6}{(1-x)^4}+e^x\right|\\
&\le \frac{1}{24}\cdot\frac{1}{4^4}\cdot\left\{\frac{6}{\left(\dfrac{3}{4}\right)^4}+\sqrt[4]{e}\right\}<\frac{1}{24}\cdot\left\{\frac{2}{27}+\frac{2}{4^4}\right\}\\
&\approx 0,00341<0,004=\frac{1}{250}.
\end{aligned}
$$

3) Finally,

$$
\begin{aligned}
\lim_{x\to 0+}\frac{f(x)-1}{x^3} &= \lim_{x\to 0+}\frac{-\dfrac{1}{6}\,x^3+x^3\varepsilon(x)}{x^3}\\
&= -\lim_{x\to 0+}\left\{-\frac{1}{6}+\varepsilon(x)\right\}=-\frac{1}{6}.
\end{aligned}
$$

Example 4.30 *Find*

$$\lim_{x\to 0}\frac{x\sin x-\ln(1+x^2)}{2\cosh x-2-x^2}.$$

(Hint: Use Taylor's formula.)

A. Limit process.

D. Start by expanding the denominator in order to find the order of the expansion.

I. The denominator has the expansion

$$
\begin{aligned}
2\cosh x-2-x^2 &= 2\left\{1+\frac{1}{2}\,x^2+\frac{1}{24}\,x^4+x^4\varepsilon(x)\right\}-2-x^2\\
&= \frac{1}{12}\,x^4+x^4\varepsilon(x)=x^4\left\{\frac{1}{12}+\varepsilon(x)\right\}.
\end{aligned}
$$

This shows that the order of expansion is 4. Then we expand the numerator up to order 4,

$$\begin{aligned}
x\,\sin x-\ln\left(1+x^2\right) &= x\left\{x-\frac{1}{6}\,x^3+x^3\varepsilon(x)\right\}-\left\{x^2-\frac{1}{2}\,x^4+x^4\varepsilon(x)\right\}\\
&= x^2-\frac{1}{6}\,x^4-x^2+\frac{1}{2}\,x^4+x^4\varepsilon(x)\\
&= \frac{1}{3}\,x^4+x^4\varepsilon(x).
\end{aligned}$$

Finally, by insertion,

$$\lim_{x\to 0}\frac{x\sin x-\ln(1+x^2)}{2\cosh x-2-x^2}=\lim_{x\to 0}\frac{x^4\left\{\dfrac{1}{12}+\varepsilon(x)\right\}}{x^4\left\{\dfrac{1}{3}+\varepsilon(x)\right\}}=\frac{1}{4}.$$

Example 4.31 *Find*

$$\lim_{x\to 0}\left(\frac{1-e^{3x^2}}{x\cdot\sin(2x)}\right).$$

A. A limit process.

D. The denominator has a zero of order 2 at $x_0=0$. Therefore, we shall find the Taylor expansions of the numerator and the denominator of order 2.

Alternatively and more troublesome we can also apply l'Hospital's rule.

I. First variant. From

$$1-e^{3x^2}=1-\left\{1+3x^2+x^2\varepsilon(x)\right\}=-3x^2+x^2\varepsilon(x),$$

and

$$x\cdot\sin(2x)=x\cdot\{2x+x\cdot\varepsilon(x)\}=2x^2+x^2\varepsilon(x),$$

we get

$$\lim_{x\to 0}\left(\frac{1-e^{3x^2}}{x\cdot\sin(2x)}\right)=\lim_{x\to 0}\frac{-3x^2+x^2\varepsilon(x)}{2x^2+x^2\varepsilon(x)}=\lim_{x\to 0}\frac{-3+\varepsilon(x)}{2+\varepsilon(x)}=-\frac{3}{2}.$$

Second variant. If we put

$$T(x)=1-e^{3x^2}\qquad\text{and}\qquad N(x)=x\cdot\sin(2x),$$

then both $T(x)$ and $N(x)$ are of class C^∞ and $T(0)=0$ and $N(0)=0$. Then by a couple of differentiations

$$\begin{aligned}
&T'(x)=-6x\cdot e^{3x^2}, && T'(0)=0,\\
&N'(x)=\sin(2x)+2x\cdot\cos(2x), && N'(0)=0,
\end{aligned}$$

og

$$\begin{aligned}
&T''(x)=-6\cdot e^{3x^2}-36x^2, && T''(0)=-6,\\
&N''(x)=4\cos(2x)-4x^2\sin(2x), && N''(0)=4.
\end{aligned}$$

By repeating l'Hospital's rule we get

$$\lim_{x \to 0} \frac{1 - e^{3x^2}}{x \cdot \sin(2x)} = \lim_{x \to 0} \frac{T(x)}{N(x)} = \lim_{x \to 0} \frac{T'(x)}{N'(x)} = \lim_{x \to 0} \frac{T''(x)}{N''(x)} = \frac{T''(0)}{N''(0)} = \frac{-6}{4} = -\frac{3}{2}.$$

5 Asymptotes

Example 5.1 *Find the possible asymptotes for the functions*

$$(1)\ \ f(x)=x^2\,\frac{1-e^x}{1+e^x}, \qquad (2)\ \ f(x)=\ln\left(e^{2x}+e^{-2x}\right).$$

A. Asymptotes.

D. The functions are defined for every x. Therefore, we shall only consider what happens for $x\to+\infty$ and for $x\to-\infty$.

I. 1) It follows from

$$\frac{1-e^x}{1+e^x}=\frac{2}{1+e^x}-1,$$

that this expression tends towards -1 for $x\to+\infty$ (cf. the latter expression), and towards $+1$ for $x\to-\infty$ (cf. the former expression). This means that $f(x)\sim -x^2$ for $x\to+\infty$ and $f(x)\sim x^2$ for $x\to-\infty$, and we do not have any asymptote.

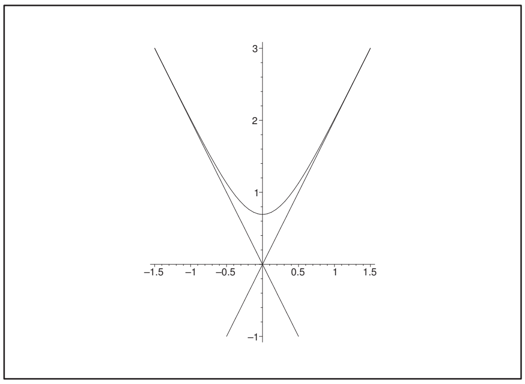

Figure 17: The function $f(x)=\ln(e^{2x}+e^{-2x})$ and its two asymptotes.

2) We conclude from

$$\begin{aligned} f(x) &= \ln(e^{2x}+e^{-2x})=\left(e^{2x}\left\{1+e^{-4x}\right\}\right)\\ &= 2x+\ln\left(1+e^{-4x}\right)=2x+e^{-4x}+e^{-4x}\varepsilon\left(e^{-4x}\right), \end{aligned}$$

that $f(x)$ has the asymptote $y=2x$ for $x\to+\infty$.

3) From

$$\begin{aligned} f(x) &= \ln(e^{-2x}\{e^{4x}+1\})=-2x+\ln(1+e^{4x})\\ &= -2x+e^{4x}+e^{4x}\varepsilon\left(e^{4x}\right), \end{aligned}$$

follows that $f(x)$ has the asymptote $y=-2x$ for $x\to-\infty$.

Example 5.2 *Find the possible asymptotes for the function*

$$f(x) = \sqrt{x^2 - 5x + 6}.$$

A. Asymptotes.

D Remove the "squared term" from the expression inside the square root, and find the Taylor expansion of the rest.

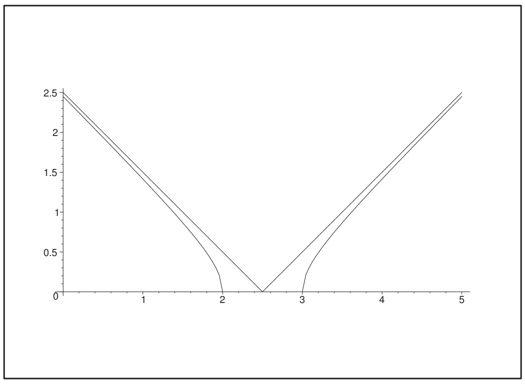

Figure 18: The function $f(x) = \sqrt{x^2 - 5x + 6}$ and its two asymptotes.

I. It follows from

$$f(x) = \sqrt{x^2 - 5x + 6} = \sqrt{(x-2)(x-3)} = \sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}}$$

that the function is defined in the two intervals $]-\infty, 2]$ and $[3, +\infty[$. Since the function is continuously defined in the end points $x = 2$ and $x = 3$, we do not have asymptotes at these points, though we of course have vertical half tangents.

1) If $x \in [3, +\infty[$, then in particular $x - \dfrac{5}{2} > 0$, thus

$$\begin{aligned} f(x) &= \sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}} = \left(x - \frac{5}{2}\right)\sqrt{1 - \frac{1}{(2x-5)^2}} \\ &= \left(x - \frac{5}{2}\right)\left\{1 - \frac{1}{2} \cdot \frac{1}{(2x-5)^2} + \frac{1}{x^2}\,\varepsilon\left(\frac{1}{x}\right)\right\} \\ &= x - \frac{5}{2} - \frac{1}{4} \cdot \frac{1}{2x-5} + \frac{1}{x}\,\varepsilon\left(\frac{1}{x}\right), \end{aligned}$$

and we conclude that $y = x - \dfrac{5}{2}$ is an asymptote for $x \to +\infty$.

2) If $x \in]-\infty, 2[$, then $\left|x - \dfrac{5}{2}\right| = \dfrac{5}{2} - x$, hence

$$\begin{aligned} f(x) &= \sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}} = -\left(x - \frac{5}{2}\right)\sqrt{1 - \frac{1}{(2x-5)^2}} \\ &= -x + \frac{5}{2} + \frac{1}{4} \cdot \frac{1}{2x-5} + \frac{1}{x}\,\varepsilon\left(\frac{1}{x}\right), \end{aligned}$$

and we conclude that $y = -x + \dfrac{5}{2}$ is an asymptote for $x \to -\infty$.

Example 5.3 *Find the possible asymptotes for the function*

$$f(x) = \frac{x^2 + x + 1}{\ln(1 + e^x)}.$$

A. Asymptotes.

D. The function is defined for all $x \in \mathbb{R}$. Investigate what happens for $x \to +\infty$ and $x \to -\infty$, separately.

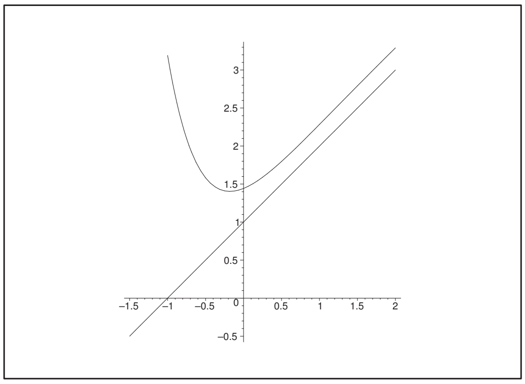

Figure 19: The function $f(x) = \dfrac{x^+ x + 1}{\ln(1 + e^x)}$ and its asymptote $y = x + 1$.

I. 1) When $x \to +\infty$, we use that

$$\ln(1 + e^x) = \ln\left(e^x\left\{1 + e^{-x}\right\}\right) = x + \ln\left(1 + e^{-x}\right) = x + e^{-x} + e^{-x}\,\varepsilon\left(e^{-x}\right),$$

so

$$\begin{aligned} f(x) &= \frac{x^2+x+1}{\ln(1+e^x)} = \frac{x^2+x+1}{x+e^{-x}+e^{-x}\,\varepsilon\,(e^{-x})} \\ &= \frac{x+1+\dfrac{1}{x}}{1+\dfrac{1}{x}\,e^{-x}+\dfrac{1}{x}\,e^{-x}\,\varepsilon\,(e^{-x})} \\ &= \left\{x+1+\frac{1}{x}\right\}\left\{1-\frac{1}{x}\,e^{-x}+\frac{1}{x}\,e^{-x}\,\varepsilon\left(e^{-x}\right)\right\} \\ &= x+1+\frac{1}{x}-e^{-x}-\frac{1}{x}\,e^{-x}+\frac{1}{x}\,e^{-x}\,\varepsilon\left(e^{-x}\right), \end{aligned}$$

and thus,

$$f(x)-(x+1)\to 0 \qquad \text{for } x\to+\infty,$$

and we see that $y=x+1$ is an asymptote for $x\to+\infty$.

2) If $x \to -\infty$, then $\ln(1+e^x) = e^x + e^x\,\varepsilon\,(e^x)$, where $e^x \to 0$, hence

$$\begin{aligned} f(x) &= \frac{x^2+x+1}{\ln(1+e^x)} = \frac{x^2+x+1}{e^x+e^x\,\varepsilon\,(e^x)} \\ &= \frac{(x^2+x+1)e^{-x}}{1+\varepsilon(e^x)} = \frac{(x^2+x+1)e^{|x|}}{1+\varepsilon(e^x)}. \end{aligned}$$

Clearly, $f(x)$ does not have an asymptote for $x \to -\infty$.

Example 5.4 *Find the possible asymptotes for the function*

$$f(x) = \left(x^3-3x^2+2\right)\sinh\frac{2}{x} - 2x^2.$$

A. Asymptotes.

D. The function is not defined at $x=0$, so there is a possibility of asymptotes for $x \to 0$, for $x \to +\infty$, and for $x \to -\infty$.

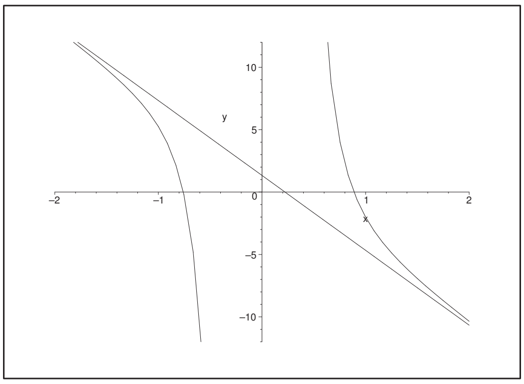

Figure 20: The graphs of the function $f(x) = (x^3-3x^2+2)\sinh\left(\frac{2}{x}\right) - 2x^2$ and its asymptote $y = -6x + \frac{4}{3}$. Different scales on the axes.

I. 1) If $x \to 0$, the factor x^3-3x^2+2 tends towards 2, and $\sinh\left(\frac{2}{x}\right) \to \pm\infty$ for $x \to 0\pm$. Thus we have a vertical asymptote $x=0$.

2) If $x \to \pm\infty$, then

$$\sinh\left(\frac{2}{x}\right) = \frac{2}{x} + \frac{1}{3!}\left(\frac{2}{x}\right)^3 + \left(\frac{1}{x}\right)^2\varepsilon\left(\frac{1}{x}\right) = \frac{2}{x} + \frac{4}{3}\cdot\frac{1}{x^3} + \frac{1}{x}^3\,\varepsilon\left(\frac{1}{x}\right),$$

hence

$$\begin{aligned}
f(x) &= \left(x^3-3x^2+2\right)\sinh\left(\frac{2}{x}\right)-2x^2\\
&= \left(x^3-3x^2+2\right)\left\{\frac{2}{x}+\frac{4}{3}\cdot\frac{1}{x^3}+\frac{1}{x^3}\,\varepsilon\left(\frac{1}{x}\right)\right\}-2x^2\\
&= 2x^2-6x+\frac{4}{x}+\frac{4}{3}-4\cdot\frac{1}{x}+\frac{8}{3}\cdot\frac{1}{x^3}+\varepsilon(x)-2x^2\\
&= -6x+\frac{4}{3}+\varepsilon,
\end{aligned}$$

and we conclude again that the function has the asymptote $y=-6x+\dfrac{4}{3}$ for $x\to\pm\infty$.

Example 5.5 *Check whether the function*

$$f(x)=x\,\text{Arctan}\,x,\qquad x\in\mathbb{R},$$

has asymptotes for $x\to-\infty$ and $x\to\infty$, resp..

A. Asymptotes; the example is almost the same as Example 5.7 (b).

D. Re-write the terms containing Arctan x etc., and then take the Taylor expansion.

I. If we put $g(x)=\text{Arctan}\,x+\text{ Arctan}\,\dfrac{1}{x}$, $x\neq 0$, then

$$g'(x)=\frac{1}{1+x^2}+\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right)=0,$$

thus $g(x)$ is a constant in each of the intervals $x>0$ and $x<0$. From $g(1)=\dfrac{\pi}{4}+\dfrac{\pi}{4}=\dfrac{\pi}{2}$ and $g(-1)=-\dfrac{\pi}{4}-\dfrac{\pi}{4}=-\dfrac{\pi}{2}$, we get

$$\text{Arctan}\,x=\begin{cases}\dfrac{\pi}{2}-\text{ Arctan}\,\dfrac{1}{x}, & \text{for } x>0,\\[2mm] -\dfrac{\pi}{2}-\text{ Arctan}\,\dfrac{1}{x}, & \text{for } x<0.\end{cases}$$

Furthermore,

$$\text{Arctan}\,y=y-\frac{1}{3}\,y^3+y^3\varepsilon(y),\qquad y\to 0.$$

1) If $x\to+\infty$, then $\dfrac{1}{x}\to 0+$, hence

$$\begin{aligned}
f(x) &= x\cdot\text{Arctan}\,x=x\left\{\frac{\pi}{2}-\text{ Arctan}\,\frac{1}{x}\right\}\\
&= x\left\{\frac{\pi}{2}-\frac{1}{x}+\frac{1}{3}\,\frac{1}{x^3}+\frac{1}{x^3}\,\varepsilon\left(\frac{1}{x}\right)\right\}\\
&= \frac{\pi}{2}\,x-1+\frac{1}{x}\,\varepsilon\left(\frac{1}{x}\right),
\end{aligned}$$

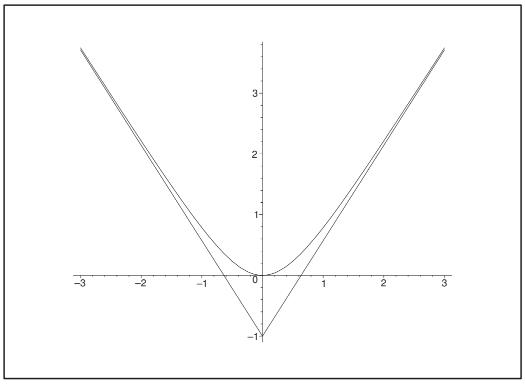

Figure 21: The graphs of $y = x\text{Arctan}\, x$ and its two asymptotes.

and we conclude that $y = \dfrac{\pi}{2}\,x - 1$ is an asymptote for $x \to +\infty$.

2) If $x \to -\infty$, then $\dfrac{1}{x} \to 0-$, hence

$$\begin{aligned} f(x) &= x \cdot \text{Arctan}\, x = x\left\{-\frac{\pi}{2} - \text{ Arctan}\, \frac{1}{x}\right\} \\ &= -\frac{\pi}{2}\,x - 1 + \frac{1}{x}\,\varepsilon\left(\frac{1}{x}\right), \end{aligned}$$

and we conclude that $y = -\dfrac{\pi}{2}\,x - 1$ is an asymptote for $x \to -\infty$.

Example 5.6 *Given the function*

$$f(x) = \ln(1 + \sinh 2x).$$

1) Find the approximating polynomial of at most second degree for f with the point of expansion $x_0 = 0$.

2) Find the limit

$$\lim_{x \to 0} \frac{\ln(1 + \sinh 2x) + 2e^{-x} - 2}{x + \ln(1 - x)}.$$

3) Prove that the graph of f has two asymptotes, and find an equation for each of these.

A. Approximating polynomial; limit; asymptotes. The example is very similar to Example 4.12.

D. Use some Taylor expansions.

I. 1) The function $f(x)$ is defined, if and only if $1+\sinh 2x > 0$, i.e. if and only if $x > -\dfrac{1}{2}\,\ln(\sqrt{2}+1)$, (cf. the calculation below). It follows from

$$\begin{aligned} f(x) &= \ln(1 + \sinh 2x), && f(0) = 0, \\ f'(x) &= \frac{2\cosh 2x}{1 + \sinh 2x}, && f'(0) = 2, \\ f''(x) &= -\frac{4\cosh^2 2x}{(1 + \sinh 2x)^2} + \sinh(2x) \cdot \{\cdots\}, && f''(0) = -4, \end{aligned}$$

that the approximating polynomial is

$$P_2(x) = \frac{2}{1!}\,x - \frac{4}{2!}\,x^2 = 2x - 2x^2,$$

and since $f(x)$ is of class C^∞ in its domain, we get

$$f(x) = 2x - 2x^2 + x^2\varepsilon(x).$$

2) From

$$f(x) = 2x - 2x^2 + x^2\varepsilon(x),$$

$$2e^{-x} = 2 - 2x + x^2 + x^2\varepsilon(x),$$

$$\ln(1-x) = -x - \frac{1}{2}\,x^2 + x^2\varepsilon(x),$$

we get by insertion,

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+\sinh 2x)+2e^{-x}-2}{x+\ln(1-x)} &= \lim_{x\to 0}\frac{2x-2x^2+2-2x+x^2-2+x^2\varepsilon(x)}{x-x-\dfrac{1}{2!}\,d^2+x^2\varepsilon(x)}\\ &= \lim_{x\to 0}\frac{-x^2+x^2\varepsilon(x)}{-\dfrac{1}{2}\,x^2+x^2\varepsilon(x)}=2.\end{aligned}$$

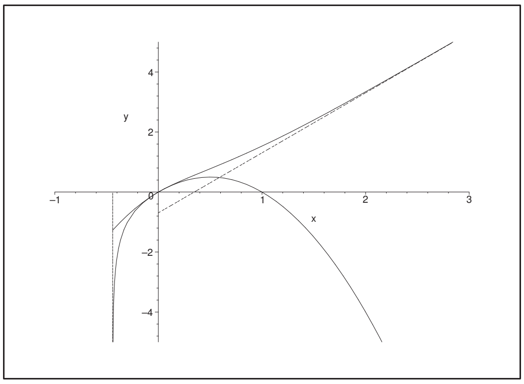

Figure 22: The graphs of $y=\ln(1+\sinh 2x)$ and its approximating polynomial $y=2x-2x^2$ and the two asymptote (both dotted).

3) First notice that $1+\sinh 2x=0$, when

$$\begin{aligned}0 &= e^{2x}+2-e^{-2x}=e^{-2x}\left\{e^{4x}+2e^{2x}+1-2\right\}\\ &= e^{-2x}\left\{\left(e^{2x}+1\right)^2-2\right\},\end{aligned}$$

i.e. when $e^{2x}=-1+\sqrt{2}\ (>0)$, thus $\ln(1+\sinh 2x)$ is defined if and only if

$$x>\frac{1}{2}\,\ln(\sqrt{2}-1)=-\frac{1}{2}\,\ln(\sqrt{2}+1).$$

Now,

$$f(x)=\ln(1+\sinh 2x)\to-\infty \qquad \text{for } x\to-\frac{1}{2}\,\ln(\sqrt{2}+1)+,$$

so the vertical line $x=-\dfrac{1}{2}\,\ln(\sqrt{2}+1)$ is an asymptote for $f(x)$.

If instead $x\to+\infty$, then we get by the definition the following calculations,

$$\begin{aligned}f(x) &= \ln(1+\sinh 2x)=\ln\left(1+\frac{1}{2}\,\left\{e^{2x}-e^{-2x}\right\}\right)\\ &= \ln\left(e^{2x}+2-e^{-2x}\right)-\ln 2\\ &= \ln\left(e^{2x}\right)-\ln 2+\ln\left(1+\frac{2-e^{-2x}}{e^{2x}}\right)\\ &= 2x-\ln 2+\ln\left(1+\frac{2-e^{-2x}}{e^{2x}}\right).\end{aligned}$$

Since

$$\ln\left(1+\frac{2-e^{-2x}}{e^{2x}}\right) \to 0 \qquad \text{for } x \to +\infty,$$

it follows that $y = 2x - \ln 2$ is also an asymptote.

The two asymptotes of $f(x)$ are

$$y = 2x - \ln 2 \qquad \text{and} \qquad x = -\frac{1}{2}\,\ln(\sqrt{2}+1).$$

Example 5.7 *1) Check in the following two cases whether the graph of f has an asymptote for $x \to +\infty$, and in case of an asymptote, an equation of it.*

(a) $f(x) = x\,\tanh x$, $\qquad (b)$ $f(x) = x\,\mathrm{Arctan}\,x$.

2) Prove that

$$\left(\frac{1}{x^2}+\frac{2}{x}\right)\ln(1+x) = \frac{1}{x}+\frac{3}{2}+\varepsilon\left(\frac{1}{x}\right), \qquad \textit{for } x \to 0,$$

and then show that the graph of

$$f(t) = \left(t^2+2t\right)\ln\left(1+\frac{1}{t}\right)$$

has an asymptote for both $t \to +\infty$ and $t \to -\infty$.

Find the equation of the asymptote.

A. Asymptotes.

D. Use Taylor expansions.

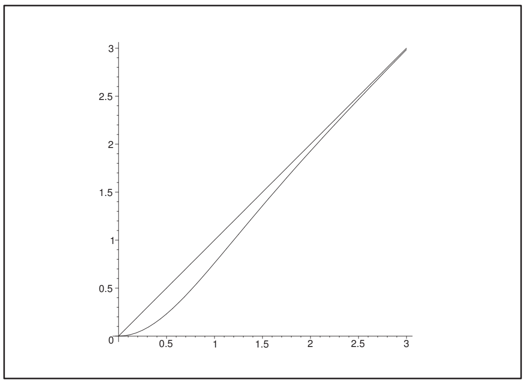

Figure 23: The graphs of $y = x\,\tanh x$ and its asymptote $y = x$.

I. 1) a) It follows from

$$\begin{aligned} f(x) &= x \tanh x = x \cdot \frac{e^x - e^{-x}}{e^x} + e^{-x} \\ &= x \cdot \frac{1 - e^{-2x}}{1 + e^{-2x}} = x \left\{ 1 - 2 \cdot \frac{e^{-2x}}{1 + e^{-2x}} \right\} \end{aligned}$$

that

$$|f(x) - x| = \frac{2}{1 + e^{-2x}} \cdot \frac{x}{e^{2x}} \to 0 \qquad \text{for } x \to +\infty,$$

and we conclude that $y = x$ is an asymptote for $f(x) = x \cdot \tanh x$, when $x \to +\infty$.

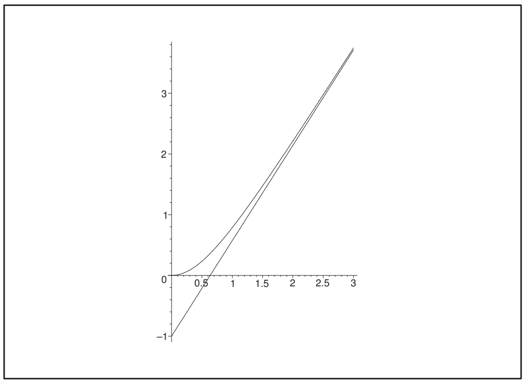

Figure 24: The graphs of $y = x \, \text{Arctan}\, x$ and its asymptote $y = \dfrac{\pi}{2}\, x - 1$.

b) The function

$$\varphi(x) = \ \text{Arctan}\ x + \ \text{Arctan}\ \frac{1}{x}, \qquad x > 0,$$

has the derivative

$$\varphi'(x) = \frac{1}{1 + x^2} + \frac{1}{1 + \left(\dfrac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2} \right) = 0,$$

thus φ is a constant. If we put $x = 1$, we obtain the constant $\dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2}$, i.e.

$$\text{Arctan}\ x + \ \text{Arctan}\ \frac{1}{x} = \frac{\pi}{2},$$

hence

$$\text{Arctan}\ x = \frac{\pi}{2} - \ \text{Arctan}\ \frac{1}{x}.$$

Finally,

$$\begin{aligned} f(x) &= x \cdot \text{Arctan}\ x) x \left\{ \frac{\pi}{2} - \ \text{Arctan}\ \frac{1}{x} \right\} \\ &= x \left\{ \frac{\pi}{2} - \frac{1}{x} + \frac{1}{x}\, \varepsilon\left(\frac{1}{x}\right) \right\} = \frac{\pi}{2}\, x - 1 + \varepsilon\left(\frac{1}{x}\right), \end{aligned}$$

and the asymptote has the equation $y = \dfrac{\pi}{2}\, x - 1$.

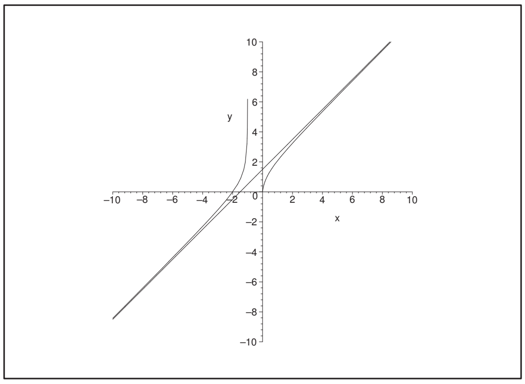

Figure 25: The graphs of $y = (x^2 + 2x)\ln\left(1 + \dfrac{1}{x}\right)$ and its asymptote $y = x + \dfrac{3}{2}$.

2) From

$$\ln(1+x) = x - \frac{1}{2}\,x^2 + x^2\varepsilon(x),$$

follows that

$$\begin{aligned}\left(\frac{1}{x^2}+\frac{2}{x}\right)\ln(1+x) &= \left(\frac{1}{x^2}+\frac{2}{x}\right)\left\{x - \frac{1}{2}\,x^2 + x^2\varepsilon(x)\right\}\\ &= \frac{1}{x} - \frac{1}{2} + \varepsilon(x) + 2 - x + x\varepsilon(x)\\ &= \frac{1}{x} + \frac{3}{2} + \varepsilon(x) \qquad \text{for } x \to 0.\end{aligned}$$

Thus by the substitution $t = \dfrac{1}{x}$

$$\begin{aligned} f(t) &= \left(t^2+2t\right)\ln\left(1+\frac{1}{t}\right) = \left(\frac{1}{x^2}+\frac{2}{x}\right)\ln(1+x)\\ &= \frac{1}{x} + \frac{3}{2} + \varepsilon(x) = t + \frac{3}{2} + \varepsilon\left(\frac{1}{t}\right),\end{aligned}$$

and we conclude that $y = t + \dfrac{3}{2}$ is an asymptote for $f(t)$ for both $t \to +\infty$ (i.e. for $x \to 0+$), and $t \to -\infty$ (i.e. for $x \to 0-$).

6 Improper integrals

Example 6.1 *One shall in the following integrals*

1) find the domain of the integrand,

2) sketch the graph of the integrand in the interval of integration,

3) check whether the integral is convergent or divergent,

4) in case of convergence, find the value of the integral,

$$(a)\quad \int_0^{+\infty} x\,e^{-x}\,dx, \qquad (b)\quad \int_0^1 \frac{2x+1}{x^2+x-2}\,dx.$$

A. Improper integrals.

D. Find the indefinite integral.

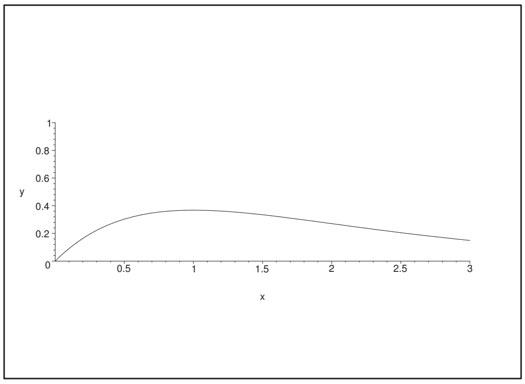

Figure 26: The graph of the integrand $x\,e^{-x}$.

I. (a) We see that $f(x) = x\,e^{-x}$ is defined for every $x \in \mathbb{R}$. We conclude from

$$f'(x) = (1-x)e^{-x},$$

that $f(x)$ has at maximum for $x = 1$, and that clearly $f(x) \to 0$ for $x \to +\infty$, and that the integrand is ≥ 0 everywhere.

We find an indefinite integral by partial integration,

$$\int x\,e^{-x}\,dx = x\left(-e^{-x}\right) - \int 1 \cdot \left(-e^{-x}\right)\,dx = -x\,e^{-x} + \int e^{-x}\,dx = -(x+1)e^{-x}.$$

This function converges towards 0 for $x \to +\infty$, hence the integral is convergent, and we get the value

$$\int_0^{+\infty} x\,e^{-x}\,dx = \lim_{x \to +\infty} \left\{-(x+1)e^{-x}\right\} + 1 = 0 + 1 = 1.$$

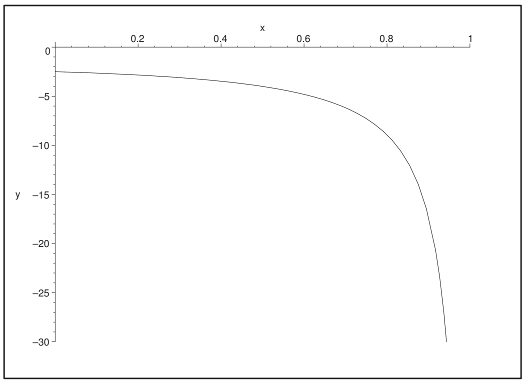

Figure 27: The graph of the integrand $f(x) = \dfrac{2x+1}{x^2+x-2}$.

(b) The function $f(x) = \dfrac{2x+1}{x^2+x-2}$ is defined for $x \neq 1$ and $x \neq 2$. It follows by a decomposition that

$$f(x) = \frac{1}{x+2} + \frac{1}{x-1} < 0 \qquad \text{for } x \in [0,1[,$$

so an integral is

$$\int \frac{1}{x+2}\,dx + \int \frac{1}{x-1}\,dx = \ln(x+2) + \ln(1-x),$$

where ALTERNATIVELY one could have noticed directly that the numerator is the derivative of the dominator, thus

$$\int \frac{2x+1}{x^2+x-2}\,dx = \ln\left|x^2+x-2\right|.$$

Now, $\ln(1-x) \to -\infty$ for $x \to 1-$, so the integral is divergent.

Example 6.2 *One shall in each of the following cases*

1) find the range of the integrand,

2) sketch the graph of the integrand in the interval of integration,

3) check whether the integral is convergent or divergent,

4) in case of convergence, one shall find the value of the integral,

$$(a)\quad \int_0^{+\infty} \sin x\,dx, \qquad (b)\quad \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx.$$

A. Improper integrals.

D. First find an indefinite integral.

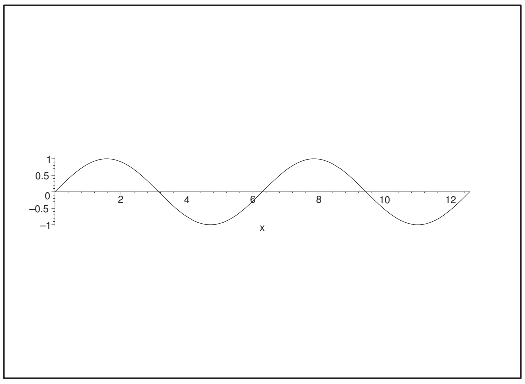

Figure 28: The graph of $f(x) = \sin x$.

I. (a) The function $f(x) = \sin x$ is defined for every $x \in \mathbb{R}$ with the indefinite integral $-\cos x$, so

$$\int_0^x \sin x \, dx = 1 - \cos x.$$

Since $1 - \cos x$ does not have a limit for $x \to +\infty$, the improper integral is divergent.

REMARK. The correct procedure of dealing with improper integrals is *always* first to split the integrand into a positive part $f^+ \geq 0$ and a "negative part" $f^- \geq 0$, such that

$$f(x) = f^+(x) - f^-(x),$$

where more precisely

$$f^+(x) = \max\{f(x), 0\} \qquad \text{and} \qquad f^-(x) = \max\{-f(x), 0\}.$$

The improper integral is convergent, if and only if

$$\int f^+(x)\,dx \qquad \text{og} \qquad \int f^-(x)\,dx$$

are both convergent, and if so, then

$$\int f(x)\,dx = \int f^+(x)\,dx - \int f^-(x)\,dx.$$

I am often missing this clarification in elementary textbooks. In the case above we may of course make a shortcut, because it is obvious that the limit does not exist. One can, however, construct cases, in which the limit exists for the indefinite integral, and where the improper integral does not exist in the strict sense given above. $\diamondsuit$

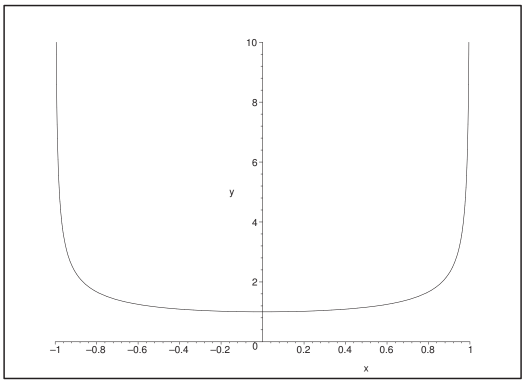

Figure 29: The graph of the integrand $\dfrac{1}{\sqrt{1-x^2}}$ for $x \in \,]-1,1[$.

(b) The function $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ is defined and positive for $x \in \,]-1,1[$. An integral is Arcsin x, thus the improper integral is convergent, and we find

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = [\text{Arcsin } x]_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Example 6.3 *One shall in each of the following integrals*

1) find the domain of the integrand,

2) sketch the graph of the integrand in the interval of integration,

3) check whether the integral is convergent or divergent,

4) in case of convergence find the value of the integral,

$$(a)\quad \int_0^{+\infty} \frac{1}{x^2+4}\,dx, \qquad (b)\quad \int_1^3 \frac{1}{x^2-1}\,dx.$$

A. Improper integrals.

D. Check the indefinite integral.

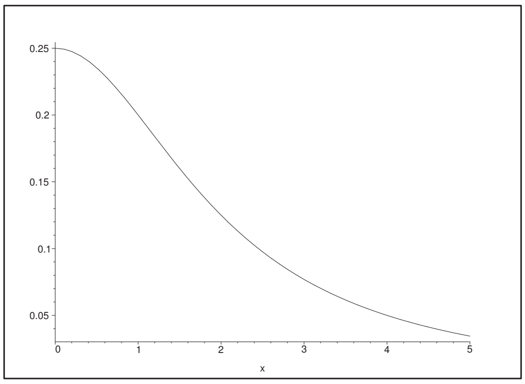

Figure 30: The graph of $f(x) = \dfrac{1}{x^2+4}$.

I. (a) The integrand $f(x) = \dfrac{1}{x^2+4}$ is defined and positive in $\mathbb{R}$. An indefinite integral is

$$\int_0^x \frac{1}{t^2+4}\,dt = \left[\frac{1}{2}\operatorname{Arctan}\frac{t}{2}\right]_0^x = \frac{1}{2}\operatorname{Arctan}\frac{z}{2},$$

which is clearly convergent for $x \to +\infty$, so its value is

$$\int_0^{+\infty} \frac{1}{x^2+4}\,dx = \lim_{x\to+\infty} \frac{1}{2}\operatorname{Arctan}\frac{x}{2} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

(b) The integrand

$$f(x) = \frac{1}{x^2-1} = \frac{1}{2}\,\frac{1}{x-1} - \frac{1}{2}\,\frac{1}{x+1}$$

is defined for $x \in \mathbb{R}\setminus\{-1,1\}$, and it is positive for $x \in \,]1,3]$. An indefinite integral is

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|,$$

which clearly is divergent for $x \to 1+$.

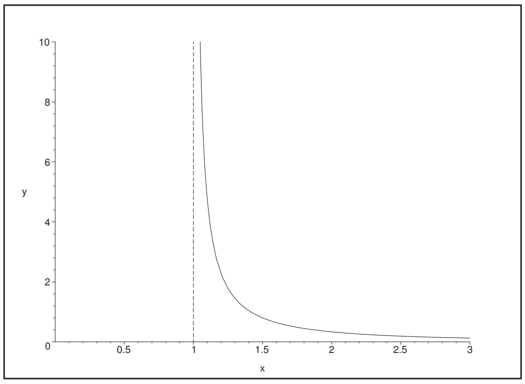

Figure 31: The graph of $\dfrac{1}{x^2-1}$ for $x \in]1,3]$.

Example 6.4 *We shall for the following integrals*

1) find the domain of the integrand,

2) sketch the graph of the integrand in the interval of integration,

3) check whether the integral is convergent or divergent,

4) in case of convergence, find the value of the integral,

$$(a) \quad \int_0^1 \frac{1}{x\,\ln x}\,dx, \qquad (b) \quad \int_0^{+\infty} \frac{x}{\sqrt{x^2}}\,dx.$$

A. Improper integrals.

D. Follow the guidelines of the example.

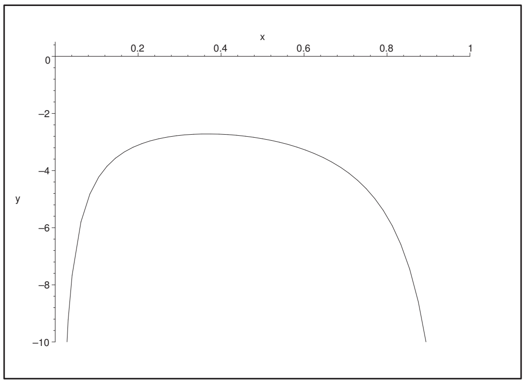

Figure 32: The graph of $\dfrac{1}{x\,\ln x}$, $x \in]0,1[$.

I. (a) Put $f(x) = \dfrac{1}{x\,\ln x}$. From $\dfrac{d}{dx}\{x\,\ln x\} = \ln x + 1$ follows that the function has a maximum for $x = \dfrac{1}{e}$, corresponding to $f\left(\dfrac{1}{e}\right) = -e$. Furthermore, $x\,\ln x \to 0-$ for $x \to 0+$, and for $x \to 1-$. We therefore conclude that $f(x) \to -\infty$ by these limit processes, and $f(x)$ is negative everywhere in $]0,1[$.

An integral of $f(x)$ is

$$\int \frac{1}{x\,\ln x}\,dx = \ln|\ln x|.$$

We conclude from $\ln|\ln x| \to -\infty$ for $x \to 1-$, and $\ln|\ln x| \to +\infty$ for $x \to 0+$, that the improper integral is divergent.

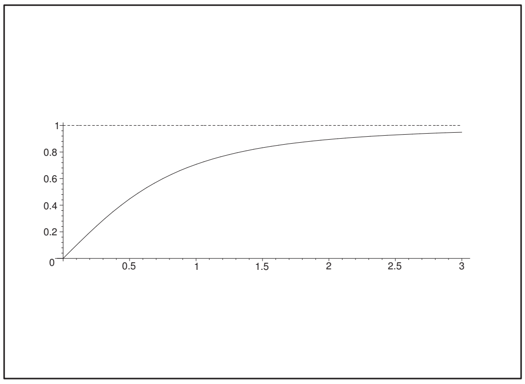

Figure 33: The graph of $\dfrac{x}{\sqrt{1+x^2}}$.

(b) Let $f(x) = \dfrac{x}{\sqrt{1+x^2}}$. Then $f(0) = 0$, and

$$f(x) = \frac{1}{\sqrt{1+\dfrac{1}{x^2}}} \to 1 \neq 0 \qquad \text{for } x \to +\infty,$$

and we see that the improper integral is divergent.

Example 6.5 *One shall in each of the following cases*

1) find the domain of the integrand,

2) sketch the graph of the integrand in the interval of integration,

3) check whether the integral is convergent or divergent,

4) in case of convergence, find the value of the integral,

$$(a)\quad \int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx, \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{2x}{1+x^2}\,dx.$$

A. Improper integrals.

D. Find an integral.

I. (a) The function $f(x) = \dfrac{e^{-\sqrt{x}}}{\sqrt{x}}$ is defined and positive for $x > 0$. We get by the monotone substitution $y = \sqrt{x}$, i.e. $x = y^2$, $dx = 2y\,dy$, the integral

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \int_{u=\sqrt{x}} \frac{e^{-y}}{y} \cdot 2y\,dy = -2e^{-\sqrt{x}},$$

which has the limits -2 for $x \to 0+$ and 0 for $x \to +\infty$, hence the improper integral is convergent, and its value is

$$\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = 0 - (-2) = 2.$$

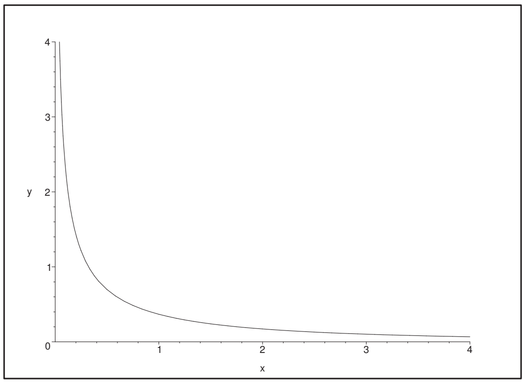

Figure 34: The graph of $\dfrac{e^{-\sqrt{x}}}{\sqrt{x}}$, $x > 0$.

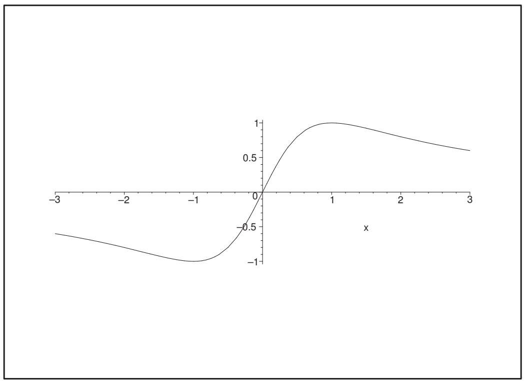

Figure 35: The graph of $f(x) = \dfrac{2x}{1+x^2}$.

(b) The function $f(x) = \dfrac{2x}{1+x^2}$ is defined for every $x \in \mathbb{R}$, and

$$f'(x) = 2\,\frac{1-x^2}{(1+x^2)^2},$$

thus we have a maximum for $x = 1$, corresponding to $f(1) = 1$.

The integrand is positive for $x > 0$ and negative for $x < 0$. An integral is

$$\int \frac{2x}{1+x^2}\,dx = \ln(1+x^2).$$

Let us only consider the positive part. Here,

$$\int_0^x \frac{2t}{1+t^2}\,dt = \ln(1+x^2) \to +\infty, \qquad \text{for } x \nearrow +\infty,$$

and we conclude that the improper integral is divergent.

Remark. There is a nasty trap here. One was actually guided towards the following fallacy: When we integrate over the symmetric interval $[-x, x]$, $x > 0$, then

$$\int_{-x}^{x} \frac{2t}{1+t^2}\,dt = \left[\ln(1^+t^2)\right]_{-x}^{x} = 0 \to 0 \qquad \text{for } x \to +\infty,$$

and we would wrongly conclude that we got "convergence" and the "value" 0. $\Diamond$

Example 6.6 *One shall in each of the following cases*

1) find the domain of the integrand,

2) sketch the graph of the integrand in the interval of integration,

3) check whether the integral is convergent or divergent,

4) in case of convergence, find the value of the integral.

$$(1)\quad \int_1^{+\infty} \frac{\ln x}{x^2}\,dx, \qquad (2)\quad \int_0^{\frac{\pi}{2}} \cot x\,dx.$$

A. Improper integrals.

D. Find an integral.

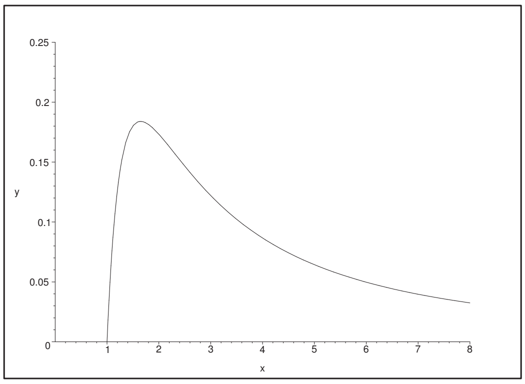

Figure 36: The graph of $y = \dfrac{\ln x}{x^2}$, $x \geq 1$.

I. 1) The function $f(x) = \dfrac{\ln x}{x^2}$ is define and differentiable for $x > 0$, and $f(x) \geq 0$ for $x \geq 1$, i.e. in the interval of integration. We conclude from

$$f'(x) = \frac{1 - 2\ln x}{x^3} = 0 \qquad \text{for } x = \sqrt{e},$$

and $f(1) = 0$ and $f(x) \to 0$ for $x \to +\infty$, that $x = \sqrt{e}$ corresponds to a global maximum. The value is here $f(\sqrt{e}) = \dfrac{1}{2e}$.

By a partial integration,

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{1+\ln x}{x}.$$

The integrand is ≥ 0 everywhere in the interval of integration, and the integral $-\dfrac{1+\ln x}{x} \to 0$ for $x \to +\infty$. Thus, we conclude that the improper integral is convergent, and its value is

$$\int_1^{+\infty} \frac{\ln x}{x^2}\,dx = \lim_{x\to+\infty}\left\{-\frac{1+\ln x}{x}\right\} + \frac{1+\ln 1}{1} = 1.$$

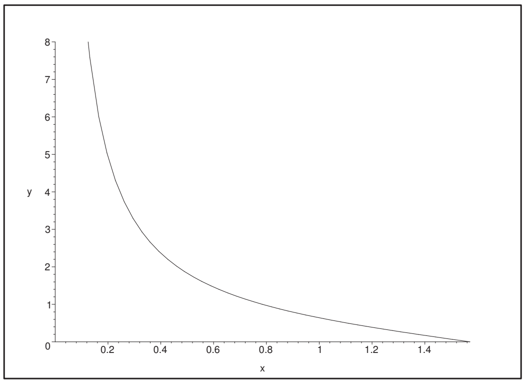

Figure 37: The graph of $y = \cot x$, $x \in \left]0, \frac{\pi}{2}\right]$.

2) The function $f(x) = \cot x$ is defined and differentiable for $x \neq p\pi$, $p \in \mathbb{Z}$. It is ≥ 0 in the interval $\left]0, \frac{\pi}{2}\right]$, hence we shall only find an integral and then consider the limit process.

Now $\sin x > 0$ for $0 < x < \frac{\pi}{2}$, thus an integral is given by

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln |\sin x| = \ln \sin x, \qquad x \in \left]0, \frac{\pi}{2}\right].$$

Since $\ln \sin x \to -\infty$ for $x \to 0+$, the improper integral is divergent.

Example 6.7 *Consider the improper integral*

$$\int_{-\infty}^{+\infty} \frac{e^{\frac{1}{3}x}}{1+e^x} \, dx.$$

1) Find the domain of the integrand.

2) Sketch the graph of the integrand in the interval of integration.

3) Is the improper integral convergent or divergent?

4) In case of convergence, find the value of the integral.

A. Improper integral.

D. Check the sign and find an integral.

I. The function $f(x) = \frac{e^{\frac{1}{3}x}}{1+e^x}$ is defined and differentiable and strictly positive for every $x \in \mathbb{R}$. It follows from

$$f'(x) = \frac{1}{3} \frac{e^{\frac{1}{3}x}}{(1+e^x)^2} \{1 - 2e^x\},$$

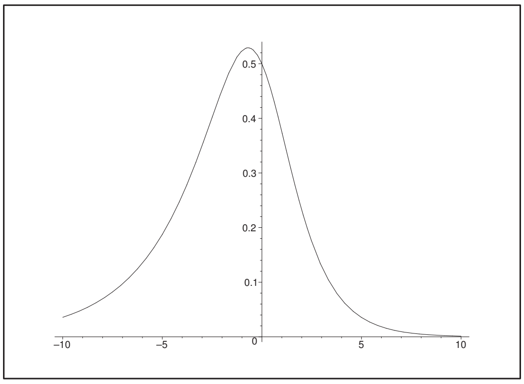

Figure 38: The graph of $y = \dfrac{e^{\frac{1}{3}x}}{1+e^x}$.

that the function has a global maximum for $x = -\ln 2$, since the function clearly tends to 0 for $x \to \pm\infty$, and because $x = -\ln 2$ is the only point for which $f'(x) = 0$. The value of the function at this point is

$$f(-\ln 2) = \frac{2}{3}\sqrt[3]{\frac{1}{2}}.$$

We get the integral by using the substitution $y = e^{\frac{1}{3}x}$,

$$\int \frac{e^{\frac{1}{3}x}}{1+e^x}\,dx = 3\int_{y=exp(\frac{1}{3}x)} \frac{dy}{1+y^3}.$$

Now, $y^3+1 = (y+1)(y^2-y+1)$, so by a decomposition,

$$\frac{1}{y^2+1} = \frac{a}{y+1} + \frac{by+c}{y^2-y+1},$$

where

$$a = \frac{1}{1+1+1} = \frac{1}{3},$$

hence

$$\begin{aligned}\frac{by+c}{y^2-y+1} &= \frac{1-\frac{1}{3}y^2+\frac{1}{3}y-\frac{1}{3}}{(y+1)(y^2-y+1)} = -\frac{1}{3}\cdot\frac{y^2-y-2}{(y+1)(y^2-y+1)} \\ &= -\frac{1}{3}\cdot\frac{y-2}{y^2-y+1} = -\frac{1}{6}\cdot\frac{2y-1}{y^2-y+1} + \frac{1}{2}\cdot\frac{1}{y^2-y+1}.\end{aligned}$$

Then by insertion,

$$\begin{aligned}
\int \frac{e^{\frac{1}{3}x}}{1+e^x}\,dx &= \int_{y=e^{\frac{1}{3}x}} \left\{ \frac{1}{y+1} - \frac{1}{2}\,\frac{2y-1}{y^2-y+1} + \frac{3}{2}\,\frac{1}{\left(y-\frac{1}{2}\right)^2+\frac{3}{4}} \right\} dy \\
&= \left[\ln(y+1) - \frac{1}{2}\ln(y^2-y+1) + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\,\mathrm{Arctan}\left(\frac{y-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_{y=e^{\frac{1}{3}x}} \\
&= \frac{1}{2}\ln\left\{\frac{(1+e^{\frac{1}{3}x})^2}{e^{\frac{2}{3}x}-e^{\frac{1}{3}x}+1}\right\} + \sqrt{3}\,\mathrm{Arctan}\left\{\frac{1}{\sqrt{3}}\left(2e^{\frac{1}{3}x}-1\right)\right\} \\
&= \frac{1}{2}\ln\left\{\frac{e^{\frac{2}{3}x}+2e^{\frac{1}{3}x}+1}{e^{\frac{2}{3}x}-e^{\frac{1}{3}x}+1}\right\} + \sqrt{3}\,\mathrm{Arctan}\left\{\frac{2e^{\frac{1}{3}x}-1}{\sqrt{3}}\right\} \\
&= \frac{1}{2}\ln\left\{1+3\cdot\frac{1}{e^{\frac{1}{3}x}-1+e^{-\frac{1}{3}x}}\right\} + \sqrt{3}\,\mathrm{Arctan}\left\{\frac{2e^{\frac{1}{3}x}-1}{\sqrt{3}}\right\} \\
&= \frac{1}{2}\ln\left\{1+\frac{3}{2\cosh\left(\frac{x}{3}\right)-1}\right\} + \sqrt{3}\,\mathrm{Arctan}\left\{\frac{2e^{\frac{1}{3}x}-1}{\sqrt{3}}\right\}.
\end{aligned}$$

It follows clearly from the latter rearrangement that the logarithmic term tends to 0 for $x \to \pm\infty$, and that the Arctan term also has limit values for $x \to \pm\infty$. We therefore conclude that the improper integral is convergent, and its value is

$$\int_{-\infty}^{+\infty} \frac{e^{\frac{1}{3}x}}{1+e^x}\,dx = \sqrt{3}\cdot\frac{\pi}{2} - \sqrt{3}\cdot\mathrm{Arctan}\left(-\frac{1}{\sqrt{3}}\right) = \sqrt{3}\cdot\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}\,\pi.$$

Example 6.8 *One shall in the following cases*

1) find the domain of the integrand,

2) sketch the graph of the integrand in the interval of integration,

3) check whether the integral is convergent or divergent,

4) in case of convergence, find the value of the integral.

$$(1)\quad \int_0^{+\infty} \frac{1}{x^2+2x+1}\,dx, \qquad (2)\quad \int_0^{+\infty} \frac{x}{x^2-2x+2}\,dx.$$

A. Improper integrals.

D. Check the sign and find an integral; then take the limit.

I. 1) Clearly,

$$f(x) = \frac{1}{x^2+2x+2} = \frac{1}{(x+1)^2+1}.$$

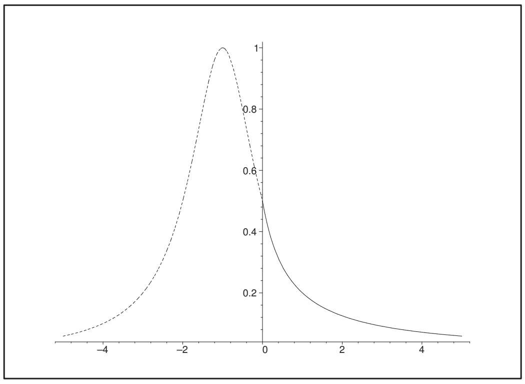

Figure 39: The graph of $y = \dfrac{1}{x^2+2x+2}$.

is defined and positive and differentiable for every $x \in \mathbb{R}$, and $f(x) \to 0$ for $x \to \pm\infty$, and $f(x)$ has a global maximum for $x = -1$, where the denominator is smallest.

The integral is

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx = \text{ Arctan}(x+1),$$

and we conclude that the improper integral is convergent, and its value is

$$\int_0^{+\infty} \frac{1}{x^2+2x+2}\,dx = [\text{Arctan}(x+1)]_0^{x\to+\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

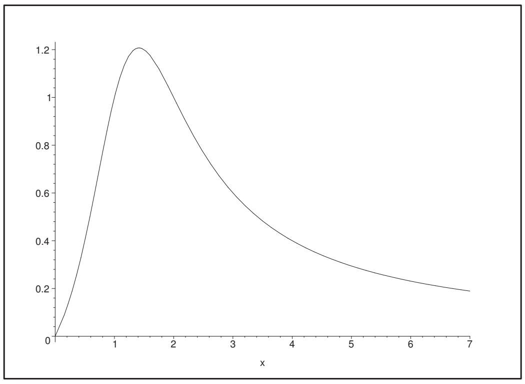

Figure 40: The graph of $y = \dfrac{x}{x^2-2x+2}$, $x > geq 0$.

2) The function

$$f(x) = \frac{x}{x^2-2x+2} = \frac{x}{(x-1)^2+1},$$

is defined and differentiable for every $x \in \mathbb{R}$, and $f(x) \geq 0$ for $x \geq 0$ where $f(0) = 0$, and $f(x) \to 0+$ for $x \to +\infty$. Now,

$$f'(x) = \frac{-x^2+2}{(x^2-2x+x)^2}$$

is 0 for $x = \sqrt{2} > 0$, so this corresponds to a maximum,

$$f(\sqrt{2}) = \frac{1+\sqrt{2}}{2}.$$

We get from the decomposition

$$f(x) = \frac{x-1}{(x-1)^2+1} + \frac{1}{(x-1)^2+1}$$

the integral

$$\int \frac{x}{x^2-2x+2}\,dx = \frac{1}{2}\ln\left\{(x-1)^2+1\right\} + \text{ Arctan}(x+1),$$

where the logarithmic term tends to $+\infty$ for $x \to +\infty$, while the Arctan term is bounded. It follows that the improper integral is divergent.

Example 6.9 *Prove that the improper integral below is convergent, and find its value.*

$$\int_2^{+\infty} \frac{4x+4}{x^4+4x^2}\,dx.$$

A. Improper integral.

D. Decompose and find an integral.

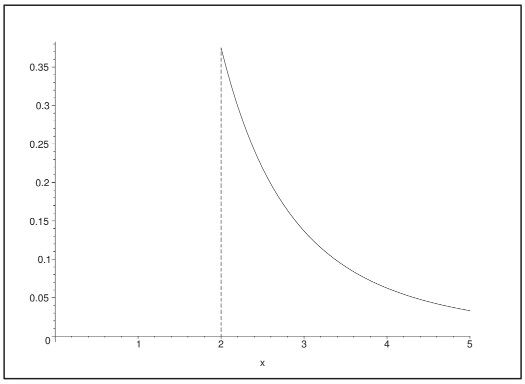

Figure 41: The graph of $y = \dfrac{4x+4}{x^4+4x^2}$, $x \geq 2$.

I. It follows from the factorization $x^4+4x^2 = x^2(x^2+4)$ that the integrand is defined and differentiable for $x \neq 0$. It is positive for $x \geq 2$. We shall only find an integral and then go to the limit.

We get by decomposition,

$$\begin{aligned}\frac{4x+4}{x^4+4x^2} &= (x+1)\cdot\frac{4}{x^2(x^2+4)} = (x+1)\left\{\frac{1}{x^2}-\frac{1}{x^2+4}\right\}\\ &= \frac{1}{x}+\frac{1}{x^2}-\frac{x}{x^2+4}-\frac{1}{x^2+4}.\end{aligned}$$

When $x \geq 2$, an integral is given by

$$\begin{aligned}\int \frac{4x+4}{x^4+4x^2}\,dx &= \int\frac{1}{x}\,dx+\int\frac{1}{x^2}\,dx-\int\frac{x}{x^2+4}\,dx-\int\frac{1}{x^2+4}\,dx\\ &= \ln x-\frac{1}{x}-\frac{1}{2}\ln(x^2+4)-\frac{1}{2}\,\text{Arctan}\left(\frac{x}{2}\right)\\ &= -\frac{1}{x}-\frac{1}{2}\,\text{Arctan}\left(\frac{x}{2}\right)-\frac{1}{2}\ln\left(\frac{x^2+4}{x^2}\right)\\ &= -\frac{1}{x}-\frac{1}{2}\,\text{Arctan}\left(\frac{x}{2}\right)-\frac{1}{2}\ln\left(1+\frac{4}{x^2}\right).\end{aligned}$$

If $x \to +\infty$ this expression converges towards $-\dfrac{\pi}{4}$, and the improper integral is convergent, and its value is

$$\begin{aligned}\int_2^{+\infty} \frac{4x+4}{x^4+4x^2}\,dx &= -\frac{\pi}{4}+\frac{1}{2}+\frac{1}{2}\,\text{Arctan}\;1+\frac{1}{2}\,\ln 2\\ &= -\frac{\pi}{4}+\frac{1}{2}+\frac{\pi}{8}+\frac{1}{2}\,\ln 2\\ &= \frac{1}{2}\,(1+\ln 2)-\frac{\pi}{8}.\end{aligned}$$

Example 6.10 *1) Decompose the fraction*

$$\frac{8x^2+24}{(x-1)^2(x^2+2x+5)}.$$

2) Prove that the integral

$$\int_2^{+\infty} \frac{8x^2+24}{(x-1)^2(x^2+2x+5)}\,dx$$

is convergent, and find its value.

A. Decomposition, where the degree of the denominator is 2 + the degree of the numerator. Improper integral.

D. Decompose.

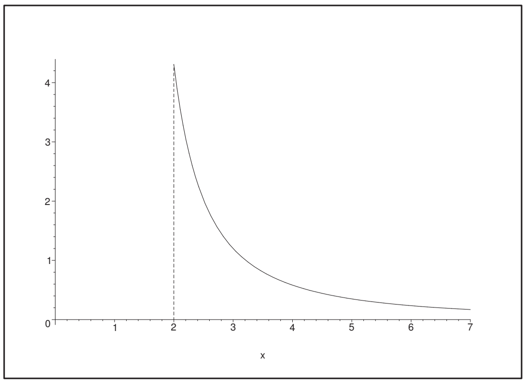

Figure 42: The graph of $y = \dfrac{8x^2+24}{(x-1)^2(x^2+2x+5)}$, $x \geq 2$.

I. 1) We get by decomposition,

$$\begin{aligned}\frac{8x^2+24}{(x-1)^2(x^2+2x+5)} &= \frac{4}{(x-1)^2}+\frac{8x^2+24-4x^2-8x-20}{(x-1)^2(x^2+2x+5)}\\ &= \frac{4}{(x-1)^2}+\frac{4x^2-8x+4}{(x-1)^2(x^2+2x+5)}\\ &= \frac{4}{(x-1)^2}+\frac{4}{(x+1)^2+4}.\end{aligned}$$

2) An integral is

$$\int \frac{8x^2+24}{(x-1)^2(x^2+2x+5)}\,dx = -\frac{4}{x-1} + 2\,\mathrm{Arctan}\left(\frac{x+1}{2}\right).$$

The singular point $x=1$ does not belong to the interval $[2,+\infty[$, thus the improper integral is convergent, and its value is

$$\int_2^{+\infty} \frac{8x^2+24}{(x-1)^2(x^2+2x+5)}\,dx = 4+\pi-1\,\mathrm{Arctan}\,\frac{3}{2}.$$

Example 6.11 *Prove that the improper integral*

$$\int_{\ln 4}^{\infty} \frac{1}{e^x-3}\,dx$$

is convergent, and find its value.

A. Improper integral.

D. The integrand is defined and positive in the interval of integration. Use the substitution $t = e^x$ to find an integral.

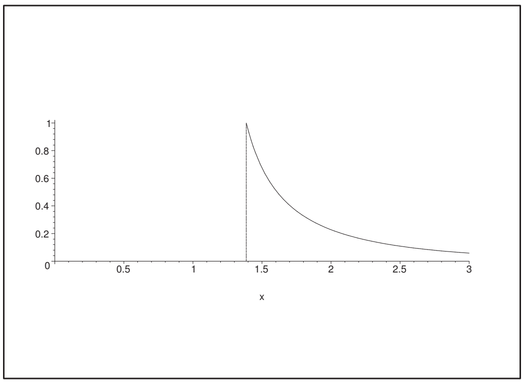

Figure 43: The graph of for $y = \dfrac{1}{e^x-3}$, $x \geq \ln 4$.

I. If we substitute $t = e^x$, $t \geq e^{\ln 4} = 4$, we get the integral

$$\begin{aligned}\int \frac{1}{e^x-3}\,dx &= \int_{t=e^x} \frac{1}{t(t-3)}\,dt = \int_{t=e^x} \left\{-\frac{1}{3}\cdot\frac{1}{y} + \frac{1}{3}\cdot\frac{1}{t-3}\right\} dt \\ &= \left[\frac{1}{3}\ln\left(\frac{t-3}{t}\right)\right]_{t=e^x} = \frac{1}{3}\ln\left(1-\frac{3}{e^x}\right).\end{aligned}$$

The integrand is positive in the interval of integration, and the indefinite integral

$$\int \frac{1}{e^x-3}\,dx = \frac{1}{3}\ln\left(1-\frac{3}{e^x}\right)$$

is defined for $x \geq \ln 4$, and it tends to 0 for $x \to +\infty$. Hence we conclude that the improper integral is convergent, and its value is

$$\int_{\ln 4}^{+\infty} \frac{1}{e^x-3}\,dx = \left[\frac{1}{3}\ln\left(1-\frac{3}{e^x}\right)\right]_{\ln 4}^{+\infty} = 0 - \frac{1}{3}\ln\left(1-\frac{3}{4}\right) = -\frac{1}{3}\ln\frac{1}{4} = \frac{2}{3}\ln 2.$$

Example 6.12 *1) Decompose the rational function*

$$F(x) = \frac{11x^2 - 30x + 9}{x(x-1)(x^2+9)}.$$

2) Prove that the improper integral

$$\int_3^{+\infty} F(x)\,dx$$

is convergent, and find its value.

A. Decomposition and improper integral.

D. Use the standard procedures.

I. 1) We get by decomposition,

$$\begin{aligned}
F(x) &= \frac{11x^2 - 30x + 9}{x(x-1)(x^2+9)} \\
&= -\frac{1}{x} - \frac{1}{x-1} + \frac{11x^2-30x+9+(x-1)(x^2+9)+x(x^2+9)}{x(x-1)(x^2+9)} \\
&= -\frac{1}{x} - \frac{1}{x-1} + \frac{11x^2-30x+99x-9+(x-1)x^2+x^3+9x}{x(x-1)(x^2+9)} \\
&= -\frac{1}{x} - \frac{1}{x-1} + \frac{x^3+11x^2-12x+x^2(x-1)}{x(x-1)(x^2+9)} \\
&= -\frac{1}{x} - \frac{1}{x-1} + \frac{x(x-1)(x+12)+x(x-1)x}{x(x-1)(x^2+9)} \\
&= -\frac{1}{x} - \frac{1}{x-1} + \frac{2x+12}{x^2+9}.
\end{aligned}$$

C. Test:

$$\begin{aligned}
&-\frac{1}{x} - \frac{1}{x-1} + \frac{2x+12}{x^2+9} \\
&= \frac{1}{x(x-1)(x^2+9)}\left\{-(x-1)(x^2+9)-x(x^2+9)+(2x+12)(x^2-x)\right\} \\
&= \frac{1}{x(x-1)(x^2+9)}\left\{-x^3-9x+x^2+9-x^3-9x+2x^3-2x^2+12x^2-12x\right\} \\
&= \frac{11x^2-30x+9}{x(x-1)(x^2+9)}. \qquad \text{Q.E.D.}
\end{aligned}$$

2) If $x > 1$, then an integral is

$$\begin{aligned}
\int F(x)\,dx &= \int \left\{-\frac{1}{x} - \frac{1}{x-1} + \frac{2x}{x^2+9} + \frac{12}{x^2+9}\right\} dx \\
&= -\ln x - \ln(x-1) + \ln(x^2+9) + \frac{12}{3}\,\text{Arctan}\left(\frac{x}{3}\right) \\
&= \ln\left\{\frac{x^2+9}{x(x-1)}\right\} + 4\,\text{Arctan}\left(\frac{x}{3}\right) \\
&= \ln\left\{1 + \frac{x+9}{x(x-1)}\right\} + 4\,\text{Arctan}\left(\frac{x}{3}\right).
\end{aligned}$$

The latter expression of the integral clearly converges towards $4 \cdot \dfrac{\pi}{2} + 0 = 2\pi$ for $x \to +\infty$. We conclude that the improper integral is convergent, and its value is

$$\begin{aligned}\int_3^{+\infty} F(x)\,dx &= 2\pi - \ln\left(\frac{3^2+9}{3\cdot 2}\right) - 4\,\mathrm{Arctan}\left(\frac{3}{3}\right)\\ &= 2\pi - \ln 3 - 4\cdot\frac{\pi}{4} = \pi - \ln 3,\end{aligned}$$

where we have used that the integrand is positive, and that the integral has a limit for $x \to +\infty$.

Example 6.13 *Prove that the integral*

$$\int_0^1 \frac{x+3}{\sqrt{1-x^2}}\,dx$$

is convergent, and find its value.

A. Improper integral.

D. The integrand is positive in $[0,1[$, but it is not defined for $x=1$. Find an integral and take the limit $x \to 1-$.

I. An integral is

$$\begin{aligned}\int \frac{x+3}{\sqrt{1-x^2}}\,dx &= \int \frac{x}{\sqrt{1-x^2}}\,dx + 3\int \frac{1}{\sqrt{1-x^2}}\,dx\\ &= 3\,\mathrm{Arcsin}\,x - \sqrt{1-x^2}.\end{aligned}$$

This shows clearly that the improper integral is convergent, and its value is

$$\int_0^1 \frac{x+3}{\sqrt{1-x^2}}\,dx = \frac{3\pi}{2} + 1.$$

Example 6.14 *Prove that the improper integral*

$$\int_1^{+\infty} \frac{1}{x\left\{(\ln x)^3 + (\ln x)^2 + \ln x + 1\right\}}\,dx$$

is convergent, and find its value.

A. Improper integral.

D. Use the substitution $y = \ln x$, and then find an integral.

I. The integrand is clearly positive in the interval of integration, so it suffices to apply the substitution, followed by a limit process.

By the substitution $y = \ln x$ (≥ 0), $dy = \dfrac{1}{x}\,dx$, the integral is written

$$\int \frac{2}{x\left\{(\ln x)^3 + (\ln x)^2 + \ln x + 1\right\}}\,dx = \int_{y=\ln x} \frac{2}{y^3+y^2+y+1}\,dy.$$

Then by a decomposition,

$$\begin{aligned}\frac{2}{y^3+y^2+y+1} &= \frac{2}{(y+1)\left(y^2+1\right)} = \frac{1}{y+1} + \frac{2-\left(y^2+1\right)}{(y+1)\left(y^2+1\right)} \\ &= \frac{1}{y+1} - \frac{y}{y^2+1} + \frac{1}{y^2+1},\end{aligned}$$

hence,

$$
\begin{aligned}
&\int \frac{2}{x\left\{(\ln x)^3+(\ln x)^2+\ln x+1\right\}}\,dx \\
&\quad = \int_{y=\ln x}\left\{\frac{1}{y+1}-\frac{y}{y^2+1}+\frac{1}{y^2+1}\right\}dy \\
&\quad = \left[\ln(y+1)-\frac{1}{2}\,\ln\left(y^2+1\right)+\ \operatorname{Arctan}\,y\right]_{y=\ln x} \\
&\quad = \left[\frac{1}{2}\,\ln\left\{\frac{(y+1)^2}{y^2+1}\right\}+\ \operatorname{Arctan}\,y\right]_{y=\ln x} \\
&\quad = \left[\frac{1}{2}\,\ln\left\{1+\frac{2y}{y^2+1}\right\}+\ \operatorname{Arctan}\,y\right]_{y=\ln x} \\
&\quad = \frac{1}{2}\,\ln\left\{1+\frac{2\ln x}{(\ln x)^2+1}\right\}+\ \operatorname{Arctan}(\ln x).
\end{aligned}
$$

This expression is clearly convergent for $x \to +\infty$, and when $x = 1$ the integrand is continuous, thus the improper integral is convergent, and its value is

$$\int_1^{+\infty} \frac{2}{x\left\{(\ln x)^3+(\ln x)^2+\ln x+1\right\}}\,dx = 0+\frac{\pi}{2}-0-0=\frac{\pi}{2}.$$

Example 6.15 *Let $n \in \mathbb{N}$ be a natural number. Prove that the integral*

$$\int_0^{+\infty} x^3 e^{-nx}\,dx$$

is convergent, and find its value.

A. Improper integral.

D. Find an integral, either by partial integration or by guessing.

I. The integrand is defined and non-negative for all $x \geq 0$, so it suffices to find an integral and then perform the limit process.

First variant. We guess the integral of the form

$$F(x) = \left(\alpha x^3+\beta x^2+\gamma x+\delta\right) e^{-nx}.$$

Then

$$
\begin{aligned}
f(x) &= F'(x) \\
&= -n\left(\alpha x^3+\beta x^2+\gamma x+\delta\right) e^{-nx}+\left(3\alpha x^2+2\beta x+\gamma\right) e^{-nx} \\
&= -n\alpha x^3 e^{-nx}+(3\alpha-n\beta)x^2 e^{-nx}+(2\beta-n\gamma)xe^{-nx}+(\gamma-n\delta)e^{-nx},
\end{aligned}
$$

which is equal to x^3e^{-nx} for

$$-n\alpha=1,\quad 3\alpha-n\beta=0,\quad 2\beta-n\gamma=0\quad \text{og}\quad \gamma-n\delta=0,$$

thus

$$\alpha = -\frac{1}{n}, \quad \beta = \frac{3\alpha}{n} = -\frac{3}{n^2}, \quad \gamma = \frac{2\beta}{n} = -\frac{6}{n^3}, \quad \delta = \frac{\gamma}{n} = -\frac{6}{n^4}.$$

An integral is

$$\int x^3 e^{-nx}\,dx = -\frac{1}{n^4}\left(n^3x^3 + 3n^2x^2 + 6nx + 6\right)e^{-nx}.$$

Second variant. By successive partial integrations we get

$$\begin{aligned}
\int x^3 e^{-nx}\,dx &= -\frac{1}{n}\,x^3e^{-nx} + \frac{3}{n}\int x^2 e^{-nx}\,dx \\
&= -\frac{1}{n}\,x^3e^{-nx} - \frac{3}{n^2}\,x^2e^{-nx} + \frac{6}{n^2}\int x\,e^{-nx}\,dx \\
&= -\frac{1}{n}\,x^3e^{-nx} - \frac{3}{n^2}\,x^2e^{-nx} - \frac{6}{n^3}\,xe^{-nx} + \frac{6}{n^3}\int e^{-nx}\,dx \\
&= -\frac{1}{n}\,x^3e^{-nx} - \frac{3}{n^2}\,x^2e^{-nx} - \frac{6}{n^3}\,xe^{-nx} - \frac{6}{n^4}\,e^{-nx} \\
&= -\frac{1}{n^4}\left(n^3x^3 + 3n^2x^2 + 6nx + 6\right)e^{-nx}.
\end{aligned}$$

An exponential dominates every polynomial, so the integral above converges towards 0 for $x \to +\infty$, and the improper integral is convergent with the value

$$\int_0^{+\infty} x^3 e^{-nx}\,dx = 0 - \left(-\frac{6}{n^4}\right) = \frac{6}{n^4}.$$

Example 6.16 *1) Find the approximating polynomial of at most third degree with the point of expansion $t_0 = 0$ of the solution of the solution of differential equation*

$$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} + 2x = \cos 2t, \qquad t \in \mathbb{R},$$

which satisfies the initial conditions

$$x(0) = 0, \qquad x'(0) = 1, \qquad x''(0) = 1.$$

2) Prove that the improper integral

$$\int_0^{+\infty} \frac{2x+2}{x^3 - x^2 + 2}\,dx$$

is convergent, and find its value.

A. Approximating polynomial of a solution of a differential equation, and an improper integral.

D. Rearrange the differential equation and put $x = 0$. The improper integral is treated in the usual way.

I. 1) By a rearrangement of the differential equation and insertion of $x = 0$ we get

$$x^{(3)}(0) = \cos(2 \cdot 0) + x''(0) - 2x(0) = 1 + 1 - 0 = 2,$$

so

$$P_3(t) = x(0) + \frac{x'(0)}{1!}\, t + \frac{x''(0)}{2!}\, t^2 + \frac{x^{(3)}(0)}{3!}\, t^3 = t + \frac{1}{2}\, t^2 + \frac{1}{3}\, t^3.$$

2) The denominator

$$x^3 - x^2 + 2 = (x+1)\{(x-1)^2 + 1\}$$

is only 0 for $x = -1 \notin [0, +\infty[$, thus the integrand is defined and positive in i $[0, +\infty[$. By reduction

$$\frac{2x+2}{x^3 - x^2 + 2} = \frac{2x+2}{(x+1)\{(x-1)^2+1\}} = \frac{2}{(x-1)^2+1},$$

hence

$$\begin{aligned}\int_0^n \frac{2x+2}{x^3-x^2+2}\, dx &= 2\int_0^n \frac{dx}{(x-1)^2+1} = [2\,\text{Arctan}(x-1)]_0^n \\ &= 2\,\text{Arctan}(n-1) + 2\,\text{Arctan}\, 1 \\ &\to 2 \cdot \frac{\pi}{2} + 2 \cdot \frac{\pi}{4} = \frac{3\pi}{2}, \qquad \text{for } n \to +\infty,\end{aligned}$$

and the improper integral is convergent with the value

$$\int_0^{+\infty} \frac{2x+2}{x^3-x^2+2}\, dx = \frac{3\pi}{2}.$$

Example 6.17 *(Cf. Example 6.18)*

1) Prove that

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}.$$

2) Prove that the integral

$$\int_1^{+\infty} \frac{1}{x^2} \sqrt{1 - \frac{1}{x^2}}\, dx$$

is convergent, and find its value.

HINT. *Use some substitution to prove that*

$$\int_1^k \frac{1}{x^2} \sqrt{1 - \frac{1}{x^2}}\, dx = \int_{\frac{1}{k}}^1 \sqrt{1-u^2}\, du.$$

3) Prove that the integral

$$I_k = \int_1^{+\infty} \frac{1}{x} \sqrt{\frac{1}{x^{2k}} - \frac{1}{x^{4k}}} \, dx$$

is convergent for every $k \in \mathbb{N}$, and find its value.

A. Improper integrals.

D. Either use an area consideration, or some substitution.

I. 1) **First variant.** A graphical consideration shows that the integral can be interpreted as the area of one quarter of the unit disc, hence

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}.$$

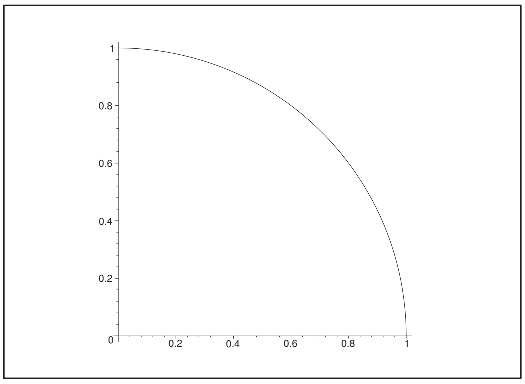

Figure 44: The graph of for $y = \sqrt{1-x^2}$, $x \in [0,1]$.

Second variant. If we instead apply the monotonous substitution

$$x = \sin t, \qquad t \in \left[0, \frac{\pi}{2}\right],$$

then

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}\,dx &= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t} \cdot \cos t\,dt \\
&= \int_0^{\frac{\pi}{2}} +\cos t \cdot \cos t\,dt \\
&= \int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \int_0^{\frac{\pi}{2}} \frac{\cos 2t + 1}{2}\,dt \\
&= \frac{1}{2} \cdot \frac{\pi}{2} + \left[\frac{\sin 2t}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} + 0 = \frac{\pi}{4}.
\end{aligned}$$

2) Choosing the substitution $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$, $u \in]0,1]$, we get

$$\int_1^k \frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\,dx = -\int_1^{\frac{1}{k}} \sqrt{1-u^2}\,du = \int_{\frac{1}{k}}^1 \sqrt{1-u^2}\,du.$$

The integrand is positive, so by taking the limit $k \to +\infty$,

$$\begin{aligned}
&\int_1^{+\infty} \frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\,dx = \lim_{x\to+\infty} \int_1^k \frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\,dx \\
&\qquad = \lim_{k\to+\infty} \int_{\frac{1}{k}}^1 \sqrt{1-u^2}\,du = \int_0^1 \sqrt{1-u^2}\,du = \frac{\pi}{4}.
\end{aligned}$$

3) Since

$$0 \le \frac{1}{x}\sqrt{\frac{1}{x^{2k}} - \frac{1}{x^{4k}}} < \frac{1}{x^{k+1}} \qquad \text{for } x \in [1,+\infty[, \quad k \in \mathbb{N},$$

the improper integral is convergent, because the improper integral of $1/x^{k+1}$ is convergent.

We find by the substitution

$$u=\frac{1}{x^k},\qquad du=-\frac{k}{x^{k+1}}\,dx,\qquad u\in\,]0,1[,$$

that

$$\begin{aligned}I_k &= \int_1^{+\infty}\frac{1}{x}\sqrt{\frac{1}{x^{2k}}-\frac{1}{x^{4k}}}\;dx=\int_1^{+\infty}\frac{1}{x^{k+1}}\sqrt{1-\frac{1}{x^{2k}}}\;dx\\ &= -\frac{1}{k}\int_1^0\sqrt{1-u^2}\,du=\frac{\pi}{4k}.\end{aligned}$$

Example 6.18 *(Cf. Example 6.17)*

1) Calculate the integral

$$\int_0^1\sqrt{1-x^2}\,dx.$$

2) Prove that the integral

$$\int_1^{+\infty}\frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\;dx$$

is convergent, and find its value.

HINT. *Use a substitution to prove that*

$$\int_1^k\frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\;dx=\int_{\frac{1}{k}}^1\sqrt{1-u^2}\,du.$$

3) Find the Taylor polynomial $P_6(t)$ of order 6 and point of expansion $t_0=0$ for the function

$$\varphi(t)=t^2\sqrt{1-t^2},\qquad t\in[-1,1].$$

Replace the integrand in

$$\int_1^{+\infty}\frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\;dx$$

by the function $P_6\left(\dfrac{1}{x}\right)$, and calculate this approximation of the integral from (2).

A. Improper integrals, and a Taylor expansion and an approximation of an integral.
The first two bullets are the same as the first two bullets of Example 6.17.

D. Either use an area consideration, or some substitution. Then a Taylor expansion, followed by a partial integration.

I. 1) **First variant.** A graphical consideration shows that the integral can be interpreted as the area of a quarter of the unit disc, so

$$\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4}.$$

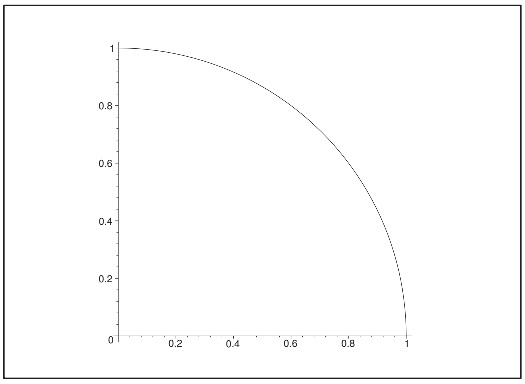

Figure 45: The graph of $y = \sqrt{1-x^2}$, $x \in [0,1]$.

Second variant. If we instead use the monotonous substitution

$$x = \sin t, \qquad t \in \left[0, \frac{\pi}{2}\right],$$

then

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,dx &= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cdot \cos t\,dt \\ &= \int_0^{\frac{\pi}{2}} +\cos t \cdot \cos t\,dt \\ &= \int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \int_0^{\frac{\pi}{2}} \frac{\cos 2t + 1}{2}\,dt \\ &= \frac{1}{2}\cdot\frac{\pi}{2} + \left[\frac{\sin 2t}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} + 0 = \frac{\pi}{4}.\end{aligned}$$

2) If we choose the substitution $u = \dfrac{1}{x}$, $du = -\dfrac{1}{x^2}\,dx$, $u \in]0,1]$, then

$$\int_1^k \frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\,dx = -\int_1^{\frac{1}{k}} \sqrt{1-u^2}\,du = \int_{\frac{1}{k}}^1 \sqrt{1-u^2}\,du.$$

The integrand is positive, and we get by taking the limit $k \to +\infty$ that

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\,dx &= \lim_{x\to+\infty}\int_1^k \frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\,dx \\ &= \lim_{k\to+\infty}\int_{\frac{1}{k}}^1 \sqrt{1-u^2}\,du = \int_0^1 \sqrt{1-u^2}\,du = \frac{\pi}{4}.\end{aligned}$$

3) From

$$\varphi(t) = t^2\sqrt{1-t^2} = t^2\left\{1 - \frac{1}{2}\,t^2 - \frac{1}{8}\,t^4 + t^4\,\varepsilon(t)\right\},$$

follows that

$$P_6(t) = t^2 - \frac{1}{2}\,t^4 - \frac{1}{8}\,t^6.$$

Then by insertion,

$$
\begin{aligned}
\int_1^{+\infty} \frac{1}{x^2}\sqrt{1-\frac{1}{x^2}}\,dx &\approx \int_1^{+\infty} P_6\left(\frac{1}{x}\right) dx \\
&= \int_1^{+\infty} \left\{\frac{1}{x^2} - \frac{1}{2}\,\frac{1}{x^4} - \frac{1}{8}\,\frac{1}{x^6}\right\} dx \\
&= \left[-\frac{1}{x} + \frac{1}{6}\,\frac{1}{x^3} + \frac{1}{40}\,\frac{1}{x^5}\right]_1^{+\infty} = 1 - \frac{1}{6} - \frac{1}{40} \\
&= \frac{1}{120}(120-20-3) = \frac{97}{120} \approx 0,808\,333.
\end{aligned}
$$

For comparison it was shown in Example 6.17 that the true value is $\frac{\pi}{4} \approx 0,785\,398$, so the error is $< 3\,\%$.

Example 6.19 *1) Decompose the fraction*

$$\frac{P(x)}{Q(x)} = \frac{x^2-10x-10}{(x-1)^2(x^2+6x+12)}, \qquad x \in \mathbb{R}\setminus\{1\}.$$

2) Prove that the integral

$$\int_2^{+\infty} \frac{P(x)}{Q(x)}\,dx$$

is convergent, and find its value.

A. Decomposition followed by an improper integral.

D. Decompose successively. Then consider the difference in degrees and apply the decomposition from (1).

I. 1) We first see that

$$x^2+6x+12 = (x+3)^2+3 \geq 3,$$

and the fraction is already written in its canonical form.

Then by decomposition,

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{1-10-10}{1+6+12}\cdot\frac{1}{(x-1)^2} + \frac{x^2-1+x-10}{(x-1)^2(x^2+6x+12)} + \frac{1}{(x-1)^2}\\
&= -\frac{1}{(x-1)^2} + \frac{x^2-10x-10+x^2+6x+12}{(x-1)^2(x^2+6x+12)}\\
&= -\frac{1}{(x-1)^2} + \frac{2x^2-4x+2}{(x-1)^2(x^2+6x+12)}\\
&= -\frac{1}{(x-1)^2} + \frac{2(x-1)^2}{(x-1)^2(x^2+6x+12)}\\
&= -\frac{1}{(x-1)^2} + \frac{2}{x^2+6x+12}.
\end{aligned}$$

C. TEST:

$$\begin{aligned}
-\frac{1}{(x-1)^2} + \frac{2}{x^2+6x+12} &= \frac{-x^2-6x-12+2x^2-4x+2}{(x-1)^2(x^2+6x+12)}\\
&= \frac{x^2-10x-10}{(x-1)^2(x^2+6x+12)}. \qquad \text{Q.E.D.}
\end{aligned}$$

2) We now make the following very practical rearrangement of the fraction,

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= -\frac{1}{(x-1)^2} + \frac{2}{x^2+6x+12} = -\frac{1}{(x-1)^2} + \frac{2}{(x+3)^2+3}\\
&= -\frac{1}{(x-1)^2} + \frac{2}{\sqrt{3}}\cdot\frac{1}{1+\left(\frac{x+3}{\sqrt{3}}\right)^2}\cdot\frac{1}{\sqrt{3}}.
\end{aligned}$$

Clearly, both terms can be integrated to infinity, and the singularity $x = 1$ does not lie in the interval $[2, +\infty[$. We conclude that the improper integral is convergent. Finally, its value is calculated in the following way,

$$\begin{aligned}
\int_2^{+\infty} \frac{P(x)}{Q(x)}\,dx &= -\int_2^{+\infty} \frac{1}{(x-1)^2}\,dx + \frac{2}{\sqrt{3}} \int_2^{+\infty} \frac{1}{1+\left(\dfrac{x+3}{\sqrt{3}}\right)^2} \cdot \frac{1}{\sqrt{3}}\,dx \\
&= \left[\frac{1}{x-1}\right]_2^{+\infty} + \frac{2}{\sqrt{3}} \left[\operatorname{Arctan}\left(\frac{x+3}{\sqrt{3}}\right)\right]_2^{+\infty} \\
&= -1 + \frac{2}{\sqrt{3}} \left\{\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{5}{\sqrt{3}}\right)\right\} \\
&= \frac{\pi}{\sqrt{3}} - \frac{2}{\sqrt{3}} \operatorname{Arctan}\left(\frac{5}{\sqrt{3}}\right) - 1.
\end{aligned}$$

Example 6.20 *Check for each of the following four integrals, whether it is convergent or divergent:*

1) $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{x}}\,dx$,

2) $\int_0^{\frac{\pi}{2}} \tan x\,dx$,

3) $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx$,

4) $\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{x}} + \tan x - \frac{1}{\cos x}\right) dx$.

A. Convergence/divergence of improper integrals.

D. Find the indefinite integrals and then take the limits.

I. 1) The integrand is here always positive, and an integral is

$$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} \qquad \text{for } x > 0.$$

It follows that

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0+} \int_a^{\frac{\pi}{2}} \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0+} \left[2\sqrt{x}\right]_a^{\frac{\pi}{2}} = \lim_{a\to 0+} \left\{2\sqrt{\frac{\pi}{2}} - 2\sqrt{a}\right\} = \sqrt{2\pi},$$

and we have convergence.

2) If $x \in \left[0, \frac{\pi}{2}\right[$, then $\tan x$ is positive, and an integral is

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln\cos x.$$

If follows from

$$\lim_{x\to\frac{\pi}{2}} \{-\ln\cos x\} = +\infty,$$

that the integral is divergent.

3) Since

$$\frac{1}{\cos x} > \frac{\sin x}{\cos x} = \tan x > 0 \qquad \text{for } 0 < x < \frac{\pi}{2},$$

and since $\int_0^{\frac{\pi}{2}} \tan x \, dx$ is divergent according to (2), the larger integral $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x} \, dx$ is also divergent.

ALTERNATIVELY an integral is

$$\begin{aligned}
\int \frac{1}{\cos x} \, dx &= \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx \\
&= \frac{1}{2} \int \left(\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} \right) d \sin x \\
&= \frac{1}{2} \{ \ln |1 + \sin x| - \ln |1 - \sin x| \} \\
&= \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right),
\end{aligned}$$

where

$$\frac{1}{2}\,\ln\left(\frac{1+\sin x}{1-\sin x}\right) \to +\infty \qquad \text{for } x \to \frac{\pi}{2}-,$$

and the integral is divergent.

4) Based on the results of (2) and (3) one might be misled to conclude that the present integral is divergent. This is not true. Let $x \in \left]0, \frac{\pi}{2}\right[$. Then an integral is

$$\begin{aligned}
&\int \left(\frac{1}{\sqrt{x}} + \tan x - \frac{1}{\cos x}\right) dx \\
&\quad = 2\sqrt{x} - \ln\cos x - \frac{1}{2}\,\ln\left(\frac{1+\sin x}{1-\sin x}\right) \\
&\quad = 2\sqrt{x} - \frac{1}{2}\,\ln\cos^2 x - \frac{1}{2}\,\ln\left(\frac{1+\sin x}{1-\sin x}\right) \\
&\quad = 2\sqrt{x} - \frac{1}{2}\,\ln\left(\frac{1+\sin x}{1-\sin x}\cdot(1-\sin^2 x)\right) \\
&\quad = 2\sqrt{x} - \frac{1}{2}\,\ln\left\{(1+\sin x)^2\right\} = 2\sqrt{x} - \ln(1+\sin x),
\end{aligned}$$

hence

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \left\{\frac{1}{\sqrt{x}} + \tan x - \frac{1}{\cos x}\right\} dx \\
&\quad = \lim_{a\to\frac{\pi}{2}-}\ \lim_{b\to 0+} \int_b^a \left\{\frac{1}{\sqrt{x}} + \tan x - \frac{1}{\cos x}\right\} dx \\
&\quad = \lim_{a\to\frac{\pi}{2}-}\ \lim_{b\to 0+} \left[2\sqrt{x} - \ln(1+\sin x)\right]_b^a \\
&\quad = 2\sqrt{\frac{\pi}{2}} - \ln 2 - 0 = \sqrt{2\pi} - \ln 2,
\end{aligned}$$

and the integral is convergent.

Remark. One shall strictly speaking also check the variation of the sign of the integrand before we go to the limit. This will here be left to the reader. ◊

Example 6.21 *Check if the improper integrals*

$$\int_0^1 \frac{1}{\tan x}\,dx, \qquad \int_0^1 \left(\frac{1}{\tan x} - \frac{1}{x}\right) dx,$$

are convergent. If so, find the value.

A. Improper integrals.

D. What is "wrong" in the integral? Check the sign of the integrand. Truncate the interval of integration and integrate. Finally, take the limit.

I. 1) Clearly, $\tan x > 0$ for $x \in]0,1]$, and the questionable point is $x = 0$. If we truncate by $\varepsilon > 0$, we get

$$\int_\varepsilon^1 \frac{1}{\tan x}\,dx = \int_\varepsilon^1 \frac{\cos x}{\sin x}\,dx = [\ln \sin x]_\varepsilon^1 = \ln \in 1 - \ln \sin \varepsilon.$$

Since $\ln \sin \varepsilon \to -(-\infty) = +\infty$ for $\varepsilon \to 0+$, this improper integral is divergent.

2) Since $\tan x > x$ for $x \in]0,1[$, we get $\dfrac{1}{\tan x} - \dfrac{1}{x} < 0$ in the same interval. The questionable point is $x = 0$. If we truncate by $\varepsilon > 0$, we get

$$\begin{aligned}\int_\varepsilon^1 \left(\frac{1}{\tan x} - \frac{1}{x}\right) &= [\ln \sin x - \ln x]_\varepsilon^1 = \left[\ln \frac{\sin x}{x}\right]_\varepsilon^1 \\ &= \ln \sin 1 - \ln \frac{\sin \varepsilon}{\varepsilon} \to \ln \sin 1 \quad \text{for } \varepsilon \to 0+,\end{aligned}$$

where we have used the well-known result

$$\lim_{\varepsilon \to 0+} \frac{\sin \varepsilon}{\varepsilon} = 1.$$

Example 6.22 *Check if the improper integral*

$$\int_1^{+\infty} \frac{\sin \frac{1}{x}}{x^2}\, dx$$

is convergent or divergent.

A. Improper integral.

D. If $x > \frac{1}{\pi}$, then the integrand is positive. By a finite truncation and the substitution $t = \frac{1}{x}$ we get

$$\begin{aligned}\int_1^n \frac{\sin \frac{1}{x}}{x^2}\, dx &= \int_{\frac{1}{n}}^1 \sin t\, dt = [-\cos t]_{\frac{1}{n}}^1 \\ &= \cos \frac{1}{n} - \cos 1 \to 1 - \cos 1 \qquad \text{for } n \to +\infty,\end{aligned}$$

thus the improper integral is convergent, and its value is

$$\int_1^{+\infty} \frac{\sin \frac{1}{x}}{x^2}\, dx = 1 - \cos 1.$$

Remark. Strictly speaking one is only asked about the convergence or the divergence. Therefore, the following is sufficient:

The integrand is continuous in the closed and bounded interval $\left[1, \frac{1}{\pi}\right]$, hence the integral exists in this interval.
For $x \in \left[\frac{1}{\pi}, +\infty\right[$ we get the estimate

$$0 \le \frac{\sin \frac{1}{x}}{x^2} \le \frac{1}{x^2}.$$

Since $\frac{1}{x^2}$ can be integrated to infinity, the same holds for the smaller integrand, and the improper integral is convergent. $\Diamond$