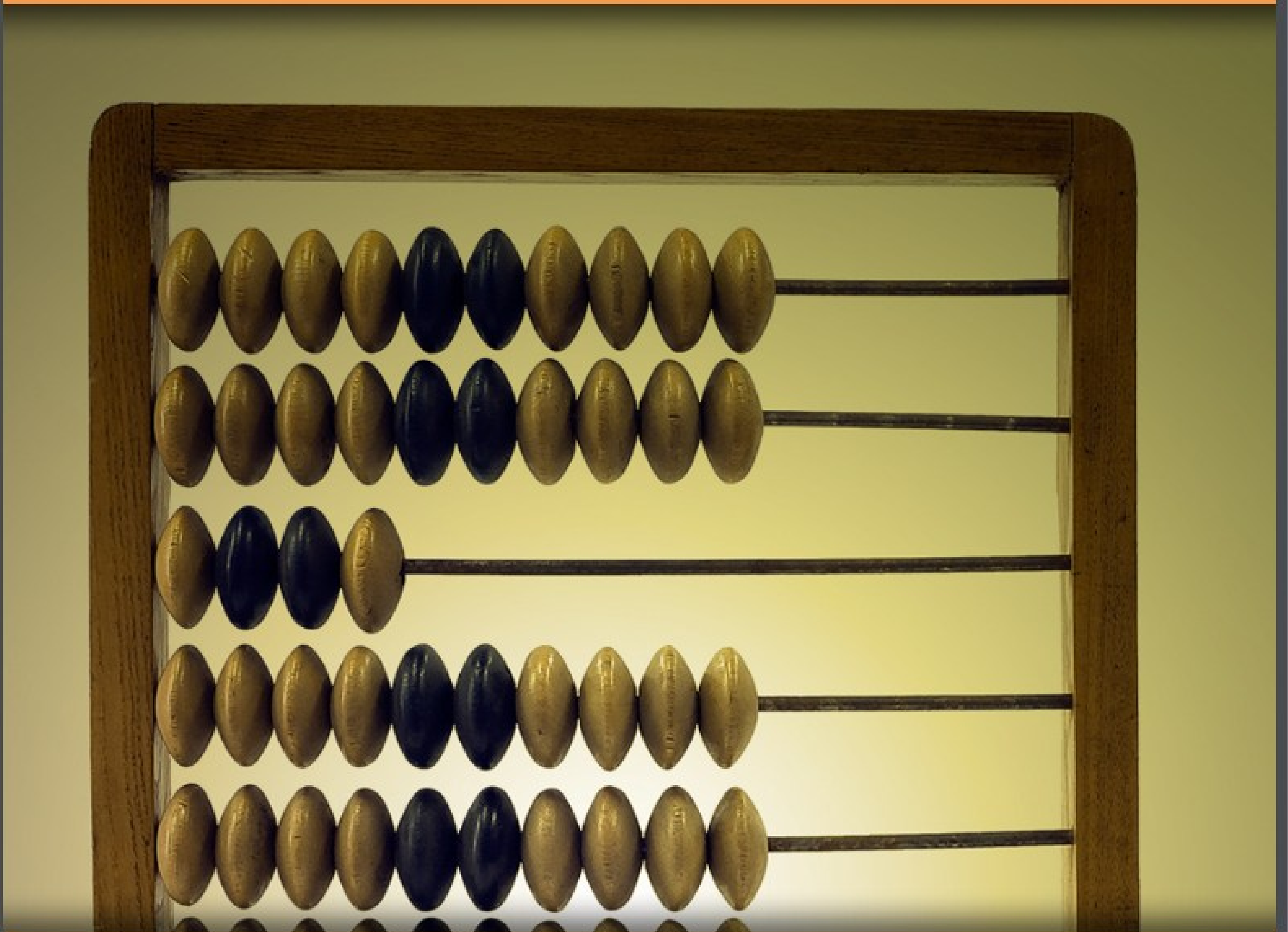
Real Functions in One Variable - Integrals...

Leif Mejlbro



Leif Meilbro

Real Functions in One Variable

Examples of Integrals

Calculus 1-c3

Real Functions in One Variable – Examples of Integrals – Calculus 1c-3

ISBN 978-87-7681-238-6

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Preface

In this volume I present some examples of *Integrals*, cf. also *Calculus 1a, Functions of One Variable*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
24th July 2007

1 Partial integration

Example 1.1 *Calculate the integrals*

$$(1)\quad \int x\,e^{x}\,dx, \qquad (2)\quad \int x\,e^{x^2}\,dx.$$

A. Integration.

D. Apply partial integration in (1), and integration by a substitution in (2).

I. 1) We get by a partial integration

$$\int x\,e^{x}\,dx = x\,e^{x} - \int 1\cdot e^{x}\,dx = x\,e^{x} - e^{x} = (x-1)\,e^{x}.$$

2) Applying the substitution $u = x^2$, $du = 2x\,dx$, we get

$$\int x\,e^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\cdot 2x\,dx = \frac{1}{2}\int e^{x^2}\,d\left(x^2\right) = \frac{1}{2}\,e^{x^2}.$$

C. Test. We get by a differentiation

1) $\dfrac{d}{dx}\{(x-1)e^{x}\} = 1\cdot e^{x} + (x-1)e^{x} = x\,e^{x}.$

2) $\dfrac{d}{dx}\left\{\dfrac{1}{2}\,e^{x^2}\right\} = \dfrac{1}{2}\cdot 2x\cdot e^{x^2} = x\,e^{x^2}.$ Q.E.D.

Example 1.2 *Calculate the integrals*

$$(1)\quad \int 3x\,\cos^2 x\,dx, \qquad (2)\quad \int 2x^2\,\cos x\,dx.$$

A. Integration.

D. In (1) we use integration ny the substitution $u = x^2$, and in (2) we apply partial integration.

I. 1) When $u = x^2$, we see that $du = 2x\,dx$, so

$$\begin{aligned}\int 3x\,\cos x^2\,dx &= \frac{3}{2}\int_{u=x^2} \cos u\,du = \left[\frac{3}{2}\,\sin u\right]_{u=x^2}\\ &= \frac{3}{2}\,\sin x^2.\end{aligned}$$

2) In this case we get by a series of partial integrations that

$$\begin{aligned}&\int 2x^2\,\cos x\,dx = 2x^2\,\sin x - 4\int x\cdot\sin x\,dx\\ &\qquad = 2x^2\,\sin x + 4x\cdot\cos x - 4x\int 1\cdot\cos x\,dx\\ &\qquad = 2x^2\,\sin x + 4x\,\cos x - 4\,\sin x.\end{aligned}$$

C. TEST. We get by differentiation

1) $\dfrac{d}{dx}\left\{\dfrac{3}{2}\,\sin\left(x^2\right)\right\} = \dfrac{3}{2}\,\cos\left(x^2\right)\cdot 2x)3x\,\cos\left(x^2\right).$ Q.E.D.

2)

$$\begin{aligned}&\frac{d}{dx}\left\{2x^2\,\sin x + 4x\,\cos x - 4\,\sin x\right\}\\ &\qquad = 4x\,\sin x + 2x^2\,\cos x + 4\,\cos x - 4x\,\sin x - 4\,\cos x\\ &\qquad = 2x^2\,\cos x. \qquad \text{Q.E.D.}\end{aligned}$$

Example 1.3 *1) Calculate the integral*

$$\int \frac{\ln(1+u)}{u^3}\,du, \qquad u > 0,$$

by first applying partial integration.

2) Find the complete solution of the differential equation

$$\frac{dx}{dt} + \frac{1}{t}\,x = \ln\left(1+\frac{1}{t}\right), \qquad t > 0.$$

(In one of the occurring integrals one may introduce the substitution $t = \frac{1}{u}$).

A. 1) Integral.

2) Linear differential equation of first order, where one possibly should apply the result from (1).

D. 1) Apply a partial integration.

2) Solve the differential equation.

I. 1) When $u > 0$, we get by a partial integration that

$$\int \frac{1}{u^3}\,\ln(1+u)\,du = \left(-\frac{1}{2}\cdot\frac{1}{u^2}\right)\ln(1+u) - \int\left(-\frac{1}{2}\,\frac{1}{u^2}\right)\frac{1}{1+u}\,du.$$

In the latter integral we decompose the integrand

$$\frac{1}{2}\cdot\frac{1}{u^2}\cdot\frac{1}{1+u} = \frac{a}{u^2}+\frac{b}{u}+\frac{c}{1+u}.$$

It follows immediately that $c = \dfrac{1}{2}$, hence

$$\begin{aligned}\frac{a}{u^2}+\frac{b}{u} &= \frac{1}{2}\,\frac{1}{u^2(1+u)} - \frac{1}{2}\cdot\frac{1-u^2}{u^2(1+u)}\\ &= \frac{1}{2}\cdot\frac{1-u}{u^2} = \frac{1}{2}\cdot\frac{1}{u^2} - \frac{1}{2}\cdot\frac{1}{u}.\end{aligned}$$

Since $u > 0$, we get by insertion that

$$\begin{aligned}\int\frac{\ln(1+u)}{u^3}\,du &= -\frac{1}{2}\,\frac{1}{u^2}\,\ln(1+u) + \frac{1}{2}\int\left\{\frac{1}{1+u}+\frac{1}{u^2}-\frac{1}{u}\right\}du\\ &= -\frac{1}{2}\,\frac{1}{u^2}\,\ln(1+u) + \frac{1}{2}\,\ln(1+u) - \frac{1}{2}\,\frac{1}{u} - \frac{1}{2}\,\ln u\\ &= \frac{1}{2}\left(1-\frac{1}{u^2}\right)\ln(1+u) - \frac{1}{2}\,\frac{1}{u} - \frac{1}{2}\,\ln u.\end{aligned}$$

C. Test. By differentiation we get for $u > 0$,

$$\begin{aligned}&\frac{d}{du}\left\{\frac{1}{2}\left(1-\frac{1}{u^2}\right)\ln(1+u) - \frac{1}{2}\,\frac{1}{u} - \frac{1}{2}\,\ln u\right\}\\ &= \frac{1}{2}\cdot\frac{2}{u^3}\,\ln(1+u) + \frac{1}{2}\,\frac{1}{u^2} - \frac{1}{2}\,\frac{1}{u}\\ &= \frac{\ln(1+u)}{u^3} + \frac{1}{2}\left\{\frac{u^2-1}{u^2(u+1)} + \frac{1}{u^2}\,(1-u)\right\}\\ &= \frac{\ln(1+u)}{u^3} + \frac{1}{2}\left\{\frac{u-1}{u^2}+\frac{1-u}{u^2}\right\} = \frac{\ln(1+u)}{u^3}. \qquad \text{Q.E.D.}\end{aligned}$$

2) When the differential equation is multiplied by $t > 0$, we get

$$\frac{d}{dt}\,(t\cdot x) = t\,\frac{dx}{dt} + 1\cdot x = t\,\ln\left(1+\frac{1}{t}\right),$$

hence the corresponding homogeneous equation has the solutions $\dfrac{c}{t}$, $c \in \mathbb{R}$ and $t \neq 0$, and a particular integral is given by

$$\begin{aligned}
\frac{1}{t}\int t\,\ln\left(1+\frac{1}{t}\right)dt &= \frac{1}{t}\int_{u=\frac{1}{t}} \frac{1}{u}\,\ln(1+u)\cdot\left(-\frac{1}{u^2}\right)du \\
&= -\frac{1}{t}\int_{u=\frac{1}{t}} \frac{\ln(1+u)}{u^3}\,du \\
&= \frac{1}{2t}\left[\left(\frac{1}{u^2}-1\right)\ln(1+u)+\frac{1}{u}+\ln u\right]_{u=\frac{1}{t}} \\
&= \frac{1}{2t}\left\{(t^2-1)\ln\left(\frac{1+t}{t}\right)+t-\ln t\right\} \\
&= \frac{1}{2t}\,(t^2-1)\ln(t+1)-\frac{1}{2}\,(t^2-1)\ln t+\frac{1}{2t}\cdot t-\frac{1}{2t}\,\ln t \\
&= \frac{1}{2}\left(t-\frac{1}{t}\right)\ln(t+1)-\frac{1}{2}\,t\,\ln t+\frac{1}{2}.
\end{aligned}$$

The complete solution is then

$$x=\frac{1}{2}\left(t-\frac{1}{t}\right)\ln(t+1)-\frac{1}{2}\,t\,\ln t+c\cdot\frac{1}{t},\qquad t>0,\quad c\in\mathbb{R}.$$

C. Test. With the x above we get

$$\begin{aligned}
\frac{dx}{dt}+\frac{x}{t} &= \frac{1}{2}\left(1+\frac{1}{t^2}\right)\ln(t+1)+\frac{1}{2}\left(t-\frac{1}{t}\right)\cdot\frac{1}{t+1}-\frac{1}{2}\,\ln t-\frac{1}{2}-\frac{c}{t^2} \\
&\qquad +\frac{1}{2}\left(1-\frac{1}{t^2}\right)\ln(t+1)-\frac{1}{2}\,\ln t+\frac{1}{2}\,\frac{1}{t}+\frac{c}{t^2} \\
&= \ln(t+1)-\ln t+\frac{1}{2}\left\{\frac{t^2-1}{t}\cdot\frac{1}{t+1}-1+\frac{1}{t}\right\} \\
&= \ln\left(1+\frac{1}{t}\right)+\frac{1}{2}\left\{\frac{t-1}{t}-\frac{t-1}{t}\right\}=\ln\left(1+\frac{1}{t}\right),. \qquad \text{Q.E.D.}
\end{aligned}$$

2 Integration by simple substitutions

Example 2.1 *Calculate the integrals*

$$(1)\quad \int \frac{1}{\sqrt{4x^2-1}}\,dx, \quad x > \frac{1}{2}, \qquad (2)\quad \int \frac{1}{\sqrt{1-4x^2}}\,dx, \quad |x| < \frac{1}{2}.$$

Write a MAPLE programme for the first integral.

A. Integral.

D. Find convenient substitutions:

1) Since the structure of the denominator is $\sqrt{u^2-1}$, choose $u = \cosh t$.

2) Since the structure of the denominator is $\sqrt{1-u^2}$, choose by analogy $u = \cos t$, (where $u = \sin t$ also would give us the result).

I. a) If we put $x = \dfrac{1}{2}\cosh t$, $t > 0$, we see that this substitution is monotonous, and $t = \ln\left(2x+\sqrt{4x^2-1}\right)$. Hence, $dx = \dfrac{1}{2}\sinh t\,dt$, $\sinh t > 0$, i.e.

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2-1}}\,dx &= \int \frac{\frac{1}{2}\sinh t}{\sqrt{4\cdot\frac{1}{4}\cosh^2 t - 1}}\,dt = \frac{1}{2}\int \frac{\sinh t}{\sqrt{\cosh^2 t - 1}}\,dt \\ &= \frac{1}{2}\int \frac{\sinh t}{+\sinh t}\,dt = \frac{t}{2} = \frac{1}{2}\ln\left(2x+\sqrt{4x^2-1}\right).\end{aligned}$$

A possible MAPLE programme is the following

```
expr = 1/(sqrt(4*x^2-1));
```

$$\begin{cases} 0 & x < \dfrac{1}{2} \\ \\ \dfrac{1}{\sqrt{4x^2-1}} & \text{otherwise} \end{cases} = \frac{1}{\sqrt{4x^2-1}}$$

which suffices in our case, though it is wrong for $x < \frac{1}{2}$. Then continue by the command `int(expr,x);` by which we get the resultat

$$\begin{cases} 0 & x \le \dfrac{1}{2} \\ \\ \dfrac{1}{4}\ln(x\sqrt{4}+\sqrt{4x^2-1})\sqrt{4} & x > \dfrac{1}{2} \end{cases}$$

This is acceptable, because we shall only need the result for $x > \frac{1}{2}$. Notice that one still must reduce $\sqrt{4} = 2$ oneself.

b) If we put $x = \frac{1}{2}\cos t$, $t \in]0,\pi[$, this substitution is monotonous, $t = \text{Arccos}(2x)$, and we get that $dx = -\frac{1}{2}\sin t\,dt$, $\sin t > 0$, $t \in]0,\pi[$, i.e.

$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \int \frac{-\dfrac{1}{2}\sin t}{\sqrt{1-4\cdot\dfrac{1}{4}\cos^2 t}}\,dt = -\frac{1}{2}\int \frac{\sin t}{\sqrt{1-\cos^2 t}}\,dt \\ &= -\frac{1}{2}\int \frac{\sin t}{+\sin t}\,dt = -\frac{t}{2} = -\frac{1}{2}\,\text{Arccos}(2x).\end{aligned}$$

C. Test.

a) When $x = \frac{1}{2}\ln\left(2x+\sqrt{4x^2-1}\right)$, $x > \frac{1}{2}$, fås

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2}\cdot\frac{1}{2x+\sqrt{4x^2-1}}\left(2+\frac{1}{8}\,\frac{8x}{\sqrt{4x^2-1}}\right) \\ &= \frac{1}{2x+\sqrt{4x^2-1}}\left(1+\frac{2x}{\sqrt{4x^2-1}}\right) \\ &= \frac{1}{2x+\sqrt{4x^2-1}}\cdot\frac{\sqrt{4x^2-1}+2x}{\sqrt{4x^2-1}} \\ &= \frac{1}{\sqrt{4x^2-1}}. \qquad \text{Q.E.D.}\end{aligned}$$

b) When $x = -\frac{1}{2}\,\text{Arccos}(2x)$, $x \in \left]-\frac{1}{2},\frac{1}{2}\right[$, fås

$$\frac{dx}{dt} = -\frac{1}{2}\cdot\frac{-1}{\sqrt{1-(2x)^2}}\cdot 2 = \frac{1}{\sqrt{1-4x^2}}. \qquad \text{Q.E.D.}$$

Example 2.2 *Calculate the integrals*

$$(1)\quad \int \frac{1}{\sqrt{1-2x^2}}\,dx, \quad (2)\quad \int \frac{1}{\sqrt{x^2+2}}\,dx, \quad (3)\quad \int \frac{1}{x^2+2}\,dx.$$

A. Integrals.

D. The methods here are:

1) Substitute, such that the denominator takes on the form $k\sqrt{1-t^2}$. Then integrate.
2) Substitute, such that the denominator takes on the form $k\sqrt{1+t^2}$. Then integrate.
3) Substitute, such that the denominator takes on the form $k(1+t^2)$. Then integrate.

I. 1) We must require that $2x^2 < 1$, i.e. $|x| < \dfrac{1}{\sqrt{2}}$. Then we substitute $t = \sqrt{2}\,x$, i.e. $x = \dfrac{1}{\sqrt{2}}\,t$,

$$\begin{aligned}\int \frac{1}{\sqrt{1-2x^2}}\,dx &= \int_{t=\sqrt{2}\,x} \frac{1}{\sqrt{1-t^2}}\cdot\frac{1}{\sqrt{2}}\,dt = \left[\frac{1}{\sqrt{2}}\,\text{Arcsin}\,t\right]_{t=\sqrt{2}\,x}\\ &= \frac{1}{\sqrt{2}}\,\text{Arcsin}\left(\sqrt{2}\,x\right).\end{aligned}$$

2) Here we get for every $x \in \mathbb{R}$ that

$$\int \frac{1}{\sqrt{x^2+2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{1+\left(\dfrac{x}{\sqrt{2}}\right)^2}}\,dx = \int_{t=\frac{x}{\sqrt{2}}} \frac{1}{\sqrt{1+t^2}}\,dt.$$

Now

$$\frac{d(\text{Arsinh}\,y)}{dy} = \frac{1}{\sqrt{1+y^2}},$$

thus

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2}}\,dx &= [\text{Arsinh}\,t]_{t=\frac{x}{\sqrt{2}}} = \ln\left(\frac{x}{\sqrt{2}}+\sqrt{\frac{x^2}{2}+1}\right)\\ &= \ln\left(x+\sqrt{x^2+2}\right) - \frac{1}{2}\,\ln 2.\end{aligned}$$

3) Here we get

$$\begin{aligned}\int \frac{1}{x^2+2}\,dx &= \frac{1}{2}\int \frac{1}{1+\left(\dfrac{x}{\sqrt{2}}\right)^2}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{1+\left(\dfrac{x}{\sqrt{2}}\right)^2}\,d\left(\frac{x}{\sqrt{2}}\right)\\ &= \frac{1}{\sqrt{2}}\,\text{Arctan}\left(\frac{x}{\sqrt{2}}\right).\end{aligned}$$

C. Test.

1) When

$$f(x) = \frac{1}{\sqrt{2}}\,\text{Arcsin}(\sqrt{2}\,x),$$

we get

$$f'(x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1-(\sqrt{2}\,x)^2}} \cdot \sqrt{2} = \frac{1}{\sqrt{1-2x^2}},$$

Q.E.D.

2) When

$$f(x) = \ln(x+\sqrt{x^2+2}) - \frac{1}{2}\,\ln 2,$$

we get

$$\begin{aligned} f'(x) &= \frac{1}{x+\sqrt{x^2+2}}\left\{1+\frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+2}}\right\} \\ &= \frac{1}{x+\sqrt{x^2+2}} \cdot \frac{\sqrt{x^2+2}+x}{\sqrt{x^2+2}} = \frac{1}{\sqrt{x^2+2}}, \end{aligned}$$

Q.E.D.

3) When

$$f(x) = \frac{1}{\sqrt{2}}\,\text{Arctan}\left(\frac{x}{\sqrt{2}}\right),$$

we get

$$f'(x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{1+\left(\dfrac{x}{\sqrt{2}}\right)^2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \cdot \frac{1}{1+\dfrac{x^2}{2}} = \frac{1}{2+x^2},$$

Q.E.D.

Example 2.3 *Calculate the integral*

$$\int \left\{ \frac{3}{\sqrt{2x^2+1}} + \frac{4}{(x^2+4)^2} \right\} dx.$$

A. Integral

D. Split the integral into a sum of two integrals. In the former one we use the substitution $t = \sqrt{2}\,x$, and in the latter one we use the substitution $t = \dfrac{x}{2}$.

I. We first split the integral in the following way,

$$\int \left\{ \frac{3}{\sqrt{2x^2+1}} + \frac{4}{\left(x^2+4\right)^2} \right\} dx = \int \frac{3}{\sqrt{2x^2+1}}\, dx + \int \frac{4}{\left(x^2+4\right)^2}\, dx = I_1 + I_2.$$

Then perform the following separate calculations,

$$\begin{aligned} I_1 &= \int \frac{3}{\sqrt{2x^2}}\, dx = \int_{t=\sqrt{2}\,x} \frac{3}{\sqrt{t^2+1}} \cdot \frac{1}{\sqrt{2}}\, dt \\ &= \frac{3}{\sqrt{2}} \left[\text{Arsinh}\ t \right]_{t=\sqrt{2}\,x} = \frac{3}{\sqrt{2}} \ln \left(\sqrt{2}\,x + \sqrt{2x^2+1} \right) \\ &= \frac{3}{\sqrt{2}} \ln \left(x + \sqrt{x^2 + \frac{1}{2}} \right) - \frac{3}{2\sqrt{2}} \ln 2, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int \frac{4}{\left(x^2+4\right)^2}\, dx = \frac{1}{4} \int \frac{1}{\left(1 + \left(\frac{x}{2}\right)^2\right)^2}\, dx \\ &= \frac{1}{2} \int_{t=\frac{x}{2}} \frac{1}{\left(1+t^2\right)^2}\, dt = \frac{1}{4} \left[\frac{t}{t^2+1} + \text{ Arctan}\ t \right]_{t=\frac{x}{2}} \\ &= \frac{1}{2}\, \frac{x}{x^2+4} + \frac{1}{4} \text{Arctan}\ \frac{x}{2}. \end{aligned}$$

Suppressing the arbitrary constants we finally get

$$\int \left\{ \frac{3}{\sqrt{2x^2+1}} + \frac{4}{\left(x^2+4\right)^2} \right\} dx = \frac{3}{\sqrt{2}} \,\ln\left(x + \sqrt{x^2+\frac{1}{2}} \right) + \frac{1}{2}\,\frac{x}{x^2+4} + \frac{1}{4}\,\text{Arctan}\,\frac{x}{2}.$$

C. In the TEST it will ease matters if we check each subresult:

I_1**.** When

$$f(x) = \frac{3}{\sqrt{2}}\,\ln\left(x + \sqrt{x^2+\frac{1}{2}} \right),$$

we get

$$\begin{aligned} f'(x) &= \frac{3}{\sqrt{2}} \cdot \frac{1}{x+\sqrt{x^2+\dfrac{1}{2}}} \cdot \left(1 + \frac{x}{\sqrt{x^2+\dfrac{1}{2}}} \right) \\ &= \frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{x^2+\dfrac{1}{2}}} = \frac{3}{2x^2+1}. \end{aligned}$$

I_2**.** When

$$g(x) = \frac{1}{2} \cdot \frac{x}{x^2+4} + \frac{1}{4}\,\text{Arctan}\,\frac{x}{2},$$

we get

$$\begin{aligned} g'(x) &= \frac{1}{2} \cdot \frac{1\cdot(x^2+4) - x\cdot 2x}{\left(x^2+4\right)^2} + \frac{1}{4} \cdot \frac{1}{1+\dfrac{x^2}{4}} \cdot \frac{1}{2} \\ &= \frac{1}{2} \cdot \frac{-x^2+4}{\left(x^2+4\right)^2} + \frac{1}{2} \cdot \frac{1}{x^2+4} = \frac{1}{2} \cdot \frac{(-x^2+4)+(x^2+4)}{(x^2+4)^1} \\ &= \frac{4}{(x^2+4)^2}. \end{aligned}$$

Example 2.4 *Calculate the integral*

$$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx, \qquad x>0,$$

by introducing the substitution $x = t^2$*.*

A. Integral.

D. Introduce the suggested substitution, which is nothing but naming the unpleasant $\sqrt{x}$ something different, here t. In one of the variants one may start by reducing the integrand before the integration.

I. First variant. When $x \neq 1$, $x > 0$, we choose $t = +\sqrt{x} > 0$ as our monotonous substitution, and it follows that $dx = 2r\,dt$. Then by insertion for $x > 0$,

$$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx = \int_{t=\sqrt{x}} \frac{1-t}{1+t} \cdot 2t\,dt = 2\int_{t=\sqrt{x}} \frac{t-t^2}{1+t}\,dt.$$

Since

$$t - t^2 = -\left(t^2+1\right) + 2(t+1) - 2 = (t+1)(-t+2) - 2,$$

a continuation of the calculations gives

$$\begin{aligned}\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx &= 2\int_{t=\sqrt{x}} \left\{-t+2-\frac{2}{t+1}\right\}dt \\ &= 2\left[-\frac{t^2}{2}+2t-2\,\ln|t+1|\right]_{t=\sqrt{x}} \\ &= -x+4\sqrt{x}-4\,\ln(\sqrt{x}+1).\end{aligned}$$

Second variant. ALTERNATIVELY we first reduce the integrand in the following way:

$$\begin{aligned}\frac{1-\sqrt{x}}{1+\sqrt{x}} &= -1+\frac{2}{1+\sqrt{x}} = -1+\frac{1}{2\sqrt{x}}\cdot\frac{4\sqrt{x}}{1+\sqrt{x}} \\ &= -1+\frac{4}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}\cdot\frac{4}{1+\sqrt{x}}.\end{aligned}$$

Then for $x > 0$,

$$\begin{aligned}\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx &= \int \left\{-1+\frac{4}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}\cdot\frac{4}{1+\sqrt{x}}\right\}dx \\ &= -x+4\sqrt{x}-4\int \frac{d\left(\sqrt{x}+1\right)}{\sqrt{x}+1} \\ &= -x+4\sqrt{x}-4\,\ln(1+\sqrt{x}).\end{aligned}$$

C. TEST. Let

$$f(x) = -x+4\sqrt{x}-4\,\ln(\sqrt{x}+1), \qquad x > 0.$$

Then by a differentiation,

$$\begin{aligned}f'(x) &= -1+\frac{4}{2}\cdot\frac{1}{\sqrt{x}}-\frac{4}{\sqrt{x}+1}\cdot\frac{1}{2\sqrt{x}} \\ &= \frac{1}{1+\sqrt{x}}\cdot\left\{-1-\sqrt{x}+\frac{1}{\sqrt{x}}(1+\sqrt{x})-\frac{2}{\sqrt{x}}\right\} \\ &= \frac{1}{1+\sqrt{x}}\left\{-1-\sqrt{x}+2\right\} \\ &= \frac{1-\sqrt{x}}{1+\sqrt{x}}.\end{aligned}$$

Q.E.D.

Example 2.5 *Explain why*

$$\int \frac{1}{\sqrt{1-(1-x^2)}}\,d\sqrt{1-x^2} = \text{ Arcsin } \sqrt{1-x^2}.$$

This integral can be found in many ways. The following three functions can actually all be used for $x \in]0,1[$*:*

$$-\text{ Arcsin } x, \qquad -\text{ Arccos } \sqrt{1-x^2}, \qquad \text{Arccos } x.$$

Explain why a calculation of the integral can give four different results.
What is the relationship between those four functions?

A. Discuss an integral and prove that four apparent different functions are all integrals of the same integrand.

D. 1) Substitute.

2) Differentiate the claimed integrals and find their relationships for $x \in]0,1[$ by insertion of one point.

I. 1) It is obvious that we must assume that $x \in]-1,1[$. When we substitute $t = \sqrt{1-x^2}$ we get

$$\begin{aligned}\int \frac{1}{\sqrt{1-(1-x^2)}}\,d\sqrt{1-x^2} &= \int_{t=\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}} \\ &= [\text{Arcsin } t]_{t=\sqrt{1-x^2}} = \text{ Arcsin } \sqrt{1-x^2}.\end{aligned}$$

2) Now let $x \in]0,1[$. Then we get (apart from constants)

$$\begin{aligned}\int \frac{1}{\sqrt{1-(1-x^2)}}\,d\sqrt{1-x^2} &= \int \frac{1}{\sqrt{x^2}} \cdot \frac{-x}{\sqrt{1-x^2}}\,dx \\ &= -\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} -\text{ Arcsin } x, \\ \quad \text{Arccos } x.\end{cases}\end{aligned}$$

We get by another rearrangement,

$$\begin{aligned}\int \frac{1}{\sqrt{1-(1-x^2)}}\,d\sqrt{1-x^2} &= -\left\{-\int \frac{1}{\sqrt{1-(1-x^2)}}\,d\sqrt{1-x^2}\right\} \\ &= -\text{ Arccos } \sqrt{1-x^2}.\end{aligned}$$

Hence, the four functions Arcsin $\sqrt{1-x^2}$, $-$ Arcsin x, $-$ Arccos $\sqrt{1-x^2}$ and Arccos x are all integrals of the same function in $]0,1[$. They only differ from each other by a constant in the interval $]0,1[$.

When $x = \frac{1}{\sqrt{2}} \in]0,1[$, we get

$$\text{Arcsin } \sqrt{1-x^2} = \text{ Arcsin } \frac{1}{\sqrt{2}} = \frac{\pi}{4},$$

$$-\text{ Arcsin } x = -\text{ Arcsin } \frac{1}{\sqrt{2}} = -\frac{\pi}{4},$$

$$-\text{ Arccos } \sqrt{1-x^2} = -\text{ Arccos } \frac{1}{\sqrt{2}} = -\frac{\pi}{4},$$

$$\text{Arccos } x = \text{ Arccos } \frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

Thus when $x \in]0,1[$,

$$\begin{aligned}\text{Arcsin } \sqrt{1-x^2} &= \text{ Arccos } x = \frac{\pi}{2} - \text{ Arcsin } x \\ &= \frac{\pi}{2} - \text{ Arccos } \sqrt{1-x^2},\end{aligned}$$

and we have found the not surprising relationship between the four functions.

Example 2.6 *Calculate the integral*

$$\int \frac{1}{x\sqrt{x^2-1}}\, dx, \qquad x > 1,$$

by introducing the substitution $x = \dfrac{1}{\cos t}$, $t \in \left]0, \dfrac{\pi}{2}\right[$.

A. Integral.

D. Introduce the suggested substitution.

I. If we choose $x = \dfrac{1}{\cos t}$, we see that $x(t)$ runs *monotonously* through the interval $]1, +\infty[$ (a necessary condition for the substitution), when t runs through $\left]0, \dfrac{\pi}{2}\right[$. Here

$$t = \text{Arccos}\left(\frac{1}{x}\right) \qquad \text{og} \qquad dx = \frac{\sin t}{\cos^2 t}\, dt,$$

and since $\sin t > 0$ in the interval we get

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2-1}}\, dx &= \int_{t=\text{ Arccos}(\frac{1}{x})} \frac{1}{\dfrac{1}{\cos t}\cdot\sqrt{\dfrac{1}{\cos^2 t}-1}} \cdot \frac{\sin t}{\cos^2 t}\, dt \\
&= \int_{t=\text{ Arccos}(\frac{1}{x})} \frac{\sin t}{\cos^2 t} \cdot \frac{1}{\sqrt{\dfrac{1-\cos^2 t}{\cos^2 t}}}\, dt \\
&= \int_{t=\text{ Arccos}(\frac{1}{x})} dt = \text{ Arccos}\, \frac{1}{x}.
\end{aligned}$$

C. Test. When $x > 1$, we get by differentiation that

$$\begin{aligned}
\frac{d}{dx}\text{Arccos}\, \frac{1}{x} &= -\frac{1}{\sqrt{1-\dfrac{1}{x^2}}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{x^2\sqrt{\dfrac{x^2-1}{x^2}}} \\
&= \frac{1}{x\sqrt{x^2-1}} \qquad \text{Q.E.D..}
\end{aligned}$$

Example 2.7 *Calculate the integral*

$$\int \frac{1}{x\sqrt{x^2+1}}\, dx, \qquad x > 0,$$

by introducing the substitution $x = \dfrac{1}{\sinh t}$, $t > 0$.

A. Integral.

D. Apply the given substitution.

I. If we choose $x = \dfrac{1}{\sinh t}$, $t > 0$, as our substitution, we see that this is *monotonously decreasing*and that

$$t = \text{ Arsinh}\, \frac{1}{x}, \quad x > 0, \qquad \text{og} \qquad dx = -\frac{\cosh t}{\sinh^2}\, dt.$$

Then by insertion,

$$
\begin{aligned}
\int \frac{1}{x\sqrt{x^2+1}}\,dx &= \int_{t=\text{ Arsinh}(\frac{1}{x})} \frac{\sinh t}{\sqrt{\dfrac{\cosh^2 t}{\sinh^2 t}}} \cdot \left(-\frac{\cosh t}{\sinh^2 t}\right) dt \\
&= -\int_{t=\text{ Arsinh}(\frac{1}{x})} dt = -\text{ Arsinh}\,\frac{1}{x} = -\ln\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+1}\right) \\
&= -\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) = \ln\left(\frac{x}{1+\sqrt{1+x^2}}\right), \qquad x>0.
\end{aligned}
$$

C. Test. We get by differentiation

$$
\begin{aligned}
\frac{d}{dx}\left\{\ln\left(\frac{x}{1+\sqrt{1+x^2}}\right)\right\} &= \frac{d}{dx}\left\{\ln x - \ln\left(1+\sqrt{1+x^2}\right)\right\} \\
&= \frac{1}{x} - \frac{1}{1+\sqrt{1+x^2}}\cdot\frac{x}{\sqrt{1+x^2}} \\
&= \frac{\sqrt{1+x^2}+1+x^2-x^2}{x(1+\sqrt{1+x^2})\sqrt{1+x^2}} = \frac{1}{x\sqrt{1+x^2}} \qquad \text{Q.E.D.}
\end{aligned}
$$

Example 2.8 *Calculate the integral*

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx, \qquad |x|<1,$$

by introducing the substitutionen $x = \sin t$*.*

A. Integration.

D. Apply the given substitution.

I. Let $x = \sin t$, $t \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$. Then $x(t)$ is monotonous on the interval $]-1,1[$ and

$$t = \text{ Arcsin } x, \qquad dx = \cos t\,dt, \qquad \cos t > 0.$$

Hence,

$$
\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int_{t=\text{ Arcsin }x} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cdot\cos t\,dt \\
&= \int_{t=\text{ Arcsin x}} \sin^2 t\,dt = \int_{t=\text{ Arcsin x}} \frac{1-\cos 2t}{2}\,dt \\
&= \frac{1}{2}\text{Arcsin } x - \frac{1}{4}\sin(2\,\text{Arcsin } x) \\
&= \frac{1}{2}\text{Arcsin } x - \frac{1}{4}\cdot 2\cdot\sin(\text{Arcsin } x)\cdot\cos(\text{Arcsin } x) \\
&= \frac{1}{2}\text{Arcsin } x - \frac{1}{2}\,x\left(+\sqrt{1-\sin^2(\text{Arcsin } x)}\right) \\
&= \frac{1}{2}\text{Arcsin } x - \frac{1}{2}\,x\sqrt{1-x^2}.
\end{aligned}
$$

C. TEST. We get by differentiation

$$\begin{aligned}
&\frac{d}{dx}\left\{\frac{1}{2}\,\text{Arcsin}\, x - \frac{1}{2}\,x\sqrt{1-x^2}\right\}\\
&\qquad = \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}} - \frac{1}{2}\,\sqrt{1-x^2} + \frac{1}{2}\cdot x\cdot\frac{x}{\sqrt{1-x^2}}\\
&\qquad = \frac{1}{2}\,\frac{1}{\sqrt{1-x^2}}\,\left\{1-(1-x^2)+x^2\right\}\\
&\qquad = \frac{1}{2}\cdot\frac{2x^2}{\sqrt{1-x^2}} = \frac{x^2}{\sqrt{1-x^2}} \qquad \text{Q.E.D..}
\end{aligned}$$

Example 2.9 *Write the polynomial $P(x) = (x+2)(3-x)$ in the form*

$$P(x) = a^2 - (x-b)^2.$$

Calculate the integral

$$\int \frac{1}{\sqrt{(x+2)(3-x)}}\,dx.$$

A. Integration with hidden guidelines.

D. Follow the guideline and substitute. Where is the integrand defined?

I. By a rearrangement we get

$$(x+2)(3-x) = 6 + x - x^2 = 4 + \frac{1}{4} - \frac{1}{4} + 2\cdot\frac{1}{2}\,x - x^2 = \left(\frac{5}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2$$

$$\left(\frac{5}{2}\right)^2\left\{1 - \left(\frac{2}{5}\,x - \frac{1}{5}\right)^2\right\} > 0 \qquad \text{for } x \in\,]-2,3[.$$

The integrand is defined for $x \in\,]-2,3[$. In this interval we use the substitution

$$t = \frac{2}{5}\,x - \frac{1}{5}, \qquad dt = \frac{2}{5}\,dx,$$

thus

$$\begin{aligned}
\int \frac{1}{\sqrt{(x+2)(3-x)}}\,dx &= \frac{2}{5}\int \frac{1}{\sqrt{1-\left(\frac{2}{5}\,x-\frac{1}{5}\right)^2}}\,dx\\
&= \int_{t=\frac{2}{5}\,x-\frac{1}{5}} \frac{1}{\sqrt{1-t^2}}\,dt = \text{ Arcsin}\left(\frac{2}{5}\,x - \frac{1}{5}\right).
\end{aligned}$$

C. Test. By a differentiation we get

$$\frac{d}{dx}\,\text{Arcsin}\left(\frac{2}{5}\,x-\frac{1}{5}\right)=\frac{2}{5}\cdot\frac{1}{\sqrt{1-\left(\frac{2}{5}\,x-\frac{1}{5}\right)^{2}}}=\frac{2}{\sqrt{5^2-(2-1)^2}}$$

$$=\frac{2}{\sqrt{(5+2x-1)(5-2x+1)}}=\frac{2}{\sqrt{(4+2x)(6-2x)}}=\frac{1}{\sqrt{(2+x)(3-x)}} \qquad \text{Q.E.D.}$$

Example 2.10 *Calculate the integral*

$$\int \frac{4x}{\sqrt{1-x^2}\,(3+x^2)}\,dx, \qquad |x|<1,$$

by first introducing the substitution $x=\sin t$ *and then the substitution* $u=\cos t$.

A. Integration by successive substitutions.

D. Analyze the substitutions and integrate. Alternatively it is possible directly to apply the substitution $u=\sqrt{1-x^2}$, a guess which is already indicated by the structure of the integrand.

I. First variant. When $x \in]-1,1[$, we get that $x = \sin t$, $t \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, is a monotonous substitution,

$$t = \text{ Arcsin } x \qquad \text{and} \qquad dx = \cos t\, dt.$$

Since $\cos t > 0$, when $x \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, we get by an insertion that

$$\begin{aligned}
\int \frac{4x}{\sqrt{1-x^2}\,(3+x^2)}\,dx &= \int_{t=\text{ Arcsin } x} \frac{4\sin t\cdot\cos t}{\sqrt{1-\sin^2 t}\cdot(3+\sin^2 t)}\,dt \\
&= \int_{t=\text{ Arcsin } x} \frac{4\sin t}{3+\sin^2 t}\,dt = -\int_{u=\cos t} \frac{4}{4-u^2}\,du \\
&= \int_{u=\cos(\text{Arcsin } x)} \left\{\frac{1}{u-2} - \frac{1}{u+2}\right\} du = [\ln|u-2| - \ln|u+2|]_{u=+\sqrt{1-x^2}} \\
&= \ln\left(\frac{2-\sqrt{1-x^2}}{2+\sqrt{1-x^2}}\right),
\end{aligned}$$

where we have applied that $\cos(\text{Arcsin } x) = +\sqrt{1-x^2}$.

Second variant. If we put $u = \sqrt{1-x^2}$, we formally obtain that

$$x^2 = 1-u^2, \qquad \text{med } du = -\frac{x}{\sqrt{1-x^2}}\,dx.$$

This substitution is *not* monotonous. However, the factor $\dfrac{x}{\sqrt{1-x^2}}$ is already present in the integrand, so this requirement is of no importance. Hence,

$$\begin{aligned}
\int \frac{4x}{\sqrt{1-x^2}\,(3+x^2)}\,dx &= -\int_{u=\sqrt{1-x^2}} \frac{4}{3+1-u^2}\,du \\
&= \int_{u=\sqrt{1-x^2}} \frac{4}{u^2-4}\,du,
\end{aligned}$$

and then we continue as in the **first variant**.

C. TEST. When $|x| < 1$, we get by a differentiation,

$$\begin{aligned}
&\frac{d}{dx}\ln\left(\frac{2-\sqrt{1-x^2}}{2+\sqrt{1-x^2}}\right) \\
&= \frac{1}{2-\sqrt{1-x^2}}\left\{-\frac{-x}{\sqrt{1-x^2}}\right\} - \frac{1}{2+\sqrt{1-x^2}}\left\{-\frac{x}{\sqrt{1-x^2}}\right\} \\
&= \frac{x}{\sqrt{1-x^2}}\left\{\frac{1}{2-\sqrt{1-x^2}} + \frac{1}{2+\sqrt{1-x^2}}\right\} \\
&= \frac{x}{\sqrt{1-x^2}}\cdot\frac{(2+\sqrt{1-x^2})+(2-\sqrt{1-x^2})}{4-(1-x^2)} \\
&= \frac{4x}{\sqrt{1-x^2}\,(3+x^2)} \qquad \text{Q.E.D.}
\end{aligned}$$

Example 2.11 *Calculate the integral*

$$\int \frac{\sqrt{1+x^2}}{(1+x^2)^2}\,dx$$

by introducing the substitution $x = \sinh t$.

A. Integration by a substitution.

D. Introduce the substitution and integrate.

I. The substitution $x = \sinh t$, $t = \text{Arsinh}\, x$, is monotonous and $dx = \cosh t\, dt$, where $\cosh t > 0$. By this substitution we get

$$\begin{aligned}\int \frac{\sqrt{1+x^2}}{(1+x^2)^2}\,dx &= \int_{t=\text{ Arsinh } x} \frac{\sqrt{1+\sinh^2 t}}{(1+\sinh^2 t)^2}\cosh t\,dt \\ &= \int_{t=\text{ Arsinh } x} \frac{1}{\cosh^2 t}\,dt = \tanh(\text{Arsinh}\, x) \\ &= \frac{\sinh(\text{Arsinh}\, x)}{\cosh(\text{Arsinh}\, x)} = \frac{x}{+\sqrt{1+\sinh^2(\text{Arsinh}\, x)}} = \frac{x}{\sqrt{1+x^2}}.\end{aligned}$$

C. TEST. By differentiating we get

$$\begin{aligned}\frac{d}{dx}\left\{\frac{x}{\sqrt{1+x^2}}\right\} &= \frac{1}{1+x^2}\left\{1\cdot\sqrt{1+x^2} - x\cdot\frac{x}{\sqrt{1+x^2}}\right\} \\ &= \frac{1}{1+x^2}\cdot\frac{1+x^2-x^2}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}}{(1+x^2)^2} \qquad \text{Q.E.D.}\end{aligned}$$

Example 2.12 *By introducing the substitution* $x = \cos tt$ *in the interval* $x \in [-1,1]$ *we get*

$$\int \sqrt{1-x^2}\,dx = -\frac{1}{2}\,\text{Arccos}\, x + \frac{1}{2}\,x\sqrt{1-x^2}.$$

Hence the right hand side is an integral in the closed interval $[-1,1]$, *and nevertheless* Arccos x *is not differentiable for* $x = \pm 1$. *Explain this apparent contradiction.*

A. Integration. Neither Arccos x nor $x\sqrt{1-x^2}$ are differentiable for $x = \pm 1$. The right hand side er continuous in $[-1,1]$.

D. Prove that the right hand side is an integral of $\sqrt{1-x^2}$ in the open subinterval $]-1,1[$.

I. Obviously

$$f(x) = -\frac{1}{2}\,\text{Arccos}\, x + \frac{1}{2}\,x\sqrt{1-x^2}$$

is continuous for $x \in [-1,1]$ and differentiable for $x \in]-1,1[$. By differentiating we get

$$\begin{aligned} f'(x) &= -\frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right) + \frac{1}{2}\sqrt{1-x^2} - \frac{1}{2}\,\frac{x^2}{\sqrt{1-x^2}} \\ &= \frac{1}{2}\,\frac{1-x^2}{\sqrt{1-x^2}} + \frac{1}{2}\sqrt{1-x^2} = \sqrt{1-x^2}, \end{aligned}$$

proving that $f(x)$ is in fact an integral of $\sqrt{1-x^2}$ in $]-1,1[$.
By continuity this must also hold in the end points.

Note that the two "singularities" of $-\frac{1}{2}\operatorname{Arccos} x$ and $\frac{1}{2}x\sqrt{1-x^2}$ in ± 1 are cancelled by the differentiation, so the sum of them becomes differentiable.

Example 2.13 *1) Prove the formula*

$$\tanh(\operatorname{Arsinh} x) = \frac{x}{\sqrt{1+x^2}}, \qquad x \in \mathbb{R}.$$

2) Then calculate the integral

$$\int \frac{\sqrt{1+x^2}}{(1+x^2)^2}\,dx, \qquad x \in \mathbb{R},$$

by applying the substitution $x = \sinh u$.

3) Find the complete solution of the differential equation

$$(t^2+1)\,\frac{dx}{dt} + tx = \frac{1}{t^2+1}, \qquad t \in \mathbb{R}.$$

4) Indicate that solution $x = \varphi(t)$*, for which* $\varphi(0) = -1$.

A. Derive a formula. Find an integral. Solve a non-normed linear differential equation of first order. Notice that (1) and (2) have already been treated in Example 2.11.

D. Start from the beginning. Remember the tests.

I. 1) Since $\cosh y = \sqrt{1+\sinh^2 y}$, we get for $x \in \mathbb{R}$,

$$\begin{aligned} \tanh(\operatorname{Arsinh} x) &= \frac{\sinh(\operatorname{Arsinh} x)}{(\cosh(\operatorname{Arsinh} x)} \\ &= \frac{x}{\sqrt{1+\sinh^2(\operatorname{Arsinh} x)}} = \frac{x}{\sqrt{1+x^2}}. \end{aligned}$$

2) The substitution $x = \sinh u$ is monotonous, so

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}}{(1+x^2)^2}\,dx &= \int_{u=Arsinh x} \frac{\sqrt{1+\sinh^2 u}}{(1+\sinh^2 u)^2}\cosh u\,du \\
&= \int_{u=\mathrm{Arsinh}\,x} \frac{\cosh u}{\cosh^4 u}\cdot \cosh u\,du = \int_{u=\mathrm{Arsinh}\,x} \frac{1}{\cosh^2 u}\,du \\
&= \tanh(\mathrm{Arsinh}\,x) = \frac{x}{\sqrt{1+x^2}},
\end{aligned}$$

hvor vi har benyttet (1).

C. TEST. By differentiating,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right) &= \frac{1}{1+x^2}\left\{1\cdot\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}\right\} \\
&= \frac{1}{(1+x^2)\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}}{(1+x^2)^2} \qquad \text{Q.E.D.}
\end{aligned}$$

3) **First variant.** Dividing by $\sqrt{1+t^2}$ we get

$$\sqrt{t^2+1}\,\frac{dx}{dt}+\frac{t}{\sqrt{t^2+1}}\,x=\frac{d}{dt}\left(x\sqrt{t^2+1}\right)=\frac{\sqrt{t^2+1}}{(t^2+1)^2},$$

hence by (2),

$$x\sqrt{t^2+1}=\int\frac{\sqrt{t^2+1}}{(t^2+1)^2}\,dt+c=\frac{t}{\sqrt{t^2+1}}+c,$$

and the complete complete solution is obtained by a division by $\sqrt{t^2+1}$:

$$x=\frac{t}{t^2+1}+\frac{c}{\sqrt{t^2+1}},\qquad c\in\mathbb{R},\quad t\in\mathbb{R}.$$

Second variant. When the equation is normed we get

$$\frac{dx}{dt}+\frac{t}{t^2+1}\,x=\frac{1}{(t^2+1)^2},\qquad t\in\mathbb{R}.$$

Here,

$$P(t)=\int\frac{t}{t^2+1}\,dt=\frac{1}{2}\,\ln\left(t^2+1\right),$$

hence a solution of the homogeneous equation is

$$e^{-P(t)}=\frac{1}{\sqrt{t^2+1}}.$$

Thus, a particular integral is given by

$$x=\frac{1}{\sqrt{t^2+1}}\int\frac{\sqrt{t^2+1}}{(t^2+1)^2}\,dt=\frac{1}{\sqrt{t^2+1}}\cdot\frac{t}{\sqrt{t^2+1}}=\frac{t}{t^2+1},$$

where we have applied (2). The complete solution is then

$$x=\frac{t}{t^2+1}+\frac{c}{\sqrt{t^2+1}},\qquad c\in\mathbb{R},\quad t\in\mathbb{R}.$$

C. TEST. By insertion into the differential equation we get

$$\begin{aligned}(t^2+1)\,\frac{dx}{dt}&+tx\\ &=(t^2+1)\left\{\frac{1-t^2}{(t^2+1)^2}+c\left(-\frac{t}{(t^2+1)^{\frac{3}{2}}}\right)\right\}+\frac{t^2}{t^2+1}+\frac{ct}{\sqrt{t^2+1}}\\ &=\frac{1-t^2}{t^2+1}-\frac{ct}{\sqrt{t^2+1}}+\frac{t^2}{t^2+1}+\frac{ct}{\sqrt{t^2+1}}=\frac{1}{t^2+1}\qquad\text{Q.E.D.}\end{aligned}$$

4) When $t=0$, we get

$$\varphi(0)=-1=0+c,$$

hence $c=-1$, and the wanted solution is

$$\varphi(t)=\frac{t}{t^2+1}-\frac{1}{\sqrt{t^2+1}}.$$

Example 2.14 *Find an integral of the function*

$$g(x) = \frac{x-3\sqrt{x}+8}{2\sqrt{x}(x-4\sqrt{x}+5)(\sqrt{x}+3)}, \qquad x>0.$$

A. Integral.

D. Decompose, substitute and integrate.

I. When we use the substitution $u=\sqrt{x}$, we first note that $u>-3$, hence by (2)

$$\begin{aligned}\int g(x)\,dx &= \int \frac{x-3\sqrt{x}+8}{2\sqrt{x}(x-4\sqrt{x}+5)(\sqrt{x}+3)}\,dx\\ &= \int_{u=\sqrt{x}} \frac{u^2-3u+8}{(u^2-4u+5)(u+3)}\,du\\ &= \ln(\sqrt{x}+3)+\ \text{Arctan}(\sqrt{x}-2), \qquad x>0.\end{aligned}$$

Example 2.15 *Calculate the following integrals*

1) $\int 3x\,\sin x^2\,dx$,

2) $\int \dfrac{x^3}{\sqrt{1+x^4}}\,dx$,

3) $\int x^2 e^{-x}\,dx$,

4) $\int \dfrac{\cos x}{2\sin^3 x}\,dx$,

5) $\int \left(x\,e^{1-x^2}+x^2\,e^{1-x}\right)\,dx$.

A. Five integrals.

D. (1), (2) and (4) are calculated by using a convenient substitution.

(3) is calculated by partial integration.

(5) is calculated by using a substitution and then by either applying the result from (3) or by partial integration.

I. 1) Putting $t=x^2$, $dt=2x\,dx$, we immediately get

$$\int 3x\,\sin\left(x^2\right)\,dx = \frac{3}{2}\int_{t=x^2}\sin t\,dt = -\frac{3}{2}\cos\left(x^2\right).$$

2) Putting $t=1+x^4$, $dt=4x^3\,dx$, we immediately get

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int_{t=1+x^4}\frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\sqrt{1+x^4}.$$

3) By repeated partial integration we get

$$\begin{aligned}\int x^2 \cdot e^{-x}\,dx &= x^2 \cdot \left(-e^{-x}\right) - \int 2x \cdot \left(-e^{-x}\right)\,dx \\ &= -x^2 e^{-x} - 2x\,e^{-x} + \int 2e^{-x}\,dx \\ &= -\left(x^2+2x+2\right) e^{-x}.\end{aligned}$$

4) Putting $t = \sin x$, $dt = \cos x\,dx$, we immediately get for $x \neq p\pi$, $p \in \mathbb{Z}$ that

$$\int \frac{\cos x}{2\sin^3 x}\,dx = \frac{1}{2}\int_{t=\sin x} t^{-3}\,dt = -\frac{1}{4}\cdot\frac{1}{\sin^2 x}.$$

5) Here we can take advantage from the result of (3):

$$\begin{aligned}\int \left\{x\,e^{1-x^2} + x^2 e^{1-x}\right\} dx &= \int x\,e^{1-x^2}\,dx + e\int x^2 e^{-x}\,dx \\ &= -\frac{1}{2}\int_{t=1-x^2} e^t\,dt + e\left\{-(x^2+2x+2)e^{-x}\right\} \\ &= -\frac{1}{2}\,e^{1-x^2} - (x^2+2x+2)e^{1-x},\end{aligned}$$

C. Test.

1)

$$\frac{d}{dx}\left\{-\frac{3}{2}\cos\left(x^2\right)\right\} = -\frac{3}{2}\left\{-\sin\left(x^2\right)\right\}\cdot 2x = 3x\,\sin\left(x^2\right).$$

2)

$$\frac{d}{dx}\left\{\frac{1}{2}\sqrt{1+x^4}\right\} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1+x^4}}\cdot 4x^3 = \frac{x^3}{\sqrt{1+x^4}}.$$

3)

$$\frac{d}{dx}\left\{-(x^2+2x+2)e^{-x}\right\} = -(2x+1)e^{-x} + (x^2+2x+2)e^{-x} = x^2 e^{-x}.$$

4)

$$\frac{d}{dx}\left\{-\frac{1}{4}\cdot\frac{1}{\sin^2 x}\right\} = -\frac{1}{4}\cdot\left(\frac{-2}{\sin^3 x}\right)\cdot\cos x = \frac{\cos x}{2\,\sin^3 x}.$$

5)

$$\begin{aligned}\frac{d}{dx}\left\{-\frac{1}{2}\,e^{1-x^2} - (x^2+2x+2)e^{1-x}\right\} &= -\frac{1}{2}\,e^{1-x^2}\cdot(-2x) + x^2 e^{1-x} \\ &= x\,e^{1-x^2} + x^2 e^{1-x}.\end{aligned}$$

In all five cases we get the corresponding integrands. Q.E.D.

Example 2.16 *Calculate the following integrals*

1)

$$\int\left\{\frac{2}{\sqrt{1-9x^2}}-\frac{3}{\sqrt{x^2+9}}+\frac{1}{x^2+2x+6}\right\}dx, \qquad |x|<\frac{1}{3}.$$

2)

$$\int\left\{\frac{3}{\sqrt{x^2-9}}+\frac{4}{2x+6}-\frac{2}{9-x^2}\right\}dx, \qquad |x|>3.$$

A. Integrals.

D. Split the integrals and then choose a convenient substitution.

I. 1) For $|x| < \frac{1}{3}$,

$$\begin{aligned}
&\int \left\{ \frac{2}{\sqrt{1-9x^2}} - \frac{3}{\sqrt{x^2+9}} + \frac{1}{x^2+2x+6} \right\} dx \\
&= \int \frac{1}{\sqrt{1-9x^2}}\,dx - \int \frac{3}{\sqrt{x^2+9}}\,dx + \int \frac{1}{x^2+2x+6}\,dx \\
&= \frac{2}{3}\int_{t=3x} \frac{dt}{\sqrt{1-t^2}} - 3\int_{t=\frac{x}{3}} \frac{dt}{\sqrt{t^2+1}} + \int \frac{1}{(x+1)^2+5}\,dx \\
&= \frac{2}{3}\,\mathrm{Arcsin}(3x) - 3\,\mathrm{Arsinh}\left(\frac{x}{3}\right) + \frac{1}{\sqrt{5}}\int_{t=\frac{x+1}{\sqrt{5}}} \frac{dt}{1+t^2} \\
&= \frac{2}{3}\,\mathrm{Arcsin}(3x) - 3\,\mathrm{Arsinh}\left(\frac{x}{3}\right) + \frac{1}{\sqrt{5}}\,\mathrm{Arctan}\left(\frac{x+1}{\sqrt{5}}\right).
\end{aligned}$$

C. Test.

$$\begin{aligned}
&\frac{d}{dx}\left\{ \frac{2}{3}\,\mathrm{Arcsin}(3x) - 3\,\mathrm{Arsinh}\left(\frac{x}{3}\right) + \frac{1}{\sqrt{5}}\,\mathrm{Arctan}\left(\frac{x+1}{\sqrt{5}}\right) \right\} \\
&\quad = \frac{2}{3}\,\frac{3}{\sqrt{1-(3x)^2}} - 3\,\frac{\frac{1}{3}}{\sqrt{1+\left(\frac{x}{3}\right)^2}} + \frac{1}{\sqrt{5}}\,\frac{1}{1+\left(\frac{x+1}{\sqrt{5}}\right)^2}\cdot\frac{1}{\sqrt{5}} \\
&\quad = \frac{2}{\sqrt{1-9x^2}} - \frac{3}{\sqrt{9+x^2}} + \frac{1}{(x+1)^2+5} \\
&\quad = \frac{2}{\sqrt{1-9x^2}} - \frac{3}{\sqrt{9+x^2}} + \frac{1}{x^2+2x+6} \qquad \text{Q.E.D.}
\end{aligned}$$

2) For $|x| > 3$,

$$\begin{aligned}
&\int \left\{ \frac{3}{\sqrt{x^2-9}} + \frac{4}{2x+6} - \frac{2}{9-x^2} \right\} dx \\
&= \int \frac{3}{\sqrt{x^2-9}}\,dx + \int \frac{2}{x+3}\,dx + \int \frac{2}{x^2-9}\,dx \\
&= \int_{t=\frac{x}{3}} \frac{3}{\sqrt{t^2-1}}\,dt + 2\ln|x+3| + \int \frac{1}{3}\left\{ \frac{1}{x-3} - \frac{1}{x+3} \right\} dx \\
&= 3\,\ln\left| \frac{x}{3} + \sqrt{\left(\frac{x}{3}\right)^2 - 1} \right| + 2\,\ln|x+3| + \frac{1}{3}\,\ln\left|\frac{x-3}{x+3}\right| \\
&= 3\,\ln\left| x + \sqrt{x^2-9} \right| + \frac{5}{3}\,\ln|x+3| + \frac{1}{3}\,\ln|x-3| - 3\,\ln 3.
\end{aligned}$$

Note that $x + \sqrt{x^2-9}$, $x+3$ and $x-3$ are all negative, when $x < -3$.

C. TEST. By differentiating we get

$$\begin{aligned}
&\frac{d}{dx}\left\{3\,\ln|x+\sqrt{x^2-9}|+\frac{5}{3}\,\ln|x+3|+\frac{1}{3}\,\ln|x-3|-3\,\ln 3\right\}\\
&=3\,\frac{1}{x+\sqrt{x^2-9}}\left(1+\frac{x}{\sqrt{x^2-9}}\right)+\frac{5}{3}\,\frac{1}{x+3}+\frac{1}{3}\,\frac{1}{x-3}\\
&=\frac{3}{x+\sqrt{x^2-9}}\,\frac{\sqrt{x^2-9}+x}{\sqrt{x^2-9}}+\frac{2}{x+3}+\frac{1}{3}\left\{\frac{1}{x-3}-\frac{1}{x+3}\right\}\\
&=\frac{3}{\sqrt{x^2-9}}+\frac{4}{2x+6}+\frac{1}{3}\,\frac{x+3-(x-3)}{x^2-9}\\
&=\frac{3}{\sqrt{x^2-9}}+\frac{4}{2x+6}-\frac{2}{9-x^2}\qquad\text{Q.E.D.}
\end{aligned}$$

Example 2.17 *1) Calculate the integral*

$$\int\frac{1-2\sqrt{x}}{1+3\sqrt{x}}\,dx,\qquad x>0,$$

by introducing the substitution given by $t=\sqrt{x}$.

2) Calculate the integral

$$\int\frac{x+\sqrt{2x+3}}{\sqrt[4]{2x+3}}\,dx,\qquad x>-\frac{3}{2},$$

by introducing the substitution given by $t=\sqrt[4]{2x+3}$.

A. Two integrals.

D. Introduce the indicated substitutions.

I. 1) The substitution $t=\sqrt{x}$, $x>0$, $t>0$, is monotonous, $x=t^2$, and $dx=2t\,dt$. Hence

$$\int\frac{1-2\sqrt{x}}{1+3\sqrt{x}}\,dx=\int_{t=\sqrt{x}}\frac{1-2t}{1+3t}\cdot 2t\,dt=\int_{t=\sqrt{x}}\frac{2t-4t^2}{1+3t}\,dt.$$

A division by polynomials gives

$$\frac{2t-4t^2}{1+3t}=-\frac{4}{3}\,t+\frac{10}{9}-\frac{10}{9}-\frac{10}{27}\cdot\frac{1}{t+\frac{1}{3}},$$

hence by insertion,

$$\begin{aligned}
\int\frac{1-2\sqrt{x}}{1+3\sqrt{x}}\,dx &= \int_{t=\sqrt{x}}\left\{-\frac{4}{3}\,t+\frac{10}{9}-\frac{10}{27}\cdot\frac{1}{t+\frac{1}{3}}\right\}dt\\
&=\left[-\frac{2}{3}\,t^2+\frac{10}{9}\,t-\frac{10}{27}\,\ln(3t+1)\right]_{t=\sqrt{x}}\\
&=-\frac{2}{3}\,x+\frac{10}{9}\,\sqrt{x}-\frac{10}{27}\,\ln(1+3\sqrt{x}).
\end{aligned}$$

C. TEST. By differentiating we get

$$\frac{d}{dx}\left\{-\frac{2}{3}\,x+\frac{10}{9}\,\sqrt{x}-\frac{10}{27}\,\ln(1+3\sqrt{x})\right\}$$
$$=-\frac{2}{3}+\frac{5}{9}\cdot\frac{1}{\sqrt{x}}-\frac{10}{27}\cdot\frac{1}{1+3\sqrt{x}}\cdot\frac{3}{2}\cdot\frac{1}{\sqrt{x}}$$
$$=-\frac{2}{3}+\frac{5}{9}\,\frac{1}{\sqrt{x}}\left\{1-\frac{1}{1+3\sqrt{x}}\right\}$$
$$=-\frac{2}{3}+\frac{5}{9}\,\frac{1}{\sqrt{x}}\cdot\frac{1+3\sqrt{x}-1}{1+3\sqrt{x}}=-\frac{2}{3}+\frac{5}{3}\cdot\frac{1}{1+3\sqrt{x}}$$
$$=\frac{1}{3}\cdot\frac{1}{1+3\sqrt{x}}\,\{-2-6\sqrt{x}+5\}=\frac{1}{3}\cdot\frac{3-6\sqrt{x}}{1+3\sqrt{x}}=\frac{1-2\sqrt{x}}{1+3\sqrt{x}}\qquad\text{Q.E.D.}$$

2) The substitution $t=\sqrt[4]{2x+3}$, $x>-\frac{3}{2}$, $t>0$, is monotonous, and $t^4=2x+3$, thus $x=\frac{1}{2}\,t^t4-\frac{3}{2}$ and $dx=2t^3\,dt$. Then by insertion,

$$\begin{aligned}\int\frac{x+\sqrt{2x+3}}{\sqrt[4]{2x+3}}\,dx &= \int_{t=\sqrt[4]{2x+3}}\frac{\frac{1}{2}\,t^4-\frac{3}{2}+t^2}{t}\cdot 2t^3\,dt\\ &= \int_{t=\sqrt[4]{2x+3}}(t^6+2t^4-3t^2)\,dt=\left[\frac{1}{7}\,t^7+\frac{2}{5}\,t^5-t^3\right]_{t=\sqrt[4]{2x+3}}\\ &= \frac{1}{7}\,(2x+3)^{\frac{7}{4}}+\frac{2}{5}\,(2x+3)^{\frac{5}{4}}-(2x+3)^{\frac{3}{4}}.\end{aligned}$$

C. TEST. By differentiating we get

$$\frac{d}{dx}\left\{\frac{1}{7}\,(2x+3)^{\frac{7}{4}}+\frac{2}{5}\,(2x+3)^{\frac{5}{4}}-(2x+3)^{\frac{3}{4}}\right\}$$
$$=\frac{1}{7}\,\frac{7}{4}\,(2x+3)^{\frac{3}{4}}\cdot 2+\frac{2}{5}\,\frac{5}{4}\,(2x+3)^{\frac{1}{4}}\cdot 2-\frac{3}{4}\,(2x+3)^{-\frac{1}{4}}\cdot 2$$
$$=\frac{1}{2}\,\frac{2x+3}{\sqrt[4]{2x+3}}+\frac{\sqrt{2x+3}}{\sqrt[4]{2x+3}}-\frac{3}{2}\,\frac{1}{\sqrt[4]{2x+3}}=\frac{x+\sqrt{2x+3}}{\sqrt[4]{2x+3}}\qquad\text{Q.E.D.}$$

Example 2.18 *1) Calculate the integral*

$$\int\sinh^3 t\,dt$$

by introducing the substitution given by $u=\cosh t$.

2) Let $x=\sinh t$. *What is* $\cosh t$ *expressed by* x*?*

3) Calculate the integral

$$\int\frac{x^3}{\sqrt{1+x^2}}\,dx$$

by introducing the substitution $x=\sinh t$.

A. Two integrals and an hyperbolic relation.

D. 1) Introduce the indicated substitution.

2) Use that $\cosh^2 t - \sinh^2 t = 1$.

3) Introduce the substitution $x = \sinh t$.
ALTERNATIVELY one can apply the substitution $t = x^2$.

I. 1) Since

$$\sinh^3 t = \sinh^2 t \cdot \sinh t = (\cosh^2 t - 1)\sinh t,$$

we get

$$\begin{aligned}\int \sinh^3 t\,dt &= \int (\cosh^2 t - 1)\sinh t\,dt \\ &= \int_{x=\cosh t} (x^2-1)\,dx = \left[\frac{x^3}{3} - x\right]_{x=\cosh t} \\ &= \frac{1}{3}\cosh^3 t - \cosh t.\end{aligned}$$

C. TEST. By differentiating we get

$$\begin{aligned}\frac{d}{dx}\left\{\frac{1}{3}\cosh^3 t - \cosh t\right\} &= \frac{1}{3}\cdot 3\cosh^2 t \cdot \sinh t - \sinh t \\ &= (\cosh^2 t - 1)\sinh t = \sinh^3 t \qquad \text{Q.E.D.}\end{aligned}$$

2) Since $\cosh t > 0$, we get

$$\cosh t = +\sqrt{1+\sinh^2 t} = \sqrt{1+x^2}.$$

3) The substitution $x = \sinh t$ is monotonous, $t = \operatorname{Arsinh} x$, and $dx = \cosh t\,dt$, hence

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1+x^2}}\,dx &= \int_{t=\operatorname{Arsinh} x} \frac{\sinh^3 t}{\sqrt{1+\sinh^2 t}} \cdot \cosh t\,dt \\
&= \int_{t=\operatorname{Arsinh} x} \sinh^3 t\,dt \\
&= \frac{1}{3}\cosh^3(\operatorname{Arsinh} x) - \cosh(\operatorname{Arsinh} x) \\
&= \frac{1}{3}\left\{\sqrt{1+x^2}\right\}^3 - \sqrt{1+x^2} \\
&= \frac{1}{3}(x^2-2)\sqrt{1+x^2}.
\end{aligned}$$

ALTERNATIVELY apply the substitution $t = x^2+1$. Then

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1+x^2}}\,dx &= \frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}} \cdot 2x\,dx = \frac{1}{2}\int \frac{x^2+1-1}{\sqrt{1+x^2}}\,d(x^2+1) \\
&= \frac{1}{2}\int_{t=x^2+1} \left\{\sqrt{t} - \frac{1}{\sqrt{t}}\right\} dt = \frac{1}{2}\left[\frac{2}{3}\,t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right]_{t=x^2+1} \\
&= \frac{1}{3}(x^2+1)\sqrt{x^2+1} - \sqrt{x^2+1} = \frac{1}{3}(x^2-2)\sqrt{1+x^2}.
\end{aligned}$$

C. TEST. By differentiating we get

$$\begin{aligned}
&\frac{d}{dx}\left\{\frac{1}{3}(x^2-2)\sqrt{1+x^2}\right\} \\
&= \frac{1}{3}\cdot 2x \cdot \sqrt{1+x^2} + \frac{1}{3}(x^2-2) \cdot \frac{x}{\sqrt{1+x^2}} \\
&= \frac{1}{3}\,\frac{1}{\sqrt{1+x^2}}\left\{2x(1+x^2) + (x^2-2)x\right\} \\
&= \frac{1}{3}\,\frac{1}{\sqrt{1+x^2}}\left\{2x + 2x^3 + x^3 - 2x\right\} = \frac{x^3}{\sqrt{1+x^2}} \qquad \text{Q.E.D.}
\end{aligned}$$

Example 2.19 *1) Let* $x = \sin t$, $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. *Find* $\tan t$ *expressed by* x*?*

2) Calculate the integral

$$\int \frac{\sqrt{1-x^2}}{(1-x^2)^2}\,dx, \qquad -1 < x < 1,$$

by introducing the substitution $x = \sin t$, $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

A. Find a geometric relation. Then calculate the integral.

D. Follow the guidelines given above.

I. 1) When $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, we have $\cos t > 0$, thus

$$\tan t = \frac{\sin t}{\cos t} = +\frac{\sin t}{\sqrt{1-\sin^2 t}} = \frac{x}{\sqrt{1-x^2}}.$$

2) The substitution $x = \sin t$, where $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $x \in]-1,1[$, is monotonous and $t = \text{Arcsin}\, x$, $dx = \cos t\, dt$, $\cos t > 0$. Then by insertion,

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{(1-x^2)^2}\, dx &= \int_{t=\text{ Arcsin } x} \frac{\sqrt{1-\sin^2 t}}{(1-\sin^2 t)^2} \cos t\, dt \\
&= \int_{t=\text{Arcsin } x} \frac{+\cos t}{\cos^4 t} \cdot \cos t\, dt = \int_{t=\text{Arcsin } x} \frac{1}{\cos^2 t}\, dt \\
&= [\tan t]_{t=\text{Arcsin } x} = \frac{x}{\sqrt{1-x^2}},
\end{aligned}$$

where (1) has been applied.

C. Test. Let $x \in]-1,1[$. Then by differentiation

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) &= \frac{1 \cdot \sqrt{1-x^2} - x \cdot \frac{-x}{\sqrt{1-x^2}}}{1-x^2} \\
&= \frac{\sqrt{1-x^2}}{(1-x^2)^2}\left\{1-x^2-(-x^2)\right\} = \frac{\sqrt{1-x^2}}{(1-x^2)^2} \qquad \text{Q.E.D.}
\end{aligned}$$

Example 2.20 *Calculate the integral*

$$\int \frac{\sqrt{x+1}}{x+5}\, dx, \qquad x \ge -1,$$

by applying the substitution given by $t = \sqrt{x+1}$.

A. Integration by substitution.

D. Apply the substitution $t = \sqrt{x+1}$, and then calculate the integral.

I. Since $t = \sqrt{x+1} > 0$ for $x > -1$, we have $x = t^2 - 1$, and thus $dx = 2t\, dt$. Then by insertion

$$\begin{aligned}
\int \frac{\sqrt{x+1}}{x+5}\, dx &= \int \frac{\sqrt{x+1}}{(x+1)+4}\, dx = \int_{t=\sqrt{x+1}} \frac{t}{t^2+4} \cdot 2t\, dt \\
&= \int_{t=\sqrt{x+1}} \left\{2 - \frac{8}{t^2+4}\right\} dt \\
&= 2\sqrt{x+1} - 2\int_{t=\sqrt{x+1}} \frac{1}{\left(\frac{t}{2}\right)^2 + 1}\, dt \\
&= 2\sqrt{x+1} - 4\,\text{Arctan}\left(\frac{\sqrt{x+1}}{2}\right), \qquad x > 1.
\end{aligned}$$

C. TEST. We get for $x > -1$ by a differentiation

$$\begin{aligned}
&\frac{d}{dx}\left\{2\sqrt{x+1} - 4\,\mathrm{Arctan}\left(\frac{\sqrt{x+1}}{2}\right)\right\} \\
&\quad= \frac{1}{\sqrt{x+1}} - 4 \cdot \frac{1}{\left(\frac{\sqrt{x+1}}{2}\right)^2+1} \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{x+1}} \\
&\quad= \frac{1}{\sqrt{x+1}}\left\{1 - \frac{1}{\frac{x+1}{4}+1}\right\} \\
&\quad= \frac{1}{\sqrt{x+1}}\left\{1 - \frac{4}{x+5}\right\} = \frac{1}{\sqrt{x+1}} \cdot \frac{x+5-4}{x+5} \\
&\quad= \frac{1}{\sqrt{x+1}} \cdot \frac{x+1}{x+5} = \frac{\sqrt{x+1}}{x+5} \qquad \text{Q.E.D.}
\end{aligned}$$

Example 2.21 *Calculate* $\int \frac{\sqrt{x-3}}{x+6}\,dx$, $x > 3$.

A. Integral, where the integrand contains a square root.

D. Use the principal rule: Whenever an "ugly" term occur, then call this something different by a substitution.

I. Putting $u = \sqrt{x-3}$, $x > 3$, we see that $u > 0$ is monotonously increasing in x, and

$$x = u^2 + 3, \qquad dx = 2u\,du.$$

Then by insertion

$$\begin{aligned}
\int \frac{\sqrt{x-3}}{x+6}\,dx &= \int_{u=\sqrt{x-3}} \frac{u}{(u^2+3)+6} \cdot 2u\,du \\
&= 2\int_{u=\sqrt{x-3}} \frac{u^2+(9-9)}{u^2+9}\,du \\
&= 2\int_{u=\sqrt{x-3}} \left\{1 - \frac{9}{u^2+9}\right\} du \\
&= [2u]_{u=\sqrt{x-3}} - 2\int_{x=\sqrt{x-3}} \frac{du}{1+\left(\frac{u}{3}\right)^2} \\
&= 2\sqrt{x-3} - 6\,\mathrm{Arctan}\left(\frac{\sqrt{x-3}}{3}\right), \qquad x > 3.
\end{aligned}$$

C. TEST. If

$$f(x) = 2\sqrt{x-3} - 6\,\mathrm{Arctan}\left(\frac{\sqrt{x-3}}{3}\right), \qquad x > 3,$$

then by a differentiation,

$$\begin{aligned}
f'(x) &= \frac{1}{\sqrt{x-3}} - 6\cdot\frac{1}{1+\left(\frac{\sqrt{x-3}}{3}\right)^2}\cdot\frac{1}{6\sqrt{x-3}} \\
&= \frac{1}{\sqrt{x-3}} - \frac{9}{9+x-3}\cdot\frac{1}{\sqrt{x-3}} = \frac{1}{\sqrt{x-3}}\left\{1-\frac{9}{x+6}\right\} \\
&= \frac{1}{\sqrt{x-3}}\cdot\frac{x-3}{x+6} = \frac{\sqrt{x-3}}{x+6},
\end{aligned}$$

which is precisely the integrand. Q.E.D.

3 Integration by advanced substitutions

Example 3.1 *Calculate the integral*

$$\int \frac{x}{(x-1)^{\frac{4}{3}}}\,dx, \qquad x>1,$$

by introducing the substitution given by $t=\sqrt[3]{x-1}$.

A. Integration by substitution.

D. Follow the hint, i.e. we call an unpleasant term something different.

I. If we put $t=\sqrt[3]{x-1}$, then $x=t^3+1>1$, i.e. $dx=3t^2\,dt$. Then by insertion for $x>1$,

$$\begin{aligned}\int \frac{x}{(x-1)^{\frac{4}{3}}}\,dx &= \int_{t=\sqrt[3]{x-1}} \frac{1}{t^4}\left(t^3+1\right)\cdot 3t^2\,dt\\ &= 3\int_{t=\sqrt[3]{x-1}}\left(t+\frac{1}{t^2}\right)\,dt = \left[\frac{3}{2}\,t^2-\frac{3}{t}\right]_{t=\sqrt[3]{x-1}}\\ &= \frac{3}{2}\,(x-1)^{\frac{2}{3}}-\frac{3}{(x-1)^{\frac{1}{3}}} = \frac{3}{2}\cdot\frac{x-3}{\sqrt[3]{x-1}}.\end{aligned}$$

C. TEST. We get by a differentiation,

$$\begin{aligned}&\frac{d}{dx}\left\{\frac{3}{2}\cdot\frac{x-3}{\sqrt[3]{x-1}}\right\} = \frac{3}{2}\cdot\frac{1}{\sqrt[3]{x-1}}-\frac{1}{2}\cdot\frac{x-3}{(x-1)^{\frac{4}{3}}}\\ &\quad = \frac{1}{2}\cdot\frac{3(x-1)-(x-3)}{(x-1)^{\frac{4}{3}}} = \frac{x}{(x-1)^{\frac{4}{3}}} \qquad \text{Q.E.D.}\end{aligned}$$

Example 3.2 *Calculate the integral*

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx, \qquad x>0,$$

by introducing the substitution given by $t=\sqrt[6]{x}$.

A. Integration by substitution.

D. Introduce the substitution, then decompose the integrand and finally calculate the integral.

I. If $x>0$, then $t=\sqrt[6]{x}>0$ is a monotonous substitution, $x=t^6$, and $dx=6t^5\,dt$. Thus

$$\begin{aligned}\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx &= \int_{t=\sqrt[6]{x}} \frac{6t^5}{t^3+t^2}\,dt = \int_{t=\sqrt[6]{x}} \frac{6t^3}{t+1}\,dt\\ &= 6\int_{t=\sqrt[6]{x}}\left\{t^2-t+1-\frac{1}{t+1}\right\}dt\\ &= \left[2t^3-3t^2+6t-6\,\ln(t+1)\right]_{t=\sqrt[6]{x}}\\ &= 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\,\ln\left(1+\sqrt[6]{x}\right).\end{aligned}$$

C. TEST. Let

$$f(x) = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\,\ln\left(1+\sqrt[6]{x}\right), \qquad x > 0.$$

By a differentiation with respect to $x > 0$ we get

$$\begin{aligned}
f'(x) &= \frac{1}{\sqrt{x}} - \frac{\sqrt[3]{x}}{x} + \frac{\sqrt[6]{x}}{x} - \frac{1}{1+\sqrt[6]{x}} \cdot \frac{\sqrt[6]{x}}{x} \\
&= \frac{1}{\sqrt{x}+\sqrt[3]{x}} \left\{ \left(x^{\frac12}+x^{\frac13}\right) x^{-\frac12} - \left(x^{\frac12}+x^{\frac13}\right) x^{-\frac23} + \left(x^{\frac12}+x^{\frac13}\right) x^{-\frac56} \right. \\
&\qquad\qquad \left. \frac{x^{\frac13}\left(x^{\frac16}+1\right)}{1+x^{\frac16}} \cdot x^{-\frac56} \right\} \\
&= \frac{1}{\sqrt{x}+\sqrt[3]{x}} \left\{ 1 + x^{-\frac16} - x^{-\frac16} - x^{-\frac13} + x^{-\frac13} + x^{-\frac12} - x^{-\frac12} \right\} \\
&= \frac{1}{\sqrt{x}+\sqrt[3]{x}} \qquad \text{Q.E.D.}
\end{aligned}$$

Example 3.3 *Calculate the integral*

$$\int \frac{1}{x}\sqrt{\frac{x}{x+1}}\,dx, \qquad x > 0,$$

by introducing the substitution given by $t = \sqrt{\dfrac{x}{x+1}}$.

A. Integration by substitution. There may also occur a decomposition.

D. Check the substitution. Then apply it and calculate the integral.

I. When $x > 0$, we see that $\dfrac{x}{x+1} = 1 - \dfrac{1}{x+1} > 0$ is increasing in x with the range $]0,1[$. Therefore, the substitution $t = \sqrt{\dfrac{x}{x+1}}$ can be applied, and we get

$$x = \frac{1}{1-t^2} - 1 = \frac{t^2}{1-t^2}, \qquad dx = \frac{2y}{(1-t^2)^2}\,dt, \qquad t \in\,]0,1[.$$

Then by insertion for $x > 0$,

$$\begin{aligned}
\int \frac{1}{x}\sqrt{\frac{x}{x+1}}\,dx &= \int_{t=\sqrt{\frac{x}{x+1}}} \frac{1-t^2}{t^2}\cdot t \cdot \frac{2t}{(1-t^2)^2}\,dt \\
&= \int_{t=\sqrt{\frac{x}{x+1}}} \frac{2}{1-t^2}\,dt = \int_{t=\sqrt{\frac{x}{x+1}}} \left\{-\frac{1}{t-1}+\frac{1}{t+1}\right\} dt \\
&= -\ln\left|\sqrt{\frac{x}{x+1}}-1\right| + \ln\left|\sqrt{\frac{x}{x+1}}+1\right| \\
&= \ln\left(\frac{1+\sqrt{\frac{x}{x+1}}}{1-\sqrt{\frac{x}{x+1}}}\right) \\
&= \ln\left(\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right) = \ln\left\{\left(\sqrt{x+1}+\sqrt{x}\right)^2\right\} \\
&= 2\,\ln\left(\sqrt{x+1}+\sqrt{x}\right).
\end{aligned}$$

C. Test. By differentiation we get for $x > 0$,

$$\begin{aligned}
&\frac{d}{dx}\left\{2\,\ln\left(\sqrt{x+1}+\sqrt{x}\right)\right\} = \frac{2}{\sqrt{x+1}+\sqrt{x}}\left\{\frac{1}{2}\,\frac{1}{\sqrt{x+1}}+\frac{1}{2}\,\frac{1}{\sqrt{x}}\right\} \\
&\quad = \frac{1}{\sqrt{x+1}+\sqrt{x}}\cdot\frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x+1}\cdot\sqrt{x}} = \frac{1}{x}\sqrt{\frac{x}{x+1}} \qquad \text{Q.E.D.}
\end{aligned}$$

Example 3.4 *Find an integral of the function*

$$f(x) = \frac{1}{(x+2)^2}\left\{\left(\sqrt[4]{\frac{x+1}{x+2}}\right)^3 - 2 + \left(\sqrt[4]{\frac{x+2}{x+1}}\right)^3\right\}, \qquad -1 < x < \infty.$$

A. Integration.

D. A substitution followed by an integration.

I. We shall apply the monotonous substitution

$$u = \sqrt[4]{\frac{x+1}{x+2}} \in\,]0,1[, \quad \text{i.e.} \quad x = \frac{2u^4-1}{1-u^4},$$

and

$$\begin{aligned}
dx &= \left\{\frac{8u^3(1-u^4)+4u^3(2u^4-1)}{(1-u^4)^2}\right\} du \\
&= 4u^3\cdot\frac{2-2u^4+2u^4-1}{(1-u^4)^2}\,du = \frac{4u^3}{(1-u^4)^2}\,du.
\end{aligned}$$

Since $\dfrac{1}{x+2} = 1 - u^4$, we get

$$\int \frac{1}{(x+2)^2}\left\{\left(\sqrt[4]{\frac{x+1}{x+2}}\right)^3 - 2 + \left(\sqrt[4]{\frac{x+2}{x+1}}\right)^3\right\} dx$$

$$= \int_{u=\sqrt[4]{\frac{x+1}{x+2}}} (1-u^4)^2\left\{u^3 - 2 + u^{-3}\right\} \cdot \frac{4u^3}{(1-u^4)^2}\, du$$

$$= \int_{u=\sqrt[4]{\frac{x+1}{x+2}}} \left\{4u^6 - 8u^3 + 4\right\} du = \left[\frac{4}{7}u^7 - 2u^4 + 4u\right]_{u=\sqrt[4]{\frac{x+1}{x+2}}}$$

$$= \frac{4}{7}\left(\sqrt[4]{\frac{x+1}{x+2}}\right)^7 - 2\cdot\frac{x+1}{x+2} + 4\sqrt[4]{\frac{x+1}{x+2}}.$$

C. Test. By a differentiation we get

$$\frac{4}{7}\cdot\frac{7}{2}\left(\sqrt[4]{\frac{x+1}{x+2}}\right)^3 \cdot \frac{1}{(x+2)^2} - \frac{2}{(x+2)^2} + 4\cdot\frac{1}{4}\left(\sqrt[4]{\frac{x+1}{x+2}}\right)^{-3} \cdot \frac{1}{(x+2)^2}$$

$$= \frac{1}{(x+2)^2}\left\{\left(\sqrt[4]{\frac{x+1}{x+2}}\right)^3 - 2 + \left(\sqrt[4]{\frac{x+2}{x+1}}\right)^3\right\}. \qquad \text{Q.E.D.}$$

Example 3.5 *We shall here eventually calculate the integral*

(1) $\displaystyle\int \sqrt{\frac{x}{3-x}}\,dx, \qquad 0 \le x < 3.$

1) Prove that $t = \sqrt{\dfrac{x}{3-x}}$ *has an inverse function* $x = \varphi(t)$*. Find the domain and the derivative of* $\varphi(t)$*.*

2) Introduce the substitution $x = \varphi(t)$ *into (1), and then calculate the integral.*

A. An integration with some guidelines.

D. Follow the guidelines.

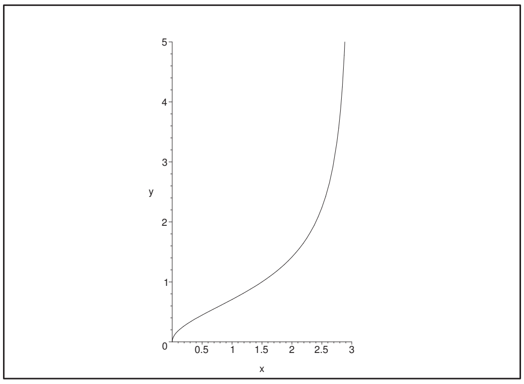


Figure 1: The graph of the function $t = \sqrt{\dfrac{x}{3-x}}$, $x \in [0,3[$.

I. 1) Let $x \in [0,3[$. Then $\dfrac{x}{3-x}$ is continuous. Since the numerator is strictly increasing and the denominator is strictly decreasing, we conclude that the function itself is strictly increasing, hence strictly monotonous. This means that $t = \sqrt{\dfrac{x}{3-x}}$ has an inverse function $\varphi(t)$, the domain of which is the range $[0,+\infty[$ of $t = \sqrt{\dfrac{x}{3-x}}$, $x \in [0,3[$. By squaring followed by an inversion we get

$$\frac{3-x}{x} = \frac{3}{x} - 1 = \frac{1}{t^2}, \quad \text{i.e.} \quad \frac{3}{x} = 1 + \frac{1}{t^2} = \frac{1+t^2}{t^2},$$

hence,

$$x = \varphi(t) = \frac{3t^2}{1+t^2} = 3 - \frac{3}{1+t^2}$$

where

$$\varphi'(t) = \frac{6t}{(1+t^2)^2}.$$

2) When we introduce the substitution $x = \varphi(t)$ into (1), we get

$$\begin{aligned}
\int \sqrt{\frac{x}{3-x}}\,dx &= \int_{t=\sqrt{\frac{x}{3-x}}} t\cdot\varphi'(t)\,dt \\
&= \int_{t=\sqrt{\frac{x}{3-x}}} \frac{6t^2}{(1+t^2)^2}\,dt \\
&= 3\int_{t=\sqrt{\frac{x}{3-x}}} t\cdot\frac{2t}{(1+t^2)^2}\,dt \\
&= \left[3t\cdot\frac{-1}{1+t^2}\right]_{t=\sqrt{\frac{x}{3-x}}} - 3\int_{t=\sqrt{\frac{x}{3-x}}} 1\cdot\frac{-1}{1+t^2}\,dt \\
&= -\frac{3\sqrt{\dfrac{x}{3-x}}}{\dfrac{x}{3-x}+1} + 3\,\text{Arctan}\left(\sqrt{\frac{x}{3-x}}\right) \\
&= 3\,\text{Arctan}\left(\sqrt{\frac{x}{3-x}}\right) - \sqrt{x(3-x)}, \qquad x \in [0,3[.
\end{aligned}$$

C. TEST. When $x \in]0,3[$ we get by differentiation

$$\begin{aligned}
&\frac{d}{dx}\left\{3\,\text{Arctan}\left(\sqrt{\frac{x}{3-x}}\right) - \sqrt{x(3-x)}\right\} \\
&= \frac{3}{\dfrac{x}{3-x}+1}\cdot\frac{1}{2}\sqrt{\frac{3-x}{x}}\cdot\frac{1\cdot(3-x)+x}{(3-x)^2} - \frac{1}{2}\,\frac{3-2x}{\sqrt{x(3-x)}} \\
&= \frac{3-x}{2}\,\sqrt{\frac{3-x}{x}}\cdot\frac{3}{(3-x)^2} - \frac{1}{2}\cdot\frac{3-2x}{\sqrt{x(3-x)}} \\
&= \frac{3-x}{2}\,\sqrt{\frac{3-x}{x}}\cdot\frac{3}{(3-x)^2} - \frac{1}{2}\cdot\frac{3-2x}{\sqrt{x(3-x)}} \\
&= \frac{1}{2}\,\frac{1}{\sqrt{x(3-x)}}\,\{3-(3-2x)\} = \frac{x}{\sqrt{x(3-x)}} = \sqrt{\frac{x}{3-x}},
\end{aligned}$$

where we use that $x > 0$. Q.E.D.

Example 3.6 *Let the function* $g(x)$ *be given by*

$$g(x) = \sqrt{\frac{x}{x+2}}, \qquad x > 0.$$

1) Prove that $g(x)$ *is monotonous and find the inverse function of* $g(x)$.

2) Denote the inverse function by $g^{-1}(t)$. *Prove that*

$$\frac{d}{dt}\left\{g^{-1}(t)\right\} = \frac{4t}{(1-t^2)^2}.$$

3) Calculate the integral

$$\int \frac{1}{x}\sqrt{\frac{x}{x+2}}\,dx, \qquad x > 0,$$

by an application of the substitution given by

$$t = \sqrt{\frac{x}{x+2}}.$$

A. An integration with a disguised substitution.

D. Follow the hints in the guidelines.

I. 1) The easiest way is simply to show that the equation

$$(2) \quad t = \sqrt{\frac{x}{x+2}} \in \,]0,1[,$$

has a unique solution $x > 0$, and find this solution.

Since $t > 0$, we see that (2) is equivalent to that equation which is obtained by squaring,

$$t^2 = \frac{x}{x+2} = \frac{x+2-2}{x+2} = 1 - \frac{2}{x+2},$$

from which

$$\frac{2}{x+2} = 1 - t^2, \qquad \text{dvs.} \qquad x+2 = \frac{2}{1-t^2},$$

and thus

$$x = \frac{2}{1-t^2} - 2 = 2\cdot\frac{t^2}{1-t^2} > 0 \qquad \text{for } t \in \,]0,1[,$$

and the claim is proved.

2) We have according to (1),

$$x = g^{-1}(t) = \frac{2t^2}{1-t^2}, \qquad t \in \,]0,1[.$$

Thus we immediately get

$$\frac{dx}{dt} = \frac{d}{dt}\left\{g^{-1}(t)\right\} = \frac{4t(1-t^2)-2t^2(-2t)}{(1-t^2)^2} = \frac{4t}{(1-t^2)^2}.$$

3) When we apply the substitution $t = \sqrt{\dfrac{x}{x+2}}$, it follows from the above that

$$\int \frac{1}{x}\sqrt{\frac{x}{x+2}}\,dx = \int_{t=\sqrt{\frac{x}{x+2}}} \frac{1-t^2}{2t^2}\cdot t\cdot \frac{4t}{(1-t^2)^2}\,dt = \int_{t=\sqrt{\frac{x}{x+2}}} \frac{2}{1-t^2}\,dt$$

$$= \int_{t=\sqrt{\frac{x}{x+2}}} \left\{-\frac{1}{t-1}+\frac{1}{t+1}\right\} dt = \left[\ln\left|\frac{t+1}{t-1}\right|\right]_{t=\sqrt{\frac{x}{x+2}}}$$

$$= \ln\left\{\frac{1+\sqrt{\dfrac{x}{x+2}}}{1-\sqrt{\dfrac{x}{x+2}}}\right\} = \ln\left(\frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}-\sqrt{x}}\right)$$

$$= \ln\left(\frac{(\sqrt{x+2}+\sqrt{x})^2}{x+2-x}\right) = 2\,\ln\left(\sqrt{x+2}+\sqrt{x}\right) - \ln 2$$

$$= 2\,\mathrm{Artanh}\left(\sqrt{\frac{x}{x+2}}\right), \qquad x > 0,$$

where we have used that $t = \sqrt{\dfrac{x}{x+2}} \in \,]0,1[$ by the above.

C. TEST. We get e.g. by a differentiation

$$\begin{aligned}
&\frac{d}{dx}\left\{2\,\text{Artanh}\left(\sqrt{\frac{x}{x+2}}\right)\right\} = \frac{2}{1-\dfrac{x}{x+2}}\,\frac{d}{dx}\left(\sqrt{\frac{x}{x+2}}\right)\\
&\quad = (x+2)\,\frac{d}{dx}\left(\sqrt{\frac{x}{x+2}}\right) = \frac{x+2}{2}\cdot\frac{1}{\sqrt{\dfrac{x}{x+2}}}\cdot\frac{d}{dx}\left\{1-\frac{2}{x+2}\right\}\\
&\quad = \frac{(x+2)^2}{2x}\sqrt{\frac{x}{x+2}}\cdot\frac{2}{(x+2)^2} = \frac{1}{x}\sqrt{\frac{x}{x+2}} \qquad \text{Q.E.D.}
\end{aligned}$$

If we instead choose the integral

$$f(x) = 2\,\ln\left(\sqrt{x+2}+\sqrt{x}\right) - \ln 2,$$

then

$$\begin{aligned}
\frac{df}{dt} &= \frac{2}{\sqrt{x+2}+\sqrt{x}}\left(\frac{1}{2}\,\frac{1}{\sqrt{x+2}}+\frac{1}{2}\,\frac{1}{\sqrt{x}}\right) = \frac{\sqrt{x+2}-\sqrt{x}}{x+2-x}\left(\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{x}}\right)\\
&= \frac{1}{2}\left\{1-\sqrt{\frac{x}{x+2}}+\sqrt{\frac{x+2}{x}}-1\right\} = \frac{1}{2}\sqrt{\frac{x}{x+2}}\left\{-1+\frac{x+2}{x}\right\}\\
&= \frac{1}{2}\cdot\frac{1}{x}\sqrt{\frac{x}{x+2}}\{-x+(x+2)\} = \frac{1}{x}\sqrt{\frac{x}{x+2}} \qquad \text{Q.E.D.}
\end{aligned}$$

4 Decomposition

Example 4.1 *Write the following polynomials as a product of polynomials of first degree and of second degree without real roots:*

$$(1)\quad P(x) = x^4 - 3x^3 - 6x^2 + 8x, \qquad (2)\quad P(x) = x^4 - 1.$$

Decompose the fraction

$$\frac{P(x)}{Q(x)} = \frac{x^2+3x+8}{x^3-x^2-2x}.$$

Write a MAPLE programme which gives the decomposition.

A. Splitting of polynomials followed by a decomposition.

D. Guess the roots in (1) and (2). In the latter fraction we factorize the denominator and then decompose.

I. 1) It is immediately seen by inspection that the roots are $x = 0$ and $x = 1$, hence $x(x-1) = x^2 - x$ is a divisor in the polynomial:

$$x^4 - 3x^3 - 6x^2 + 8x = x(x-1)(x^2-2x-8).$$

Since $x^2 - 2x - 8 = (x-4)(x+2)$, we get the factorization

$$x^4 - 3x^3 - 6x^2 + 8x = (x+2)x(x-1)(x-4),$$

and the four simple roots are -2, 0, 1 and 4.

2) By a small rearrangement we get

$$x^4 - 1 = \left(x^2\right)^2 - 1 = \left(x^2-1\right)\left(x^2+1\right) = (x+1)(x-1)\left(x^2+1\right).$$

The simple, real roots are ± 1, and we have furthermore the two simple, complex roots $\pm i$.

$$(x+1)(x-1)(x^2+1).$$

3) First find the factors of the denominator $Q(x)$,

$$Q(x) = x^3 - x^2 - 2x = x(x^2-x-2) = x(x+1)(x-2).$$

Since the numerator $P(x)$ has lower degree than the denominator $Q(x)$, we get by the standard procedure that

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{x^2+3x+8}{(x+1)x(x-2)} \\
&= \frac{(-1)^2+3(-1)+8}{(-1)(-1-2)} \cdot \frac{1}{x+1} + \frac{0^2+3\cdot 0+8}{(0+1)(0-2)} \cdot \frac{1}{x} \\
&\qquad + \frac{2^2+3\cdot 2+8}{(2+1)\cdot 2} \cdot \frac{1}{x-2} \\
&= \frac{1-3+8}{3} \cdot \frac{1}{x+1} + \frac{8}{-2} \cdot \frac{1}{x} + \frac{4+6+8}{6} \cdot \frac{1}{x-2} \\
&= \frac{2}{x+1} - \frac{4}{x} + \frac{3}{x-2}.
\end{aligned}$$

A corresponding MAPLE programme could be

```
f := (x^2+3*x+8)/(x^3-x^2-2*x);
```

$$\frac{x^2+3x+8}{x^3-x^2-2x}.$$

```
convert(f,parfrac);
```

$$-\frac{4}{x}+\frac{2}{x+1}+\frac{3}{x-2}.$$

Example 4.2 *Decompose the fraction*

$$\frac{P(x)}{Q(x)} = \frac{1}{x^3-x^2+x}.$$

A. Decomposition.

D. Dissolve the denominator into its factors and then decompose.

I. Since

$$Q(x) = x^3-x^2+x = x(x^2-x+1) = x^2\left\{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right\},$$

we conclude that the structure of the real decomposition is

$$\frac{P(x)}{Q(x)} = \frac{1}{x(x^2-x+1)} = \frac{a}{x}+\frac{bx+c}{x^2-x+1},$$

hence by the standard procedure we get $a = 1$. Hence by reduction,

$$\begin{aligned}\frac{bx+c}{x^2-x+1} &= \frac{1}{x(x^2-x+1)}-\frac{1}{x}\\ &= \frac{1-x^2+x-1}{x(x^2-x+1)} = \frac{-x+1}{x^2-x+1}.\end{aligned}$$

When $x \neq 0$ we get

$$\frac{P(x)}{Q(x)} = \frac{1}{x^3-x^2+x} = \frac{1}{x}+\frac{-x+1}{x^2-x+1}.$$

Example 4.3 *Consider a fraction*

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-1)^2(x-2)},$$

where the degree of $P(x)$ *is at most 2, and where the numerator and the denominator do not have common factors.*

1) Prove that it can never happen that

$$\frac{P(x)}{Q(x)} = \frac{c_1}{x-1} + \frac{c_2}{x-2}.$$

2) Assume that

$$\frac{x^2-x+1}{(x-1)^2(x-2)} = \frac{c_1}{(x-1)^2} + \frac{c_2}{x-2}.$$

Find c_1 *and* c_2*.*

3) When we put c_1 *and* c_2 *in the equation (2), we get a false identity. Show this and explain what is wrong.*

A. Investigation of a decomposition.

D. "Decompose" unconsciously and then find out what is wrong.

I. 1) We miss one term of the type $\dfrac{c}{(x-1)^2}$, where

$$c = \frac{P(1)}{1-2} = -P(1) \neq 0.$$

2) By the usual decomposition technique we get

$$c_1 = \frac{1-1+1}{+1-2} = -1, \qquad c_2 = \frac{2^2-2+1}{(2-1)^2} = 3.$$

Then by a rearrangement,

$$\begin{aligned}\frac{x^2-x+1}{(x-1)^2(x-2)} &+ \frac{1}{(x-1)^2} - \frac{3}{x-2}\\ &= \frac{(x^2-x+1)+(x-2)-3(x^2-2x+1)}{(x-1)^2(x-2)}\\ &= \frac{-2x^2+6x-4}{(x-1)(x^2-3x+2)} = -\frac{2}{x-1}.\end{aligned}$$

All things considered we finally get

$$\frac{x^2-x+1}{(x-1)^2(x-2)} = -\frac{1}{(x-1)^2} - \frac{2}{x-1} + \frac{3}{x-2},$$

3) and we see from this that we have forgotten the term

$$\frac{c}{x-1} = -\frac{2}{x-1}$$

in (2).

REMARK. Roughly speaking $\dfrac{c_1}{(x-1)^2}$ shades the terms of lower order $\dfrac{c}{x-1}$, when we apply the standard procedure on the factor in the denominator under consideration. The message is that one should always first reduce and then repeat the method on the reduced expressions, in which the degree of the denominator has become smaller. ◊

Example 4.4 *Decompose the fractions*

$$(1)\quad \frac{P(x)}{Q(x)} = \frac{x^2+3x+8}{x^3-x^2-2x}, \qquad (2)\quad \frac{P(x)}{Q(x)} = \frac{2x^6+x^4-2x^2}{x^4-1}.$$

A. Decomposition; in (2) the degree of the numerator is bigger than the degree of the denominator.

D. Factorize the polynomials of the denominators and decompose by the standard procedure. In (2) we first perform a division by polynomials.

I. 1) When the denominator is factorized we get

$$x^3-x^2-2x = x(x^2-x-2) = x(x+1)(x-2).$$

Thus, if $x \neq -1$, 0, 2, then

$$\frac{P(x)}{Q(x)} = \frac{x^2+3x+8}{x^3-x^2-2x} = \frac{x^2+3x+8}{x(x+1)(x-2)} = \frac{8}{x} + \frac{2}{x+1} + \frac{3}{x-2}.$$

2) Here the denominator is factorized in the following way,

$$x^4 - 1 = (x^2)^2 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1).$$

First perform a division by polynomials, then use the trick of first decomposing after x^2 and finally after x. For $x \neq \pm 1$ we get

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{2x^6+x^4-2x^2}{x^4-1} = \frac{2x^2(x^4-1)+(x^4-1)+1}{x^4-1} \\
&= 2x^2+1+\frac{1}{(x^2-1)(x^2+1)} \\
&= 2x^2+1+\frac{1}{2}\cdot\frac{1}{x^2-1}-\frac{1}{2}\cdot\frac{1}{x^2+1} \\
&= 2x^2+1+\frac{1}{2}\cdot\frac{1}{(x-1)(x+1)}-\frac{1}{2}\cdot\frac{1}{x^2+1} \\
&= 2x^2+1+\frac{1}{4}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{x+1}-\frac{1}{2}\cdot\frac{1}{x^2+1}.
\end{aligned}$$

Example 4.5 *Assume that the polynomial*

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has the roots $\alpha_1, \alpha_2, \dots, \alpha_n$. Prove that

$$a_{n-1} = -(\alpha_1+\alpha_2+\cdots+\alpha_n) \quad \textit{and} \quad a_0 = (-1)^n\alpha_1\cdot\alpha_2\cdots\alpha_n.$$

A. A theoretical investigation of a polynomial. When $n=2$ the formulæ are known from high school .

D. Write $P(x)$ in two ways and identify the coefficients.

I. By a simple combinatoric argument we get

$$\begin{aligned}
P(x) &= (x-\alpha_1)(x-\alpha_2)\cdot(x-\alpha_n) \\
&= x^2-(\alpha_1+\cdots+\alpha_n)x^{n-1}+\cdots+(-1)^n\alpha_1\cdots\alpha_n \\
&= x^2+a_{n-1}x^{n-1}+\cdots+a_1x+a_0.
\end{aligned}$$

Since the coefficients of two identical polynomials agree we get

$$a_{n-1} = -(\alpha_1+\cdots+\alpha_n) \quad \text{og} \quad a_0 = (-1)^n\alpha_1\cdots\alpha_n,$$

i.e. the sum of the roots is always equal to $-a_{n-1}$, and the product of the roots is always equal to $(-1)^n a_0$.

Example 4.6 *Decompose the fraction*

$$\frac{x^3+x^2+1}{x(x^2+1)^2}.$$

A. Decomposition.

D. Use the standard method and reduce.

I. The polynomial of the numerator is of lower degree than the polynomial of the denominator, so if we decompose with respect to x followed by a reduction we get

$$\begin{aligned}\frac{x^3+x^2+1}{x(x^2+1)^2} &= \frac{1}{x}+\frac{x^3+x^2+1-(x^4+2x^2+1)}{x(x^2+1)^2} = \frac{1}{x}+\frac{-x^4+x^3-x^2}{x(x^2+1)^2}\\ &= \frac{1}{x}-\frac{x^3-x^2+x}{(x^2+1)^2} = \frac{1}{x}-\frac{x(x^2+1)-(x^2+1)+1}{(x^2+1)^2}\\ &= \frac{1}{x}-\frac{x-1}{x^2+1}-\frac{1}{(x^2+1)^2}.\end{aligned}$$

C. TEST:

$$\begin{aligned}\frac{1}{x}+\frac{1.x}{x^2+1}-\frac{1}{(x^2+1)^2} &= \frac{1}{x(x^2+1)^2}\left\{(x^2+1)^2+(1-x)x(x^2+1)-x\right\}\\ &= \frac{1}{x(x^2+1)}\left\{x^4+2x^2+1-x^4-x^2+x^3+x-x\right\}\\ &= \frac{x^3+x+1}{x(x^2+1)^2} \qquad \text{Q.E.D.}\end{aligned}$$

Example 4.7 *Decompose the fraction*

$$\frac{14x+23}{(x^2+1)(x+2)^2}, \qquad x\in\mathbb{R}\setminus\{-2\}.$$

A. Decomposition.

D. Use the standard procedure and reduce.

I. The polynomial of the numerator is of lower degree than the polynomial of the denominator, hence

$$
\begin{aligned}
\frac{14x+23}{(x^2+1)(x+2)^2} &= \frac{14\cdot(-2)+23}{(4+1)(x+2)^2}+\frac{14x+23}{(x^2+1)(x+2)^2}-\frac{14(-2)+23}{(4+1)(x+1)^2}\\
&= -\frac{1}{(x+2)^2}+\frac{(14x+23)+(x^2+1)}{(x^2+1)(x+2)^2}\\
&= -\frac{1}{(x+2)^2}+\frac{x^2+14x+24}{(x^2+1)(x+2)^2}=-\frac{1}{(x+2)^2}+\frac{x+12}{(x^2+1)(x+2)}\\
&= -\frac{1}{(x+2)^2}+\frac{2}{x+2}+\frac{(x+12)-2(x^2+1)}{(x^2+1)(x+2)}\\
&= -\frac{1}{(x+2)^2}+2x+2-\frac{2x^2-x-10}{(x^2+1)(x+2)}\\
&= -\frac{1}{(x+2)^2}+\frac{2}{x+2)}-\frac{(x+2)(2x-5)}{(x^2+1)(x+2)}\\
&= -\frac{1}{(x+2)^2}+\frac{2}{x+2}-\frac{2x-5}{x^2+1}.
\end{aligned}
$$

C. Test:

$$
\begin{aligned}
&-\frac{1}{(x+2)^2}+\frac{2}{x+2}-\frac{2x-5}{x^2+1}\\
&\quad=\frac{1}{(x^2+1)(x+2)^2}\left\{-(x^2+1)+2(x^2+1)(x+2)-(2x-5)(x+2)^2\right\}\\
&\quad=\frac{1}{(x^2+1)(x+2)^2}\left\{-x^2-1+2x^3+4x^2+2x+4-(2x-5)(x^2+4x+4)\right\}\\
&\quad=\frac{1}{(x^2+1)(x+2)^2}\left\{2x^3+3x^2+2x+3-2x^3-8x^2-8x+5x^2+20x+20\right\}\\
&\quad=\frac{1}{(x^2+1)(x+2)^2}\,(14x+23) \qquad \text{Q.E.D.}
\end{aligned}
$$

Example 4.8 *Decompose the fraction*

$$\frac{2x^2+2x}{(x-1)^2(x+3)}.$$

A. Decomposition.

D. Apply the standard procedure and reduce.

I. Since the polynomial of the numerator is of lower degree than the polynomial of denominator, we get by the standard procedure and a reduction that

$$
\begin{aligned}
\frac{2x^2+2x}{(x-1)^2(x+3)} &= \frac{1}{(x-1)^2}+\frac{3}{4}\,\frac{1}{x+3}+\frac{2x^2+2x-x-3-\dfrac{3}{4}\,(x-1)^2}{(x-1)^2(x+3)}\\
&= \frac{1}{(x-1)^2}+\frac{3}{4}\,\frac{1}{x+3}+\frac{2x^2+2x-x-3-\dfrac{3}{4}(x-1)^2}{(x-1)^2(x+3)}\\
&= \frac{1}{(x-1)^2}+\frac{3}{4}\,\frac{1}{x+3}+\frac{2x+3-\dfrac{3}{4}(x-1)}{(x-1)^2(x+3)}\\
&= \frac{1}{(x-1)^2}+\frac{3}{4}\,\frac{1}{x+3}+\frac{1}{4}\,\frac{8x+12-3x+3}{(x-1)(x+3)}\\
&= \frac{1}{(x-1)^2}+\frac{5}{4}\,\frac{1}{x-1}+\frac{3}{4}\,\frac{1}{x+3}.
\end{aligned}
$$

Example 4.9 *Decompose the fraction*

$$\frac{2x^2+2x}{(x-1)^2(x^2+3)},$$

A. Decomposition.

D. Apply the standard procedure and reduce.

I. Since the polynomial of the numerator is of lower degree that the polynomial of the denominator, we get by the standard procedure and a reduction that

$$\begin{aligned}
\frac{2x^2+2x}{(x-1)^2(x^2+3)} &= \frac{1}{(x-1)^2}+\frac{2x^2+2x-x^2-3}{(x-1)^2(x^2+3)} \\
&= \frac{1}{(x-1)^2}+\frac{x^2+2x-3}{(x-1)^2(x^2+3)} \\
&= \frac{1}{(x-1)^2}+\frac{(x-1)(x+3)}{(x-1)^2(x^2+3)} \\
&= \frac{1}{(x-1)^2}+\frac{x+3}{(x-1)(x^2+3)} \\
&= \frac{1}{(x-1)^2}+\frac{1}{x+1}+\frac{x+3-x^2-3}{(x-1)(x^2+3)} \\
&= \frac{1}{(x-1)^2}+\frac{1}{x-1}-\frac{x}{x^2+3}. \qquad \Diamond
\end{aligned}$$

Example 4.10 *1) Prove that if $x=\alpha$ is a treble root in the polynomial $P(x)$, then*

(3) $P(\alpha)=P'(\alpha)=P''(\alpha)=0.$

2) Conversely, prove that if (3) holds, then $x=\alpha$ is a root of $P(x)$ of at least multiplicity 3.

3) The condition (3) holds for every root, the multiplicity of which is bigger than or equal to 3. How is it possible by means of the derivatives of $P(x)$ to decide whether the multiplicity of the root $x=\alpha$ is precisely 3?

4) The figure 3 is in this connection not special. Formulate the general result.

A. Multiplicity of a root of a polynomial.

D. Go straight to the general case, because $n=3$ is only a special case.

I. Let $x=\alpha$ be a root of the polynomial of multiplicity $n\in\mathbb{N}$, i.e. there exists a polynomial $Q(x)$, such that

$$P(x)=(x-\alpha)^n\,Q(x), \qquad Q(\alpha)\neq 0.$$

Then by successive differentiations,

$$\begin{aligned}
P'(x) &= n(x-\alpha)^{n-1}Q(x)+(x-\alpha)^nQ_1(x),\\
P''(x) &= n(n-1)(x-\alpha)^{n-2}Q(x)+(x-\alpha)^{n-1}Q_2(x),\\
&\vdots\\
P^{(n-1)}(x) &= n(n-1)\cdots 2\cdot(x-\alpha)Q(x)+(x-\alpha)^2Q_{n-1}(x),\\
P^{(n)}(x) &= n!\,Q(x)+(x-\alpha)Q_n(x),
\end{aligned}$$

where $Q_1(x), \cdots, Q_n(x)$ are polynomials occurring by the differentiations. Due to the factor $x-\alpha$ it follows that

$$P(\alpha)=P'(\alpha)=P''(\alpha)=\cdots=P^{(n-1)}(\alpha)=0,$$

and

$$P^{(n)}(\alpha)=n!\,Q(\alpha)\neq 0.$$

Thus we have proved that *if* $x=\alpha$ is a root of multiplicity $n\in\mathbb{N}$ of the polynomial $P(x)$, *then*

(4) $P(\alpha)=P'(\alpha)=\cdots=P^{(n-1)}(\alpha)=0 \quad\text{and}\quad P^{(n)}(\alpha)\neq 0.$

Conversely, if (4) holds, then $x=\alpha$ is a root (because $P(\alpha)=0$), hence $x=\alpha$ must have a multiplicity $m\in\mathbb{N}$. From the result above follows that

$$P(\alpha)=P'(\alpha)=\cdots=P^{(m-1)}(\alpha)=0 \quad\text{og}\quad P^{(m)}(\alpha)\neq 0.$$

Since we assumed (4), this is only true if $m=n$. Thus,

- $x=\alpha$ is a root of multiplicity $n\in\mathbb{N}$ in the polynomial $P(x)$, if and only if

$$P(\alpha)=P'(\alpha)=\cdots=P^{(n-1)}(\alpha)=0 \quad\text{oand}\quad P^{(n)}(\alpha)\neq 0.$$

Remark. Another and more direct method is to develop the Taylor expansion from $x=\alpha$. Then

$$\begin{aligned}
P(x) &= P(\alpha)+\frac{P'(\alpha)}{1!}(x-\alpha)+\cdots+\frac{P^{(n-1)}(\alpha)}{(n-1)!}(x-\alpha)^{n-1}\\
&\quad+\frac{P^{(n)}(\alpha)}{n!}(x-\alpha)^n+\cdots .
\end{aligned}$$

It follows immediately, that if

$$P(\alpha)=P'(\alpha)=\cdots=P^{(n-1)}(\alpha)=0 \quad\text{and}\quad P^{(n)}(\alpha)\neq 0,$$

then $(x-\alpha)^n$ is a divisor in $P(x)$, hence $x=\alpha$ is a root of precisely order n, because $P^{(n)})(\alpha)\neq 0$. On the other hand, if $x=\alpha$ is a root of order n, then the Taylor expansion above shows that

$$P(\alpha)=P'(\alpha)=\cdots=P^{(n-1)}(\alpha)=0 \quad\text{and}\quad P^{(n)}(\alpha)\neq 0. \quad\Diamond$$

Example 4.11 *1) Find all real roots of the polynomial*

$$Q(x) = x^4 - 2x^3 + 2x^2 - 2x + 1.$$

2) Decompose the den rationale function

$$\frac{P(x)}{Q(x)} = \frac{4x^3 - x^2 - 4x + 5}{x^4 - 2x^3 + 2x^2 - 2x + 1}.$$

A. Decomposition.

D. Factorize the denominator and then decompose.

I. 1) It follows by inspection that

$$\begin{aligned} Q(x) &= x^4 - 2x^3 + 2x^2 - 2x + 1 \\ &= (x^4 - 2x^3 + x^2) + (x^2 - 2x + 1) \\ &= (x^2 + 1)(x - 1)^2. \end{aligned}$$

Hence, the double root $x = 1$ is the only real root.

2) Since $\deg(P) < \deg(Q)$, we get by (1) that the structure is given by

$$\frac{P(x)}{Q(x)} = \frac{4x^3-x^2-4x+5}{(x^2+1)(x-1)^2} = \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{cx+d}{x^2+1}.$$

We multiply this equation by $(x-1)^2$ and then take the limit $x \to 1$. Thus

$$a = \frac{1}{2}\{4-1-4+5\} = 2.$$

Then by a rearrangement,

$$\begin{aligned}
\frac{b}{x-1} + \frac{cx+d}{x^2+1} &= \frac{4x^3-x^2-4x+5}{(x^2+1)(x-1)^2} - \frac{2}{(x-1)^2}\cdot\frac{x^2+1}{x^2+1} \\
&= \frac{4x^3-x^2-4x+5-2x^2-2}{(x^2+1)(x-1)^2} = \frac{4x^3-3x^2-4x+3}{(x^2+1)(x-1)^2} \\
&= \frac{4x(x^2-1).3(x^2-1)}{(x^2+1)(x-1)^2} = \frac{4x(x+1)-3(x+1)}{(x^2+1)(x-1)} \\
&= \frac{4x^2+x-3}{(x^2+1)(x-1)}.
\end{aligned}$$

This equation is multiplied by $x-1$, followed by the limit process $x \to 1$. Then

$$b = \frac{1}{2}\{4+1-3\} = 1.$$

By a rearrangement,

$$\begin{aligned}
\frac{cx+d}{x^2+1} &= \frac{4x^2+x-3}{(x^2+1)(x-1)} - \frac{1}{x-1}\cdot\frac{x^2+1}{x^2+1} = \frac{4x^2+x-3-x^2-1}{(x^2+1)(x-1)} \\
&= \frac{3x^2+x-4}{(x^2+1)(x-1)} = \frac{(3x+4)(x-1)}{(x^2+1)(x-1)} = \frac{3x+4}{x^2+1}.
\end{aligned}$$

When all things are put together we get for $x \neq 1$,

$$\frac{P(x)}{Q(x)} = \frac{4x^3-x^2-4x+5}{x^4-2x^3+2x^2-2x+1} = \frac{2}{(x-1)^2} + \frac{1}{x-1} + \frac{3x+4}{x^2+1}.$$

Remark. Decomposition is boring in itself. It looks like there is missing a question, in which we shall use the decomposition. We shall here add the task of finding an integral of the fraction. This is here

$$\begin{aligned}
&\int \frac{4x^3-x^2-4x+5}{x^4-2x^3+2x^2-2x+1}\,dx \\
&\qquad = -\frac{2}{x-1} + \ln|x-1| + \frac{3}{2}\,\ln(x^2+1) + 4\,\mathrm{Arctan}\,x, \qquad x \neq 1. \quad \diamond
\end{aligned}$$

Example 4.12 *Consider a proper rational function*

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)},$$

where α_1, α_2 and α_3 are the three different roots of $Q(x)$. We put

$$\begin{aligned} Q_1(x) &= (x-\alpha_2)(x-\alpha_3), \\ Q_2(x) &= (x-\alpha_1)(x-\alpha_3), \\ Q_3(x) &= (x-\alpha_1)(x-\alpha_2). \end{aligned}$$

According to the theorem of decomposition there exist constants k_1, k_2 and k_3, such that

$$\frac{P(x)}{Q(x)} = \frac{k_1}{x-\alpha_1} + \frac{k_2}{x-\alpha_2} + \frac{k_3}{x-\alpha_3}.$$

1) Prove that

$$k_i = \frac{P(\alpha_i)}{Q_i(\alpha_i)}, \qquad i = 1,\, 2,\, 3.$$

2) Prove that

$$k_i = \frac{P(\alpha_i)}{Q'(\alpha_i)}, \qquad i = 1,\, 2,\, 3.$$

3) One has a similar general result, when the denominator $Q(x)$ can be written as a product of n different polynomials of first degree. Formulate this general result.

A. Theory of decomposition; the standard procedure; Heaviside's expansion theorem.

D. Apply the definitions and rpve the claims.

I. From

$$\frac{P(x)}{Q(x)} = \frac{k_1}{x-\alpha_1} + \frac{k_2}{x-\alpha_2} + \frac{k_3}{x-\alpha_3}$$

follows for $i = 1, 2, 3$,

1)

$$\begin{aligned} k_i &= \lim_{x\to\alpha_i} (x-\alpha_i)\cdot\frac{P(x)}{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)} \\ &= \lim_{x\to\alpha_i} \frac{P(x)}{Q_i(x)} = \frac{P(\alpha_i)}{Q_i(\alpha_1)}, \end{aligned}$$

(you blind that factor in the denominator, which becomes 0 for $x = \alpha_i$ and then put $x = \alpha_i$ into the rest), and

2)

$$k_i = \lim_{x\to\alpha_i} (x-\alpha_i)\,\frac{P(x)}{Q(x)} = \lim_{x\to\alpha_i} \frac{P(x)}{\dfrac{Q(x)-Q(\alpha_i)}{x-\alpha_i}} = \frac{P(\alpha_i)}{Q'(\alpha_i)},$$

which is also called Heaviside's expansion theorem.

3) If $Q(x) = (x-\alpha_1)\cdots(x-\alpha_n)$ has n different roots α_i, and $\deg P(x) < n$, we put $Q_i(x) = \dfrac{Q(x)}{x-\alpha_i}$. Then

$$\frac{P(x)}{Q(x)} = \frac{k_1}{x-\alpha_1} + \cdots + \frac{k_n}{x-\alpha_n},$$

where

$$k_i = \frac{P(\alpha_i)}{Q_i(\alpha_i)} = \frac{P(\alpha_i)}{Q'(\alpha_i)}, \qquad i = 1, \cdots, n.$$

The proofs are following the same lines as in (1) and (2) above.

Example 4.13 *Decompose the fraction*

$$\frac{3x^2-2x-1}{(x-1)^2(x+1)}, \qquad x \neq \pm 1.$$

A. Decomposition.

D. Use the standard procedure (blind the factor which is 0 for $x = \pm 1$ in the denominator and insert the chosen x in the rest). Finally, reduce.

I. First we note that the degree of the polynomial of the numerator is lower than the degree of the polynomial of the denominator. Then we see that

$$3x^2 - 2x - 1 = (x-1)(3x+1),$$

hence by reduction and the standard procedure of decomposition described above we get for $x \neq \pm 1$,

C. Test.

$$\begin{aligned} \frac{2}{x-1} + \frac{1}{x+1} &= \frac{1}{(x-1)^2(x+1)}\left\{2(x-1)(x+1) + (x-1)^2\right\} \\ &= \frac{1}{(x-1)^2(x+1)}\left\{2x^2 - 2 + x^2 - 2x + 1\right\} \\ &= \frac{3x^2-2x-1}{(x-1)^2(x+1)} \qquad \text{Q.E.D.} \end{aligned}$$

Example 4.14 *Decompose the fraction*

$$\frac{3x^2+2x+1}{(x-2)(x^2+4x+5)}.$$

A. Decomposition.

D. Find the term $\dfrac{a}{x-2}$ by the standard procedure.

I. The degree of the numerator is lower than the degree of the denominator. Furthermore,

$$x^2+4x+5=(x+2)^2+1\geq 1,$$

and the fraction is put on its canonical form. Then by the standard procedure,

$$\begin{aligned}\frac{3x^2+2x+1}{(x-2)(x^2+4x+5)} &= \frac{1}{x-2}+\frac{3x^2+2x+1-x^2-4x-5}{(x-2)(x^2+4x+5)}\\ &= \frac{1}{x-2}+\frac{2x^2-2x-4}{(x-2)(x^2+4x+5)}\\ &= \frac{1}{x-2}+\frac{2x+2}{x^2+4x+5}.\end{aligned}$$

C. TEST.

$$\begin{aligned}&\frac{1}{x-2}+\frac{2x+2}{x^2+4x+5}\\&\quad=\frac{1}{(x-2)(x^2+4x+5)}\left\{x^2+4x+5+(x-2)(2x+2)\right\}\\&\quad=\frac{1}{(x-2)(x^2+4x+5)}\left\{x^2+4x+5+2x^2-2x-4\right\}\\&\quad=\frac{3x^2+2x+1}{(x-2)(x^2+4x+5)}\qquad\text{Q.E.D.}\end{aligned}$$

Example 4.15 *Decompose the fraction*

$$\frac{12x+7}{(x-1)(x^2+6x+12)}.$$

A. Decomposition.

D. Apply the standard procedure and then reduce.

I. The degree of the numerator is lower than the degree of the denominator. Furthermore,

$$x^2+6x+12=(x+3)^2+3\geq 3,$$

so the fraction is put into its canonical form.

By the method of blinding a factor of the denominator the coefficient of $\frac{1}{x-1}$ becomes

$$\frac{12+7}{1+6+12}=1,$$

hence

$$\begin{aligned}\frac{P(x)}{Q(x)}&=\frac{12x+7}{(x-1)(x^2+6x+12)}=\frac{1}{x-1}+\frac{12x+7-(x^2+6x+12)}{(x-1)(x^2+6x+12)}\\&=\frac{1}{x-1}+\frac{-x^2+6x-5}{(x-1)(x^2+6x+12)}=\frac{1}{x-1}+\frac{-x+5}{x^2+6x+12}\\&=\frac{1}{x-1}-\frac{x-5}{x^2+6x+12}.\end{aligned}$$

C. TEST.

$$\begin{aligned}\frac{1}{x-1}-\frac{x-5}{x^2+6x+12}&=\frac{x^2+6x+12-(x-1)(x-5)}{(x-1)(x^2+6x+12)}\\&=\frac{x^2+6x+12-x^2+6x-5}{(x-1)(x^2+6x+12)}\\&=\frac{12x+7}{(x-1)(x^2+6x+12)}\qquad\text{Q.E.D.}\end{aligned}$$

Example 4.16 *Decompose*

$$\frac{3x^2+2x+5}{(x+2)(x^2-2x+5)}, \qquad x \neq -2.$$

A. Decomposition.

D. "Blind the factors, etc." and reduce.

I. Since $x^2-2x+5=(x+1)^2+4 \geq 4$, and the numerator is of lower degree than the denominator, we know that the structure of the decomposition is

$$\frac{3x^2+2x+5}{(x+2)(x^2-2x+5)} = \frac{A}{x+2}+\frac{Bx+C}{x^2-2x+5},$$

where

$$\begin{aligned} A &= \left[\frac{3x^2+2x+5}{x^2-2x+5}\right]_{x=-2} = \frac{3(-2)^2+2(-2)+5}{(-2)^2-2(-2)+5} \\ &= \frac{3\cdot 4-4+5}{4+4+5} = \frac{13}{13} = 1, \end{aligned}$$

hence

$$\begin{aligned} \frac{Bx+C}{x^2-2x+5} &= \frac{3x^2+2x+5}{(x+2)(x^2-2x+5)} - \frac{1}{x+2} = \frac{3x^2+2x+5-x^2+2x-5}{(x+2)(x^2-2x+5)} \\ &= \frac{2x^2+4x}{(x+2)(x^2-2x+5)} = \frac{2x}{x^2-2x+5}. \end{aligned}$$

The decomposition is

$$\frac{3x^2+2x+5}{(x+2)(x^2-2x+5)} = \frac{1}{x+2}+\frac{2x}{x^2-2x+5}.$$

5 Integration by decomposition

Example 5.1 *Calculate the integrals*

$$(1)\quad \int \frac{1}{4-x^2}\,dx, \quad x>2, \qquad (2)\quad \int \frac{1}{4+x^2}\,dx.$$

A. Simple integrals.

D. 1) Decompose the integrand and integrate.

2) Substitute conveniently before the integration.

I. 1) By the decomposition ("blind the factor, etc.") we get

$$\frac{1}{4-x^2} = -\frac{1}{(x-2)(x+2)} = -\frac{1}{4}\cdot\frac{1}{x-2} + \frac{1}{4}\cdot\frac{1}{x+2}.$$

Then for $x>2$,

$$\begin{aligned}\int \frac{1}{4-x^2}\,dx &= \frac{1}{4}\int\left(\frac{1}{x+2}-\frac{1}{x-2}\right)dx = \frac{1}{4}\left\{\ln|x+2|-\ln|x+2|\right\}\\ &= \frac{1}{4}\left\{\ln(x+2)-\ln(x-2)\right\} = \frac{1}{4}\ln\frac{x+2}{x-2}, \qquad x>2.\end{aligned}$$

2) When we substitute $t=\dfrac{x}{2}$, $x=2t$, we get

$$\begin{aligned}\int \frac{1}{4+x^2}\,dx &= \frac{1}{4}\int\frac{2}{1+t^2}\,dt = \frac{1}{2}\operatorname{Arctan} t\\ &= \frac{1}{2}\operatorname{Arctan}\left(\frac{x}{2}\right), \qquad x\in\mathbb{R}.\end{aligned}$$

C. TEST.

1) If

$$f(x) = \frac{1}{4}\ln\frac{x+2}{x-2} = \frac{1}{4}\left\{\ln(x+2)-\ln(x-2)\right\}, \qquad x>2,$$

then

$$f'(x) = \frac{1}{4}\left\{\frac{1}{x+2}-\frac{1}{x-2}\right\} = \frac{1}{4}\cdot\frac{(x-2)-(x+2)}{(x+2)(x-2)} = -\frac{1}{x^2-4} = \frac{1}{4-x^2},$$

Q.E,D.

2) If $f(x)=\dfrac{1}{2}\operatorname{Arctan}\left(\dfrac{x}{2}\right)$, $x\in\mathbb{R}$, then

$$f'(x) = \frac{1}{2}\cdot\frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2} = \frac{1}{4}\cdot\frac{1}{1+\frac{x^2}{4}} = \frac{1}{4+x^2}, \qquad \text{Q.E.D.}$$

Example 5.2 *Calculate the integrals*

$$(1)\quad \int \frac{1}{x^2+4x+8}\,dx, \qquad (2)\quad \int \frac{1}{\left(4x^2+9\right)^2}\,dx.$$

A. Simple integrals.

D. Analyze the denominators and choose convenient substitutes. When a factor in the denominator consists of only two terms, the trick is to norm it such that the constant becomes 1.

I. 1) Since

$$x^2+4x+8=(x+2)^2+4=4\left\{1+\left(\frac{x+2}{2}\right)^2\right\},$$

we choose the substitution

$$t=\frac{x+2}{2}=1+\frac{x}{2}, \quad \text{dvs.} \quad x=2t-2.$$

Then

$$\int \frac{1}{x^2+4x+8}\,dx = \int \frac{1}{4\left\{+\left(\dfrac{x+2}{2}\right)^2\right\}}\,dx$$

$$= \frac{1}{4}\left[\int \frac{2}{1+t^2}\,dt\right]_{t=\frac{x+2}{x}} = \frac{1}{2}[\text{Arctan } t]_{t=\frac{x+2}{2}}$$

$$= \frac{1}{2}\,\text{Arctan}\left(\frac{x+2}{2}\right) = \frac{1}{2}\,\text{Arctan}\left(1+\frac{x}{2}\right).$$

2) Since

$$4x^2+9 = 9\left\{1+\left(\frac{2}{3}\,x\right)^2\right\},$$

we choose the substitution

$$t = \frac{2}{3}\,x, \qquad x = \frac{3}{2}\,t.$$

Then

$$\int \frac{1}{(4x^2+9)^2}\,dx = \frac{1}{9^2}\int \frac{1}{\left\{1+\left(\dfrac{2}{3}\,x\right)^2\right\}^2}\,dx$$

$$= \frac{1}{81}\cdot\frac{3}{2}\int_{t=\frac{2}{3}\,x} \frac{1}{(1+t^2)^2}\,dt$$

$$= \frac{1}{54}\left\{\frac{t}{2\,(t^2+1)}+\frac{1}{2}\,\text{Arctan } t\right\}_{t=\frac{2}{3}\,x}$$

$$= \frac{1}{108}\cdot\frac{\dfrac{2}{3}\,x}{1+\left(\dfrac{2}{3}\,x\right)^2}+\frac{1}{108}\,\text{Arctan}\left(\frac{2}{3}\,x\right)$$

$$= \frac{1}{108}\cdot\frac{9\cdot\dfrac{2}{3}\,x}{9+4x^2}+\frac{1}{108}\,\text{Arctan}\left(\frac{2}{3}\,x\right)$$

$$= \frac{1}{18}\cdot\frac{x}{9+4x^2}+\frac{1}{108}\,\text{Arctan}\left(\frac{2}{3}\,x\right).$$

C. TEST.

1) If

$$f(x) = \frac{1}{2}\,\text{Arctan}\left(\frac{x+2}{2}\right),$$

then

$$f'(x) = \frac{1}{2}\cdot\frac{1}{1+\left(\dfrac{x+2}{2}\right)^2}\cdot\frac{1}{2} = \frac{1}{4+(x+2)^2} = \frac{1}{x^2+4x+8}. \qquad \text{Q.E.D.}$$

2) If

$$f(x) = \frac{1}{18} \cdot \frac{x}{9+4x^2} + \frac{1}{108} \operatorname{Arctan}\left(\frac{2}{3}\,x\right),$$

then

$$\begin{aligned} f'(x) &= \frac{1}{18} \cdot \frac{1 \cdot (9+4x^2) - x \cdot 8x}{(9+4x^2)^2} + \frac{1}{108} \cdot \frac{1}{1+\left(\frac{2}{3}\,x\right)^2} \cdot \frac{2}{3} \\ &= \frac{1}{18} \cdot \frac{9-4x^2}{(9+4x^2)^2} + \frac{6}{108} \cdot \frac{1}{9+4x^2} \\ &= \frac{1}{8} \cdot \frac{(9-4x^2)+(9+4x^2)}{(9+4x^2)^2} = \frac{1}{(9+4x^2)^2}, \end{aligned}$$

Q.E.D.

Example 5.3 *Decompose the fraction*

$$\frac{P(x)}{Q(x)} = \frac{4x^2-36}{x^4-2x^2+1}.$$

Then calculate the integral

$$\int \frac{P(x)}{Q(x)}\,dx, \qquad x \in]-1,1[.$$

A. A decomposition where the degree of the numerator is lower than the degree of the denominator, followed by an integration.

D. Dissolve the denominator into factors and decompose, e.g. by “blinding the factors”.

I. Since

$$x^4 - 2x^2 + 1 = \left(x^2-1\right)^2 = (x-1)^2(x+1)^2,$$

and the degree of the numerator is lower than the degree of the numerator, the structure of the decomposition is given by

$$\begin{aligned} \frac{P(x)}{Q(x)} &= \frac{4x^2-36}{x^4-2x^2+1} = \frac{4x^2-36}{(x-1)^2(x+1)^2} \\ &= \frac{a}{(x-1)^2} + \frac{b}{(x+1)^2} + \frac{c}{x-1} + \frac{d}{x+1}. \end{aligned}$$

We multiply by $(x-1)^2$ (“blind the factor”) and perform the limit process $x \to 1$. Then

$$a = \lim_{x\to 1} \frac{4x^2-36}{(x+1)^2} = \frac{4-36}{(1+1)^2} = -\frac{32}{4} = -8.$$

By a multiplication by $(x+1)^2$ ("blind the factor"), followed by the limit $x \to -1$, we get

$$b = \lim_{x \to -1} \frac{4x^2-36}{(x-1)^2} = \frac{4(-1)^2-36}{(-1-1)^2} = -8.$$

When these constants are inserted, we get by a rearrangement and a reduction,

$$\begin{aligned}
\frac{c}{x-1}+\frac{d}{x+1} &= \frac{4x^2-36}{(x-1)^2(x+1)^2}+\frac{8}{(x-1)^2}+\frac{8}{(x+1)^2} \\
&= \frac{4x^2-36}{(x-1)^2(x+1)^2}+8\cdot\frac{(x+1)^2+(x-1)^2}{(x-1)^2(x+1)^2} \\
&= \frac{4x^2-36+16x^2+16}{(x-1)(x+1)(x^2-1)} = \frac{20x^2-20}{(x-1)(x+1)(x^2-1)} \\
&= \frac{20}{(x-1)(x+1)} = \frac{10}{x-1}-\frac{10}{x+1}.
\end{aligned}$$

We see that the decomposition is given by

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{4x^2-36}{x^4-2x^2+1} \\
&= -\frac{8}{(x-1)^2}+\frac{10}{x-1}-\frac{8}{(x+1)^2}-\frac{10}{x+1}.
\end{aligned}$$

Then by an integration for $x \in]-1,1[$,

$$\begin{aligned}
\int \frac{P(x)}{Q(x)}\,dx &= \frac{8}{x-1}+10\,\ln|x-1|+\frac{8}{x+1}-10\,\ln|x+1| \\
&= \frac{16}{x^2-1}+10\,\ln(1-x^2).
\end{aligned}$$

C. TEST. We shall only check the decomposition:

$$\begin{aligned}
&-\frac{8}{(x-1)^2}+\frac{10}{x-1}-\frac{8}{(x+1)^2}-\frac{10}{x+1} = \frac{-8\cdot 2(x^2+1)}{(x^2-1)^2}+\frac{10\cdot 2}{x^2-1} \\
&\qquad = \frac{-16x^2-16+20(x^2-1)}{(x^2-1)^2} = \frac{4x^2-36}{x^4-2x^2+1} \qquad \text{Q.E.D.}
\end{aligned}$$

Example 5.4 *Calculate the integral*

$$\int \frac{2x^5-2x^4+1}{x^4-x^3}\,dx.$$

A. Integral with a latent decomposition.

D. First check where the integrand is defined. Since the degree of the denominator is lower than the degree of the numerator, we shall first divide by the denominator in order to find the polynomial, before we decompose the residual fraction. Once the fraction has been decomposed we integrate.

Since $x^4 - x^3 = x^3(x-1)$, the integrand is defined for $x \neq 0$, 1, i.e. in the three intervals

$$]-\infty, 0[, \qquad]0,1[, \quad \text{and} \quad]1,+\infty[.$$

If $x \neq 0$, 1, then

$$\begin{aligned}
\frac{2x^5 - 2x^4 + 1}{x^4 - x^3} &= 2x + \frac{1}{x^4 - x^3} = 2x + \frac{1}{x^3)x-1)} \\
&= 2x + \frac{1}{x-1} + \frac{1 - x^3}{x^3(x-1)} = 2zx + \frac{1}{x-1} - \frac{x^2+x+1}{x^3} \\
&= 2x + \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3}.
\end{aligned}$$

All these fractions can immediately be integrated, hence

$$\begin{aligned}
\int \frac{2x^5 - 2x^4 + 1}{x^4 - x^3}\,dx &= \int \left\{ 2x + \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} \right\} dx \\
&= x^2 + \ln|x-1| - \ln|x| + \frac{1}{x} + \frac{1}{2}\,\frac{1}{x^2}.
\end{aligned}$$

C. Test. If we put

$$f(x) = x^2 + \ln|x-1| - \ln|x| + \frac{1}{x} + \frac{1}{2}\,\frac{1}{x^2},$$

we get

$$\begin{aligned} f'(x) &= 2x + \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} \\ &= 2x + \frac{1}{x-1} - \frac{x^2+x+1}{x^3} = 2x + \frac{x^3-(x^3-1)}{(x-1)x^3} \\ &= 2x + \frac{1}{x^4-x^3} = \frac{2x^5-2x^4+1}{x^4-x^3} \qquad \text{Q.E.D.} \end{aligned}$$

Example 5.5 *Calculate the integral*

$$\int \frac{1}{x^5+2x^3+x}\,dx$$

in each of the intervals, in which the integrand is continuous.

A. Integration based on a decomposition and possibly some substitution (this will depend on the variant).

D. Dissolve the denominator into factors and find the domain. Then decompose before the integration. We have two variants.

I. From

$$x^5+2x^3+x = x\left\{x^4+2x^2+1\right\} = x\left(x^2+1\right)^2,$$

follows that the intervals of the domain are

$$]-\infty,0[\quad \text{and} \quad]0,+\infty[.$$

First variant. When $x \neq 0$ we get by the substitution $t = x^2$ that

$$\int \frac{1}{x(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{2x}{x^2(x^2+1)^2}\,dx = \frac{1}{2}\int_{t=x^2} \frac{1}{t(t+1)^2}\,dt.$$

By decomposition,

$$\frac{1}{t(t+1)^2} = \frac{A}{t} + \frac{B}{(t+1)^2} + \frac{C}{t+1},$$

we first get by "blinding the factor" that $A = 1$ and $B = -1$, hence by a reduction,

$$\begin{aligned} \frac{C}{t+1} &= \frac{1}{t(t+1)^2} - \frac{1}{t} + \frac{1}{(t+1)^2} = \frac{1-(t+1)^2+t}{t(t+1)^2} \\ &= \frac{(t+1)(1-t-1)}{t(t+1)} = -\frac{1}{t+1}. \end{aligned}$$

With these values of the constants we get the integral

$$\begin{aligned}
\int \frac{1}{x(x^2+1)}\,dx &= \frac{1}{2}\int_{t=x^2} \frac{1}{t(t+1)^2}\,dt \\
&= \frac{1}{2}\int_{t=x^2} \left\{\frac{1}{t} - \frac{1}{(t+1)^2} - \frac{1}{t+1}\right\} dt \\
&= \frac{1}{2}\left[\ln|t| + \frac{1}{t+1} - \ln|t+1|\right]_{t=x^2} \\
&= \frac{1}{2}\ln\left(x^2\right) - \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\,\frac{1}{x^2+1} \\
&= \frac{1}{2}\,\frac{1}{x^2+1} + \frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right).
\end{aligned}$$

Second variant. By a direct decomposition,

$$\frac{1}{x(x^2+1)^2} = \frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{kx+\ell}{(x^2+1)^2},$$

we get immediately that $a = 1$, and then

$$\begin{aligned}
\frac{1}{x(x^2+1)^2} - \frac{1}{x} &= \frac{1-(x^2+1)^2}{x(x^2+1)^2} = \frac{(x^2+2)\cdot(-x^2)}{x(x^2+1)^2} \\
&= -x \cdot \frac{x^2+2}{(x^2+1)^2} = -x\left\{\frac{1}{x^2+1} + \frac{1}{(x^2+1)^2}\right\},
\end{aligned}$$

i.e.

$$\frac{1}{x(x^2+1)^2} = \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}.$$

By insertion into the integral we get for $x \neq 0$,

$$\begin{aligned}
\int \frac{1}{x(x^2+1)}\,dx &= \int \left\{\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right\} dx \\
&= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\,\frac{1}{x^2+1} \\
&= \frac{1}{2}\,\frac{1}{x^2+1} + \frac{1}{2}\ln\frac{x^2}{x^2+1}.
\end{aligned}$$

C. TEST. If $x \neq 0$, then

$$\begin{aligned}
&\frac{d}{dx}\left\{\frac{1}{2}\,\frac{1}{x^2+1} + \frac{1}{2}\ln\frac{x^2}{x^2+1}\right\} \\
&\quad = \frac{1}{2}\left\{-\frac{2x}{(x^2+1)^2} + \frac{x^2+1}{x^2}\cdot\frac{2x(x^2+1)-2x(x^2)}{(x^2+1)^2}\right\} \\
&\quad = -\frac{x}{(x^2+1)^2} + \frac{x}{(x^2+1)^2} + \frac{1}{x^2}\cdot\frac{x}{(x^2+1)^2} \\
&\quad = \frac{1}{x(x^4+2x^2+1)} = \frac{1}{x^5+2x^3+x}. \qquad \text{Q.E.D.}
\end{aligned}$$

Example 5.6 *Calculate the integral*

$$\int \frac{x^2+4x-7}{x^3-4x^2+9x-10}\,dx$$

in each of the intervals, in which the integrand is continuous.

A. Integration by decomposition.

D. Dissolve the denominator and decompose before the integration.

I. First consider the denominator. The only *possible* rational roots are

$$x = \pm 1,\ \pm 2,\ \pm 5 \text{ and } \pm 10.$$

From this we get by inspection that

$$\begin{aligned} x^3-4x^2+9x-10 &= (x^3-4x^2+4x)+(5x-10) \\ &= x\{x^2-4x+4\}+5(x-2) = (x-2)\{x(x-2)+5\} \\ &= (x-2)\{x^2-2x+1+4\} = (x-2)\{(x-1)^2+4\}. \end{aligned}$$

The integrand is defined and continuous in each of the intervals

$$]-\infty, 2[\qquad \text{and} \qquad]2, +\infty[.$$

By the decomposition we must have the structure

$$\begin{aligned} \frac{x^2+4x-7}{x^3-4x^2+9x-10} &= \frac{x^2+4x-7}{(x-2)(x^2-2x+5)} \\ &= \frac{a}{x-2}+\frac{bx+c}{x^2-2x+5}. \end{aligned}$$

It follows immediately ("blind the factor") that

$$a = \frac{2^2+4\cdot 2-7}{2^2-2\cdot 2+5} = 1,$$

hence

$$\begin{aligned} \frac{bx+c}{x^2-2x+5} &= \frac{x^2+4x-7}{(x-2)(x^2-2x+5)} - \frac{1}{x-2}\cdot\frac{x^2-2x+5}{x^2-2x+5} \\ &= \frac{x^2+4x-7-x^2+2x-5}{(x-2)(x^2-2x+5)} \\ &= \frac{6x-12}{(x-2)(x^2-2x+5)} = \frac{6}{(x-1)^2+2^2}. \end{aligned}$$

When $x \neq 2$, we get by insertion that

$$\begin{aligned} \int \frac{x^2+4x-7}{x^3-4x^2+9x-10}\,dx &= \int \left\{\frac{1}{x-2}+\frac{6}{(x-1)^2+2^2}\right\}dx \\ &= \ln|x-2| + 3\,\text{Arctan}\left(\frac{x-1}{2}\right). \end{aligned}$$

C. Test. Let

$$f(x) = \ln |x-2| + 3 \operatorname{Arctan}\left(\frac{x-1}{2}\right), \qquad x \neq 2.,$$

Then by differentiation for $x \neq 2$,

$$\begin{aligned} f'(x) &= \frac{1}{x-2} + 3 \cdot \frac{1}{1+\left(\frac{x-1}{2}\right)^2} \cdot \frac{1}{2} = \frac{1}{x-2} + \frac{6}{x^2-2x+5} \\ &= \frac{x^2-2x+5+6x-12}{x^3-4x^2+9x-1=} = \frac{x^2+4x-7}{x^3-4x^2+9x-10}. \qquad \text{Q.E.D.} \end{aligned}$$

Example 5.7 *Find the complete solution of the differential equation*

$$(1-t^4)\,\frac{dx}{dt}+4tx=2t(1-t^4),\qquad t>1.$$

A. A non-normed, linear, inhomogeneous differential equation of first order, where the method of guessing is not obvious.

D. First norm the equation. Then find the complete solution of the homogeneous equation, and finally a particular solution by means of a solution formula. One shall use a decomposition during these calculations.

I. A division by $1-t^4<0$ gives the equivalent normed differential equation

$$\frac{dx}{dt}+\frac{4t}{1-t^4}\,x=2t,\qquad t>1.$$

From

$$\begin{aligned}P(t) &= \int\frac{4t}{1-t^4}\,dt=\int\left(\frac{1}{1-t^2}+\frac{1}{1+t^2}\right)2t\,dt\\ &= \int_{u=t^2}\left(-\frac{1}{u-1}+\frac{1}{u+1}\right)du=\ln\left|\frac{t^2+1}{t^2-1}\right|\\ &= \ln\left(\frac{t^2+1}{t^2-1}\right)\end{aligned}$$

follows that the complete solution of the homogeneous equation is

$$c\cdot\exp\left(-\ln\left(\frac{t^2+1}{t^2-1}\right)\right)=c\cdot\frac{t^2-1}{t^2+1},\qquad t>1,\quad c\in\mathbb{R}.$$

If $\varphi(t)=\dfrac{t^2-1}{t^2+1}$, $t>1$, then a particular integral is given by

$$\begin{aligned}x_0(t) &= \varphi(t)\int\frac{2t}{\varphi(t)}\,dt=\frac{t^2-1}{t^2+1}\int\frac{t^2+1}{t^2-1}\cdot 2t\,dt\\ &= \frac{t^2-1}{t^2+1}\int_{u=t^2}\frac{u+1}{u-1}\,du=\frac{t^2-1}{t^2+1}\int_{u=t^2}\left\{1+\frac{2}{u-1}\right\}du\\ &= \frac{t^2-1}{t^2+1}\left\{t^2+2\,\ln\left(t^2-1\right)\right\}\\ &= t^2-2+\frac{2}{t^2+1}+2\left(1-\frac{2}{t^2+1}\right)\ln\left(t^2-1\right).\end{aligned}$$

Hence the complete solution is

$$x=t^2-2+\frac{2}{t^2+1}+2\left(1-\frac{2}{t^2+1}\right)\ln\left(t^2-1\right)+c_1\cdot\frac{t^2-1}{t^2+1},\quad t>1,$$

where $c_1\in\mathbb{R}$ is an arbitrary constant.

Since
$$t^2-2+\frac{2}{t^2+1}=t^2-1-\frac{t^2-1}{t^2+1},$$
this expression can also be written
$$x=t^2-1+2\cdot\frac{t^2-1}{t^2+1}\,\ln\left(t^2-1\right)+c\cdot\frac{t^2-1}{t^2+1},\qquad t>1,\quad c\in\mathbb{R},$$
where $c=c_1-1$.

C. TEST. Let
$$\begin{aligned}x &= t^2-1+2\cdot\frac{t^2-1}{t^2+1}\,\ln\left(t^2-1\right)+c\cdot\frac{t^2-1}{t^2+1}\\ &= t^2-1+2\left(1-\frac{2}{t^2+1}\right)\ln\left(t^2-1\right)+c\cdot\frac{t^2-1}{t^2+1},\qquad t>1,\quad c\in\mathbb{R}.\end{aligned}$$

Then
$$\begin{aligned}&(1-t^4)\,\frac{dx}{dt}+4tx\\ &= (1-t^4)\left\{2t+\frac{8t}{(t^2+1)^2}\ln(t^2-1)+2\,\frac{t^2-1}{t^2+1}\cdot\frac{2t}{t^2-1}+c\cdot\frac{4t}{(t^2+1)^2}\right\}\\ &\qquad +4t^3-4t+8t\cdot\frac{t^2-1}{t^2+1}\,\ln(t^2-1)+4t\cdot c\cdot\frac{t^2-1}{t^2+1}\\ &= 2t(1-t^4)+8t\cdot\frac{1-t^2}{t^2+1}\,\ln(t^2-1)-4t(t^2-1)+4ct\cdot\frac{1-t^2}{1+t^2}\\ &\qquad +4t(t^2-1)+8t\cdot\frac{t^2-1}{t^2+1}\,\ln(t^2-1)+4tc\cdot\frac{t^2-1}{t^2+1}\\ &= 2t(1-t^4).\qquad \text{Q.E.D.}\end{aligned}$$

Example 5.8 *1) Decompose the fraction*
$$f(x)=\frac{x^2-3x+8}{(x^2-4x+5)(x+3)},\qquad x>-3.$$
2) Find an integral of f for $x>-3$.

A. Decomposition and integration.

D. Decomposition, substitution and integration.

I. 1) Since $x^2-4x+5=(x-2)^2+1\geq 1$, we get by the standard method,
$$\begin{aligned}f(x) &= \frac{x^2-3x+8}{(x^2-4x+5)(x+3)}\\ &= \frac{9+9+8}{9+12+5}\,\frac{1}{x+3}+\frac{x^2-3x+8-x^2+4x-5}{(x^2-4x+5)(x+3)}\\ &= \frac{1}{x+3}+\frac{1}{x^2-4x+5}=\frac{1}{x+3}+\frac{1}{(x-2)^2+1}.\end{aligned}$$

2) The integral is

$$\int f(x)\,dx = \ln(x+3) + \text{ Arctan}(x-2), \qquad x > -3.$$

Example 5.9 *1) Given that the polynomial*

$$Q(x) = x^5 - 2x^4 + 2x^3 - 4x^2 + x - 2$$

has the root 2. Write $Q(x)$ as a product of factors of first degree and irreducible factors of second degree.

2) Decompose the fraction

$$\frac{P(x)}{Q(x)} = \frac{2x^4 - 2x^3 + 3x^2 - 3x + 3}{Q(x)},$$

and calculate

$$\int \frac{P(x)}{Q(x)}\,dx.$$

A. Integration via a decomposition.

D. 1) Divide by $x-2$ and analyze the resulting polynomial for further roots.

2) Use the splitting from the decomposition (1). The integrate.

I. 1) It is immediately seen that

$$\begin{aligned} Q(x) &= (x^5 - 2x^4) + (2x^3 - 4x^2) + (x-2) \\ &= (x-2)(x^4 + 2x^2 + 1) = (x-2)(x^2+1)^2. \end{aligned}$$

2) The structure of decomposition is

$$\begin{aligned} \frac{P(x)}{Q(x)} &= \frac{2x^4 - 2x^3 + 3x^2 - 3x + 3}{(x-2)(x^2+1)^2} \\ &= \frac{a}{x-2} + \frac{bx+c}{x^2+1} + \frac{kx+\ell}{(x^2+1)^2}, \end{aligned}$$

because deg $P <$ deg Q. It is immediately seen that

$$a = \frac{2\cdot 2^4 - 2\cdot 2^3 + 3\cdot 2^2 - 3\cdot 2 + 3}{(2^2+1)^2} = \frac{32 - 16 + 12 - 6 + 3}{25} = 1.$$

Then by a rearrangement,

$$\begin{aligned} &\frac{bx+c}{x^2+1} + \frac{kx+\ell}{(x^2+1)^2} \\ &\quad= \frac{2x^4 - 2x^3 + 3x^2 - 3x + 3}{(x-2)(x^2+1)^2} - \frac{1}{x-2}\cdot\frac{(x^2+1)^2}{(x^2+1)^2} \\ &\quad= \frac{2x^4 - 2x^3 + 3x^2 - 3x + 3 - x^4 - 2x^2 - 1}{(x-2)(x^2+1)^2} \\ &\quad= \frac{x^4 - 2x^3 + x^2 - 3x + 2}{(x-2)(x^2+1)^2} = \frac{x^3(x-2) + (x-2)(x-1)}{(x-2)(x^2+1)^2} \\ &\quad= \frac{x^3 + x - 1}{(x^2+1)^2} = \frac{x}{x^2+1} - \frac{1}{(x^2+1)^2}. \end{aligned}$$

All things put together we get

$$\frac{P(x)}{Q(x)} = \frac{1}{x-2} + \frac{x}{x^2+1} - \frac{1}{(x^2+1)^2}.$$

For $x \neq 2$ we get

$$\int \frac{1}{x-2}\,dx = \ln|x-2| \qquad \int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(1+x^2),$$

hence

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{x}{2(x^2+1)} + \frac{1}{2}\,\text{Arctan}\,x.$$

When these subresults are inserted we get for $x \neq 2$,

$$\begin{aligned}\int \frac{P(x)}{Q(x)}\,dx &= \int\left\{\frac{1}{x-2} + \frac{x}{x^2+1} - \frac{1}{(x^2+1)^2}\right\}dx \\ &= \ln|x-2| + \frac{1}{2}\ln(1+x^2) - \frac{1}{2}\,\frac{x}{1+x^2} - \frac{1}{2}\,\text{Arctan}\,x.\end{aligned}$$

C. TEST. By differentiation we get

$$\begin{aligned}
&\frac{d}{dx}\left\{\ln|x-2|+\frac{1}{2}\ln(1+x^2)-\frac{1}{2}\frac{x}{1+x^2}-\frac{1}{2}\operatorname{Arctan} x\right\}\\
&= \frac{1}{x-2}+\frac{x}{1+x^2}-\frac{1}{2}\frac{1}{1+x^2}+\frac{x^2}{(1+x^2)^2}-\frac{1}{2}\frac{1}{1+x^2}\\
&= \frac{1}{x-2}+\frac{x}{1+x^2}+\frac{x^2-(1+x^2)}{(1+x^2)^2}=\frac{1}{x-2}+\frac{x(1+x^2)-1}{(1+x^2)^2}\\
&= \frac{x^4+2x^2+1+(x-2)(x^3+x-1)}{(x-2)(x^4+2x^2+1)}\\
&= \frac{x^4+2x^2+1+x^4-2x^3+x^2-3x+2}{x^5-2x^4+2x^3-4x^2+x-2}\\
&= \frac{2x^4-2x^3+3x^2-3x+3}{x^5-2x^4+2x^3-4x^2+x-2}. \qquad \text{Q.E.D.}
\end{aligned}$$

Example 5.10 *Find an integral of the function*

$$f(x)=\frac{x^2-5x+5}{(x^2-6x+10)(x-3)}, \qquad x>3.$$

A. Integration via decomposition.

D. Start by a decomposition. We give three variants. Then integrate.

I. Since the degree of the numerator is lower than the degree of the denominator in the rational function $f(x)$, the decomposition must have the structure

$$f(x)=\frac{x^2-5x+4}{(x^2-6x+10)(x-3)}=\frac{A}{x-3}+\frac{Bx+C}{x^2-6x+10},$$

because $x^2-6x+10=(x-3)^2+1\geq 1$ for every $x\in\mathbb{R}$.

We give three variants of the solution:

First variant. By "blinding the factor, etc." $x-3$, we get

$$A=\left[\frac{x^2-5x+5}{x^2-6x+10}\right]_{x=3}=\frac{9-15+5}{9-18+10}=-1.$$

Putting $A=-1$ we get by a rearrangement,

$$\begin{aligned}
\frac{Bx+C}{x^2-6x+10} &= \frac{x^2-5x+5}{(x^2-6x+10)(x-3)}-\frac{A}{x-3}\\
&= \frac{(x^2-5x+5)+(x^2-6x+10)}{(x^2-6x+10)(x-3)}\\
&= \frac{2x^2-11x+15}{(x^2-6x+10)(x-3)}\\
&= \frac{2x-5}{x^2-6x+10}.
\end{aligned}$$

Hence,

$$f(x) = \frac{x^2-5x+5}{(x^2-6x+10)(x-3)} = -\frac{1}{x-3} + \frac{2x-5}{x^2-6x+10}.$$

Second variant. When we multiply by $(x^2-6x+10)(x-3)$ we get

$$\begin{aligned} x^2-5x+5 &= A(x^2-6x+10)+(Bx+C)(x-3) \\ &= (A+B)x^2+(-6A-3B+C)x+(10A-3C). \end{aligned}$$

An identification of the coefficients gives

$$\left\{ \begin{array}{rcrcrcr} A & + & B & & & = & 1, \\ -6A & - & 3B & + & C & = & -5, \\ 10A & & & - & 3C & = & 5, \end{array} \right.$$

from which we get either by fumbling or by using Linear Algebra,

$$A=-1, \qquad B=2 \quad \text{og} \quad C=-5,$$

i.e.

$$f(x) = \frac{x^2-5x+5}{(x^2-6x+10)(x-3)} = -\frac{1}{x-3} + \frac{2x-5}{x^2-6x+10}.$$

Third variant. From $x^2-6x+10=(x-3)^2+1$ we get by the substitution $u=x-3$, $x=u+3$, that

$$\begin{aligned} \frac{x^2-5x+5}{(x^2-6x+10)(x-3)} &= \frac{(x-3)^2+(x-3)-1}{\{(x-3)^2+1\}(x-3)} = \frac{u^2+u-1}{(u^2+1)u} \\ &= \frac{u^2+u-1}{(u^2+1)u} + \frac{1}{u} - \frac{1}{u} = \frac{u^2+u-1+u^2+1}{(u^2+1)u} - \frac{1}{u} \\ &= \frac{2u+1}{u^2+1} - \frac{1}{u} = -\frac{1}{x-3} + \frac{2x-5}{x^2-6x+10}. \end{aligned}$$

Fourth variant. (Sketch). Insert three different x-values and solve the system of linear equations in A, B and C. It is possible to get through by this method, but it is nevertheless the most difficult one, and since it does not have a unique variant, it shall not be given here in all details.

Integration. From $x^2-6x+10=(x-3)^2+1>0$ follows for $x>3$ via the decomposition above,

$$\begin{aligned} \int \frac{x^2-5x+5}{(x^2-6x+10)(x-3)}\,dx &= -\int \frac{1}{x-3}\,dx + \int \frac{2x-5}{x^2-6x+10}\,dx \\ &= -\ln|x-3| + \int \frac{2(x-3)+1}{(x-3)^2+1}\,dx \\ &= -\ln(x-3) + \int \frac{2(x-3)}{(x-3)^2+1}\,dx + \int \frac{1}{(x-3)^2+1}\,dx \\ &= -\ln(x-3) + \ln\{(x-3)^2+1\} + \operatorname{Arctan}(x-3) \\ &= -\ln(x-3) + \ln(x^2-6x+10) + \operatorname{Arctan}(x-3) \\ &= \ln\left\{x-3+\frac{1}{x-3}\right\} + \operatorname{Arctan}(x-3), \end{aligned}$$

where some variants of the final result are given.

Example 5.11 *Calculate the integral*

$$\int \frac{1}{x^3+x^2+x+1}\,dx.$$

A. Integration by decomposition.

D. Decompose and integrate.

I. Since $x^3+x^2+x+1=(x+1)(x^2+1)$, we get for $x \neq -1$,

$$\begin{aligned}\frac{1}{x^3+x^2+x+1} &= \frac{1}{(x+1)(x^2+1)} = \frac{1}{2}\,\frac{1}{x+1} + \frac{-\frac{1}{2}\,x^2-\frac{1}{2}+1}{(x+1)(x^2+1)}\\ &= \frac{1}{2}\,\frac{1}{x+1} - \frac{1}{2}\,\frac{(x+1)(x-1)}{(x+1)(x^2+1)} = \frac{1}{2}\,\frac{1}{x+1} - \frac{1}{2}\,\frac{x-1}{x^2+1}.\end{aligned}$$

C. Test.

$$\begin{aligned}\frac{1}{2}\,\frac{1}{x+1} - \frac{1}{2}\,\frac{x-1}{x^2+1} &= \frac{1}{2}\cdot\frac{1}{(x+1)(x^2+1)}\left\{x^2+1-x^2+1\right\}\\ &= \frac{1}{x^3+x^2+x+1}. \qquad \text{Q.E.D.}\end{aligned}$$

I. For $x \neq -1$ it follows by the decomposition that

$$\begin{aligned}&\int \frac{1}{x^3+x^2+x+1}\,dx\\ &\quad= \frac{1}{2}\int\frac{1}{x+1}\,dx - \frac{1}{2}\int\frac{x}{x^2+1}\,dx + \frac{1}{2}\int\frac{1}{x^2+1}\,dx\\ &\quad= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\,\text{Arctan}\,x.\end{aligned}$$

Example 5.12 *Calculate the integral*

$$\int \frac{5x^2}{4x^3-4x^2+1-1}\,dx.$$

A. An integration where one must assume that $x \neq 1$.

D. Start by a decomposition.

I. Now,

$$4x^3-4x^2+x-1=(x-1)(4x^2+1),$$

and the degree of the numerator is lower than the degree of the denominator. Therefore, by a decomposition for $x \neq 1$,

$$\begin{aligned}
\frac{5x^2}{4x^3-4x^2+x-1} &= \frac{5x^2}{(x-1)(4x^2+1)} \\
&= \frac{5}{5}\cdot\frac{1}{x-1}+\left\{\frac{5x^2}{(x-1)(4x^2+1)}-\frac{1}{x-1}\right\} \\
&= \frac{1}{x-1}+\frac{5x^2-4x^2-1}{(x-1)(4x^2+1)} = \frac{1}{x-1}+\frac{x+1}{4x^2+1}.
\end{aligned}$$

Assuming that $x \neq 1$, we get the integral

$$\begin{aligned}
\int \frac{5x^2}{4x^3-4x^2+x-1}\,dx &= \int\frac{dx}{x-1}+\int\frac{x}{4x^2+1}\,dx+\int\frac{1}{4x^2+1}\,dx \\
&= \ln|x-1|+\frac{1}{8}\ln(4x^2+1)+\frac{1}{2}\operatorname{Arctan}(2x).
\end{aligned}$$

C. Test. By a differentiation of the result we get

$$\frac{1}{x-1}+\frac{1}{8}\cdot\frac{8x}{4x^2+1}+\frac{1}{2}\cdot\frac{2}{4x^2+1}=\frac{1}{x-1}+\frac{x+1}{4x^2+1}$$
$$=\frac{4x^2+1+x^2-1}{(x-1)(4x^2+1)}=\frac{5x^2}{4x^3-4x^2+x-1}. \qquad \text{Q.E.D.}$$

Example 5.13 *1) Decompose the fraction*

$$\frac{t^2+t+3}{(t-1)(t^2+4)}.$$

2) Then find the complete solution of the differential equation

$$\frac{dx}{dt}+\frac{2t+1}{t^2+t+3}\,x=\frac{1}{(t-1)(t^2+4)}, \qquad t>1.$$

A. Decomposition, and a linear, inhomogeneous differential equation of variable coefficients.

D. By the decomposition we "blind the factor" $t-1$ in the denominator and reduce. Then solve the differential equation, either by a formula or by multiplying by the integrating factor $t^2+t+3>0$.

I. 1) By the "blinding of a factor" we get for $t>1$,

$$\begin{aligned}\frac{t^2+t+3}{(t-1)(t^2+4)} &= \frac{1+1+3}{1+4}\cdot\frac{1}{t-1}+\left\{\frac{t^2+t+3}{(t-1)(t^2+4)}-\frac{1}{t-1}\right\}\\ &= \frac{1}{t-1}+\frac{t^2+t+3-t^2-4}{(t-1)(t^2+4)}=\frac{1}{t-1}+\frac{1}{t^2+4},\end{aligned}$$

where we immediately can check the result. (Mental arithmetic!)

2) **First variant.** When we multiply the equation by $t^2+t+3>0$, we obtain the equivalent equation

$$\begin{aligned}\frac{1}{t-1}+\frac{1}{t^2+4} &= \frac{t^2+t+3}{(t-1)(t^2+4)}=(t^2+t+3)\,\frac{dx}{dt}+(2t+1)\cdot x\\ &= \frac{d}{dt}\left\{\left(t^2+t+3\right)x\right\},\end{aligned}$$

hence by integration, where $t>1$,

$$\begin{aligned}(t^2+t+3)x &= c+\int\frac{dt}{t-1}+\frac{1}{4}\int\frac{dt}{1+\left(\dfrac{t}{2}\right)^2}\\ &= c+\ln(t-1)+\frac{1}{2}\,\text{Arctan}\,\frac{t}{2}.\end{aligned}$$

Division by t^2+t+3 gives the complete solution

$$x=\frac{\ln(t-1)+\dfrac{1}{2}\,\text{Arctan}\,\dfrac{t}{2}}{t^2+t+3}+\frac{c}{t^2+t+3}, \qquad t>1, \quad c\in\mathbb{R}.$$

Second variant. The equation is normed, and $p(t) = \dfrac{2t+1}{t^2+t+3}$, so

$$P(t) = \int p(t)\,dt = \int \frac{2t+1}{t^2+t+3}\,dt = \ln(t^2+t+3),$$

and the complete solution is

$$\begin{aligned} x(t) &= c\cdot e^{-P(t)} + e^{-P(t)}\int e^{P(t)}q(t)\,dt \\ &= \frac{c}{t^2+t+3} + \frac{1}{t^2+t+3}\int \frac{t^2+t+3}{(t-1)(t^2+4)}\,dt \\ &= \frac{c}{t^2+t+3} + \frac{1}{t^2+t+3}\int \left\{\frac{1}{t-1} + \frac{1}{t^2+4}\right\}dt \\ &= \frac{\ln(t-1) + \frac{1}{2}\,\mathrm{Arctan}\left(\frac{x}{2}\right)}{t^2+t+3} + \frac{c}{t^2+t+3}, \qquad t > 0, \quad c \in \mathbb{R}. \end{aligned}$$

Example 5.14 *Given the differential equation*

$$\frac{dx}{dt} + \frac{2t+1}{t^2+t+3}\,x = \frac{1}{t^3-t^2+4t-4}, \qquad t > 1.$$

1) Explain why the complete solution of the homogeneous *equation is given by*

$$x_{hom}(t) = \frac{c}{t^2+t+3}, \qquad c \in \mathbb{R}.$$

2) Find the complete solution of the inhomogeneous equation.

A. This is more or less the same as Example 5.13, only written in another way. A linear, inhomogeneous differential equation of first order of variable coefficients.

D. Test the given solution in the equation and then exploit the general structure of the solution.

I. 1) Putting $x_{\text{hom}}(t) = \dfrac{c}{t^2+t+3}$ into the left hand side of the equation gives

$$\frac{dx}{dt} + \frac{2t+1}{t^2+t+3}\,x = -c\cdot\frac{2t+1}{(t^2+t+3)^2} + \frac{2t+1}{t^2+t+3}\cdot\frac{c}{t^2+t+3} = 0.$$

Thus, $x_{\text{hom}}(t)$ is a solution of the homogeneous equation. since $c \in \mathbb{R}$ is an arbitrary constant, the general structure of the solution then gives the result.

2) A particular solution is given by

$$x_0(t) = x_{\text{hom}}(t)\int \frac{q(t)}{x_{\text{hom}}(t)}\,dt = \frac{1}{t^2+t+3}\int \frac{t^2+t+3}{t^3-t^2+4t-4}\,dt.$$

Decompose the integrand, i.e. start by factorizing the denominator

$$\begin{aligned} \frac{t^2+t+3}{t^3-t^2+4t-4} &= \frac{t^2+t+3}{(t-1)(t^2+4)} = \frac{(t^2+4)+(t-1)}{(t-1)(t^2+4)} \\ &= \frac{1}{t-1} + \frac{1}{t^2+4}. \end{aligned}$$

Since $t > 1$ is given, it follows by insertion and integration that a particular solution is given by

$$\begin{aligned} x_0(t) &= \frac{1}{t^2+t+3}\int\left\{\frac{1}{t-1}+\frac{1}{t^2+4}\right\}dt \\ &= \frac{\ln(t-1)+\dfrac{1}{2}\,\text{Arctan}\,\dfrac{t}{2}}{t^2+t+3}. \end{aligned}$$

Combined with (1) we get the complete solution

$$x(t) = \frac{\ln(t-1)+\dfrac{1}{2}\,\text{Arctan}\,\dfrac{t}{2}+c}{t^2+t+3}, \qquad t>1, \quad c\in\mathbb{R}.$$

Example 5.15 *Calculate the integral*

$$\int \frac{x^2+x+1}{(x+1)(x^2+1)}\,dx, \qquad x \neq -1.$$

A. Integration via a decomposition.

D. Decompose and integrate.

I. The degree of the numerator is lower than the degree of the denominator, hence the fraction is given in its canonical form. By blinding the factor $x+1$ in the denominator we get that the coefficient of $\dfrac{1}{x+1}$ is $\dfrac{1}{2}$, hence

$$\begin{aligned} \frac{x^2+x+1}{(x+1)(x^2+1)} &= \frac{1}{2}\,\frac{1}{x+1}+\frac{x^2+x+1-\dfrac{1}{2}(x^2+1)}{(x+1)(x^2+1)} \\ &= \frac{1}{2}\,\frac{1}{x+1}+\frac{1}{2}\,\frac{(x+1)^2}{(x+1)(x^2+1)} \\ &= \frac{1}{2}\,\frac{1}{x+1}+\frac{1}{2}\,\frac{x+1}{x^2+1}. \end{aligned}$$

Thus for $x \neq -1$,

$$\begin{aligned} \int \frac{x^2+x+1}{(x+1)(x^2+1)}\,dx &= \frac{1}{2}\int\frac{1}{x+1}\,dx+\frac{1}{2}\int\frac{x}{x^2+1}\,dx+\frac{1}{2}\int\frac{1}{x^2+1}\,dx \\ &= \frac{1}{2}\ln|x+1|+\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\,\text{Arctan}\,x. \end{aligned}$$

C. Test. We get by a differentiation,

$$\frac{d}{dx}\left\{\frac{1}{2}\ln|x+1|+\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\,\text{Arctan}\,x\right\}$$

$$= \frac{1}{2}\,\frac{1}{x+1}+\frac{1}{4}\,\frac{2x}{x^2+1}+\frac{1}{2}\,\frac{1}{x^2+1} = \frac{1}{2}\left\{\frac{1}{x+1}+\frac{x+1}{x^2+1}\right\}$$

$$= \frac{1}{2}\cdot\frac{x^2+1+(x+1)^2}{(x+1)(x^2+1)} = \frac{x^2+x+1}{(x+1)(x^2+1)}. \qquad \text{Q.E.D.}$$

Example 5.16 *Find that integral $F(x)$ of*

$$f(x) = \frac{x^3 + x^2 + 4x + 1}{x^4 + 5x^2 + 4},$$

for which $F(0) = 0$.

A. Integration.

D. Start by a decomposition.

I. Since $x^4 + 5x^2 + 4 = \left(x^2+1\right)\left(x^2+4\right)$, we get by considering odd/even exponents in the numerator that

$$\begin{aligned} f(x) &= \frac{x^3 + x^2 + 4x + 1}{x^4 + 5x^2 + 4} = x \cdot \frac{x^2+4}{(x^2+1)(x^2+4)} + \frac{x^2+1}{(x^2+1)(x^2+4)} \\ &= \frac{x}{x^2+1} + \frac{1}{x^2+4}, \end{aligned}$$

hence,

$$\begin{aligned} F(x) &= \int_0^x \left\{ \frac{t}{t^2+1} + \frac{1}{t^2+4} \right\} dt \\ &= \left[\frac{1}{2} \ln\left(t^2+1\right) + \frac{1}{2} \operatorname{Arctan}\left(\frac{t}{2}\right) \right]_0^x \\ &= \frac{1}{2} \ln\left(x^2+1\right) + \frac{1}{2} \operatorname{Arctan} \frac{x}{2}. \end{aligned}$$

Example 5.17 *Decompose the function*

$$f(x) = \frac{x^2 - 4x + 5}{(x-1)(x^2+2)}, \qquad x > 1,$$

and then find all the integrals of the function $f(x)$.

A. Decomposition.

D. Blind the factor $(x-1)$ in the denominator; remove the corresponding term and reduce.

I. We get by the standard procedure,

$$\begin{aligned}
\frac{x^2-4x+5}{(x-1)(x^2+2)} &= \frac{2}{3}\,\frac{1}{x-1} + \frac{1}{3}\left\{\frac{3x^2-12x+15}{(x-1)(x^2+2)} - \frac{2x^2+4}{(x-1)(x^2+2)}\right\} \\
&= \frac{2}{3}\,\frac{1}{x-1} + \frac{1}{3}\,\frac{x^2-12x+11}{(x-1)(x^2+2)} \\
&= \frac{2}{3}\,\frac{1}{x-1} + \frac{1}{3}\,\frac{x-11}{x^2+2} \\
&= \frac{2}{3}\,\frac{1}{x-1} + \frac{1}{3}\,\frac{x}{x^2+2} - \frac{11}{3}\,\frac{1}{x^2+2}.
\end{aligned}$$

Hence for $x > 1$,

$$\begin{aligned}
\int \frac{x^2-4x+5}{(x-1)(x^2+2)}\,dx &= \frac{2}{3}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{x}{x^2+2}\,dx - \frac{11}{3}\int \frac{dx}{x^2+2} \\
&= \frac{2}{3}\,\ln(x-1) + \frac{1}{6}\,\ln\left(x^2+2\right) - \frac{11}{3\sqrt{2}}\,\operatorname{Arctan}\left(\frac{x}{\sqrt{2}}\right).
\end{aligned}$$

6 Trigonometric integrals

Example 6.1 *Prove that*

$$\int \frac{1}{\sin x}\,dx = \ln\left|\tan\frac{x}{2}\right|, \qquad x \in \,]p\pi,(p+1)\pi[, \quad p \in \mathbb{Z}.$$

Then prove that

$$\int \frac{1}{\cos x}\,dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|, \qquad x \in \left]-\frac{\pi}{2}+p\pi, \frac{\pi}{2}+p\pi\right[, \quad p \in \mathbb{Z}.$$

A. Check two integrations.

D. Is the integrand, resp. the right hand side defined in the given interval?

Differentiate the right hand side in order to get the integrand, i.e. test the indicated solutione. The second result can be derived from the first one.

I. and C. 1) If $x \in \,]p\pi,(p+1)\pi[$, $p \in \mathbb{Z}$, then $\sin x \neq 0$ and $\tan\dfrac{x}{2}$ are defined and $\neq 0$. The two sides of the equation are there both defined in the given interval.

Now let

$$f(x) = \ln\left|\tan\frac{x}{2}\right|, \qquad x \in \,]p\pi,(p+1)\pi[, \quad p \in \mathbb{Z}.$$

Then by a differentiation

$$f'(x) = \frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2} = \frac{1}{2\,\sin\frac{x}{2}\,\cos\frac{x}{2}} = \frac{1}{\sin x},$$

so we have tested our solution, and we have proved the first formula.

2) If $x \in \left]-\dfrac{\pi}{2}+p\pi, \dfrac{\pi}{2}+p\pi\right[$, $p \in \mathbb{Z}$, then

$$x + \frac{\pi}{2} \in \,]p\pi,(p+1)\pi[, \qquad p \in \mathbb{Z}.$$

Thus we can apply the result if (1) with x replaced by $x+\dfrac{\pi}{2}$. Then by insertion,

$$\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| = \int \frac{1}{\sin\left(x+\frac{\pi}{2}\right)}\,dx = \int \frac{1}{\cos x}\,dx,$$

and the formula is proved.

Example 6.2 *Calculate the integrals*

$$(1) \quad \int \cos^3 x \, \sin x \, dx, \qquad (2) \quad \int \cos^3 x \, dx.$$

A. Simple Integrations.

D. In both cases we introduce a substitution, $t = \cos x$ in (1), and $t = \sin x$ in (2).

I. 1) If $t = \cos x$, then $dt = -\sin x \, dx$, hence

$$\int \cos^3 x \, \sin x \, dx = -\int_{t=\cos x} t^3 \, dt = -\frac{1}{4} \cos^4 x.$$

2) If $t = \sin x$, then $dt = \cos x \, dx$, hence

$$\begin{aligned} \int \cos^3 x \, dx &= \int (1 - \sin^2 x) \cos x \, dx \\ &= \int_{t? \sin x} (1 - t^2) \, dt \\ &= \sin x - \frac{1}{3} \sin^3 x. \end{aligned}$$

C. The tests here are just mental arithmetics.

Example 6.3 *The following calculation is not correct. Indicate what is wrong and then calculate the correct answer.*

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} \cot x \, dx &= \int_{y=\tan \frac{\pi}{4}}^{y=\tan \frac{5\pi}{6}} \cot(\text{ Arctan } y) \, d \text{ Arctan } y \\ &= \int_1^{\sqrt{3}} \frac{1}{y(1+y^2)} \, dy = \int_1^{\sqrt{3}} \left(\frac{1}{y} - \frac{y}{1+y^2} \right) dy = \frac{1}{2} \ln \frac{3}{8}. \end{aligned}$$

A. This is an example of the notorious task: "Find the error in the following, correct the error and then perform the calculations."

D. Check if the application of the substitution is legal.

I. We cannot apply the substitution $x = \text{Arctan } y \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$ in the interval $\left[\frac{\pi}{4}, \frac{5\pi}{6} \right]$, thus, already the first equality sign is wrong. There are also other errors. E.g. the correct value of $\tan \frac{5\pi}{6}$ is $-\frac{1}{\sqrt{3}}$.

It is possible to save a lot of this wrong calculation, if we just notice that $\cot x$ is defined in the interval and that

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cot x \, dx = 0,$$

because

$$\cot\left(\frac{\pi}{2}+x\right) = -\cot\left(\frac{\pi}{2}-x\right).$$

Then

$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} \cot x\,dx &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cot x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \cot x\,dx \\
&= 0 + \int_{y=\tan\frac{\pi}{4}}^{y=\tan\frac{\pi}{6}} \cot(\text{Arctan } y)\,d(\text{Arctan } y) \\
&= \int_1^{\frac{1}{\sqrt{3}}} \frac{1}{y(1+y^2)}\,dy = \int_1^{\frac{1}{\sqrt{3}}} \left\{\frac{1}{y} - \frac{y}{1+y^2}\right\} dy \\
&= \left[\ln y - \frac{1}{2}\ln\left(1+y^2\right)\right]_{y=1}^{\frac{1}{\sqrt{3}}} = \frac{1}{2}\left[\ln\left(\frac{y^2}{1+y^2}\right)\right]_{y=1}^{\frac{1}{\sqrt{3}}} \\
&= \frac{1}{2}\left\{\ln\left(\frac{\frac{1}{3}}{1+\frac{1}{3}}\right) - \ln\left(\frac{1}{2}\right)\right\} = -\frac{1}{2}\ln 2.
\end{aligned}$$

REMARK. The chosen substitution is not the most convenient one. Instead we note that $\sin x > 0$ in the total interval of integration, and then we get

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} \cot x\,dx &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} \frac{1}{\sin x}\cos x\,dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} \frac{d\sin x}{\sin x} \\ &= \ln\sin\frac{5\pi}{6} - \ln\sin\frac{\pi}{4} = \ln\frac{1}{2} - \frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2. \quad \diamond\end{aligned}$$

Example 6.4 *1) Apply the substitution given by* $t = \tan\frac{x}{2}$ *in order to calculate the integral*

$$\int \frac{1}{2\sin x - \cos x + 5}\,dx.$$

2) Find the exact values of the following two integrals

$$\int_{-\pi}^{\pi} \frac{1}{2\sin x - \cos x + 5}\,dx, \qquad \int_0^{2\pi} \frac{1}{2\sin x - \cos x + 5}\,dx.$$

A. Integration by the substitution $t = \tan\frac{x}{2}$. This is only valid for $x \in]-\pi,\pi[$. Therefore, there is a trap in the second question of (2).

D. If possible, apply a table. In (2) we use that the integrand is periodic of period 2π. We may possibly use a decomposition.

I. 1) When we use the substitution

$$t = \tan\frac{x}{2}, \quad x = 2\,\text{Arctan}\,t, \quad x \in]-\pi,\pi[, \quad t \in]-\infty,+\infty[,$$

$$dx = \frac{2}{1+t^2}\,dt, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad \sin x = \frac{2t}{1+t^2},$$

we get *for* $\underline{x \in]-\pi,\pi[}$,

$$\begin{aligned}&\int \frac{1}{2\sin x - \cos x + 5}\,dx \\ &= \int_{t=\tan\frac{x}{2}} \frac{1}{\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} + 5} \cdot \frac{2}{1+t^2}\,dt \\ &= \int_{t=\tan\frac{x}{2}} \frac{2}{4t - 1 + t^2 + 5 + 5t^2}\,dt = \int_{t=\tan\frac{x}{2}} \frac{2}{6t^2+4t+4}\,dt \\ &= \frac{1}{3}\int_{t=\tan\frac{x}{2}} \frac{1}{t^2 + \frac{2}{3}t + \frac{2}{3}}\,dt = \frac{1}{3}\int_{t=\tan\frac{x}{2}} \frac{1}{\left(t+\frac{1}{3}\right)^2 + \frac{5}{9}}\,dt \\ &= \frac{1}{3}\cdot\frac{3}{\sqrt{5}}\,\text{Arctan}\left(\frac{3}{\sqrt{5}}\left\{\tan\frac{x}{2} + \frac{1}{3}\right\}\right) \\ &= \frac{1}{\sqrt{5}}\,\text{Arctan}\left(\frac{3}{\sqrt{5}}\tan\frac{x}{2} + \frac{1}{\sqrt{5}}\right).\end{aligned}$$

which only holds for $x \in]-\pi,\pi[$.

C. TEST. By differentiation we get

$$\begin{aligned}
&\frac{d}{dx}\left\{\frac{1}{\sqrt{5}}\,\mathrm{Arctan}\left(\frac{3}{\sqrt{5}}\,\tan\frac{x}{2}+\frac{1}{\sqrt{5}}\right)\right\}\\
&= \frac{1}{\sqrt{5}}\cdot\frac{1}{1+\left\{\frac{3}{\sqrt{5}}\,\tan\frac{x}{2}+\frac{1}{\sqrt{5}}\right\}^2}\cdot\frac{3}{\sqrt{5}}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\\
&= \frac{3}{2}\cdot\frac{1}{5+\left\{3\tan\frac{x}{2}+1\right\}^2}\cdot\frac{1}{\cos^2\frac{x}{2}}\\
&= \frac{3}{2}\cdot\frac{1}{5\cos^2\frac{x}{2}+\left\{3\sin\frac{x}{2}+\cos\frac{x}{2}\right\}^2}\\
&= \frac{3}{2}\cdot\frac{1}{5\cos^2\frac{x}{2}+9\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+6\sin\frac{x}{2}\cos\frac{x}{2}}\\
&= \frac{3}{2}\cdot\frac{1}{6\cos^2\frac{x}{2}+9\sin^2\frac{x}{2}+3\sin x}\\
&= \frac{1}{4\cos^2\frac{x}{2}+6\sin^2\frac{x}{2}+2\sin x}\\
&= \frac{1}{2(1+\cos x)+3(1-\cos x)+2\sin x}\\
&= \frac{1}{5-\cos x+2\sin x}. \qquad \text{Q.E.D.}
\end{aligned}$$

2) In the first case we can use the substitution $t=\tan\frac{x}{2}$, and we find according to (1),

$$\begin{aligned}
&\int_{-\pi}^{\pi}\frac{1}{2\sin x-\cos x+5}\,dx\\
&= \left[\frac{1}{\sqrt{5}}\,\mathrm{Arctan}\left(\frac{3}{\sqrt{5}}\,\tan\frac{x}{2}+\frac{1}{\sqrt{5}}\right)\right]_{x=-\pi+}^{x=\pi-}\\
&= \frac{1}{\sqrt{5}}\left\{\lim_{x\to\pi-}\ \mathrm{Arctan}\left(\frac{3}{\sqrt{5}}\,\tan\frac{x}{2}+\frac{1}{\sqrt{5}}\right)\right.\\
&\qquad\left.-\lim_{x\to-\pi+}\ \mathrm{Arctan}\left(\frac{3}{\sqrt{5}}\tan\frac{x}{2}+\frac{1}{\sqrt{5}}\right)\right\}\\
&= \frac{1}{\sqrt{5}}\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\}=\frac{\pi}{\sqrt{5}}.
\end{aligned}$$

In the second case the substitution $t=\tan\frac{x}{2}$ is no longer legal. However, the integrand is periodic of period 2π, hence

$$\int_0^{2\pi}\frac{1}{2\sin x-\cos x+5}\,dx=\int_{-\pi}^{\pi}\frac{1}{2\sin x-\cos x+5}\,dx=\frac{\pi}{\sqrt{5}}.$$

Example 6.5 *Calculate the value of the integral*

$$\int_0^{\frac{\pi}{2}} \cos^2 t \, \sin^3 t \, dt.$$

A. A trigonometric integral. Here I shall take the liberty of demonstrating that I have found 12 variants. The purpose is to show that even if a textbook indicates a *standard substitution*, this may not be the most convenient one to use (cf. **7th** and **8th variant**). The easiest method is of course **1st variant**, where a substitution known already from high school is applied.

D. Just go through all the variants.

I. 1st variant. By the substitution $u = \cos t$, $du = -\sin t \, dt$, and the fundamental trigonometric relation we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^2 t \, \sin^3 t \, dt &= \int_0^{\frac{\pi}{2}} \cos^2 t \cdot (1-\cos^2 t) \sin t \, dt \\ -\int_1^0 u^2(1-u^2)\, du &= \int_0^1 \left\{u^2 - u^4\right\} du = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.\end{aligned}$$

REMARK. This is the simplest variant. $\Diamond$

2nd variant. The substitution $u = \cos t$, $du = -\sin t \, dt$, followed by a couple of partial integrations give

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^2 t \, \sin^3 t \, dt &= -\int_{t=0}^{\frac{\pi}{2}} \cos^2 t \, \sin^2 t \, d\cos t \\ &= \left[-\frac{1}{3}\cos^3 t \cdot \sin^2 t\right]_0^{\frac{\pi}{2}} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos^4 t \, \sin t \, dt \\ &= -\frac{2}{3}\int_0^{\frac{\pi}{2}} \cos^4 t \, d\cos t = \left[-\frac{2}{15}\cos^5 t\right]_0^{\frac{\pi}{2}} = \frac{2}{15}.\end{aligned}$$

3rd variant. By some trigonometric pottering, where we also transform into the double angle a couple of times and an antilogarithmic formula, we get

$$\begin{aligned}\cos^2 t \, \sin^3 t &= \{\cos t \cdot \sin t\}^2 \sin t = \left\{\frac{1}{2}\sin 2t\right\}^2 \sin t \\ &= \frac{1}{4}\sin^2 2t \cdot \sin t = \frac{1}{4} \cdot \frac{1-\cos 4t}{2} \cdot \sin t \\ &= \frac{1}{8}\{\sin t - \cos 4t \cdot \sin t\} \\ &= \frac{1}{8}\left\{\sin t - \frac{1}{2}(\sin 5t - \sin 3t)\right\} \\ &= -\frac{1}{16}\sin 5t + \frac{1}{16}\sin 3t + \frac{1}{8}\sin t.\end{aligned}$$

Then by integration,

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \cos^2 t\, \sin^3 t\, dt \\
&= -\frac{1}{16}\int_0^{\frac{\pi}{2}} \sin 5t\, dt + \frac{1}{16}\int_0^{\frac{\pi}{2}} \sin 3t\, dt + \frac{1}{8}\int_0^{\frac{\pi}{2}} \sin t\, dt \\
&= \frac{1}{80}\left[\cos 5t\right]_0^{\frac{\pi}{2}} - \frac{1}{48}\left[\cos 3t\right]_0^{\frac{\pi}{2}} - \frac{1}{8}\left[\cos t\right]_0^{\frac{\pi}{2}} = 0 - \frac{1}{80} + \frac{1}{48} + \frac{1}{8} \\
&= \frac{1}{16}\left(-\frac{1}{5}+\frac{1}{3}+2\right) = \frac{32}{16\cdot 15} = \frac{2}{15}.
\end{aligned}$$

4th variant. By transforming into the double angle a couple of times and a partial integration we get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^2 t\, \sin^3 t\, dt &= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2t\, \sin t\, dt \\
&= \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4t)\sin t\, dt \\
&= \frac{1}{8}\int_0^{\frac{\pi}{2}} \sin t\, dt - \frac{1}{8}\int_0^{\frac{\pi}{2}} \cos 4t\, \sin t\, dt.
\end{aligned}$$

Here,

$$\int_0^{\frac{\pi}{2}} \sin t\,dt = [-\cos t]_0^{\frac{\pi}{2}} = 1.$$

By two partial integrations (NB, in the right succession) we get

$$\begin{aligned}\int \cos 4t\cdot\sin t\,dt &= -\cos 4t\cdot\cos t - 4\int \sin 4t\cdot\cos t\,dt\\ &= -\cos 4t\cdot\cos t - 4\sin 4t\cdot\sin t + 16\int\cos 4t\cdot\sin t\,dt,\end{aligned}$$

hence by a rearrangement;

$$15\int\cos 4t\,\sin t\,dt = \cos 4t\cdot\cos t + 4\sin 4t\cdot\sin t,$$

i.e.

$$\int_0^{\frac{\pi}{2}}\cos 4t\cdot\sin t\,dt = \frac{1}{15}\left[\cos 4t\cdot\cos t + 4\sin 4t\cdot\sin t\right]_0^{\frac{\pi}{2}} = -\frac{1}{15}.$$

Then by insertion,

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\cos^2 t\cdot\sin^3 t\,dt &= \frac{1}{8}\int_0^{\frac{\pi}{2}}\sin t\,dt - \frac{1}{8}\int_0^{\frac{\pi}{2}}\cos 4t\cdot\sin t\,dt\\ &= \frac{1}{8}\left(1+\frac{1}{15}\right) = \frac{1}{8}\cdot\frac{16}{15} = \frac{2}{15}.\end{aligned}$$

5th variant. An application of Euler's formulæ gives

$$\begin{aligned}&\cos^2 t\,\sin^3 t = \left\{\frac{e^{it}+e^{-it}}{2}\right\}^2\left\{\frac{e^{it}-e^{-it}}{2i}\right\}^3\\ &= \frac{1}{4}\left(e^{2it}+2+e^{-2it}\right)\cdot\left(-\frac{1}{8i}\right)\left(e^{3it}-3e^{it}+3e^{-it}-e^{-3it}\right)\\ &= -\frac{1}{32i}\left\{e^{5it}-e^{3it}-2e^{it}+2e^{-it}+e^{-3it}-e^{-5it}\right\}\\ &= -\frac{1}{16}\left\{\frac{e^{5it}-e^{-5it}}{2i}-\frac{e^{3it}-e^{-3it}}{2i}-2\cdot\frac{e^{it}-e^{-it}}{2i}\right\}\\ &= -\frac{1}{16}\left\{\sin 5t-\sin 3t-2\sin t\right\}\\ &= -\frac{1}{16}\,\sin 5t+\frac{1}{16}\,\sin 3t+\frac{1}{8}\,\sin t.\end{aligned}$$

Then by an integration,

$$\begin{aligned}\int\cos^2 t\,\sin^3 t\,dt &= -\frac{1}{16}\int\sin 5t\,dt+\frac{1}{16}\int\sin 3t\,dt+\frac{1}{8}\int\sin t\,dt\\ &= \frac{1}{80}\,\cos 5t-\frac{1}{48}\,\cos 3t-\frac{1}{8}\,\cos t,\end{aligned}$$

thus

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^2 t\, \sin^3 t\, dt &= \left[\frac{1}{80}\,\cos 5t - \frac{1}{48}\,\cos 3t - \frac{1}{8}\,\cos t\right]_0^{\frac{\pi}{2}} \\ &= 0 - \frac{1}{80} + \frac{1}{48} + \frac{1}{8} = \frac{1}{16}\left(-\frac{1}{5}+\frac{1}{3}+2\right) \\ &= \frac{32}{16\cdot 15} = \frac{2}{15}.\end{aligned}$$

6th variant. We again apply Euler's formulæ, cf. the **5th variant**,

$$cos^2 t\, \sin^3 t = -\frac{1}{32i}\left(e^{5it} - e^{3it} - 2\,e^{it} + 2\,e^{-it} + e^{-3it} - e^{-5it}\right).$$

When this is integrated with respect to the real parameter t, we get

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \cos^2 t\, \sin^3 t\, dt \\ &= -\frac{1}{32i}\left\{\int_0^{\frac{\pi}{2}} e^{5it}dt - \int_0^{\frac{\pi}{2}} e^{3it}dt - 2\int_0^{\frac{\pi}{2}} e^{it}dt\right. \\ &\qquad\left. +2\int_0^{\frac{\pi}{2}} e^{-it}dt + \int_0^{\frac{\pi}{2}} e^{-3it}dt - \int_0^{\frac{\pi}{2}} e^{-5it}dt\right\},\end{aligned}$$

thus

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \cos^2 t\, \sin^3 t\, dt \\ &= -\frac{1}{32i}\left\{\frac{1}{5i}\left[e^{5it}\right]_0^{\frac{\pi}{2}} - \frac{1}{3i}\left[e^{3it}\right]_0^{\frac{\pi}{2}} - \frac{2}{i}\left[e^{it}\right]_0^{\frac{\pi}{2}}\right. \\ &\qquad\left. -\frac{2}{i}\left[e^{-it}\right]_0^{\frac{\pi}{2}} - \frac{1}{3i}\left[e^{-3it}\right]_0^{\frac{\pi}{2}} + \frac{1}{5i}\left[e^{-5it}\right]_0^{\frac{\pi}{2}}\right\} \\ &= -\frac{1}{32i}\left\{-\frac{i}{5}(i-1)+\frac{i}{3}(-i-1)+2i(i-1)+2i(-i-1)+\frac{i}{3}(i-1)-\frac{i}{5}(-i-1)\right\} \\ &= -\frac{1}{32i}\left\{-\frac{i}{5}\cdot(-2)+\frac{i}{3}\cdot(-2)-4i\right\} = \frac{1}{32}\left\{-\frac{2}{5}+\frac{2}{3}+4\right\} \\ &= \frac{1}{16}\left\{-\frac{1}{5}+\frac{1}{3}+2\right\} = \frac{1}{16}\cdot\frac{-3+5+30}{15} = \frac{1}{16}\cdot\frac{32}{15} = \frac{2}{15}.\end{aligned}$$

7th variant. Since $\left[0,\frac{\pi}{2}\right] \subset]-\pi,\pi[$, we can also apply the "standard" substitution, recommended by some textbooks as being *the* method,

$$x = \tan\frac{t}{2}, \qquad t = 2\,\text{Arctan}\, x, \qquad t \in]-\pi,\pi[, \qquad x \in \mathbb{R},$$

$$\frac{dt}{dx} = \frac{2}{1+x^2}, \qquad \cos t = \frac{1-x^2}{1+x^2}, \qquad \sin t = \frac{2x}{1+x^2}.$$

We shall see below that this standard method in the given case implies a lot of unnecessary calculation. We first get by insertion,

$$\int_0^{\frac{\pi}{2}} \cos^2 t \, \sin^3 t \, dt = \int_0^{\tan\frac{\pi}{4}} \left(\frac{1-x^2}{1+x^2}\right)^2 \cdot \left(\frac{2x}{1+x^2}\right)^3 \cdot \frac{2}{1+x^2}\, dx$$
$$= \int_0^1 \frac{(1-x^2)^2 \cdot 8x^3 \cdot 2}{(1+x^2)^6}\, dx = 16 \int_0^1 \frac{(1-x^2)^2 x^3}{(1+x^2)^6}\, dx,$$

where the integrand is a rational function, so we can decompose it.

However, it will here be smarter first to apply the substitution $u = x^2$, $du = 2x\, dx$. Then we get somewhat easier,

$$16 \int_0^1 \frac{(1-x^2)^2 x^3}{(1+x^2)^6}\, dx = 8 \int_0^1 \frac{(1-u)^2 u}{(1+u)^6}\, du = 8 \int_0^1 \frac{u^3 - 2u^2 + u}{(u+1)^6}\, du$$
$$= \int_0^1 \left\{ -\frac{32}{(u+1)^6} + \frac{64}{(u+1)^5} - \frac{10}{(u+1)^4} + \frac{8}{(u+1)^3} \right\} du$$
$$= \left[\frac{32}{5} \cdot \frac{1}{(u+1)^5} - \frac{64}{4} \cdot \frac{1}{(u+1)^4} + \frac{40}{3} \cdot \frac{1}{(u+1)^3} - \frac{8}{2} \cdot \frac{1}{(u+1)^2} \right]_0^1$$
$$= \left\{ \frac{32}{5} \cdot \frac{1}{32} - 16 \cdot \frac{1}{16} + \frac{40}{3} \cdot \frac{1}{8} - 4 \cdot \frac{1}{4} \right\} - \left\{ \frac{32}{5} - 16 + \frac{40}{3} - 4 \right\}$$
$$= \frac{1}{5} - 1 + \frac{5}{3} - 1 - \frac{32}{5} + 16 - \frac{40}{3} + 4$$
$$= \frac{1}{5} - \frac{1}{3} - \left(6 + \frac{2}{5}\right) + 20 - \left(13 + \frac{1}{3}\right)$$
$$= 1 - \frac{1}{5} - \frac{2}{3} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.$$

8th variant. First we apply the same "standard" substitution as in the **7th variant**. As before we get

$$\int_0^{\frac{\pi}{2}} \cos^2 t \, \sin^3 t \, dt = 16 \int_0^1 \frac{(1-x^2)^2 x^3}{(1+x^2)^6}\, dx.$$

This time I shall show in all its horror what happens if one starts by unconsciously decomposing without using the substitution $u = x^2$. Since the denominator $(1+x^2)^6$ cannot be reduced further in the real, we must also perform a division. Then we get by "adding something and then subtract it again" from the numerator that

$$\begin{aligned}
16\cdot\frac{(1-x^2)^2x^3}{(1+x^2)^6} &= 16\cdot\frac{x^7-2x^5+x^3}{(1+x^2)^6}\\
&= 16\cdot\frac{(x^7+x^5)-3(x^5+x^3)+4(x^3+x)-4x}{(1+x^2)^6}\\
&= 16\cdot\frac{(x^2+1)(x^5-3x^3+4x)-4x}{(1+x^2)^6}\\
&= -\frac{64x}{(1+x^2)^6}+16\cdot\frac{x^5-3x^3+4x}{(1+x^2)^5}\\
&= -\frac{64x}{(1+x^2)^6}+16\cdot\frac{(x^5+x^3)-4(x^3+x)+8x}{(1+x^2)^5}\\
&= -\frac{64x}{(1+x^2)^6}+16\cdot\frac{(x^2+1)(x^3-4x)+8x}{(1+x^2)^5}\\
&= -\frac{64x}{(1+x^2)^6}+\frac{128x}{(1+x^2)^5}+16\cdot\frac{x^3-4x}{(1+x^2)^4},
\end{aligned}$$

hence

$$\begin{aligned}16\cdot\frac{(1-x^2)^2x^3}{(1+x^2)^6} &= -\frac{64x}{(1+x^2)^6}+\frac{128x}{(1+x^2)^5}+16\cdot\frac{(x^3+x)-5x}{(1+x^2)^4}\\ &= -\frac{64x}{(1+x^2)^6}+\frac{128x}{(1+x^2)^5}-\frac{80x}{(1+x^2)^4}+\frac{16x}{(1+x^2)^3},\end{aligned}$$

and then by integration

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\cos^2 t\,\sin^3 t\,dt = 16\int_0^1\frac{(1-x^2)^2x^3}{(1+x^2)^6}\,dx\\ &\quad=\int_0^1\left\{-\frac{64x}{(1+x^2)^6}+\frac{128x}{(1+x^2)^5}-\frac{80x}{(1+x^2)^4}+\frac{16x}{(1+x^2)^3}\right\}dx\\ &\quad=\left[+\frac{64}{2\cdot 5}\cdot\frac{1}{(1+x^2)^5}-\frac{128}{2\cdot 4}\cdot\frac{1}{(1+x^2)^4}\right.\\ &\qquad\qquad\left.+\frac{40}{2\cdot 3}\cdot\frac{1}{(1+x^2)^3}-\frac{16}{2\cdot 2}\cdot\frac{1}{(1+x^2)^2}\right]_0^1\\ &\quad=\frac{2}{15},\end{aligned}$$

where the final calculations of course must follow the same pattern as in the **7th variant**.

9th variant. *Some pottering, starting by the substitution* $u=\tan t$.

Since $u=\tan t$ is a one-to-one map of $\left[0,\frac{\pi}{2}\right[$ onto $[0,+\infty[$ where $du=(1+\tan^2 t)dt$, and

$$\cos t=+\frac{1}{\sqrt{1+\tan^2 t}},\qquad \sin t=+\frac{\tan t}{\sqrt{1+\tan^2 t}},\qquad t\in\left[0,\frac{\pi}{2}\right[,$$

it follows from the rearrangement

$$\begin{aligned}\cos^2 t\,\sin^3 t &= \cos^5 t\cdot\tan^3 t\\ &= \frac{\tan^3 t}{(1+\tan^2 t)^3\sqrt{1+\tan^2 t}}\cdot(1+\tan^2 t),\end{aligned}$$

that

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\cos^2 t\,\sin^3 t\,dt &= \int_0^{\frac{\pi}{2}}\frac{\tan^3 t}{(1+\tan^2 t)^3\sqrt{1+\tan^2 t}}\,(1+\tan^2 t)\,dt\\ &= \int_0^{+\infty}\frac{u^3}{(1+u^2)^3\sqrt{1+u^2}}\,du.\end{aligned}$$

The presence of the factor $\sqrt{1+u^2}$ in the denominator invites one to apply the substitution $u=\sinh x$, $\sqrt{1+u^2}=+\cosh x$, $du=\cosh x\,dx$, hence

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\cos^2 t\,\sin^3 t\,dt=\int_0^{+\infty}\frac{u^3}{(1+u^2)^3\sqrt{1+u^2}}\,du=\int_0^{+\infty}\frac{\sinh^3 x}{\cosh^6 x}\cdot\frac{\cosh x}{\cosh x}\,dx\\ &\quad=\int_0^{+\infty}\frac{\cosh^2 x-1}{\cosh^6 x}\cdot\sinh x\,dx=\int_{x=0}^{+\infty}\left(\frac{1}{\cosh^4 x}-\frac{1}{\cosh^6 x}\right)d\cosh x\\ &\quad=\left[-\frac{1}{3}\cdot\frac{1}{\cosh^3 x}+\frac{1}{5}\cdot\frac{1}{\cosh^5 x}\right]_0^{+\infty}=0+\frac{1}{3}-\frac{1}{5}=\frac{2}{15}.\end{aligned}$$

10th variant. Start like in the **9th variant** and derive the formula

$$\int_0^{\frac{\pi}{2}} \cos^2 t\,\sin^3 t\,dt = \frac{1}{2}\int_0^{+\infty} \frac{u^2}{(1+u^2)^{\frac{7}{2}}}\cdot 2u\,du.$$

Then by the substitution $y = 1+u^2$, $dy = 2u\,du$, $u^2 = y-1$, at

$$\int_0^{\frac{\pi}{2}} \cos^2 t\,\sin^3 t\,dt = \frac{1}{2}\int_1^{+\infty} \frac{y-1}{y^{\frac{7}{2}}}\,dy = \frac{1}{2}\int_1^{+\infty}\left\{y^{-\frac{5}{2}} - y^{-\frac{7}{2}}\right\}dy$$

$$= \frac{1}{2}\left[\frac{1}{-\frac{3}{2}}\cdot y^{-\frac{3}{2}} - \frac{1}{-\frac{5}{2}}\cdot y^{-\frac{5}{2}}\right]_1^{+\infty} = \frac{1}{2}\left\{0+\frac{2}{3}-\frac{2}{5}\right\} = \frac{2}{15}.$$

11th variant. It is also possible to use the so-called *Beta integral*, but this lies outside what can be expected by the reader, so we skip this variant.

12th variant. If one by some accident should have chosen the "wrong" substitution

$$u = \sin t, \qquad du = \cos t\,dt,$$

it is still possible to go through the calculations, because $\cos t = +\sqrt{1-u^2}$ for $t \in \left[0,\frac{\pi}{2}\right]$. Then

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^2 t\cdot\sin^3 t\,dt &= \int_0^{\frac{\pi}{2}} \sin^3 t\cdot\sqrt{1-\sin^2 t}\cdot\cos t\,dt \\
&= \int_0^1 u^3\sqrt{1-u^2}\,du = \frac{1}{2}\int_0^1 u^2\sqrt{1-u^2}\cdot 2u\,du \\
&= \frac{1}{2}\int_0^1 y\sqrt{1-y}\,dy = \frac{1}{2}\int_0^1 (1-v)\sqrt{v}\,dv \\
&= \frac{1}{2}\int_0^1\left\{v^{\frac{1}{2}} - v^{\frac{3}{2}}\right\}dv = \frac{1}{2}\left[\frac{2}{3}v^{\frac{3}{2}} - \frac{2}{5}v^{\frac{5}{2}}\right]_0^1 \\
&= \frac{1}{3}-\frac{1}{5} = \frac{2}{15}.
\end{aligned}$$

Example 6.6 *Calculate the integral*

$$\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx.$$

A. Integration.

D. Apply the substitution $u = \sin x$, $du = \cos x\,dx$.

I. By the substitution above we get

$$\begin{aligned}
\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx &= \int_{u=\sin x} \frac{du}{\sqrt{1+u^2}} = [\text{Arsinh } u]_{u=\sinh x} \\
&= [\ln(u+\sqrt{1+u^2})]_{u=\sin x} = \ln\left(\sin x + \sqrt{1+\sin^2 x}\right).
\end{aligned}$$

C. Test. A differentiation gives

$$\begin{aligned}
&\frac{d}{dx}\,\ln(\sin x+\sqrt{1+\sin^2 x})\\
&\qquad = \frac{1}{\sin x+\sqrt{1+\sin^2 x}}\left\{\cos x+\frac{2\sin c\cdot\cos x}{2\sqrt{1+\sin^2 x}}\right\}\\
&\qquad = \frac{\cos x}{\sin x+\sqrt{1+\sin^2 x}}\cdot\frac{\sqrt{1+\sin^2 x}+\sin x}{\sqrt{1+\sin^2 x}} = \frac{\cos x}{\sqrt{1+\sin^2 x}}. \qquad \text{Q.E.D.}
\end{aligned}$$

Example 6.7 *Calculate by means of Euler's formulæ*

$$\int_0^\pi \sin^2 x\cos^2 3x\,dx.$$

A. Trigonometric integral.

D. First apply Euler's formulæ on the integrand.

I. The integrand becomes

$$\begin{aligned}
\sin^2 x\cos^2 3x &= \left\{\frac{1}{2i}\left(e^{ix}-e^{-ix}\right)\right\}^2\left\{\frac{1}{2}\left(e^{3ix}+e^{-3ix}\right)\right\}^2\\
&= -\frac{1}{4}\left\{e^{2ix}-2+e^{-2ix}\right\}\cdot\frac{1}{4}\left\{e^{6ix}+2+e^{-6ix}\right\}\\
&= -\frac{1}{16}\left\{e^{8ix}-2e^{6ix}+e^{4ix}+2e^{2ix}-4+2e^{-2ix}+e^{-4ix}-2e^{-6ix}+e^{-8x}\right\}\\
&= -\frac{1}{8}\{\cos 8x-2\cos 6x+\cos 4x+2\cos 2x-2\},
\end{aligned}$$

hence by insertion,

$$\begin{aligned}
\int_0^\pi \sin^2 x\cos^2 3x\,dx &= \frac{1}{8}\int_0^\pi\{-\cos 8x+2\cos 6x-\cos 4x-2\cos 2x+2\}dx\\
&= \frac{1}{8}\cdot 2\cdot\pi = \frac{\pi}{4},
\end{aligned}$$

because

$$\int_0^\pi \cos nx\,dx = \left[\frac{1}{n}\,\sin nx\right]_0^\pi = 0.$$

Remark. A calculation without using Euler's formulæ is the following

$$\begin{aligned}
\sin^2 x\cos^2 3x &= \frac{1}{2}\{1-\cos 2x\}\cdot\frac{1}{2}\{1+\cos 6x\}\\
&= \frac{1}{4}\{1-\cos 2x+\cos 6x-\cos 6x\cdot\cos 2x\}\\
&= \frac{1}{4}\left\{1-\cos 2x+\cos 6x-\frac{1}{2}(\cos 8x+\cos 4x)\right\},
\end{aligned}$$

and we obtain again

$$\int_0^{\pi} \sin^2 x \cos^2 3x \, dx = \frac{\pi}{4} + 0 + 0 + 0 + 0 = \frac{\pi}{4}. \qquad \Diamond$$

7 Riemann sums

Example 7.1 *Given a natural number n, consider the Riemann sum*

$$S(n)=\sum_{i=1}^{n}\frac{1}{n+i}.$$

1) Calculate in decimals $S(2)$, $S(3)$ and $S(4)$.

2) Find the limit

$$\lim_{n\to\infty} S(n).$$

Hint: Apply the rearrangement

$$\sum_{i=1}^{n}\frac{1}{n+i}=\sum_{i=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\frac{i}{n}}.$$

A. Riemann sum.

D. Follow the text and consider the function $f(x)=\dfrac{1}{1+x}$, $x\in[0,1]$.

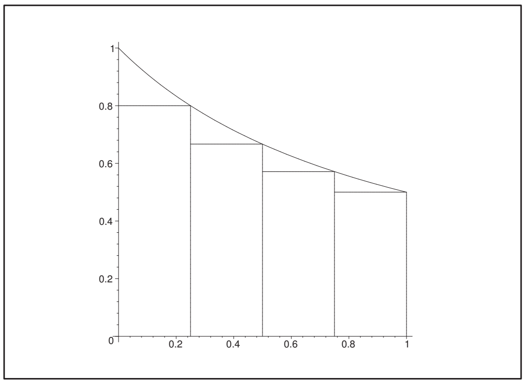


Figure 2: Graphic interpretation for $n=4$.

I. For $n=2$ we get

$$S(2)=\sum_{i=1}^{2}\frac{1}{2+i}=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}=0,5833\dots.$$

For $n=3$ we get

$$\begin{aligned}S(3) &= \sum_{i=1}^{3}\frac{1}{3+i}=\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{1}{60}\{15+12+10\}\\ &= \frac{37}{60}=0,6167\dots.\end{aligned}$$

For $n = 4$ we get

$$\begin{aligned} S(4) &= \sum_{i=1}^{4} \frac{1}{4+i} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{1}{840}\{168 + 140 + 120 + 105\} \\ &= \frac{533}{840} = 0,6345\dots. \end{aligned}$$

By choosing the function $f(x) = \dfrac{1}{1+x}$, we see that

$$\sum_{i=1}^{n} \frac{1}{n+i} = \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1 + \dfrac{i}{n}}$$

is a Riemann sum for $\int_0^1 \dfrac{dx}{1+x} = \ln 2$, and since the length of the interval $\dfrac{1}{n} \to 0$ for $n \to \infty$, we get by taking the limit that

$$\lim_{n\to\infty} S(n) = \int_0^1 \frac{dx}{1+x} = \ln 2.$$

Example 7.2 *The number*

$$M_n = \sum_{i=1}^{n} \frac{i + 2in}{n^3}$$

can be considered as a Riemann sum for some function $f(x)$. Apply the main theorem of the differential and integration calculus to find

$$\lim_{n\to\infty} M_n.$$

A. Consideration of a Riemann sum. Find the continuous function f which corresponds to this Riemann sum,

D. Choose some length of the interval and corresponding subintervals. Then find the corresponding function f.

I. Since

$$M_n = \sum_{i=1}^{n} \frac{i + 2in}{n^3} = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{2n+1}{n^2},$$

we choose the length of the interval in the n-th step,

$$\Delta x_{i,n} = \frac{2n+1}{n^2} \to 0 \qquad \text{for } n \to +\infty.$$

The i-th subinterval corresponding to the n-th step is

$$I_{i,n} = \left[\frac{2n+1}{n^2}(i-1), \frac{2n+1}{n^2} \cdot i\right], \qquad i = 1, \dots, n.$$

It follows from the inequality

$$\frac{2n+1}{n^2}\,(i-1) = \frac{2i}{n} + \frac{i-2n-1}{n^2} < \frac{2i}{n} < \frac{2n+1}{n}\cdot i,$$

that $x_{i,n} = \dfrac{2i}{n} \in I_{i,n}$. Putting

$$f(x_{i,n}) = f\left(\frac{2i}{n}\right) = \frac{i}{n} = \frac{1}{2}\,x_{i,n},$$

we see that we can choose the function $f(x) = \dfrac{x}{2}$, independently of all the $n \in \mathbb{N}$.

Consider

$$M_n = \sum_{i=1}^{n} \frac{i+2in}{n^3} = \sum_{i=1}^{n} f(x_{i,n})\Delta x_{i,n}$$

as a Riemann sum for $f(x)$ over the variable interval

$$\bigcup_{i=1}^{n} \left[\frac{2n+1}{n^2}\,(i-1), \frac{2n+1}{n^2}\,i\right] = \left[0, \frac{2n+1}{n^2}\,n\right] = \left[0, 2+\frac{1}{n}\right].$$

Since $f(x) = \dfrac{x}{2}$ is continuous and $2 + \dfrac{1}{n} \to 2$ for $n \to \infty$, it follows from the main theorem that

$$\begin{aligned} M_n &= \sum_{i=1}^{n} \frac{i+2in}{n^3} = \sum_{i=1}^{n} f(x_{i,n})\, \Delta x_{i,n} \\ &\to \int_0^2 \frac{x}{2}\, dx = \left[\frac{x^2}{4}\right]_0^2 = 1 \qquad \text{for } n \to +\infty, \end{aligned}$$

because the contribution from the additional interval $\left[2, 2+\dfrac{1}{n}\right]$ is bounded from above by $\leq 2 \cdot \dfrac{1}{n} \to 0$ for $n \to +\infty$.

C. TEST. It is possible to check the result, if we start by proving that

$$(5) \quad \sum_{i=1}^{n} i = \frac{1}{2}\, n(n+1).$$

Formula (5) is obvious for $n = 1$ and $n = 2$.
Assume that (5) holds for some $n \in \mathbb{N}$. Then for $n+1$,

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{1}{2}\, n(n+1) + (n+1) = \frac{1}{2}\,(n+1)(n+2),$$

and the claim then follows by induction.

We conclude from (5) that

$$\begin{aligned} M_n &= \sum_{n=1}^{n} \frac{i+2in}{n^3} = \frac{2n+1}{n^3} \sum_{0=1}^{n} i = \frac{2n+1}{n^3} \cdot \frac{1}{2}\, n)n+1) \\ &= \frac{(2n+1)(n+1)}{2n^2} = \left(1 + \frac{1}{2n}\right)\left(1 + \frac{1}{n}\right) \to 1 \qquad \text{for } n \to +\infty. \end{aligned}$$

Example 7.3 *Let $x_0 < x_1 < \cdots < x_{n-1} < x_n$ be a subdivision of the interval $[0,2]$, and put $\Delta x_i = x_i - x_{i-1}$. Find the limit*

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i \cdot \Delta x_i \cdot \sqrt{4 - x_i^2},$$

by choosing a sequence of subdivisions, for which $\Delta(m) \to 0$ for $m \to \infty$, where $\Delta(m)$ denotes the length of the longest interval in the m-th subdivision.

A. A Riemann sum for some continuous function.

D. Find the function, and interpret the limit of the Riemann sum as an integral.

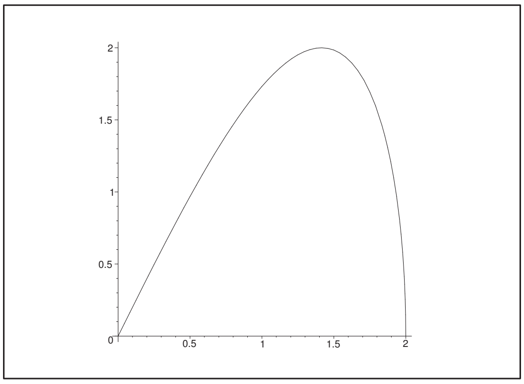


Figure 3: The graph of $f(x) = x\sqrt{4-x^2}$, $x \in [0,2]$.

I. When $f(x) = x\sqrt{4-x^2}$, $x \in [0,2]$, it follows that f is continuous, thus

$$\begin{aligned}\sum_{i=1}^{n} x_i \Delta x_i \sqrt{4-x_i^2} &= \sum_{i=1}^{n} f(x_i)\Delta x_i \to \int_0^2 f(x)\,dx \\ &= \int_0^2 x\sqrt{4-x^2}\,dx = \frac{1}{2}\int_0^{2^2} \sqrt{4-t}\,dt \\ &= \frac{1}{2}\left[-\frac{2}{3}(4-t)^{\frac{3}{2}}\right]_{t=0}^{4} = \frac{1}{2}\cdot\frac{2}{3}\cdot 4^{\frac{3}{2}} = \frac{8}{3}.\end{aligned}$$

Example 7.4 *Let $x_0 < x_1 < \cdots < x_{n-1} < x_n$ be a subdivision of the interval $[1,4]$, and put $\Delta x_i = x_i - x_{i-1}$. Find the limit*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3x_i\Delta x_i + 4\Delta x_i}{x_i^2 + 3x_i + 2},$$

by choosing a sequence of subdivisions for which $\Delta(m) \to 0$ for $m \to \infty$, where $\Delta(m)$ denotes the length of the longest interval in the m-th subdivision.

A. Limit for a Riemann sum.

D. Isolate Δx_i, and then choose a convenient continuous function f, such the the sum can be interpreted as a Riemann sum of f. Then calculate the limit by interpreting it as an integral. Here we shall also make use of a decomposition.

I. First

$$\sum_{i=1}^{n} \frac{3x_i\Delta x_i + 4\Delta x_i}{x_i^2 + 3x_i + 2} = \sum_{i=1}^{n} \frac{3x_i + 4}{x_i^2 + 3x_i + 2}\,\Delta x_i.$$

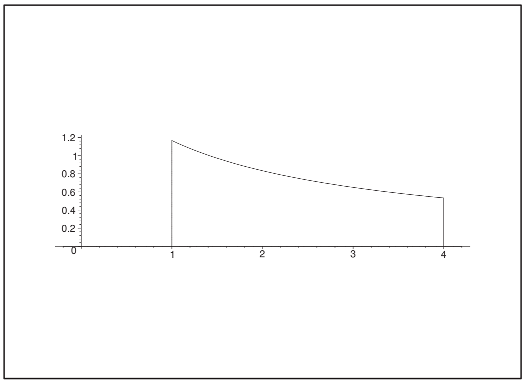


Figure 4: The graph of $f(x) = \dfrac{3x+4}{x^2+3x+2} = \dfrac{1}{x+1} + \dfrac{2}{x+2}$, $x \in [1,4]$.

Therefore, choose the function

$$f(x) = \frac{3x+4}{x^2+3x+2} = \frac{3x+4}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{2}{x+2}.$$

Assuming that $\Delta(n) \to 0$ for $n \to \infty$, and interpreting the sum as a Riemann sum for the continuous function $f(x)$ over $[1,4]$, we obtain by taking the limit that

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3x_i\Delta x_i + 4\Delta x_i}{x_i^2+3x_i+2} &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \int_1^4 f(x)\,dx \\ &= \int_1^4 \left\{\frac{1}{x+1} + \frac{2}{x+2}\right\} dx = [\ln(x+1) + 2\,\ln(x+2)]_1^4 \\ &= \ln 5 + 2\ln 6 - \ln 2 - 2\ln 3 = \ln\frac{5\cdot 6^2}{2\cdot 3^2} = \ln 10.\end{aligned}$$

Example 7.5 *Given the rather incalculable sum*

$$S_n = \frac{1}{1+(\frac{0}{n})^2}\cdot 1n + \frac{1}{1+(\frac{1}{n})^2}\cdot\frac{1}{n} + \frac{1}{1+(\frac{2}{n})^2}\cdot\frac{1}{n} + \cdots + \frac{1}{1+(\frac{2n-1}{n})^2}\cdot\frac{1}{n}.$$

1) How many terms does S_n contain? And how does each term behave when $n \to \infty$?

2) Calculate (in decimals) S_1, S_2 and S_3.

3) Find a function, an interval and a subdivision of this interval, such that S_n is a Riemann sum for function in the interval corresponding to the given subdivision. Sketch S_1, S_2 and S_3 as areas of polygons.

4) How does S_n behave for $n \to \infty$?

A. An incalculable sum interpreted as a Riemann sum.

D. Follow the description above. In (3) we interpret S_n as a Riemann sum of an integral.

I. 1) From

$$S_n = \sum_{j=0}^{2n-1} \frac{1}{1+\left(\frac{j}{m}\right)^2} \cdot \frac{1}{n} = \sum_{j=0}^{2n-1} \frac{n}{n^2+j^2},$$

it is seen that S_n contains $2n$ terms.

Furthermore,

$$\frac{1}{5}\cdot\frac{1}{n} < \frac{1}{1+\left(\frac{2n-1}{n}\right)^2}\cdot\frac{1}{n} \le \frac{1}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{1}{n} \le \frac{1}{1+\left(\frac{0}{n}\right)^2}\cdot\frac{1}{n} = \frac{1}{n},$$

so

$$\frac{1}{5n} \le \frac{1}{1+\left(\frac{j}{n}\right)^2}\cdot\frac{1}{n} \le \frac{1}{n}.$$

Since we have $2n$ terms in S_n, all satisfying these estimates, we get

$$\frac{2}{5} < S_n < 2.$$

2) Then by a calculation (yawn!)

$$\begin{aligned}
S_1 &= 1 \cdot \left\{ \frac{1}{1+0^2} + \frac{1}{1+1^2} \right\} = 1,5, \\
S_2 &= 2 \cdot \left\{ \frac{1}{4+0^2} + \frac{1}{4+2^2} + \frac{1}{4+3^2} \right\} = 2 \left\{ \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{13} \right\} \\
&\approx 1,303846, \\
S_3 &= 3 \cdot \left\{ \frac{1}{9+0^2} + \frac{1}{9+1^2} + \frac{1}{9+2^2} + \frac{1}{9+3^2} + \frac{1}{9+4^2} + \frac{1}{9+5^2} \right\} \\
&= 3 \left\{ \frac{1}{9} + \frac{1}{10} + \frac{1}{13} + \frac{1}{18} + \frac{1}{25} + \frac{1}{34} \right\} \approx 1,239005.
\end{aligned}$$

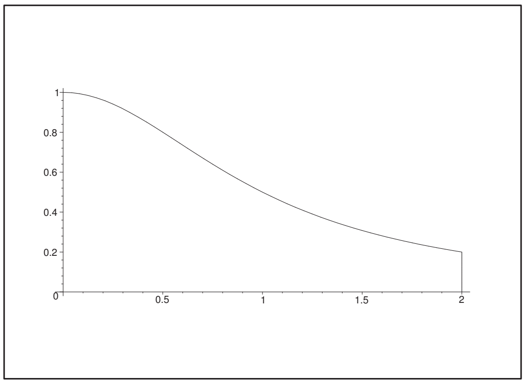


Figure 5: The graph of the function $f(x) = \dfrac{1}{1+x^2}$, $x \in [0,2]$.

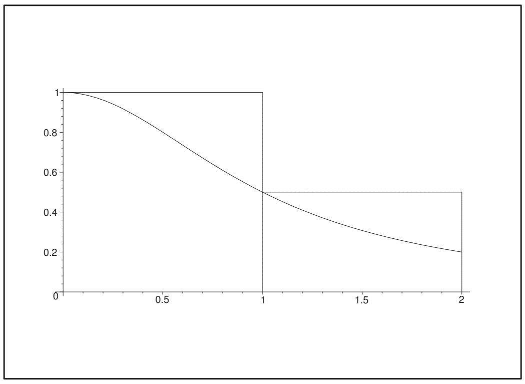


Figure 6: The polygon corresponding to S_1.

3) Put $f(x) = \dfrac{1}{1+x^2}$, and

$$x_{j,n} = \frac{j}{n}, \qquad \Delta x_{j,n} = x_{j+1,n} - x_{j,n} = \frac{1}{n}, \qquad j = 0, 1, \dots, 2n-1.$$

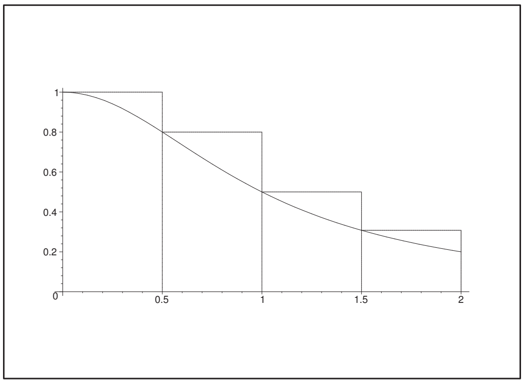


Figure 7: The polygon corresponding to S_2.

Then

$$0 = x_{0,n} < x_{1,n} < \cdots < x_{2n-1,n} = 2 - \frac{1}{n} > 2 = x_{2n,n}$$

is a subdivision of the interval $[0, 2]$, and we can interpret

$$S_n = \sum_{j=0}^{2n-1} f(x_{j,n}) \Delta x_{j,n}$$

as a Riemann sum of the integral $\int_0^2 \frac{1}{1+x^2}\, dx$. I have here sketched the polygons for S_1 and S_2. The sketch of the polygon for S_3 is left to the reader.

4) If $\Delta x_{j,n} = \frac{1}{n} \to 0$, i.e. if $n \to \infty$, then

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{\Delta x_{j,n}\to 0} \sum_{j=0}^{2n-1} f(x_{j,n}) \Delta x_{j,n} \\ &= \int_0^2 \frac{1}{1+x^2}\, dx = \text{Arctan } 2 \quad (\approx 1,107149). \end{aligned}$$

Example 7.6 *Derive by means of the main theorem of the differential and integration calculus the formula of the area of a disc of radius R. This is done by dividing the disc into small parallel strips and then continue in two different ways: First use an angle as the variable of integration, and then use a distance as a variable of integration.*

A. Derivation of the area of a disc.

D. We may assume that $R = 1$, because the general result then is obtained by a multiplication by R^2.

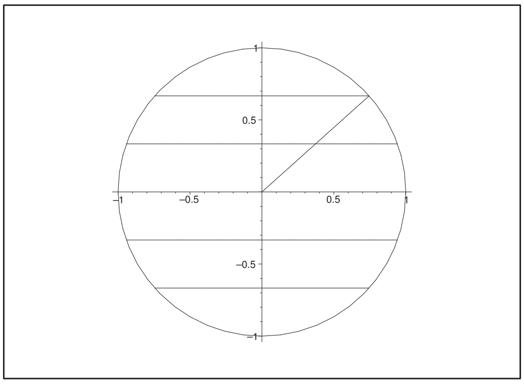


Figure 8: Subdivision of the unit disc, partly into strips, partly by means of a radius, given by an angle.

I. At the height $y = \frac{j}{n}$, $j = 0, \dots, n-1$, we cut a parallel strip out of the disc of area

$$\approx \frac{2}{n}\sqrt{1-\left(\frac{j}{n}\right)^2}.$$

Since the union of all these strips cover the upper half disc which only a small extra area (tending towards 0 for $n \to +\infty$) we get

$$\text{area } \approx 2\sum_{j=0}^{n-1}\frac{2}{n}\sqrt{1-\left(\frac{j}{n}\right)^2} = 4\sum_{j=0}^{n}\frac{1}{n}\sqrt{1-\left(\frac{j}{n}\right)^2} \to 4\int_0^1 \sqrt{1-x^2}\,dx.$$

Expressed by the angle we get

$$\sqrt{1-\left(\frac{j}{n}\right)^2} = \cos t_j,$$

where the height is

$$\in t_{j+1} - \sin t_j \approx \cos t_j \cdot (t_{j+1} - t_j),$$

hence

$$\begin{aligned}\text{area} \quad &\approx \quad 4\sum_{j=1}^{n}\frac{1}{n}\sqrt{1-\left(\frac{j}{n}\right)^2} \approx 4\sum_{j=1}^{n}\cos t_j \cdot \cos t_j \cdot (t_{j+1}-t_j)\\ &\to \quad 4\int_0^{\frac{\pi}{2}}\cos^2 t\,dt = 4\int_0^{\frac{\pi}{2}}\left\{\frac{1}{2}+\frac{1}{2}\cos 2t\right\}dt\\ &= \quad 4\left[\frac{1}{2}t+\frac{1}{4}\sin 2t\right]_0^{\frac{\pi}{2}} = 2\cdot\frac{\pi}{2}+0 = \pi.\end{aligned}$$

We conclude that the unit disc has the area π, hence the disc of radius R has the area πR^2.

Example 7.7 *Find by means of the main theorem of the differential and integral calculus the length L of the parabolic arc $y = x^2$, $0 \le x \le 1$.*

A. Length of an arc.

D. Approximate the parabolic arc by a piecewise linear curve and find the length of the latter by means of Pythagoras's theorem. When the subdivision is made finer one gets better approximations of the curve length.

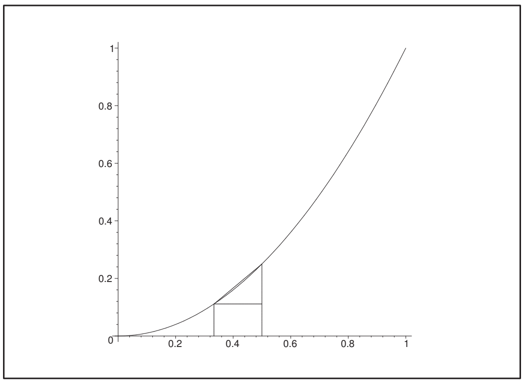


Figure 9: The curve $y = x^2$ with one single subinterval and approximating segment.

I. The secant between the points of the curve

$$\left(\frac{j}{n}, \frac{j^2}{n^2}\right) \quad \text{og} \quad \left(\frac{j+1}{n}, \frac{(j+1)^2}{n^2}\right), \qquad j = 0, \dots, n-1,$$

has by Pythagoras's theorem the length

$$\sqrt{\frac{1}{n^2} + \left\{\frac{(j+1)^2 - j^2}{n^2}\right\}^2} = \frac{1}{n}\sqrt{1 + \frac{(2j+1)^2}{n^2}},$$

hence the arc length is approximated by

$$L \approx \sum_{j=0}^{n-1} \frac{1}{n}\sqrt{1 + \frac{(2j+1)^2}{n^2}} \approx \sum_{j=0}^{n-1} \frac{1}{n}\sqrt{1 + 4\left(\frac{j}{n}\right)^2},$$

because

$$\sqrt{1 + \frac{(2j+1)^2}{n^2}} - \sqrt{1 + 4\left(\frac{j}{n}\right)^2}$$

$$\frac{\dfrac{(2j+1)^2}{n^2} - \dfrac{(2j)^2}{n^2}}{\sqrt{1 + \dfrac{(2j+1)^2}{n^2}} + \sqrt{1 + 4\left(\dfrac{j}{n}\right)^2}}$$

$$\le \frac{1}{2n^2}(4j+1) \le \frac{4n+1}{2n^2} = \frac{2}{n} + \frac{1}{2n^2} \to 0 \quad \text{for } n \to \infty.$$

Now,

$$\sum_{j=0}^{n-1} \frac{1}{n} \sqrt{1+4\left(\frac{j}{n}\right)^2} \to \int_0^1 \sqrt{1+4x^2}\,dx \qquad \text{for } n \to \infty.$$

By choosing the monotonous substitution $x = \dfrac{1}{2}\sinh t$, we get the arc length

$$\begin{aligned}
L &= \int_0^1 \sqrt{1+4x^2}\,dx = \int_{x=0}^1 \sqrt{1+\sinh^2 t} \cdot \frac{1}{2}\cosh t\,dt \\
&= \frac{1}{2}\int_{x=0}^1 \cosh^2 t\,dt = \frac{1}{4}\int_{x=0}^1 (1+\cosh 2t)\,dt \\
&= \frac{1}{4}\left[\operatorname{Arsinh} 2x\right]_0^1 + \frac{1}{8}\left[\sinh 2t\right]_{x=0}^1 \\
&= \frac{1}{4}\left[\ln\left(2x+\sqrt{4x^2+1}\right)\right]_0^1 + \frac{1}{4}\left[\sinh t \cdot \cosh t\right]_{x=0}^1 \\
&= \frac{1}{4}\ln(2+\sqrt{5}) + \frac{1}{4}\left[2x\sqrt{1+4x^2}\right]_0^1 \\
&= \frac{1}{4}\ln(2+\sqrt{5}) + \frac{\sqrt{5}}{2}.
\end{aligned}$$

Example 7.8 *Let P be a point in the first quadrant of an XY coordinate system with origo O. Put $OP = r$, and let v denote the angle (measured in radians) from the X axis to OP. The set of all points P, for which $r = v$, defines the curve, which is shown on the figure.*

Find the length s of this curve. This is done in the following way: Consider two neighbouring points on the curve, P and Q, where the angle from the X axis to OP is v, and the angle from OP to OQ is Δv, where $\Delta v > 0$, i.e. small. First prove that the arc PQ approximatively has the length $\Delta s = \sqrt{1+v^2}\,\Delta v$, and then find s by a summation.

A. Curve length by a Riemann sum and a limit.

D. Sketch and analyze the figure. Find an approximative value of the length of a curve segment. Notice that the curve is given in *polar coordinates*. Set up a Riemann sum and go to the limit. Finally, calculate the integral by choosing a substitution.

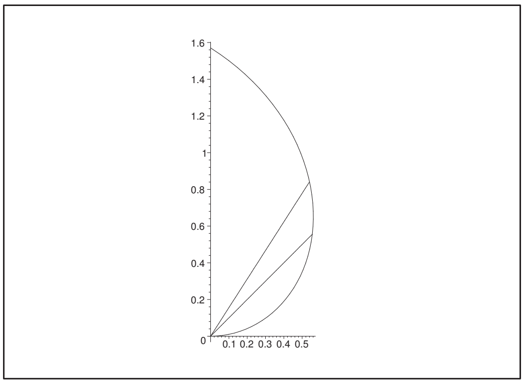


Figure 10: The curve given in polar coordinates by $r = v$.

I. The curve is given in polar coordinates by

$$r = v, \qquad v \in \left[0, \frac{\pi}{2}\right].$$

Choose $v \in \left[0, \frac{\pi}{2}\right[$ and $0 < \Delta v << 1$, such that $v + \Delta v \leq \frac{\pi}{2}$. Then $P(v)$ is given in *rectangular coordinates* by

$$r(\cos v, \sin v) = (v\,\cos v, v\,\sin v),$$

and $P(v + \Delta v)$ has the rectangular coordinates

$$\begin{aligned}&(r+\Delta r)\cdot(\cos(v+\Delta v), \sin(v+\Delta v))\\ &\qquad = (\{v+\Delta v\}\cos\{v+\Delta v\}, \{v+\Delta v\}\sin\{v+\Delta v\})\,.\end{aligned}$$

The length of the arc between $P(v)$ and $P(v+\Delta v)$ is approximated by the length of the cord between these two points:

$$\begin{aligned}
\Delta\ell &\approx |P(v+\Delta v)-P(v)| \\
&= \sqrt{\{(v+\Delta v)\cos(v+\Delta v)-v\,\cos v\}^2+\{(v+\Delta v)\sin(v+\Delta v)\sin(v+\Delta v)-v\,\sin v\}^2} \\
&= \sqrt{(v+\Delta v)^2+v^2-2v(v+\Delta v)\{\cos(v+\Delta v)\cos v+\sin(v+\Delta v)\sin v\}} \\
&= \sqrt{(v+\Delta v)^2+v^2-2v(v+\Delta v)\cos(v+\Delta v-v)} \\
&= \sqrt{(v+\Delta v)^2+v^2-2v(v+\Delta v)\cos(\Delta v)} \\
&\approx \sqrt{(v+\Delta v)^2+v^2-2v(v+\Delta v)\left\{1-\frac{1}{2}(\Delta v)^2\right\}} \\
&= \sqrt{2v^2+2v\Delta v+(\Delta v)^2-2v^2-2v\Delta v+v(v+\Delta v)(\Delta v)^2} \\
&= \Delta v\,\sqrt{1+v^2+v\,\Delta v} \approx \Delta v\,\sqrt{1+v^2},
\end{aligned}$$

i.e.

$$\Delta\ell \approx \sqrt{1+v^2}\,\Delta v.$$

Let

$$0 = v_0 < v_1 < \cdots < v_n < v_{n+1} = \frac{\pi}{2},$$

be a subdivision of $\left[0,\frac{\pi}{2}\right]$, and put

$$\Delta v_i = v_{i+1}-v_i, \qquad i=0,1,\dots,n.$$

Then the curve length ℓ is approximatively given by

$$\ell \approx \sum_{i=0}^{n}\Delta\ell_i \approx \sum_{i=0}^{n}\sqrt{1+v_i^2}\,\Delta v_i.$$

the latter expression is a Riemann sum for $\int_0^{\frac{\pi}{2}}\sqrt{1+v^2}\,dv$, hence

$$\ell = \lim_{n\to+\infty}\sum_{i=0}^{n}\sqrt{1+v_i^2}\,\Delta v_i = \int_0^{\frac{\pi}{2}}\sqrt{1+v^2}\,dv,$$

where we have assumed that $\Delta v_i \to 0$ by this limit.

We calculate the integral by using the monotonous substitution

$$v=\sinh t, \qquad t \in \left[0,\operatorname{Arsinh}\frac{\pi}{2}\right] = \left[0,\ln\left(\frac{\pi}{2}+\sqrt{1+\frac{\pi^2}{4}}\right)\right].$$

Thus

$$\begin{aligned}
\ell &= \int_0^{\frac{\pi}{2}} \sqrt{1+v^2}\,dv = \int_0^{\text{Arsinh}\,\frac{\pi}{2}} \sqrt{1+\sinh^2 t}\,\cosh t\,dt \\
&= \int_0^{\text{Arsinh}\,\frac{\pi}{2}} \cosh^2 t\,dt = \int_0^{\text{Arsinh}\,\frac{\pi}{2}} \frac{1}{2}\left\{\cosh 2t+1\right\}dt \\
&= \frac{1}{2}\left[\frac{1}{2}\sinh 2t + t\right]_0^{\text{Arsinh}\,\frac{\pi}{2}} = \frac{1}{2}[\sinh t\cdot\cosh t+1]_0^{\text{Arsinh}\,\frac{\pi}{2}} \\
&= \frac{1}{2}\left[\sinh t\cdot\sqrt{1+\sinh^2 t}+t\right]_0^{\text{Arsinh}\,\frac{\pi}{2}} \\
&= \frac{1}{2}\left\{\frac{\pi}{2}\sqrt{1+\frac{\pi^2}{4}}+\ln\left(\frac{\pi}{2}+\sqrt{1+\frac{\pi^2}{4}}\right)\right\}.
\end{aligned}$$

Example 7.9 *A ball of radius R has the surface area $A = 4\pi R^2$ and the volume $V = \dfrac{4}{3}\pi R^3$. We note here that $\dfrac{dV}{dR} = A$. Give reasons for this formula by assuming the formula of the volume.*

A. A direct derivation of a formula.

D. Consider the volumes $V(R)$ and $V(R-\Delta R)$.

I. The volume of the shell between the radii $R-\Delta R$ and R is approximatively given by

$$V(R) - V(R-\Delta R) \approx A(R)\cdot\Delta(R),$$

from which

$$\frac{V(R)-V(R-\Delta R)}{\Delta R} \approx A(R).$$

Then by taking the limit $\Delta R \to 0+$ we get

$$\frac{dV}{dR} = A.$$

Example 7.10 *Find the Riemann sum $S(n)$ of the function $y = x^3$, $x \in [1,4]$, corresponding to a subdivision of the interval $[1,4]$ into n equal parts. Calculate $S(2)$ and $S(3)$, and take the limit $\lim_{n\to\infty} S(n)$. Write a MAPLE programme which defines the function $S(n)$.*

A. Riemann sum of a given function.

D. Start by defining the subdivision.

I. Let $n \in \mathbb{N}$ be given. Then $\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$, and the points of division are

$$1 + \frac{3j}{n}, \qquad j = 0, 1, \dots, n.$$

We get our Riemann sum by e.g. choosing the values of the function in the right end point of each subinterval, so

$$\begin{aligned} S(n) &= \sum_{j=1}^{n} f(1 + j\Delta x) \cdot \Delta x \\ &= \sum_{j=1}^{n} \left(1 + \frac{3j}{n}\right)^3 \cdot 3n = \frac{3}{n} \sum_{j=1}^{n} \left(1 + \frac{3j}{n}\right)^3. \end{aligned}$$

In particular,

$$\begin{aligned}
S(2) &= \frac{3}{2}\sum_{j=1}^{2}\left(1+\frac{3}{2}\,j\right)^3 = \frac{3}{2}\left\{\left(1+\frac{3}{2}\right)^3+\left(1+\frac{3}{2}\cdot 2\right)^3\right\} \\
&= \frac{3}{2}\left\{\left(\frac{5}{2}\right)^3+4^3\right\} = \frac{3}{2}\left\{\frac{125}{8}+64\right\} = \frac{3}{16}\,\{125+512\} \\
&= \frac{3}{16}\cdot 637 = \frac{1911}{16} = 119\,\frac{7}{16},
\end{aligned}$$

and

$$\begin{aligned}
S(3) &= \frac{3}{3}\sum_{j=1}^{3}\left(1+\frac{3j}{3}\right)^3 = \sum_{j=1}^{3}(1+j)^3 = 2^3+3^3+4^3 \\
&= 8+27+64 = 99.
\end{aligned}$$

Then

$$\lim_{n\to\infty} S(n) = \int_1^4 x^3\,dx = \left[\frac{x^4}{4}\right]_1^4 = 64-\frac{1}{4} = 63\,\frac{3}{4} = \frac{255}{4},$$

and since we always take the maximum value in each of the subintervals, it follows that $S(n)$ is decreasing.

A MAPLE programme is e.g.

```
sum((1+3*j/n)^3,j=1..n).
```

Example 7.11 *Write a MAPLE programme which calculates a Riemann sum of the function*

$$f(x) = \sqrt{1+\sin 2x}, \qquad x\in[0,\pi],$$

corresponding to a subdivision of the interval $[0,\pi]$ *into 100 equal parts.*

A. Riemann sum; MAPLE programme.

D. Describe the subdivision and choose the right end point of each subinterval.

I. Let $\Delta x = \dfrac{\pi}{100}$, and

$$x_j = \frac{j\pi}{100}, \qquad j = 0,\,1,\,2,\,\dots,\,100.$$

Then we get the Riemann sum

$$S(100) = \frac{\pi}{100}\sum_{j=1}^{100}\sqrt{1+\sin\left(\frac{j\pi}{50}\right)}$$

and the corresponding MAPLE programme is

```
evalf(Pi/(100)*Sum(sqrt(1+sin(j*Pi/50)),j=1..100));
```

By taking the limit $n \to \infty$ we get

$$\begin{aligned}
\lim_{n\to\infty} S(n) &= \int_0^\pi \sqrt{1+\sin 2x}\,dx \\
&= \int_0^\pi \sqrt{\cos^2 x + \sin^2 x + 2\cos x\,\sin x}\,dx \\
&= \int_0^\pi \sqrt{(\cos x+\sin x)^2}\,dx = \int_0^\pi |\cos x + \sin x|\,dx \\
&= \sqrt{2}\int_0^\pi \left|\frac{1}{\sqrt{2}}\,\cos x + \frac{1}{\sqrt{2}}\,\sin x\right|\,dx = \sqrt{2}\int_0^\pi \left|\cos\left(x-\frac{\pi}{4}\right)\right|\,dx \\
&= \sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} |\cos x|\,dx = \sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx - \sqrt{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos x\,dx \\
&= \sqrt{2}\,[\sin x]_{-\frac{\pi}{4}}^{\frac{\pi}{2}} - \sqrt{2}\,[\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \sqrt{2}\left\{1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}+1\right\} \\
&= 2\sqrt{2}.
\end{aligned}$$

Example 7.12 *A domain A in the plane is given on the figure.*

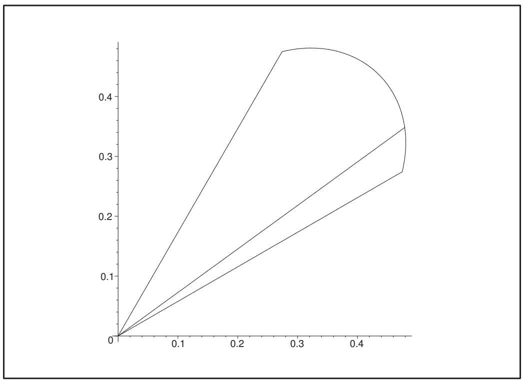


Figure 11: The domain A of radius vector $\varphi(\theta)$, corresponding to an angle $\theta \in [\alpha,\beta]$.

We are given two angles α and β, $\alpha < \beta$, $\beta - \alpha \le 2\pi$, and a continuous function $\varphi : [\alpha,\beta] \to \mathbb{R}_+$. Corresponding to every angle $\theta \in [\alpha,\beta]$ consider the half line of direction θ and the points on this half line, where the distance from O is smaller or equal to $\varphi(\theta)$. In this way we get the set A, and the task is to find a formula of the area of A.

1) *Divide the interval $[\alpha,\beta]$ by the points θ_0, θ_1, $\dots$, θ_n. The area corresponding to the interval $[\theta_{i-1},\theta_i]$ is then approximated by the circular sector given on the figure.*

   Find this approximation of the area of A.

2) *Then set up a formula of the area of A.*

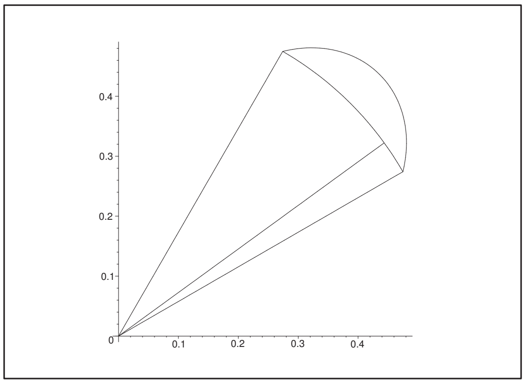


Figure 12: An approximative circular sector.

A. Area in polar coordinates with guidelines.

D. Follow the guidelines.

I. 1) Let $\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$ be the division points. the area of the sector between θ_{i-1} and θ_i of radius $\varphi(\theta_i)$ is

$$\pi\,\varphi(\theta_i)^2 \cdot \frac{\theta_i - \theta_{i-1}}{2\pi} = \frac{1}{2}\,\varphi(\theta_i)^2\,\Delta\theta_i.$$

Therefore, a Riemann sum of this area is

$$M_n = \frac{1}{2}\sum_{i=1}^{n} \varphi(\theta_i)^2\,\Delta\theta_i.$$

2) Then by taking the limit we see that this Riemann sum converges towards

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \varphi(\theta)^2\,d\theta,$$

when $n \to +\infty$ and $\Delta x_i \to 0$.

Example 7.13 *We shall here derived the formula of the length of a graph. Consider a differentiable function* $f : [a,b] \to \mathbb{R}$*, and let* ℓ *denote the length of the graph of* $f(x)$*.*

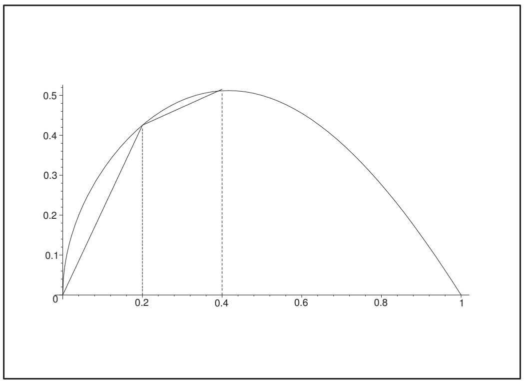


Figure 13: Approximation of a graph by a broken line.

1) Divide the interval $[a,b]$ *by the points* $x_0,\ x_1,\ \dots,\ x_n$*. Corresponding to the interval* $[x_{i-1}, x_i]$ *we approximate the length of the graph by the length of the line on the figure.*

Show that this approximation of ℓ *can be written*

$$\ell_n = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

2) Let $\Delta x_i = x_i - x_{i-1}$*. Apply the* mean value theorem *to prove that* ℓ_n *can be written*

$$\ell_n = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \{f'(\xi_i)\,\Delta x_i\}^2},$$

where $\xi_i \in\,]x_{i-1}, x_i[$.

3) Find a formula of the length ℓ expressed by means of an integral.

A. Derivation of a formula of a curve length with guidelines.

D. Follow the guidelines.

I. 1) The curve segment between $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ is according to the Pythagoras theorem approximated by a line of length

$$\sqrt{(x_i - x_{i-1})^2 + \{f(x_i) - f(x_{i-1})\}^2},$$

thus the approximation of the total length is

$$\ell_n = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + \{f(x_i) - f(x_{i-1})\}^2}.$$

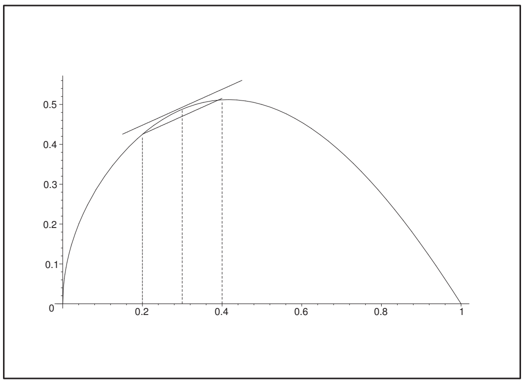


Figure 14: Geometric interpretation of the mean value theorem for a single subinterval, $\xi_i \in]x_{i-1}, x_i[$.

2) It follows from the mean value theorem for differentiable functions that

$$f(x_i) - f(x_{i-1}) = f'(\xi_i) \cdot \{x_i - x_{i-1}\}$$

for some $\xi_i \in]x_{i-1}, x_i[$. Putting $\Delta x_i = x_i - x_{i-1}$, we get by insertion

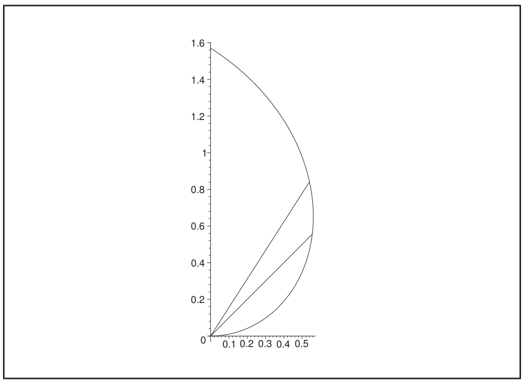
$$\begin{aligned}
\ell_n &= \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + \{f(x_i) - f(x_{i-1})\}^2} \\
&= \sum_{i=1}^{n} \sqrt{\{\Delta x_i\}^2 + \{f'(\xi_i)\,\Delta x_i\}^2} \\
&= \sum_{i=1}^{n} \sqrt{1 + \{f'(\xi)\}^2} \cdot \Delta x_i.
\end{aligned}$$

3) The latter expression of (2) indicates ℓ_n as a Riemann sum of the integral

$$\ell = \int_a^b \sqrt{1+f'(x)^2}\,dx.$$

Since $\ell_n \to \ell$, when $n \to +\infty$ and $\Delta x_i \to 0$ during this limit, we have obtained our formula of the curve length of a graph.

Example 7.14 *Let P be a point in the first quadrant of an XY coordinate system with origo O. Put $OP = r$, and let v denote the angle (measured in radians) from the X axis to OP. The set of points P, for which $r = v$, defines the curve, which is shown on the figure.*



Find the area A of the domain which is defined by the curve and the Y axis. Proceed in the following way: Divide A into "narrow" subdomains by means of line segments from O to various points P_1, P_2, ..., P_n on the curve. Approximate the area of the narrow subdomain OP_iP_{i+1} by the area of a circular sector of radius $= P_i$ and then perform a limit.

A. Area in polar coordinates.

D. Follow the given guidelines and finally take the limit.

I. A sector of radius $r(v_i) = OP_i$ has approx. the area $\frac{1}{2}\,r(v_i)^2\Delta v_i$, i.e. the total area is approximatively given by

$$\sum_i \frac{1}{2}\,r(v_i)^2\Delta v_i \to \int_0^{\frac{\pi}{2}} \frac{1}{2}\,r(v)^2\,dv \qquad \text{for } \Delta v_i \to 0.$$

We conclude that the area is given by

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}\,r(v)^2\,dv = \frac{1}{2}\int_0^{\frac{\pi}{2}} v^2\,dv = \frac{1}{6}\,\left[v^3\right]_0^{\frac{\pi}{2}} = \frac{\pi^3}{48}.$$

Example 7.15 *We are given in a rectangular coordinate system the points*

$$P:\quad (x, \cosh x) \quad og \quad Q:\quad (x+\Delta x, \cosh(x+\Delta x)),$$

where $\Delta x > 0$*. Explain that the distance* PQ *satisfies*

$$\frac{PQ}{\Delta x} \to \cosh x \qquad \text{for } \Delta x \to 0.$$

Then exploit the relation

$$PQ \approx \Delta x \cdot \cosh x \qquad \text{for } \Delta x \text{ small},$$

followed by taking a limit and an application of the main theorem of the differential and integral calculus to find the length of the graph of the function $y = \cosh x$*,* $x \in [0, \ln 2]$*.*

A. Curve length.

D. Sketch a figure, and consider a rectangular triangle. Then proceed with the guidelines.

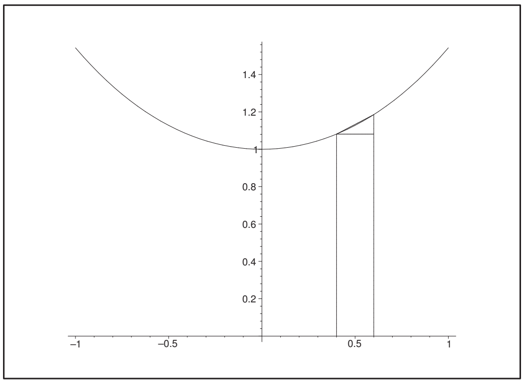


Figure 15: Approximation of the curve length by the hypothenuse of a rectangular triangle, where one of the shorter sides has length Δx and the other one $\cosh(x+\Delta x) - \cosh x$.

I. Let $\Delta x > 0$. Then by the Pythagoras theorem we get for the approximating cord PQ,

$$\begin{aligned} PQ &= \sqrt{\Delta x^2 + \{\cosh(x+\Delta x) - \cosh x\}^2} \\ &= \Delta\sqrt{1+\left\{\frac{\cosh(x+\Delta x)-\cosh x}{\Delta x}\right\}^2}. \end{aligned}$$

Since

$$\frac{\cosh(x+\Delta x)-\cosh x}{\Delta x} \to \frac{d}{dx}\cosh x = \sinh x \quad \text{for } \Delta x \to 0,$$

it follows by taking the limit that

$$\begin{aligned} \lim_{\Delta x\to 0+}\frac{PQ}{\Delta x} &= \lim_{\Delta x\to 0+}\sqrt{1+\left\{\frac{\cosh(x+\Delta x)-\cosh x}{\Delta x}\right\}^2} \\ &= \sqrt{1+\left\{\lim_{\Delta\to 0+}\frac{\cosh(x+\Delta x)-\cosh x}{\Delta x}\right\}^2} \\ &= \sqrt{1+\sinh^2 x} = \cosh x. \end{aligned}$$

Then by a convenient subdivision we see that the curve length is

$$\ell \approx \sum_j \Delta x_j P_j P_{j+1} = \sum_j \Delta x_j \, \cosh x_j \to \int_0^{\ln 2} \cosh x\, dx,$$

for $j \to \infty$ and $\Delta x_j \to 0$. This implies that the curve length is

$$\begin{aligned} \ell &= \int_0^{\ln 2}\cosh x\, dx = [\sinh x]_0^{\ln 2} \\ &= \sinh(\ln 2) = \frac{1}{2}\left(2-\frac{1}{2}\right) = \frac{3}{4}. \end{aligned}$$

8 Moment of inertia

Example 8.1 *Let I denote the moment of inertia of a body L with respect to an axis of rotation a. Divide L into two sub-bodies L_1 and L_2, the moments of inertia of which with respect to a are denoted by I_1 and I_2. Prove that $I = I_1 + I_2$.*
Use this formula the moment of inertia of a thin, homogeneous, straight rod of length ℓ and mass M with respect to an axis which is perpendicular to the rod at its midpoint.

A. Moment of inertia.

D. Exploit that the moment of inertia with respect to a fixed axis of rotation is additive with respect to a disjoint subdivision of the body.

I. Let the density at a point be given by $m(\mathbf{x})$, and let $\ell(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathbf{a})$ be the distance from $\mathbf{x}$ to the line a. Then

$$\begin{aligned} I &= \int_L m(\mathbf{x}) \cdot \ell(\mathbf{x})^2 \, d\Omega \\ &= \int_{L_1} m(\mathbf{x}) \cdot \ell(\mathbf{x})^2 \, d\Omega + \int_{L_2} m(\mathbf{x}) \cdot \ell(\mathbf{x})^2 \, d\Omega = I_1 + I_2. \end{aligned}$$

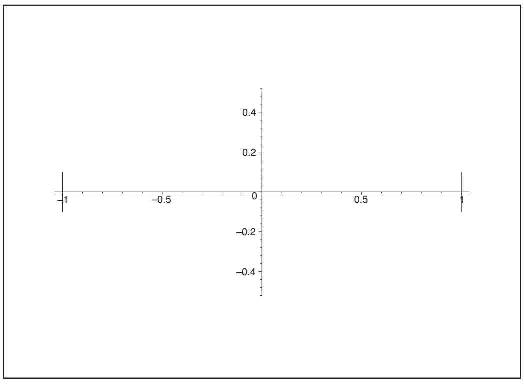


Figure 16: A homogeneous rod along the x axis of length $\ell = 2$ where the y axis is chosen as the axis of rotation through the midpoint of the rod.

Assuming that the rod is homogeneous, the density is $m(x) = \dfrac{M}{\ell}$. Place the rod along the x axis, represented by the interval $\left[-\dfrac{\ell}{2}, \dfrac{\ell}{2}\right]$.

Let $L_1 = \left[0, \dfrac{\ell}{2}\right]$ and $L_2 = \left[-\dfrac{\ell}{2}, 0\right]$. Then for symmetric reasons, $I_1 = I_2$, so

$$I = 2I_1 = 2\int_0^{\frac{\ell}{2}} \frac{M}{\ell} \cdot x^2 \, dx = 2\,\frac{M}{\ell} \cdot \frac{1}{3} \cdot \left(\frac{\ell}{2}\right)^3 = \frac{M}{12}\,\ell^2.$$

Example 8.2 *In a rectangular coordinate system XY in the plane we consider the points*

$$A: \ (0,0), \qquad B: \ (2,0) \quad og \quad C: \ (2,4).$$

1) Find the area of the domain O, which is bounded by the parabolic arc

$$y = x^2, \qquad x \in [0,2],$$

and the line segments AB and BC.

2) Consider the domain O as a thin homogeneous plate P of mass M. Find the moment of inertia of P, when X is the axis of rotation.

A. Area and moment of inertia. This example is almost the same as Example 8.5. The only difference is the axis of rotation.

D. Sketch the domain and analyze.

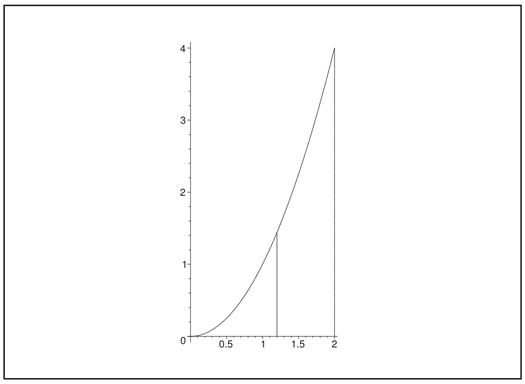


Figure 17: The domain O.

I. 1) From the sketch follows immediately that

$$\text{areal}(O) = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

2) The density is $\dfrac{M}{\text{areal}(O)} = \dfrac{3M}{8}$.

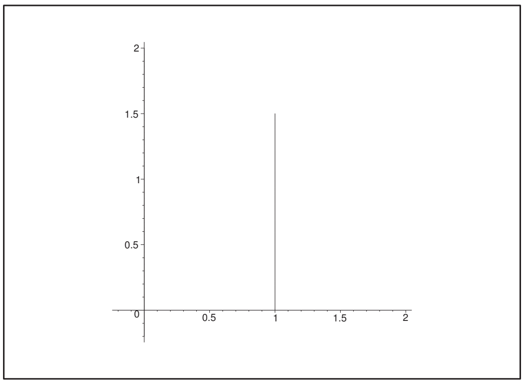
For fixed $x \in [0,2]$ we get the moment of inertia of the line segment $y \in \left[0, x^2\right]$ over x,

$$\frac{3M}{8}\int_0^{x^2} y^2\,dy = \frac{3M}{8}\left[\frac{y^3}{4}\right]_0^{x^2} = \frac{M}{8}\,x^6.$$

Then the total moment of inertia is obtained by gathering all projections of the moments of inertia onto the X axis, i.e.

$$I = \int_0^2 \frac{M}{8}\,x^6\,dx = \frac{M}{8}\left[\frac{x^7}{7}\right]_0^2 = \frac{M}{2^3}\cdot\frac{2^7}{7} = \frac{16}{7}\,M.$$

Example 8.3 *We consider in a rectangular coordinate system XY in the plane a thin homogeneous rod of length L and mass M. The end points of the rod have the coordinates $(a, 0)$ and (a, L), where $a > 0$ is some given number.*



Find the moment of inertia of the rod, when the axis of rotation is perpendicular to the XY plane in origo.

A. Moment of inertia.

D. Analyze the distance function.

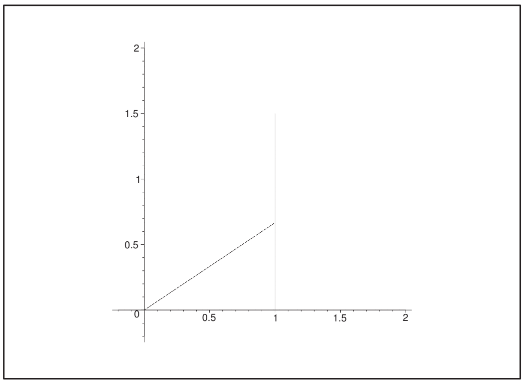
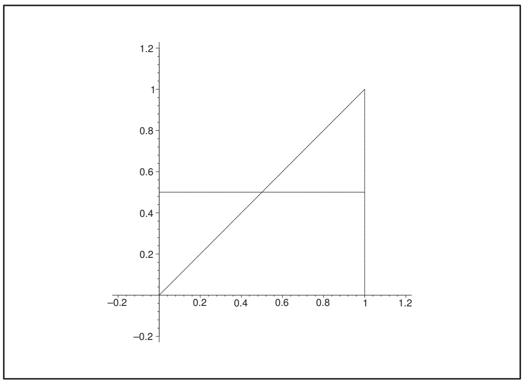


Figure 18: The distance from $(0,0)$ to a (a,y) on the rod is $\sqrt{a^2+y^2}$.

I. The density is $\dfrac{M}{L}$. A point (a,y), $y \in [0,L]$, on the rod has the distance $\sqrt{a^2+y^2}$ from $(0,0)$, thus the moment of inertia is

$$I = \frac{M}{L}\int_0^L \{a^2+y^2\}dy = \frac{M}{L}\cdot\left(a^2L+\frac{L^3}{3}\right) = M\left(a^2+\frac{L^2}{3}\right).$$

Example 8.4 *A homogeneous plate P of mass M has the shape of a rectangular triangle ABC, in which the shorter sides AC and BC both have the length a. Find the moment of inertia of P with respect to the axis of rotation ℓ through A, which is parallel with BC.*



A. Moment of inertia.

D. Find the density. Place the coordinate system, such that ℓ lies along the ordinate axis and AC along the abscissa axis. Find (apart from the factor Δy) the moment of inertia for every fixed $y \in [0, a]$ of a thin "rod", which is given by $x \in [y, a]$. Finally, we "collect" (i.e. we integrate) all these y contributions of the moment of inertia.

I. Since the area is $\frac{1}{2}\, a^2$, the density is $m = \dfrac{2M}{a^2}$.

Keep $y \in [0, a]$ fixed. Then the corresponding moment of inertia is (leaving out the factor Δy)

$$\int_y^{a^2} x^2\, dx = \frac{2M}{a^2}\left[\frac{x^3}{3}\right]_y^a = \frac{2M}{3a^2}\left\{a^3 - y^3\right\} = \frac{1}{2}\, Ma - \frac{2M}{3a^2}\, y^3.$$

When we integrate this expression with respect to y, we get the moment of inertia

$$\begin{aligned} I &= \int_0^a \left\{\frac{2}{3}\, Ma - \frac{2M}{3a^2}\, y^3\right\} dy = \frac{2}{3}\, Ma^2 - \frac{M}{6a^2}\cdot a^4 \\ &= \left(\frac{2}{3} - \frac{1}{6}\right) Ma^2 = \frac{1}{2}\, Ma^2. \end{aligned}$$

Example 8.5 *Consider in a rectangular coordinate system XY in the plane the points*

$A:\ (0,0), \qquad B:\ (2,0) \quad og \quad C:\ (2,4).$

1) Find the area of the domain O, which is bounded by the parabolic arc

$$y = x^2, \qquad x \in [0, 2],$$

and the segments AB and BC.

2) Consider the domain O as a thin homogeneous plate P of mass M. Find the moment of inertia of P, when the ordinate axis is chosen as axis of rotation.

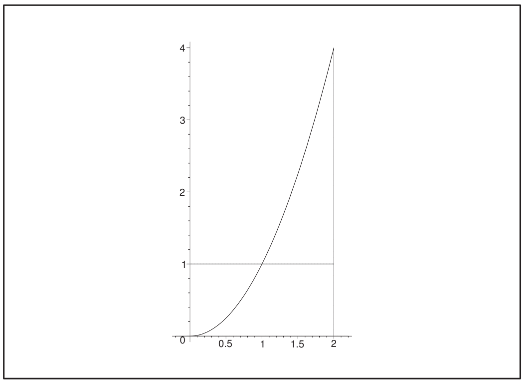
A. Area and moment of inertia. The example is very similar to Example 8.2. The only difference is the axis of rotation.

D. Sketch the domain and analyze.

I. 1) By the sketch we get

$$\text{area}(O) = \int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

2) The density is $\dfrac{M}{\text{area}(O)} = \dfrac{3M}{8}$.



We get for fixed $y \in [0,4]$ the moment of inertia of the line segment $x \in [\sqrt{y}, 2]$ over y,

$$\frac{3M}{8}\int_{\sqrt{y}}^{2} x^2\,dx = \frac{3M}{8}\left[\frac{x^3}{3}\right]_{\sqrt{y}}^{2} = M\left\{1-\frac{1}{8}\,y^{\frac{3}{2}}\right\}.$$

The total moment of inertia is obtained by collecting all the projections of the moments of inertia onto the ordinate axis i.e.

$$\begin{aligned} I &= M\int_0^4 \left\{1-\frac{1}{8}\,y^{\frac{3}{2}}\right\} dy = M\left[y-\frac{1}{8}\cdot\frac{2}{5}\cdot y^{\frac{5}{2}}\right]_0^4 \\ &= M\left\{4-\frac{1}{4\cdot 5}\cdot 32\right\} = M\left\{4-\frac{8}{5}\right\} = \frac{12}{5}\,M. \end{aligned}$$

Example 8.6 *Find the area A of the domain O in the XY plane which is bounded by the curve*

$$y = \frac{1}{x^2}, \qquad 1 \le x \le 2,$$

and the X axis and the lines $x = 1$ and $x = 2$.
Consider the domain O as a thin, homogeneous plate of mass M. Find the moment of inertia of the plate, when the line $x = 1$ is the axis of rotation.

A. Area and moment of inertia. The example is very similar to Example 8.7. The only difference is the axis of rotation.

D. Sketch a figure. Find the area by an integral. Find the moment of inertia.

I. The domain O has the area

$$A = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = 1-\frac{1}{2} = \frac{1}{2}.$$

The density is $\dfrac{M}{A} = 2M$.

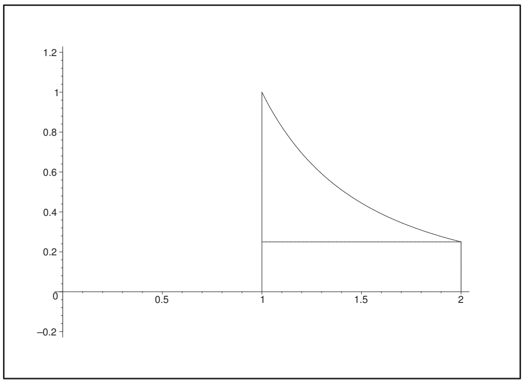


Figure 19: The domain O.

When $y \in \left[0, \frac{1}{4}\right]$ is kept fixed, then the moment of inertia of the corresponding x-interval $[1,2]$ is given by

$$2M \int_1^2 (x-1)^2 \, dx = 2M \int_0^1 t^2 \, dt = \frac{2M}{3}.$$

When $y \in \left[\frac{1}{4}, 1\right]$ is kept fixed, then the moment of inertia of the corresponding x-interval $\left[1, \frac{1}{\sqrt{y}}\right]$ is given by

$$\begin{aligned} 2M \int_1^{\frac{1}{\sqrt{y}}} (x-1)^2 dx &= 2M \int_0^{\frac{1}{\sqrt{y}}-1} t^2\,dt = \frac{2M}{3}\left\{\frac{1}{\sqrt{y}} - 1\right\}^3 \\ &= \frac{2M}{3}\left\{y^{-\frac{3}{2}} - 3\,\frac{1}{y} + 3u^{-\frac{1}{2}} - 1\right\}. \end{aligned}$$

Integrating with respect to y we get the total moment of inertia

$$\begin{aligned} I &= \int_0^{\frac{1}{4}} \frac{2M}{3}\,dy + \frac{2M}{3}\int_{\frac{1}{4}}^1 \left\{y^{-\frac{3}{2}} - \frac{3}{y} + 3y^{-\frac{1}{2}} - 1\right\} dt \\ &= \frac{2M}{3}\cdot\frac{1}{4} + \frac{2M}{3}\left[-\frac{2}{\sqrt{y}} - 3\,\ln y + 6\sqrt{y} - y\right]_{\frac{1}{4}}^1 \\ &= \frac{2M}{3}\left\{\frac{1}{4} - 2 + 6 - 1 + 4 - 3\,\ln 4 - 6\cdot\frac{1}{2} + \frac{1}{4}\right\} \\ &= \frac{2M}{3}\left\{\frac{1}{2} + 4 - 6\,\ln 2\right\} = \frac{M}{3}\,(9 - 12\,ln2) = (3 - 4\,\ln 2)M. \end{aligned}$$

Example 8.7 *Find the area A of the domain O in the XY-plane, which is bounded by the curve*

$$y = \frac{1}{x^2}, \qquad 1 \le x \le 2,$$

and the abscissa axis and the lines $x = 1$ and $x = 2$.
Consider O as a thin, homogeneous plate of mass M. Find the moment of inertia of the plate, when the line $x = 2$ is the axis of rotation.

A. Area and moment of inertia. The example is similar to Example 8.6. The only difference is the axis of rotation.

D. Sketch a figure. Indicate the area as an integral. Finally, calculate the moment of inertia.

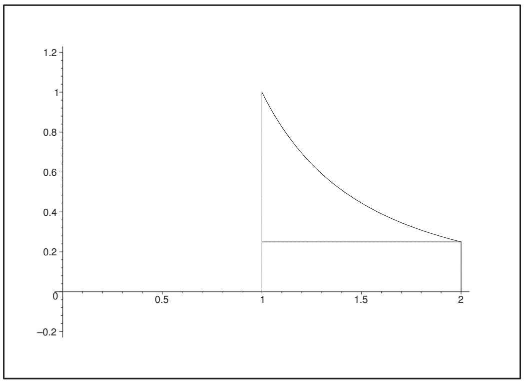
I. The domain O has the area

$$A = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}.$$

The density is $\frac{M}{A} = 2M$.

When $y \in \left[0, \frac{1}{4}\right]$ is kept fixed, then the moment of inertia of the corresponding x-interval $[1, 2]$ is given by

$$2M \int_1^2 (2-x)^2\,dx = 2M\int_0^1 t^2\,dt = \frac{2M}{3}.$$



When $y \in \left[\frac{1}{4}, 1\right]$ is kept fixed, then the moment of inertia of the corresponding x-interval $\left[1, \frac{1}{\sqrt{y}}\right]$ is given by

$$\begin{aligned} 2M \int_1^{\frac{1}{\sqrt{y}}} (x-2)^2\, dx &= \frac{2M}{3}\left[(x-2)^3\right]_1^{\frac{1}{\sqrt{y}}} = \frac{2M}{3}\left\{1 - \left(\frac{1}{\sqrt{y}} - 2\right)^3\right\} \\ &= \frac{2M}{3}\left\{1 - y^{-\frac{3}{2}} + \frac{6}{y} - 12y^{-\frac{1}{2}} + 8\right\}. \end{aligned}$$

Integrating finally with respect to y we get the total moment of inertia

$$\begin{aligned} I &= \frac{2M}{3}\cdot\frac{1}{4} + \frac{2M}{3}\int_{\frac{1}{4}}^1 \left\{9 - y^{-\frac{3}{2}} + \frac{6}{y} - 12y^{-\frac{1}{2}}\right\} dy \\ &= \frac{2M}{3}\left\{\frac{1}{4} + \frac{27}{4} + \left[\frac{2}{\sqrt{y}} + 6\,\ln y - 24\sqrt{y}\right]_{\frac{1}{4}}^1\right\} \\ &= \frac{2M}{3}\{7 + 2 - 4 + 12\,\ln 2 - 24 + 12\} \\ &= \frac{2M}{3}\{12\,\ln 2 - 7\}. \end{aligned}$$

Example 8.8 *Find the moment of inertia of a homogeneous disc of mass M and radius R with respect to an axis of rotation perpendicular to the plane of the disc through its centre.*

A. Moment of inertia.

D. Sketch a figure and analyze. Set up a Riemann sum and take the limit.

I. The area of the circle is πR^2, and the circle has a homogeneous layer of mass. Hence, the density is $\frac{M}{\pi R^2}$.

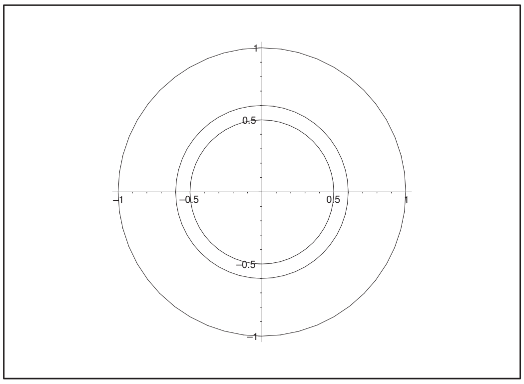


Figure 20: A thin annulus cut from the unit disc.

Consider an annulus of inner radius r and of thickness Δr, where $0 < \Delta r << r$. The mass of the annulus is

$$\frac{M}{\pi R^2}\left\{\pi(r+\Delta r)^2 - \pi r^2\right\} = \frac{M}{R^2}\left\{2r\Delta r + (\Delta r)^2\right\} \approx \frac{2M}{R^2}\, r\Delta r.$$

Then split the annulus into small pieces of e.g. the same size (with respect to the angle). Since each piece gives equal contribution to the moment of inertia, we can treat the annulus as a whole. When e.g. the annulus is split into m pieces of equal size, we get the contribution $\frac{1}{m}$ of the total from each of them, and $m \cdot \frac{1}{m} = 1$. The moment of inertia of the annulus is then approximately

$$r^2 \cdot \left\{\frac{2M}{R^2}\, r\, \Delta r\right\} = \frac{2M}{R^2}\, r^3\, \Delta r.$$

Let

$$0 = r_0 < r_1 < \cdots < r_n < r_{n+1} = R$$

be a subdivision of $[0, R]$, and let

$$\Delta r_i = r_{i+1} - r_i, \qquad i = 0, 1, \dots, n.$$

Since the disc is split into annuli of the type described above, we get

$$I \approx \sum_{i=0}^{n} \frac{2M}{R^2}\, r_i^3\, \Delta r_i = \frac{2M}{R^2} \sum_{i=0}^{n} r_i^3\, \Delta r_i.$$

This expression is a Riemann sum for some integral, hence by taking the limit $\Delta r_i \to 0$,

$$I = \lim_{\Delta r_i \to 0} \frac{2M}{R^2} \sum_{i=0}^{n} r_i^3\, \Delta r_i = \frac{2M}{R^2} \int_0^R r^3\, dr = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{MR^2}{2}.$$

Example 8.9 *Find the moment of inertia of a thin and homogeneous disc of mass M and radius R with respect to an axis of rotation which is a diameter of the disc.*

A. Moment of inertia.

D. Analyze a figure. Set up a Riemann sum for an integral and take the limit.

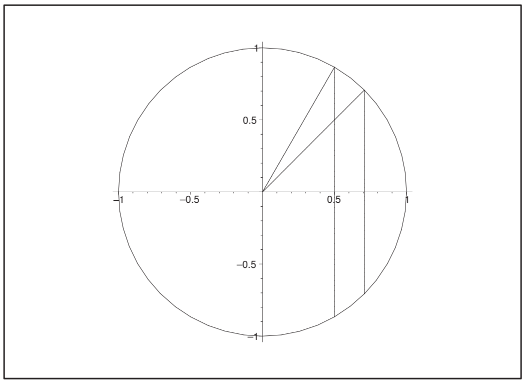


Figure 21: The unit disc is rotated around the ordinate axis.

I. The area of the circle is πR^2, and the mass M is spread homogeneously over the disc. Hence, the density is $\dfrac{M}{\pi R^2}$.

Let $v \in [0, \pi]$ and $0 < \Delta v << 1$, such that $v + \Delta v \le \pi$. Consider the domain between the two vertical lines on the figure. Then the distance to the axis of rotation is approximately $R\,\cos v$, and the mass is approximately

$$\begin{aligned}
&\frac{M}{\pi R^2} \cdot R\,\sin v\,\{R\,\cos v - R\,\cos(v+\Delta v)\}\\
&\qquad = \frac{M}{\pi}\,\sin v\,\{\cos v - \cos(v+\Delta v)\}\\
&\qquad = \frac{M}{\pi}\,\sin v\,\{\cos v - \cos v \cdot \cos(\Delta) + \sin v \cdot \sin(\Delta v)\}\\
&\qquad = \frac{M}{\pi}\,\sin v\,\{\sin v \cdot \sin(\Delta v) + \cos v\,[1 - \cos(\Delta v)]\}\\
&\qquad \approx \frac{M}{\pi}\,\sin^2 v \cdot \Delta v,
\end{aligned}$$

because

$$1 - \cos \Delta v \approx \frac{1}{2}\,(\Delta v)^2 \quad \text{og} \quad \sin \Delta v \approx \Delta v,$$

when Δv is small.

Then split the domain between the two vertical lines into small pieces of the same masse. However, since the disc is homogeneous, this step is no longer necessary. The moment of inertia of this strip is approximately

$$R^2 \cos^2 v \cdot \frac{M}{\pi}\,\sin^2 v\,\Delta v = \frac{MR^2}{4\pi}\,\sin^2 2v\,\Delta v = \frac{MR^2}{8\pi}\,(1 - \cos 4v)\,\Delta v.$$

Let

$$0 = v_0 < v_1 < \cdots < v_n < v_{n+1} = \pi$$

be a subdivision of $[0, \pi]$, and put

$$\Delta v_i = v_{i+1} - v_i, \qquad i = 0, 1, \dots, n.$$

If we split the disc into strips as above according to this subdivision, then an approximative value of the moment of inertia is given by

$$I \approx \frac{MR^2}{8\pi} \sum_{i=0}^{n} \{1 - \cos(4v_i)\} \Delta v_i.$$

When $\Delta v_i \to 0$, the Riemann sum converges towards the moment of inertia, i.e.

$$\begin{aligned} I &= \lim_{\Delta v_i \to 0} \frac{MR^2}{8\pi} \sum_{i=0}^{n} (1 - \cos 4v_i)\, \Delta v_i = \frac{MR^2}{8\pi} \int_0^\pi (1 - \cos 4v)\, dv \\ &= \frac{MR^2}{8\pi} \left[v - \frac{1}{4} \sin 4v \right]_0^\pi = \frac{MR^2}{8\pi}. \end{aligned}$$

Example 8.10 *This example is also about the moment of inertia.*

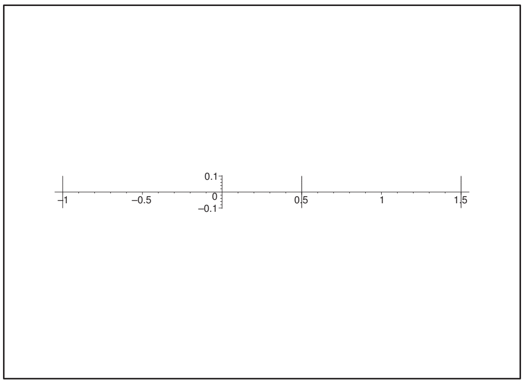


Figure 22: Three masses lying on a straight line, where 0 indicates the origo of the ordinate axis.

Given a system of finitely many masses $m_1, m_2, \dots, m_k$*, lying on a straight line, and let* r_i *denote the distance from the origo to the mass* m_i*. The moment of inertia with respect to the origo for this system is defined by*

(6) $I = \displaystyle\sum_{i=1}^{k} m_i r_i^2.$

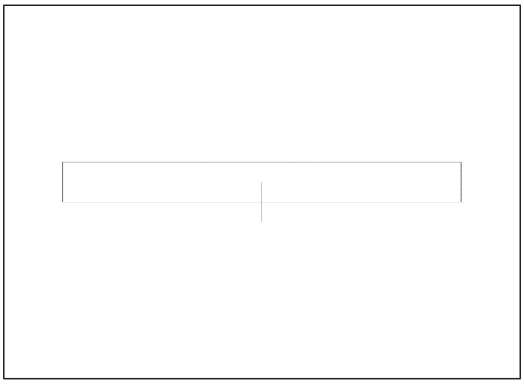


Figure 23: A thin, homogeneous rod of mass M and length L, with its midpoint 0.

Consider a thin, homogeneous rod of mass M *and length* L*. We shall find the moment of inertia* J *of this rod with respect to the midpoint 0.*

1) *Subdivide the interval* $\left[-\dfrac{L}{2}, \dfrac{L}{2}\right]$ *by the points* $x_0, x_1, \dots, x_k$*. Corresponding to each subinterval* $[x_{i-1}, x_i]$ *we let the corresponding mass be concentrated at the point* x_i*. Write by means of (6) this approximation of the moment of inertia* J*.*

2) *Then apply the main theorem of the differential and integral calculus and derive the formula of the moment of inertia* J *expressed by means of* L *and* M*.*

A. The setup of a model for calculation of the moment of inertia by means of Riemann sums. A guideline is given.

D. Follow the guideline.

D. 1) When the rod is homogeneous of mass M and length L, then the density is $\dfrac{M}{L}$. The mass in the subinterval $[x_{i-1}, x_i]$ is therefore

$$m_i = \frac{M}{L}\,(x_i - x_{i-1}) = \frac{M}{L}\,\Delta x_i.$$

This gives the approximation

$$I = \sum_{i=1}^{k} m_i r_i^2 = \sum_{i=1}^{k} \frac{M}{L}\,\Delta x_i \cdot x_i^2 = \frac{M}{L} \sum_{i=1}^{k} x_i^2\,\Delta x_i$$

of the moment of inertia J as a Riemann sum.

2) When $\Delta x_i \to 0$ for $k \to +\infty$, we see that this Riemann sum converges towards

$$J = \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\,dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-\frac{L}{3}}^{\frac{L}{2}} = \frac{M}{L}\cdot\frac{1}{3}\cdot 2\left(\frac{L}{2}\right)^3 = \frac{1}{12}\,ML^2.$$

9 Mathematical models

Example 9.1 *Send a current $i = e^{-st}$, $t \geq 0$, where s is given, through a capacitor. To time $t = 0$ there is no charging in the capacitor. Find the charging in the capacitor when the time is infinite. Use that the current of a file is the charging which passes through a cross section of the file per time unit.*

A. A mathematical model.

D. Use that $i = \dfrac{dQ}{dt}$, where $Q(t)$ is the charging to time t. In particular, $Q(0) = 0$.

I. From $i(t) = e^{-st}$ we get the equation

$$\frac{dQ}{dt} = e^{-st}, \qquad Q(0) = 0,$$

from which

$$Q(t) = \int_0^t e^{-s\tau}\, d\tau = \left[-\frac{1}{s}\, e^{-s\tau}\right]_0^t = \frac{1 - e^{-st}}{s} \to \frac{1}{s} \quad \text{for } t \to \infty.$$

Example 9.2 *Consider a particle of mass m, moving on a straight line under the influence of a so-called Coulomb potential*

$$V(x) = -\frac{k}{x},$$

where $k > 0$. Here $x = x(t)$ denotes the distance of the particle from the point 0 to time t. (A coulomb potential can e.g. be created by an electric charging or by a mass).

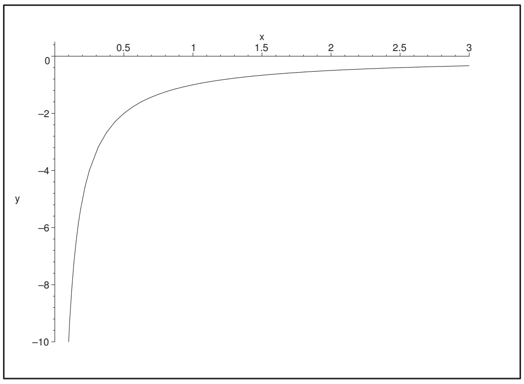


Figure 24: The graph of the potential function $V(x) = -\frac{k}{x}$, $x > 0$.

The particle moves from x_0 towards the point 0 with a negative velocity, and we assume that the particle to time $t = 0$ lies at the point x_0. It follows from the theorem of conservation of the energy that

$$(7)\quad \frac{1}{2}\,m\left\{\frac{dx}{dt}\right\}^2 + V(x) = -K, \qquad t \geq 0,$$

where K is a constant. We consider the case where $K > 0$.

We shall take all this for granted. The task is now to find the time T it takes for the particle to move from the point x_0 to the point 0.

1) *Let T_1 denote the time it takes for the particle to move from the point x_0 to the point x_1. Prove that*

$$T_1 = -\sqrt{\frac{m}{2}} \int_{x_0}^{x_1} \frac{1}{\sqrt{\frac{k}{x} - K}}\,dx.$$

2) *Calculate the integral above by using the substitution $t = \sqrt{\frac{k}{x} - K}$.*

3) *The time T is given by*

$$T = \lim_{x_1 \to 0} \left\{ -\sqrt{\frac{m}{2}} \int_{x_0}^{x_1} \frac{1}{\sqrt{\frac{k}{x} - K}}\,dx \right\}.$$

Find T.

4) The time T depends on the constant K from (7), thus it (of course) depends of the velocity of the particle at the point x_0. Derive from (7) this velocity $x'(0)$ expressed by means of K, and then find the limit of T, when $x'(0) \to 0$.

A. By (7) we are given a non-linear differential equation for x in t. We shall find an expression of the inverse function $t = t(x)$.

D. Find $\dfrac{dx}{dt}$ from (7). Show that the inverse function exists and is given by (1). Then follow the guidelines of the example.

I. 1) It follows from the assumption $\dfrac{dx}{dt} < 0$ and $V(x) = -\dfrac{k}{x}$ and (7) that

$$\left\{\frac{dx}{dt}\right\}^2 = \frac{2}{m}\{-K - V(x)\} = \frac{2}{m}\left\{\frac{k}{x} - K\right\} > 0,$$

i.e. $x \in \left]0, \dfrac{k}{K}\right[$, and

$$\frac{dx}{dt} = -\sqrt{\frac{2}{m}} \cdot \sqrt{\frac{k}{x} - K}, \qquad x \in \left]0, \frac{k}{K}\right[.$$

This shows that $x(t)$ is monotonous, so the inverse function exists, and

$$\frac{dt}{dx} = \left\{\frac{dx}{dt}\right\}^{-1} = -\sqrt{\frac{m}{2}} \cdot \frac{1}{\sqrt{\dfrac{k}{x} - K}}, \qquad x \in \left]0, \frac{k}{K}\right[.$$

Then we get by an integration starting at x_0,

$$T_1 = t(x_1) = -\sqrt{\frac{m}{2}} \int_{x_0}^{x_1} \frac{1}{\sqrt{\dfrac{k}{x} - K}}\, dx, \qquad x_0,\, x_1 \in \left]0, \frac{k}{K}\right[,$$

where $x_1 < x_0$.

2) When $t = \sqrt{\dfrac{k}{x} - K}$ for $x \in \left]0, \dfrac{k}{K}\right[$, we see that the substitution is monotonous with the range $\mathbb{R}_+$, and

$$x = \frac{k}{K + t^2}, \qquad \text{dvs.} \qquad dx = -\frac{2kt}{(K + t^2)^2}\, dt.$$

Hence,

$$\begin{aligned}
T_1 &= t(x_1) = -\sqrt{\frac{m}{2}}\int_{\sqrt{\frac{k}{x_0}-K}}^{\sqrt{\frac{k}{x_1}-K}} \frac{1}{t}\cdot\frac{-2kt}{(K+t^2)^2}\,dt \\
&= \frac{k}{K^2}\sqrt{\frac{m}{2}}\int_{\sqrt{\frac{k}{x_0}-K}}^{\sqrt{\frac{k}{x_1}-K}} \frac{2}{\left(1+\left(\frac{t}{\sqrt{K}}\right)^2\right)^2}\,dt \\
&= \frac{k}{K\sqrt{K}}\sqrt{\frac{m}{2}}\left[\frac{\frac{t}{\sqrt{K}}}{\left(\frac{t}{\sqrt{K}}\right)^2+1} + \text{ Arctan}\left(\frac{t}{\sqrt{K}}\right)\right]_{t=\sqrt{\frac{k}{x_0}-K}}^{t=\sqrt{\frac{k}{x_1}-K}} \\
&= \frac{k}{K}\sqrt{\frac{m}{2}}\left[\frac{t}{t^2+K}+\frac{1}{\sqrt{K}}\,\text{Arctan}\left(\frac{t}{\sqrt{K}}\right)\right]_{t=\sqrt{\frac{k}{x_0}-K}}^{t=\sqrt{\frac{k}{x_1}-K}} \\
&= \frac{1}{K}\sqrt{\frac{m}{2}}\,x_1\sqrt{\frac{k}{x_1}-K}-\frac{1}{K}\sqrt{\frac{m}{2}}\,x_0\sqrt{\frac{k}{x_0}-K} \\
&\qquad +\frac{k}{K\sqrt{K}}\sqrt{\frac{m}{2}}\left\{\text{Arctan}\left(\sqrt{\frac{k}{K}\,\frac{1}{x_1}-1}\right)\right. \\
&\qquad\qquad\qquad \left.-\text{ Arctan}\left(\sqrt{\frac{k}{K}\,\frac{1}{x_0}-1}\right)\right\}.
\end{aligned}$$

C. Test. Let

$$\varphi(x)=\frac{1}{K}\sqrt{\frac{m}{2}}\cdot x\sqrt{\frac{k}{x}-K}+\frac{k}{K\sqrt{K}}\sqrt{\frac{m}{2}}\,\text{Arctan}\left(\sqrt{\frac{k}{K}\,\frac{1}{x}-1}\right).$$

Then

$$
\begin{aligned}
\varphi'(x) &= \frac{1}{K}\sqrt{\frac{m}{2}}\cdot\frac{\frac{k}{x}-K}{+}\frac{1}{K}\sqrt{\frac{m}{2}}\cdot x\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{k}{x}-K}}\cdot\left(-\frac{k}{x^2}\right)\\
&\quad+\frac{k}{K\sqrt{K}}\sqrt{\frac{m}{2}}\cdot\frac{1}{1+\frac{k}{K}\frac{1}{x}-1}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{k}{K}\frac{1}{x}-1}}\cdot\frac{k}{K}\cdot\left(-\frac{1}{x^2}\right)\\
&= \sqrt{\frac{m}{2}}\cdot\frac{1}{\sqrt{\frac{k}{x}-K}}\left\{\frac{1}{K}\left(\frac{k}{x}-K\right)+\frac{1}{K}\cdot\frac{x}{2}\cdot\left(-\frac{k}{x^2}\right)\right.\\
&\quad\left.+\frac{k}{K\sqrt{K}}\cdot\frac{Kx}{k}\cdot\frac{1}{2}\cdot\frac{\sqrt{\frac{k}{x}-K}}{\sqrt{\frac{k}{x}\cdot\frac{1}{x}-1}}\cdot\frac{k}{K}\cdot\left(-\frac{1}{x^2}\right)\right\}\\
&= \sqrt{\frac{m}{2}}\cdot\frac{1}{\sqrt{\frac{k}{x}-K}}\left\{-1+\frac{k}{K}\cdot\frac{1}{x}-\frac{1}{2}\frac{k}{K}\cdot\frac{1}{x}+\frac{1}{2}\frac{k}{K}\left(-\frac{1}{x}\right)\right\}\\
&= -\sqrt{\frac{m}{2}}\cdot\frac{1}{\sqrt{\frac{k}{x}-K}}. \qquad \text{Q.E.D.}
\end{aligned}
$$

3) Then by taking the limit $x_1 \to 0$,

$$\begin{aligned}
T &= \lim_{x_1 \to 0} t(x_1) \\
&= 0 - \frac{1}{K}\sqrt{\frac{m}{2}} \cdot x_0 \sqrt{\frac{k}{x_0} - K} \\
&\qquad + \frac{k}{K\sqrt{K}}\sqrt{\frac{m}{2}}\left\{\frac{\pi}{2} - \ \text{Arctan}\left(\sqrt{\frac{k}{K}\cdot\frac{1}{x_0} - 1}\right)\right\}.
\end{aligned}$$

Since Arctan x+ Arccot $x = \dfrac{\pi}{2}$, we reduce to

$$T = \frac{k}{K\sqrt{K}}\sqrt{\frac{m}{2}}\,\text{Arccot}\left(\sqrt{\frac{k}{K}\,\frac{1}{x_0} - 1}\right) - \frac{1}{K}\sqrt{\frac{m}{2}} \cdot \sqrt{kx_0 - Kx_0^2}.$$

4) When $t = 0$, it follows from (7) that

$$K = -\frac{1}{2}\,m\,\{x'(0)\}^2 + \frac{k}{x_0} > 0,$$

i.e.

$$Kx_0 = -\frac{1}{2}\,m\,x_0\{x'(0)\}^2 + k \quad \text{and} \quad x'(0) = -\sqrt{\frac{2}{m}}\sqrt{\frac{k}{x_0} - K},$$

where we have assumed that $x'(0) < 0$. Hence by insertion,

$$\begin{aligned}
T &= \frac{k}{\left\{\dfrac{k}{x_0} - \dfrac{m}{2}[x'(0)]^2\right\}^{\frac{3}{2}}}\sqrt{\frac{m}{2}} \cdot \text{Arccot}\left(\sqrt{\frac{k}{k - \dfrac{m}{2}\{x'(0)\}^2 x_0} - 1}\right) \\
&\qquad - \frac{1}{\dfrac{k}{x_0} - \dfrac{m}{2}\{x'(0)\}^2}\sqrt{\frac{m}{2}} \cdot \sqrt{kx_0 - kx_0 + \frac{1}{2}\,m\,x_0\{x'(0)\}^2},
\end{aligned}$$

which can be further reduced. However, there is no need to do this because we directly get for $x'(0) \to 0$,

$$\begin{aligned}
T &= \frac{k}{\left\{\dfrac{k}{x_0}\right\}^{\frac{3}{2}}}\sqrt{\frac{m}{2}} \cdot \text{Arccot}\left(\sqrt{\frac{k}{k} - 1}\right) - 0 \\
&= \frac{1}{\sqrt{k}}\,x_0^{\frac{3}{2}}\sqrt{\frac{m}{2}} \cdot \frac{\pi}{2} \\
&= \frac{\pi}{2} \cdot \sqrt{\frac{m}{2k}}\,x_0\sqrt{x_0}.
\end{aligned}$$

Example 9.3 *The gravity on a particle of mass m at the distance x from the centre of the Earth is given by $K \cdot \dfrac{m}{x^2}$, where $K = GM$ is the product of the constant of gravity and the mass of the Earth. What is the work needed to move the particle from the surface of the Earth to infinity? (Work = force $\cdot$ path).*

A. Mathematical model of the law of gravity.

D. First find the infinitesimal work from x to $x + \Delta x$.

I. Let R denote the radius of the Earth. There is no need to specify this further, since neither G nor M are specified by numbers. Assume that the particle m is at distance $x \geq R$ from the centre of the Earth.

The work needed to take the particle from x to $x + \Delta x$ is then

$$\Delta A \approx K \cdot \frac{m}{x^2} \cdot \Delta x, \quad \text{dvs.} \quad dA = \frac{Km}{x^2}\,dx,$$

from which

$$A(x) = \int_R^x \frac{Km}{\xi^2}\,d\xi = \left[-\frac{Km}{\xi}\right]_R^x = Km\left(\frac{1}{R} - \frac{1}{x}\right) \to \frac{Km}{R}$$

for $x \to \infty$.

Example 9.4 *Consider a particle, which falls towards the surface of the Earth from a height, only under the influence of the gravity. We shall in this example derive a formula for the time T it takes for the particle to hit the surface of the Earth.*

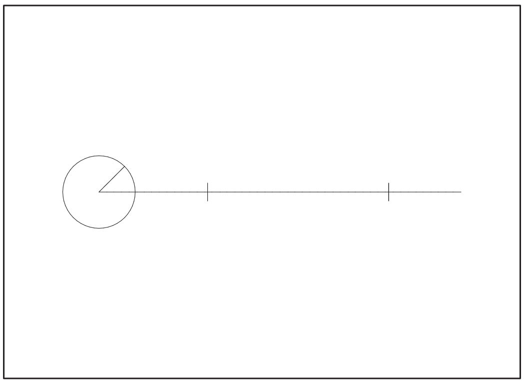


Figure 25: A particle falls from H (the rightmost point) towards the Earth (the circle).

Denote by H the distance from the centre 0 of the Earth to the place, from which the particle starts falling, and denote by R the radius of the Earth. Let x denote the actual distance from the point 0 to the particle. Then it can be shown by Newton's laws that

$$T = \lim_{y \to H} \int_y^R \frac{-1}{\sqrt{2gR^2}\sqrt{\dfrac{1}{x} - \dfrac{1}{H}}}\, dx.$$

We shall here take this for granted.

1) Calculate the integral

$$\int \frac{x}{\sqrt{Hx - x^2}}\, dx, \qquad 0 < x < H,$$

by introducing the substitution $x = \dfrac{H}{2} + \dfrac{H}{2}\cos u$.

2) Prove by means of this that

$$T = \frac{1}{R}\sqrt{\frac{H}{2g}}\left\{\sqrt{RH - R^2} + \frac{H}{2}\,\text{Arccos}\left(\frac{2R-H}{H}\right)\right\}.$$

A. By analyzing the formulation we see that the task can be reduced to

1. an integration,

2a. a rearrangement,

2b. a limit.

Some guidelines are given.

D. Follow the guidelines.

I. 1) When $x \in]0,H[$ we see that

$$x = \frac{H}{2} + \frac{H}{2}\,\cos u = \frac{H}{2}\,(1+\cos u), \qquad u \in \left]-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right[,$$

is a monotonous substitution with $dx = -\dfrac{H}{2}\,\sin u\,du$, $\sin u > 0$. The inverse function is given by

$$u = \operatorname{Arccos}\left(\frac{2x}{H}-1\right).$$

Then by insertion

$$\begin{aligned}
\int \frac{x}{\sqrt{Hx-x^2}}\,dx &= \int \frac{x}{\sqrt{x(H-x)}}\,dx \\
&= \int_{u=\operatorname{Arccos}\left(\frac{2x}{H}-1\right)} \frac{\frac{H}{2}\,(1+\cos u)\cdot\left(-\frac{H}{2}\,\sin u\right)}{\sqrt{\frac{H}{2}\,(1+\cos u)\cdot\frac{H}{2}\,(1-\cos u)}}\,du \\
&= -\int_{u=\operatorname{Arccos}\left(\frac{2x}{H}-1\right)} \frac{\left(\frac{H}{2}\right)^2(1+\cos u)\sin u}{\frac{H}{2}\sqrt{1-\cos^2 u}}\,du \\
&= -\frac{H}{2}\int_{u=\operatorname{Arccos}\left(\frac{2x}{H}-1\right)} \frac{(1+\cos u)\sin u}{+\sin u}\,du \\
&= -\frac{H}{2}\left\{\operatorname{Arccos}\left(\frac{2x}{H}-1\right)+\sin\left(\operatorname{Arccos}\left(\frac{2x}{H}-1\right)\right)\right\} \\
&= -\frac{H}{2}\left\{\operatorname{Arccos}\left(\frac{2x}{H}-1\right)+\sqrt{1-\cos^2\left(\operatorname{Arccos}\left(\frac{2x}{H}-1\right)\right)}\right\} \\
&= -\frac{H}{2}\left\{\operatorname{Arccos}\left(\frac{2x}{H}-1\right)+\sqrt{1-\left(\frac{2x}{H}-1\right)^2}\right\} \\
&= -\frac{H}{2}\left\{\sqrt{\frac{2x}{H}\left(2-\frac{2x}{H}\right)}+\operatorname{Arccos}\left(\frac{2x}{H}-1\right)\right\} \\
&= -\left\{\sqrt{x(H-x)}+\operatorname{Arccos}\left(\frac{2x}{H}-1\right)\right\}.
\end{aligned}$$

C. Test. By differentiation we get

$$
\begin{aligned}
&\frac{d}{dx}\left\{-\frac{H}{2}\left[\frac{2}{H}\sqrt{x(H-x)}+\operatorname{Arccos}\left(\frac{2x}{H}-1\right)\right]\right\}\\
&= -\frac{H}{2}\left\{\frac{2}{H}\cdot\frac{1}{2}\cdot\frac{H-2x}{\sqrt{x(H-x)}}-\frac{1}{\sqrt{1-\left(\frac{2x}{H}-1\right)^2}}\cdot\frac{2}{H}\right\}\\
&= -\frac{1}{2}\cdot\frac{H-2x}{\sqrt{x(H-x)}}+\frac{1}{\sqrt{\frac{2x}{H}\left(2-\frac{2x}{H}\right)}}\\
&= \frac{x}{\sqrt{Hx-x^2}}-\frac{H}{2}\cdot\frac{1}{\sqrt{x(H-x)}}+\frac{H}{2}\cdot\frac{1}{\sqrt{x(H-x)}}\\
&= \frac{x}{\sqrt{Hx-x^2}}. \qquad \text{Q.E.D.}
\end{aligned}
$$

2) By a rearrangement and an insertion of the result from (1) we get

$$\begin{aligned}
T &= \lim_{y\to H}\int_y^R \frac{-1}{\sqrt{2gR^2}\cdot\sqrt{\dfrac{1}{x}-\dfrac{1}{H}}}\,dx \\
&= \frac{1}{\sqrt{2gR^2}}\lim_{y\to H}\int_y^R \frac{-1}{\sqrt{\dfrac{H-x}{xH}}}\,dx \\
&= \frac{\sqrt{H}}{\sqrt{2g}\,R}\lim_{y\to H}\int_y^R \frac{-x}{\sqrt{Hx-x^2}}\,dx \\
&= \frac{1}{R}\sqrt{\frac{H}{2g}}\lim_{y\to H}\left[\sqrt{Hx-x^2}+\frac{H}{2}\operatorname{Arccos}\left(\frac{2x}{H}-1\right)\right]_{x=y}^{x=R} \\
&= \frac{1}{R}\sqrt{\frac{H}{2g}}\left\{\sqrt{HR-R^2}+\frac{H}{2}\operatorname{Arccos}\left(\frac{2R-H}{H}\right)\right\} \\
&\qquad -\frac{1}{R}\sqrt{\frac{H}{2g}}\lim_{y\to H}\left\{\sqrt{y(H-y)}+\frac{H}{2}\operatorname{Arcos}\left(\frac{2y}{H}-1\right)\right\} \\
&= \frac{1}{R}\sqrt{\frac{H}{2g}}\left\{\sqrt{HR-R^2}+\frac{H}{2}\operatorname{Arccos}\left(\frac{2R-H}{H}\right)\right. \\
&\qquad \left. -\frac{1}{R}\sqrt{\frac{H}{2g}}\cdot\frac{H}{2}\operatorname{Arccos}1\right\} \\
&= \frac{1}{R}\sqrt{\frac{H}{2g}}\left\{\sqrt{HR-R^2}+\frac{H}{2}\operatorname{Arccos}\left(\frac{2R-H}{H}\right)\right\}.
\end{aligned}$$

Example 9.5 *"Trees do not grow into the sky." This statement illustrates one of the problems of the description of the growth process of a closed ecological system. The simplest forms of growth like linear and exponential growth can only to some extent describe the actual conditions.*

*Let us consider a limited domain. The number of individuals who can live on this domain is called B (*the carrying capacity *of the domain). The* rate of birth *F is the fraction by which a given number of individuals is reproducing per time unit. The* rate of death *D is the fraction which dies per time unit. (If for instance $D = 5\ \% = 0,05$ per time unit, then 100 individuals will be reduced by 5 per time unit.)*

We shall describe the task of how the number of individuals varies in time in such a domain. We shall start by making precise the assumptions which our analysis is based on.

Let us assume that the rate of birth F is a constant, while the rate of death $D = D(t)$ depends on both the time and the number of individuals $N = N(t)$ to time t. More precisely, we assume that

(8) $D(t) = k\cdot N(t), \qquad k$ *constant.*

We shall furthermore assume that the situation in the domain is stable, which means that

$$(9)\quad \begin{cases} N(t)\to B & \text{for } t\to+\infty,\\ D(t)\to F & \text{for } t\to+\infty.\end{cases}$$

Finally, assume that $N(t)$ is a differentiable function (which therefore not always takes on natural numbers as values), and *that the change ΔN in a small time interval Δt is given by*

(10) $\Delta N = F \cdot \Delta t \cdot N - D \cdot \Delta t \cdot N.$

1) Prove that $k = \dfrac{F}{B}$.

2) Explain the contents of the assumption (10) and prove that $N(t)$ satisfies

$$(11)\quad \frac{dN}{dt} = N\left\{F - \frac{F}{B}\,N\right\}.$$

3) Solve the differential equation

$$(12)\quad \frac{dx}{dt} = x(b - ax), \qquad 0 < x < \frac{b}{a}, \quad t \geq 0, \quad a,\, b \in \mathbb{R}_+,$$

by separating the variables.

4) Prove that if $N_0 < B$, then

$$N(t) = \frac{B}{1 + \left(\dfrac{B}{N_0} - 1\right)e^{-Ft}}, \qquad t \geq 0.$$

A. A mathematical model of ecological growth, which can be formulated as a differential equation of first order, in which the variables can be separated.

D. There are given some guidelines. Follow these.

I. 1) It follows from the assumptions (8) and (9) that

$$k = \frac{D(t)}{N(t)} \to \frac{B}{F} \qquad \text{for } t \to +\infty.$$

Since k is a constant through the limit process, we have $k = \dfrac{B}{F}$.

2) The *change* ΔN of the number of population N is equal to the *rate of birth* F per time unit multiplied by the *length* Δt of the *time interval* multiplied by the *number of population* N, from which we shall subtract the analogous number for the decrease of the population, i.e. the *rate of death* D per time unit multiplied by Δt and N.

We get from (10) by dividing by $\Delta t > 0$,

$$\frac{\Delta N}{\Delta t} = F \cdot N(t) - D(t) \cdot N(t) = N\left(F - \frac{F}{B}\,N\right),$$

where we according to (1) have substituted $D = k \cdot N = \dfrac{F}{B} \cdot N$. By the limit $\Delta t \to 0+$ we get (11), i.e.

$$\frac{dN}{dt} = N\left(F - \frac{F}{B}\,N\right).$$

3) A separation of the variables then gives for $x \in \left]0, \frac{b}{a}\right[$,

$$\frac{1}{x(b-ax)}\,\frac{dx}{dt} = 1.$$

By a decomposition we get

$$\frac{1}{x(b-ax)} = \frac{1}{bx} - \frac{1}{b}\cdot\frac{1}{x-\frac{b}{a}},$$

thus by an integration,

$$\begin{aligned} t + c_1 &= \frac{1}{b}\left\{\ln|x| - \ln\left|x - \frac{b}{a}\right|\right\} \\ &= \frac{1}{b}\,\ln\left(\frac{x}{\frac{b}{a}-x}\right) = \frac{1}{b}\,\ln\frac{ax}{b-ax}, \end{aligned}$$

hence

$$\frac{ax}{b-ax} = \frac{b-(b-ax)}{b-ax} = \frac{b}{b-ax} - 1 = c_2 e^{bt}, \quad \text{hvor } c_2 = e^{bc_1},$$

which implies that

$$b - ax = \frac{b}{1 + c_2 e^{bt}}.$$

Putting $c = \frac{1}{c_2} = e^{-bc_1}$, we get

$$x = \frac{b}{a}\left\{1 - \frac{1}{1+c_2 e^{bt}}\right\} = \frac{b}{a} \cdot \frac{c_2 e^{bt}}{1+c_2 e^{bt}} = \frac{b}{a} \cdot \frac{1}{1 + c\,e^{-bt}}.$$

C. Test. When

$$x = \frac{b}{a} \cdot \frac{1}{1+c\,e^{-bt}},$$

we get by insertion,

$$\begin{aligned}
&\frac{dx}{dt} - bx + ax^2 \\
&\quad = \frac{b}{a} \cdot \frac{1}{(1+c+ce^{-bt})^2} \cdot \left(-c(-b)e^{-bt}\right) \\
&\qquad\quad -\frac{b^2}{a} \cdot \frac{1}{1+c\,e^{-bt}} + a \cdot \frac{b^2}{a^2} \cdot \frac{1}{(1+c\,e^{-bt})^2} \\
&\quad = \frac{b^2}{a} \cdot \frac{1}{(1+c\,e^{-bt})^2}\left\{c\,e^{-bt} - (1+c\,e^{-bt}) + 1\right\} = 0. \qquad \text{Q.E.D.}
\end{aligned}$$

4) Let $N(0) = N_0 < B$. If $x = N$ and $b = F$ and $a = \frac{F}{B}$, we see that (11) and (12) are identical. Thus, by insertion into the solution in (3),

$$N(t) = \frac{b}{a} \cdot \frac{1}{1+c\,e^{-bt}} = \frac{F}{\frac{F}{B}} \cdot \frac{1}{1+c\,e^{-Ft}} = \frac{B}{1+c\,e^{-Ft}}.$$

For $t = 0$ we get

$$N(0) = N_0 = \frac{B}{1+c}, \qquad \text{dvs.} \qquad c = \frac{B}{N_0} - 1,$$

hence by insertion,

$$N(t) = \frac{B}{1 + \left(\frac{B}{N_0} - 1\right) e{-}Ft}.$$

Since $N_0 < B$, we have $\frac{B}{N_0} - 1 > 0$, hence $N(t) < B = \frac{b}{a}$, and the domain of the solution is given by $t > 0$.