Examples of Power Series

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Calculus 3c-3

Examples of Power Series – Calculus 3c-3

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Contents

Introduction

Here follows a collection of general examples of *power series*. The reader is also referred to *Calculus 3b*.

The important technique of solving linear differential equations with polynomial coefficients by means of power series is postponed to the next book in this series, *Calculus 3c-4*.

It should no longer be necessary rigourously to use the ADIC-model, described in *Calculus 1c* and *Calculus 2c*, because we now assume that the reader can do this himself.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
14th May 2008

1 Power series; radius of convergence and sum

Example 1.1 *Find the radius of convergence for the power series,*

$$\sum_{n=1}^{\infty} \frac{1}{n^n} x^n.$$

Let $a_n(x) = \dfrac{1}{n^n} x^n$. Then by the **criterion of roots**

$$\sqrt[n]{|a_n(x)|} = \frac{|x|}{n} \to 0 \qquad \text{for } n \to \infty,$$

and the series is convergent for every $x \in \mathbb{R}$, hence the interval of convergence is $\mathbb{R}$.

Example 1.2 *Find the interval of convergence for the power series*

$$\sum_{n=1}^{\infty} \frac{\ln n}{3^n} x^n.$$

Let $a_n(x) = \dfrac{\ln n}{3^n} |x|^n \geq 0$. Then we get by the **criterion of roots**

$$\sqrt[n]{a_n(x)} = \sqrt[n]{\ln n} \cdot \frac{|x|}{3} \to \frac{|x|}{3} \qquad \text{for } n \to \infty.$$

The limit value is < 1, if and only if $x \in]-3,3[$, so the interval of convergence is $]-3,3[$.

Alternatively we may apply the **criterion of quotients**, when $n > 1$ and $x \neq 0$,

$$\frac{a_{n+1}(x)}{a_n(x)} = \frac{\ln(n+1)}{3^{n+1}} \cdot |x|^{n+1} \cdot \frac{3^n}{\ln n} \cdot \frac{1}{|x|^n} = \frac{\ln(n+1)}{\ln n} \cdot \frac{|x|}{3} \to \frac{|x|}{3}$$

for $n \to \infty$, because

$$\frac{\ln(n+1)}{\ln n} = \frac{\ln n + \ln\left(1+\frac{1}{n}\right)}{\ln n} = 1 + \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n} \to 1 \quad \text{for } n \to \infty.$$

Since $\dfrac{|x|}{3} < 1$ for $x \in]-3, 3[$, the interval of convergence is $]-3, 3[$.

Example 1.3 *Find the interval of convergence for the power series*

$$\sum_{n=1}^{\infty} \{1-(-2)^n\}\, x^n.$$

Put $a_n(x) = \{1-(-2)^n\}\, x^n$.

The **criterion of roots** gives the following,

$$\begin{aligned}\sqrt[n]{|a_n(x)|} &= \sqrt[n]{|1+(-1)^{n+1}\cdot 2^n|}\cdot|x| = \sqrt[n]{2^n|1+(-1)^{n+1}\cdot 2^{-n}|}\cdot|x| \\ &= 2\sqrt[n]{1+(-1)^{n+1}\cdot\frac{1}{2^n}}\cdot|x| \to 2|x| \quad \text{for } n \to \infty.\end{aligned}$$

Since $2|x| < 1$ for $|x| < \dfrac{1}{2}$, the interval of convergence is $\left]-\dfrac{1}{2}, \dfrac{1}{2}\right[$.

When $x \neq 0$, then $a_n(x) \neq 0$, so we can apply the **criterion of quotients**

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \left|\frac{1-(-2)^{n+1}}{1-(-2)^n}\right| \cdot |x| = \left|\frac{2^{n+1}\left\{1-(-1)^n\cdot\frac{1}{2^{n+1}}\right\}}{2^n\left\{1-(-1)^{n+1}\cdot\frac{1}{2^n}\right\}}\right| |x| \to 2|x|$$

for $n \to \infty$. Since $2|x| < 1$ for $|x| < \dfrac{1}{2}$, the interval of convergence is $\left]-\dfrac{1}{2}, \dfrac{1}{2}\right[$.

Remark 1.1 One can prove that the sum function is

$$\begin{aligned}\sum_{n=1}^{\infty} \{1-(-2)^n\}\, x^n &= \left\{\frac{1}{1-x}-1\right\} - \left\{\frac{1}{1+2x}-1\right\} = \frac{1}{1-x} - \frac{1}{1+2x} \\ &= \frac{3x}{(1-x)(1+2x)} \qquad \text{for } x \in \left]-\frac{1}{2}, \frac{1}{2}\right[.\end{aligned}$$

Example 1.4 *Find the interval of convergence for the power series*

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^{2n}.$$

Put

$$a_n(x) = \left| \frac{2^n}{n^2} x^{2n} \right| = \frac{2^n}{n^2} x^{2n} \geq 0.$$

1) We get by the **criterion of roots** the condition

$$\sqrt[n]{a_n(x)} = \sqrt[n]{\frac{2^n}{n^2} x^{2n}} = \frac{2x^2}{\left(\sqrt[n]{n}\right)^2} \to 2x^2 < 1 \qquad \text{for } n \to \infty.$$

The interval of convergence is given by $x^2 < \frac{1}{2}$, so it is $\left]-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right[$.

2) If we instead apply the **criterion of quotients**, we must except $x = 0$, because one must never divide by 0. However, the convergence is trivial for $x = 0$. Then we get for $x \neq 0$

$$\frac{a_{n+1}(x)}{a_n(x)} = \frac{2^{n+1}}{(n+1)^2} \cdot x^{2(n+1)} \cdot \frac{n^2}{2^n x^{2n}} = 2x^2 \left(\frac{n}{n+1} \right)^2 \to 2x^2$$

when $n \to \infty$. This limit value is < 1 for $|x| < \frac{1}{\sqrt{2}}$, so the interval of convergence is

$$\left]-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right[.$$

Remark 1.2 The sum function cannot be expressed by known elementary functions. $\Diamond$

Remark 1.3 There also exist some other methods of solution, but since they are rather sophisticated, they are not given here.

Example 1.5 *Find the interval of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)2^n}.$$

We get by the **criterion of roots**

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{\frac{|x|^n}{(n+1)2^n}} = \frac{1}{\sqrt[n]{n+1}} \cdot \frac{|x|}{2} \to \frac{|x|}{2} \qquad \text{for } n \to \infty.$$

Since $\frac{|x|}{2} < 1$ for $x \in]-2, 2[$, the interval of convergence is $]-2, 2[$.

Alternatively, when $x \neq 0$ we get by the **criterion of quotients**,

$$\left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \frac{(n+1) \cdot 2^n}{(n+2) \cdot 2^{n+1}} \cdot \frac{|x|^{n+1}}{|x|^n} = \frac{n+1}{n+2} \cdot \frac{|x|}{2} \to \frac{|x|}{2} \quad \text{for } n \to \infty$$

.

Remark 1.4 It will later be possible to prove that the series is almost a logarithmic series in the interval $0 < |x| < 2$ and with the sum

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{x}{2}\right)^n = \frac{2}{x} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x}{2}\right)^n = -\frac{2}{x} \ln\left(1 - \frac{x}{2}\right),$$

thus

$$f(x) = \begin{cases} \dfrac{2}{x} \ln\left(\dfrac{2}{2-x}\right) & \text{for } 0 < |x| < 2, \\ 1 & \text{for } x = 0. \end{cases}$$

Example 1.6 *Find the interval of convergence for the power series*

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{n} + 1\right)^n x^{3n}.$$

We get by the **criterion of roots**

$$\sqrt[n]{|a_n(x)|} = \left(\sqrt[n]{n} + 1\right) \cdot |x|^3 \to 2|x|^3 \qquad \text{for } n \to \infty.$$

Since $2|x|^3 < 1$ for $|x| < 1/\sqrt[3]{2}$, the interval of convergence becomes

$$\left] -\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} \right[.$$

Alternatively one may *try* to apply the **criterion of quotients** for $x \neq 0$. Then we get the following awkward expression

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{\left(\sqrt[n+1]{n+1} + 1\right)^{n+1}}{\left(\sqrt[n]{n} + 1\right)^n} \cdot |x|^3 = \left\{\frac{\sqrt[n+1]{n+1} + 1}{\sqrt[n]{n} + 1}\right\}^n \cdot \left(\sqrt[n+1]{n+1} + 1\right) \cdot |x|^3.$$

It is difficult, though still possible, to show that this expression tends towards $2|x|^3$ for $n \to \infty$.

Sketch of proof. First rearrange in the following way,

$$\begin{aligned} \frac{\sqrt[n+1]{n+1} + 1}{\sqrt[n]{n} + 1} &= 1 - \frac{\sqrt[n]{n} - \sqrt[n+1]{n+1}}{\sqrt[n]{n} + 1} = 1 - \frac{\exp\left(\dfrac{\ln n}{n}\right) - \exp\left(\dfrac{\ln(n+1)}{n+1}\right)}{\sqrt[n]{n} + 1} \\ &= 1 - \frac{1}{\sqrt[n]{n} + 1} \cdot \frac{\exp\left(\dfrac{\ln n}{n}\right) - \exp\left(\dfrac{\ln(n+1)}{n+1}\right)}{\dfrac{\ln n}{n} - \dfrac{\ln(n+1)}{n+1}} \cdot \left\{\frac{\ln n}{n} - \frac{\ln(n+1)}{n+1}\right\}. \end{aligned}$$

The first factor converges towards $\dfrac{1}{2}$, the second factor converges towards $\left[\dfrac{d}{dt}\exp(t)\right]_{t=0} = 1$. Finally, note that the last factor is $\left(\sim \dfrac{\ln n}{n(n+1)}\right)$, apply Taylor's formula and insert (i.e. take the n-th power). Finally, take the limit. Obviously, this method is far from the easiest one, so in this case one should avoid the **criterion of quotients** and find another possible solution method.

Example 1.7 *Find the interval of convergence for the power series*

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}\, x^n.$$

Put

$$a_n(x) = \frac{n!}{n^n}|x|^n \geq 0, \qquad \text{where } a_n(x) > 0 \quad \text{for } x \neq 0.$$

Note that the convergence is trivial for $x = 0$.

The faculty function occurs, so we are led to choose the **criterion of quotients**. When $x \neq 0$ we have $a_n(x) \neq 0$, so

$$\begin{aligned} \frac{a_{n+1}(x)}{a_n(x)} &= \frac{(n+1)!}{(n+1)^{n+1}}\,|x|^{n+1} \cdot \frac{n^n}{n!|x|^n} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)\cdot(n+1)^n} \cdot |x| \\ &= \frac{n+1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n |x| = \frac{|x|}{\left(1+\dfrac{1}{n}\right)^n} \to \frac{|x|}{e} \qquad \text{for } n \to \infty, \end{aligned}$$

because $\left(1 + \dfrac{1}{n}\right)^n \to e$ for $n \to \infty$.

The condition of convergence becomes $\dfrac{|x|}{e} < 1$, hence $|x| < e$, and the interval of convergence is $I =]-e,e[$.

Remark 1.5 The sum function in $]-e,e[$ *cannot* be expressed by elementary functions.

Example 1.8 *Find the interval of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{x^n}{a^n+b^n}, \qquad a \geq b > 0.$$

Since $a \geq b > 0$, we get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{|x|}{\sqrt[n]{a^n+b^n}} = \frac{1}{a} \cdot \frac{|x|}{\sqrt[n]{1+\left(\dfrac{b}{a}\right)^n}} \to \frac{|x|}{a}.$$

Since $\dfrac{|x|}{a} < 1$ for $|x| < a$, the interval of convergence is $]-a,a[$.

Alternatively, assuming that $x \neq 0$ and thus $a_n(x) \neq 0$,we get by the **criterion of quotients**,

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{a^n+b^n}{a^{n+1}+b^{n+1}} \cdot |x| = \frac{a^n \cdot \left\{1+\left(\dfrac{b}{a}\right)^n\right\}}{a^{n+1} \cdot \left\{1+\left(\dfrac{b}{a}\right)^{n+1}\right\}} \cdot |x| \to \frac{|x|}{a}$$

for $n \to \infty$. In fact, if $a > b > 0$, then $\left(\dfrac{b}{a}\right)^n \to 0$ for $n \to \infty$.
If instead $a = b > 0$, then

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{a^n+a^n}{a^{n+1}+a^{n+1}}\,|x| = \frac{|x|}{a}.$$

Since $\dfrac{|x|}{a} < 1$ for $|x| < a$, the interval of convergence is $]-a,a[$.

Example 1.9 *Find the interval of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n+1}.$$

We get by the **criterion of roots**

$$\sqrt[n]{|a_n(x)|} = \frac{|x|}{\sqrt[n]{2^n+1}} = \frac{|x|}{2} \cdot \frac{1}{\sqrt[n]{1+2^{-n}}} \to \frac{|x|}{2} \qquad \text{for } n \to \infty.$$

Sine $\dfrac{|x|}{2} < 1$ for $|x| < 2$, the interval of convergence is $]-2,2[$.

Alternatively, when $x \neq 0$ we get by the **criterion of quotients**,

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{2^n+1}{2^{n+1}+1}\cdot|x| = \frac{1+2^{-n}}{1+2^{-n-1}}\cdot\frac{|x|}{2} \to \frac{|x|}{2} \qquad \text{for } n\to\infty,$$

and we conclude as above that the interval of convergence is $]-2,2[$.

In this case we may also apply the **criterion of equivalence**. In fact, when $x \neq 0$, then

$$\frac{|x|^n}{2^n+1} \sim \frac{|x|^n}{2^n} = \left(\frac{|x|}{2}\right)^n,$$

and $\sum_{n=0}^{\infty}\left(\frac{|x|}{2}\right)^n$ is convergent, if and only if $x \in]-2,2[$.

Example 1.10 *Find the interval of convergence for the power series*

$$\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}}{2n}x^{2n}.$$

We get by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|} = \frac{2^2x^2}{\sqrt[n]{2n}} \to (2|x|)^2 \qquad \text{for } n\to\infty.$$

Since $(2|x|)^2<1$ for $|x|<\frac{1}{2}$, the interval of convergence $\left]-\frac{1}{2},\frac{1}{2}\right[$.

Alternatively, we get by the **criterion of quotients** for $x\neq 0$,

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{2^{2(n+1)}}{2(n+1)}\cdot|x|^{2(n+1)}\cdot\frac{2n}{2^{2n}}\cdot\frac{1}{|x|^{2n}} = \frac{n}{n+1}\cdot(2|x|)^2 \to (2|x|)^2$$

for $n\to\infty$, and we conclude as above that the interval of convergence $\left]-\frac{1}{2},\frac{1}{2}\right[$.

Remark 1.6 It can be shown later that the sum function is

$$f(x) = \sum_{n=1}^{\infty}\frac{(-1)^n2^{2n}}{2n}x^{2n} = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot(4x^2)^n = -\frac{1}{2}\ln\left(1+4x^2\right)$$

for $x\in\left]-\frac{1}{2},\frac{1}{2}\right[$.

Example 1.11 *Find the interval of convergence for the power series*

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}\, x^n.$$

Put

$$a_n(x) = \frac{(2n)!}{(n!)^2}\, |x|^n \geq 0 \qquad \text{where } a_n(x) > 0 \quad \text{for } x \neq 0.$$

Since the faculty function occurs, we apply the **criterion of quotients**.

When $x = 0$, the series is trivially convergent.
When $x \neq 0$, we get the quotient

$$\begin{aligned}\frac{a_{n+1}(x)}{a_n(x)} &= \frac{(2\{n+1\})!}{(\{n+1\}!)^2}\, |x|^{n+1} \cdot \frac{(n!)^2}{(2n)!} \cdot \frac{1}{|x|^n} = \frac{(2\{n+1\})!}{(2n)!} \cdot \left\{\frac{n!}{(n+1)!}\right\}^2 |x| \\ &= \frac{2(n+1)\cdot(2n+1)}{(n+1)\cdot(n+1)}|x| = \frac{1+\frac{1}{2n}}{1+\frac{1}{n}} \cdot 4|x| \to 4|x| \qquad \text{for } n \to \infty.\end{aligned}$$

According to the **criterion of quotients** the series is convergent for $4|x| < 1$, hence the interval of convergence is $\left]-\frac{1}{4}, \frac{1}{4}\right[$.

Example 1.12 *Find the interval of convergence for the power series*

$$\sum_{n=0}^{\infty} 3^{-n^2} x^n.$$

It follows by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{3^{-n^2}|x|^n} = \frac{|x|}{3^n} \to 0 \quad \text{for alle } x \in \mathbb{R}, \text{ når } n \to \infty.$$

Hence, the interval of convergence is $\mathbb{R}$.

Alternatively if follows by the **criterion of quotients** for $x \neq 0$ that

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{|x|^{n+1}}{3^{+(n+1)^2}} \cdot \frac{3^{+n^2}}{|x|^n} = \frac{|x|}{3^{2n+1}} \to 0 \qquad \text{for alle } x \in \mathbb{R}, \text{ når } n \to \infty.$$

We conclude again that the interval of convergence is $\mathbb{R}$.

Remark 1.7 The sum function of the series cannot be expressed by elementary functions.

Example 1.13 *Find the radius of convergence for the power series*

$$\sum_{n=1}^{\infty} n x^{2n},$$

and check if the series is absolutely convergent, conditionally convergent or divergent at the endpoints of the interval of convergence.

It follows by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{n} \cdot x^2 \to x^2 \qquad \text{for } n \to \infty.$$

As $x^2 < 1$ for $x \in]-1, 1[$, the radius of convergence is $\varrho = 1$.

Alternatively it follows by the **criterion of quotients** for $x \neq 0$ that

$$\left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \frac{n+1}{n} \cdot |x| = \left(1 + \frac{1}{n} \right) |x| \to |x| \qquad \text{for } n \to \infty.$$

Hence the interval of convergence is given by $|x| < 1$, so $\varrho = 1$.

Let r $x = \pm 1$ be anyone of the endpoints. Then

$$|a_n(\pm 1)| = n \to \infty \qquad \text{for } n \to \infty.$$

The **necessary condition** for convergence is *not* fulfilled, so the series is coarsely divergent at the endpoints of the interval of convergence.

Remark 1.8 The sum function of this series in $]-1,1[$ is found by the following argument: When $y \in]-1,1[$, then

$$\sum_{n=1}^{\infty} n y^n = y \sum_{n=1}^{\infty} n y^{n-1} = y \frac{d}{dy}\left(\sum_{n=0}^{\infty} y^n\right) = y \frac{d}{dy}\left(\frac{1}{1-y}\right) = \frac{y}{(1-y)^2}.$$

Putting $y = x^2$, $x \in]-1,1[$, we get

$$f(x) = \sum_{n=1}^{\infty} n x^{2n} = \sum_{n=1}^{\infty} n\left(x^2\right)^n = \frac{x^2}{\left(1-x^2\right)^2}.$$

Example 1.14 *Find the radius of convergence for the power series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n}\, x^n,$$

and check if the series is absolutely convergent, conditionally convergent or divergent at the endpoints of the interval of convergence.

It follows by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|} = \frac{1}{\sqrt[n]{n}} \cdot \frac{|x|}{5} \to \frac{|x|}{5} \qquad \text{for } n \to \infty.$$

Now, $\dfrac{|x|}{5} < 1$ for $|x| < 5$, thus $\varrho = 5$.

Alternatively it follows by the **criterion of quotients** for $x \neq 0$ that

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{|x|^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n \cdot 5^n}{|x|^n} = \frac{n}{n+1} \cdot \frac{|x|}{5} \to \frac{|x|}{5} \quad \text{for } n \to \infty,$$

and we conclude as above that $\varrho = 5$.

If $x = 5$, then we get the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$. This series is *alternating*, and since $\dfrac{1}{n} \to 0$ is *decreasing*, the series is (conditionally) convergent by **Leibniz's criterion**. *Conditionally convergent*, because the numerical series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent (then harmonic series).

If $x = -5$, then we get the *divergent* series $-\sum_{n=1}^{\infty} \dfrac{1}{n}$, and the series is *divergent* at $x = -5$.

Remark 1.9 One can prove that for $x \in]-5,5[$, the sum function is

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{x}{5}\right)^n = \ln\left(1+\frac{x}{5}\right).$$

Onw can also prove by applying *Abel's theorem* that if $x=5$, then

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \lim_{x\to 5-} \ln\left(1+\frac{x}{5}\right) = \ln 2.$$

Example 1.15 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{x^n}{1+n^3},$$

and check if the series is absolutely convergent, conditionally convergent or divergent at the endpoints of the interval of convergence.

It follows by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|} = \frac{|x|}{\sqrt[n]{1+n^3}} = \frac{|x|}{\left(\sqrt[n]{n}\right)^3 \sqrt[n]{1+\dfrac{1}{n^3}}} \to |x| \qquad \text{for } n\to\infty.$$

Thus, the condition for convergence is $|x|<1$, so $\varrho = 1$.

Alternatively it follows by the **criterion of quotients** for $x \neq 0$ that

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{|x|^{n+1}}{1+(n+1)^3}\cdot\frac{1+n^3}{|x|^n} = \frac{1+\dfrac{1}{n^3}}{\left(1+\dfrac{1}{n}\right)^3+\dfrac{1}{n^3}}\cdot|x| \to |x| \qquad \text{for } n\to\infty,$$

and we conclude as above that $\varrho = 1$.

Then consider the endpoints $x=\pm 1$. Using the **criterion of equivalence** we get

$$\sum_{n=0}^{\infty} \frac{1}{1+n^3}|\pm 1|^n = \sum_{n=0}^{\infty} \frac{1}{1+n^3} \sim \sum_{n=0}^{\infty} \frac{1}{n^3},$$

which is convergent, because the exponent in the denominator is $3>1$. Hence, it follows that the series is *absolutely convergent* at the endpoints of the interval of convergence.

Remark 1.10 One can prove that the sum function cannot be expressed by elementary functions.

Example 1.16 *Find the radius of convergence for the power series*

$$\sum_{n=1}^{\infty} \frac{n(n+2)}{1+(n+2)^3}\, x^n,$$

and check if the series is absolutely convergent, conditionally convergent or divergent at the endpoints of the interval of convergence.

The **criterion of equivalence**. Since $\dfrac{n(n+2)}{1+(n+2)^3} \sim \dfrac{1}{n}$, and $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$ has the radius of convergence $\varrho = 1$, we conclude that we also have $\varrho = 1$ for the given series.

Alternatively it follows by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{\frac{n(n+2)}{1+(n+2)^3}} \cdot |x| = \frac{\sqrt[n]{1+\dfrac{2}{n}}}{\sqrt[n]{n} \cdot \sqrt[n]{\left(1+\dfrac{2}{n}\right)^2 + 1}} \cdot |x| \to |x|$$

for $n \to \infty$, hence $\varrho = 1$.

Alternatively it follows by the **criterion of quotients**, when $x \neq 0$ that

$$\begin{aligned} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| &= \frac{(n+1)(n+3)}{1+(n+3)^3} \cdot \frac{1+(n+2)^3}{n(n+2)} \cdot |x| \\ &= \frac{1+(n+2)^3}{1+(n+3)^3} \cdot \frac{(n+1)(n+3)}{n(n+3)} \cdot |x| \to |x| \quad \text{for } n \to \infty, \end{aligned}$$

and we conclude that $\varrho = 1$.

Then we check the endpoints. Since $\dfrac{n(n+2)}{1+(n+2)^3} \sim \dfrac{1}{n}$, and $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent, the series is *divergent* at the point $x = 1$, and we cannot have absolute convergence at the point $x = -1$.

At the endpoint $x = -1$ we get the *alternating* series $\sum_{n=1}^{\infty} \dfrac{n(n+2)}{1+(n+2)^3}\,(-1)^n$. If we delete the change of sign $(-1)^n$, we see that the *inverse* of the remainder

$$\left\{\frac{n(n+2)}{1+(n+2)^3}\right\}^{-1} = \frac{n^3+6n^2+12n+9}{n^2+2n} = n + 4 + \frac{9}{2} \cdot \frac{1}{n} - \frac{1}{2} \cdot \frac{1}{n+2}$$

tends *increasingly* towards ∞ for $n \geq N_0$ and $n \to \infty$, hence $\dfrac{n(n+2)}{1+(n+2)^3} \to 0$ is *decreasing* eventually. Then it follows by **Leibniz's criterion** that the series is (conditionally) convergent for $x = -1$.

Example 1.17 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} \left\{ e^n \ln(3n+7) \right\} x^n,$$

and check if the series is absolutely convergent, conditionally convergent or divergent at the endpoints of the interval of convergence.

It follows by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|} = e\,\sqrt[n]{\ln(3n+7)} \cdot |x| \to e|x| \qquad \text{for } n \to \infty,$$

thus the condition of convergence $e|x| < 1$ is fulfilled for $|x| < \dfrac{1}{e}$, thus $\varrho = \dfrac{1}{e}$.

Alternatively we get by the **criterion of quotients** for $x \neq 0$ the following calculations,

$$\left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \frac{e^{n+1} \cdot \ln(3n+10)}{e^n \cdot \ln(3n+7)} \cdot |x| = \frac{\ln n + \ln\left(3 + \dfrac{10}{n}\right)}{\ln n + \ln\left(3 + \dfrac{7}{n}\right)} \cdot e|x| \to e|x| \qquad \text{for } n \to \infty,$$

and we conclude as above that $\varrho = \dfrac{1}{e}$.

At the *endpoints* $x = \pm\frac{1}{e}$ we have

$$\left|a_n\left(\pm\frac{1}{e}\right)\right| = \ln(3n+7) \to \infty \qquad \text{for } n \to \infty,$$

and the **necessary condition** for convergence is *not* satisfied.
The series is *coarsely divergent* at both endpoints.

Example 1.18 *Find the radius of convergence for the power series*

$$\sum_{n=1}^{\infty} \frac{x^{4n}}{n(n+1)}.$$

Check if the series is absolutely convergent, conditionally convergent or divergent at the endpoints of the interval of convergence.

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First solution. Breadth of view.
We have according to the laws of magnitudes that $\left|\frac{x^{4n}}{n(n+1)}\right| \to \infty$ for $|x| > 1$ and $n \to \infty$, so the series is coarsely divergent for $|x| > 1$, and we conclude that $\varrho \le 1$.
On the other hand, if $|x| \le 1$, then the series has a convergent *majoring series*

$$0 \le \sum_{n=1}^{\infty} \left|\frac{x^{4n}}{n(n+1)}\right| \le \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad [= 1] \quad \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

hence

1) $\varrho \ge 1$, thus $\varrho = 1$, since also $\varrho \le 1$.

2) The series is absolutely convergent at the endpoints of the interval.

3) The series is also *uniformly convergent* in the interval $[-1, 1]$.

Second solution. The criterion of roots.
If we put $a_n(x) = \left|\frac{x^{4n}}{n(n+1)}\right| = \frac{x^{4n}}{n(n+1)}$, then

$$\sqrt[n]{a_n(x)} = \frac{x^4}{\sqrt[n]{n(n+1)}} \to x^4 \qquad \text{for } n \to \infty.$$

Since $x^4 < 1$ for $|x| < 1$, the radius of convergence is $\varrho = 1$.
We find at the *endpoints* $x = \pm 1$ that $\left|\frac{x^{4n}}{n(n+1)}\right| = \frac{1}{n(n+1)}$. Since the sequence of segments is given by

$$s_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left\{\frac{1}{n} - \frac{1}{n+1}\right\} = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=2}^{N+1} \frac{1}{n} = 1 - \frac{1}{N+1},$$

the series is *absolutely convergent* at the endpoints of the interval, and the sum is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} s_N = 1.$$

Third solution. The criterion of quotients.
Put again $a_n(x) = \left|\frac{x^{4n}}{n(n+1)}\right| = \frac{x^{4n}}{n(n+1)}$. Then $a_n(x) > 0$ for $x \neq 0$, and [still for $x \neq 0$]

$$\frac{a_{n+1}(x)}{a_n(x)} = \frac{x^{4(n+1)}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^{4n}} = \frac{n}{n+2} \cdot x^4 \to x^4 \quad \text{for } n \to \infty.$$

Since $x^4 < 1$ for $|x| < 1$, the radius of convergence is $\varrho = 1$.
Then continue as in the **second solution**.

Remark 1.11 It is not difficult to find the sum function. First note that $f(0) = 0$ and that whenever $0 < |x| < 1$ then

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{x^{4n}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n}(x^4)^n - \sum_{n=1}^{\infty} \frac{1}{n+1}\left(x^4\right)^n \qquad \text{all series have } \varrho = 1 \\
&= \sum_{n=1}^{\infty} \frac{1}{n}\left(x^4\right)^n - \sum_{n=2}^{\infty} \frac{1}{n}\left(x^4\right)^{n-1} \qquad \text{change of indices } n \mapsto n-1 \\
&= \sum_{n=1}^{\infty} \frac{1}{n}\left(x^4\right)^n - \frac{1}{x^4}\left\{\sum_{n=1}^{\infty} \frac{1}{n}\left(x^4\right)^n - x^4\right\} \qquad \text{add and subtract} \\
&= 1 + \left(1 - \frac{1}{x^4}\right)\sum_{n=1}^{\infty}\left(x^4\right)^n \qquad \text{collecting the terms} \\
&= 1 + \frac{x^4-1}{x^4}\left\{\ln\left(\frac{1}{1-x^4}\right)\right\} \qquad \text{recognize the series of logarithm} \\
&= 1 + \frac{1-x^4}{x^4}\ln(1-x^4),
\end{aligned}$$

hence

$$f(x) = \begin{cases} 0 & \text{for } x = 0, \\[2mm] 1 + \dfrac{1-x^4}{x^4}\ln(1-x^4) & \text{for } 0 < |x| < 1, \\[2mm] 1 & \text{for } x = \pm 1. \end{cases}$$

Example 1.19 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{n+4},$$

and check if the series is absolutely convergent, conditionally convergent or divergent at the endpoints of the interval of convergence.

It follows by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|} = \frac{|x|^3}{\sqrt[n]{n+4}} \to |x|^3 \qquad \text{for } n \to \infty,$$

where the condition $|x|^3 < 1$ gives the radius of convergence $\varrho = 1$.

Alternatively we get by the **criterion of quotients** for $x \neq 0$ that

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{|x|^{3n+3}}{n+5} \cdot \frac{n+4}{|x|^{3n}} = \frac{n+4}{n+5} \cdot |x|^3 \to |x|^3 \qquad \text{for } n \to \infty,$$

and we conclude as above that $\varrho = 1$.

At the *endpoint* $x = 1$ the series $\sum_{n=0}^{\infty} \dfrac{1}{n+4} = \sum_{n=4}^{\infty} \dfrac{1}{n}$ is clearly *divergent*.

At the *endpoint* $x = -1$ we get the series $\sum_{n=0}^{\infty} \dfrac{(-1)^{3n}}{n+4} = \sum_{n=4}^{\infty} \dfrac{(-1)^n}{n}$. It is well-known that this series is *conditionally convergent*. (Apply **Leibniz's criterion**.)

Remark 1.12 When $0 < |x| < 1$ the sum function is

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{x^{3n}}{n+4} = \sum_{n=4}^{\infty} \frac{x^{3(n-4)}}{n} = \frac{1}{x^4} \sum_{n=4}^{\infty} \frac{1}{n} \left(x^3\right)^n \\
&= \frac{1}{x^4} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \left(x^3\right)^n - \frac{1}{1} x^3 - \frac{1}{2} x^6 - \frac{1}{3} x^9 \right\} \qquad \text{(læg til og træk fra)} \\
&= -\frac{1}{x^4} \ln\left(1 - x^3\right) - \frac{1}{x} - \frac{1}{2} x^2 - \frac{1}{3} x^5.
\end{aligned}$$

This expression does not make sense for $x = 0$. If we, however, insert $x = 0$ directly into the series, we get $f(0) = \dfrac{1}{4}$.

Example 1.20 *Find the radius of convergence for the power series*

$$\sum_{n=1}^{\infty}\left\{2^n n^{-2}\ln(n+2)\right\}x^n,$$

and check if the series is absolutely convergent, conditionally convergent or divergent at the endpoint of the interval of convergence.

It follows by the **criterion of roots** that

$$\sqrt[n]{|a_n(x)|}=2\cdot\frac{1}{\left(\sqrt[n]{n}\right)^2}\cdot\sqrt[n]{\ln(n+2)}\cdot|x|\to 2|x|\qquad\text{for } n\to\infty,$$

thus the condition $2|x|<1$ shows that the radius of convergence is $\varrho=\dfrac{1}{2}$.

Alternatively it follows by the **criterion of quotients** for $x\neq 0$ that

$$\begin{aligned}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| &= \frac{2^{n+1}}{(n+1)^2}\cdot\ln(n+3)\cdot|x|^{n+1}\cdot\frac{n^2}{2^n}\cdot\frac{1}{\ln(n+2)}\cdot\frac{1}{|x|^n}\\ &= \left(\frac{n}{n+1}\right)^2\cdot\frac{\ln(n+3)}{\ln(n+2)}\cdot|x|\cdot 2\to 2|x| \text{ for } n\to\infty,\end{aligned}$$

and we conclude as above that $\varrho=\dfrac{1}{2}$.

At the *endpoints* $x = \pm\dfrac{1}{2}$ of the interval of convergence we get the series

$$\sum_{n=1}^{\infty} |a_n(x)| = \sum_{n=1}^{\infty} \frac{\ln(n+2)}{n^2}.$$

Now

$$0 < \frac{\ln(n+2)}{n^2} = \frac{\ln(n+2)}{\sqrt{n}} \cdot \frac{1}{n^{3/n}} \leq \frac{c}{n^{3/2}},$$

and $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is convergent, because $\dfrac{3}{2} > 1$. Hence the series is *absolutely convergent* at both endpoint of the interval of convergence.

Example 1.21 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\, x^{2n},$$

and find its sum for each $x \in]-\varrho, \varrho[$.

Here we have several variants.

1) The **shortest version** is the following:
   For $x = 0$ the sum is 1. For $x \neq 0$ we get by a rearrangement and comparing with the logarithmic series that

   $$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\, x^{2n} = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x^2)^{n+1} = \frac{1}{x^2}\, \ln(1+x^2)$$

   for $x^2 < 1$, i.e. for $|x| < 1$, so $\varrho = 1$. The sum function is

   $$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\, x^{2n} = \begin{cases} \dfrac{1}{x^2}\ln(1+x^2) & \text{for } 0 < |x| < 1, \\[2mm] 1 & \text{for } x = 0. \end{cases}$$

2) A more traditional proof is directly to prove that $\varrho = 1$.
   a) An application of the **criterion of roots** gives

   $$\sqrt[n]{|a_n(x)|} = \frac{\sqrt[n]{x^{2n}}}{\sqrt[n]{n+1}} = \frac{x^2}{\sqrt[n]{n+1}} \to x^2 \qquad \text{for } n \to \infty.$$

   The condition of convergence is $x^2 < 1$, thus $|x| < 1$, and we see that $\varrho = 1$.
   b) We get by the **criterion of quotients** for $x \neq 0$ that $a_n(x) \neq 0$ and

   $$\left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \frac{n+1}{n+2} \cdot \frac{x^{2n+2}}{x^{2n}} = \frac{n+1}{n+2} \cdot x^2 \to x^2 \quad \text{for } n \to \infty.$$

   The condition of convergence becomes $x^2 < 1$, thus $|x| < 1$ and $\varrho = 1$.

c) An application of the **criterion of comparison** shows that

$$\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{n+1}\,x^{2n}\right| \le \sum_{n=0}^{\infty} x^{2n} \le \sum_{n=0}^{\infty}|x|^n < \infty \quad \text{for } |x|<1,$$

hence $\varrho \ge 1$.

On the other hand, if $|x| > 1$, then it follows by the rules of magnitudes that $|a_n(x)| = \dfrac{1}{n+1}|x|^{2n} \to \infty$, and the necessary condition of convergence is not fulfilled, so $\varrho \le 1$. We conclude that $\varrho = 1$.

Example 1.22 *Given the power series*

$$\sum_{n=1}^{\infty}(n+1)x^n.$$

Find its interval of convergence and its sum.

First variant. It is well-known that

$$\sum_{n=1}^{\infty}(n+1)x^n = \sum_{n=2}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} - 1 = \frac{1}{(1-x)^2} - 1 \quad \text{for } |x|<1.$$

Second variant We get by e.g. the *criterion of roots* that

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{n+1}\cdot|x| \to |x| \quad \text{for } n\to\infty.$$

The condition of convergence becomes $|x|<1$ so the interval of convergence is $]-1,1[$.

Then we get by integrating each term in $]-1,1[$ that

$$\int_0^x f(t)\,dt = \sum_{n=1}^{\infty}\int_0^x (n+1)t^n dt = \sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x}.$$

The sum function is then obtained by a differentiation,

$$f(x) = \frac{d}{dx}\left(\frac{x^2-1+1}{1-x}\right) = \frac{d}{dx}\left\{-x-1+\frac{1}{1-x}\right\} = \frac{1}{(1-x)^2} - 1,$$

which we write as

$$f(x) = \frac{1-(1-x)^2}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2} \quad \text{for } x \in]-1,1[.$$

There are of course other variants.

Example 1.23 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1},$$

and find its sum function for each $x\,]-\varrho,\varrho[$.

If we e.g. apply the **criterion of roots**, then

$$\sqrt[n]{|a_n(x)|} = \frac{|x|}{\sqrt[n]{n+1}} \to |x| \qquad \text{for } n \to \infty.$$

The condition of convergence is $|x| < 1$, so $\varrho = 1$.

The polynomial of first degree in the denominator indicates that the logarithmic function must enter somewhere in the sum function.

1) If $x = 0$ then we of course get $f(0) = 1$.

2) If $0 < |x| < 1$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n} = -\frac{1}{x}\ln(1-x).$$

We conclude that the sum function is

$$f(x) = \begin{cases} 1, & \text{for } x = 0, \\ -\dfrac{1}{x}\ln(1-x), & \text{for } 0 < |x| < 1. \end{cases}$$

Example 1.24 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{n+1}{n!}\, x^n,$$

and find its sum function.

We get by formal calculations that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{n+1}{n!}\, x^2 = \sum_{n=1}^{\infty} \frac{n}{n!}\, x^n + \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\, x^n + \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n \\ &= x\sum_{n=0}^{\infty} \frac{1}{n!}\, x^n + e^x = (x+1)e^x. \end{aligned}$$

The exponential series is convergent in $\mathbb{R}$, hence these calculations are legal, and the interval of convergence is $\mathbb{R}$, and $\varrho = \infty$.

Alternatively we get for $x \neq 0$ that

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{n+2}{(n+1)!}|x|^{n+1} \cdot \frac{n!}{n+1} \cdot \frac{1}{|x|^n} = \frac{n+2}{(n+1)^2}\,|x| \to 0 \qquad \text{for } n \to \infty.$$

It follows from the **criterion of quotients** that $\varrho = \infty$.

When each term is integrated, it then follows that

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{1}{n!}\,x^{n+1} = x\sum_{n=0}^{\infty} \frac{1}{n!}\,x^n = xe^x.$$

We obtain the sum function by a differentiation,

$$f(x) = (x+1)e^x, \qquad x \in \mathbb{R}.$$

Example 1.25 *1) Find the radius of convergence λ for the power series*

$$\sum_{n=2}^{\infty} \frac{2x^{n+2}}{n^2-1}, \qquad x \in \mathbb{R}.$$

2) Find the sum of the power series for $|x| < \lambda$.

1) The radius of convergence is here found in three different ways.

a) The **criterion of comparison**. Since

$$2x^2 \cdot \frac{|x|^n}{n^2} \le \frac{2|x|^{n+2}}{n^2-1} \le 2x^2 \cdot |x|^n, \qquad n \ge 2,$$

and since $\sum \frac{x^n}{n^2}$ and $\sum x^n$ both have the radius of convergence $\lambda = 1$, the given series must also have the radius of convergence $\lambda = 1$.

b) The **criterion of roots**. For $n > 1$ we get

$$\sqrt[n]{\frac{2|x|^{n+2}}{n^2-1}} = |x| \sqrt[n]{2x^2} \cdot \sqrt[n]{\frac{1}{1-\frac{1}{n^2}}} \cdot \frac{1}{(\sqrt[n]{n})^2} \to |x| \quad \text{for } n \to \infty,$$

so we conclude that the radius of convergence is $\lambda = 1$.

c) The **criterion of quotients**. When $x \neq 0$ we get

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{2|x|^{n+3}}{(n+1)^2-1} \cdot \frac{n^2-1}{2|x|^{n+2}} = \frac{n^2-1}{(n+1)^2-1} \cdot |x| \\ &= \frac{1-\frac{1}{n}}{\left(1+\frac{1}{n}\right)-1} \cdot |x| \to |x| \qquad \text{for } n \to \infty, \end{aligned}$$

and the radius of convergence is $\lambda = 1$.

2) The sum function is here found in two different ways.

a) **Application of a known series**. We know that

$$\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad \text{for } |x| < 1 = \lambda.$$

Then we get by a **decomposition**,

$$\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}.$$

It is now legal to split the series, when $|x| < 1$, in the following way:

$$\begin{aligned} f(x) &= \sum_{n=2}^{\infty} \frac{2x^{n+2}}{n^2-1} = \sum_{n=2}^{\infty} \frac{x^{n+2}}{n-1} - \sum_{n=2}^{\infty} \frac{x^{n+2}}{n+1} = \sum_{n=1}^{\infty} \frac{x^{n+3}}{n} - \sum_{n=3}^{\infty} \frac{x^{n+1}}{n} \\ &= x^3 \sum_{n=1}^{\infty} \frac{x^n}{n} - x \sum_{n=1}^{\infty} \frac{x^n}{n} + x^2 + \frac{x^3}{2} \\ &= (x^3 - x) \ln\left(\frac{1}{1-x}\right) + x^2 + \frac{x^3}{2}, \qquad |x| < 1. \end{aligned}$$

b) **Differentiation**. Putting

$$g(x) = 2\sum_{n=2}^{\infty} \frac{x^{n+1}}{n^2-1}, \qquad |x| < 1,$$

we see that $g(0) = 0$, and $f(x) = x \cdot g(x)$. By differentiation of each term of the series of $g(x)$ we get for $|x| < 1$ that

$$g'(x) = 2\sum_{n=2}^{\infty} \frac{(n+1)x^n}{n^2-1} = 2x\sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1} = 2x\sum_{n=1}^{\infty} \frac{x^2}{n} = -2x\ln(1-x).$$

Hence

$$\begin{aligned}
f(x) &= x \cdot g(x) = x\int_0^x g'(t)\,dt = -2x\int_0^x t \cdot \ln(1-t)\,dt \\
&= -2x\left[\frac{t^2}{2} \cdot \ln(1-t)\right]_0^x + 2x\int_0^x \frac{t^2}{2} \cdot \frac{-1}{1-t}\,dt \\
&= -x^3\ln(1-x) + x\int_0^x \left(t+1+\frac{1}{t-1}\right)dt \\
&= -x^3\ln(1-x) + \frac{x^3}{2} + x^2 + x\ln(1-x) \\
&= (x^3-x)\ln\left(\frac{1}{1-x}\right) + x^2 + \frac{x^3}{2}, \qquad |x| < 1.
\end{aligned}$$

Remark 1.13 One may of course make this method more troublesome by defining

$$h(x) = 2\sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1}, \qquad h(0) = 0,$$

and $g'(x) = x \cdot h(x)$. Since

$$h'(x) = \sum_{n=2}^{\infty} x^{n-2} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1,$$

we get $h(x) = \int_0^x h'(t)\,dt = -\ln(1-x)$, and then one continues as above.

Example 1.26 *Given the power series*

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n^2+1)}.$$

1) Find the interval of convergence $]-\varrho, \varrho[$ *for the power series.*

2) Prove that the power series is convergent at both endpoints of the interval of convergence.

3) Is the power series uniformly convergent in the interval $[-\varrho, \varrho]$*?*

1) Here there are several variants, like e.g.

a) **Criterion of comparison and magnitudes**. Since

$$\sum_{n=1}^{\infty} \frac{|x|^n}{n(n^2+1)} \le \sum_{n=1}^{\infty} |x|^n < \infty \qquad \text{for } |x| < 1,$$

the series is *at least* convergent for $|x| < 1$, i.e. $\varrho \ge 1$. On the other hand,

$$\frac{|x|^n}{n(n^2+1)} \to \infty \quad \text{for } n \to \infty, \quad \text{if } |x| > 1,$$

hence the series is coarsely divergent for $|x| > 1$, and $\varrho \le 1$.
We conclude that the interval of convergence is $]-1,1[$, and the radius of convergence is $\varrho = 1$.

b) The **criterion of quotients**. If $x \ne 0$, then $a_n(x) = \dfrac{|x|^n}{n(n^2+1)} > 0$, hence

$$\begin{aligned} \frac{a_{n+1}(x)}{a_n(x)} &= \frac{|x|^{n+1}}{(n+1)\{(n+1)^2+1\}} \cdot \frac{n(n^2+1)}{|x|^n} \\ &= \frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}} \cdot |x| \to |x| \quad \text{for } n \to \infty. \end{aligned}$$

The condition of convergence becomes $|x| < 1$, so $\varrho = 1$ and $I =]-1,1[$.

c) The **criterion of roots**. Put $a_n(x) = \dfrac{|x|^n}{n(n^2+1)} \geq 0$. Then

$$\sqrt[n]{a_n(x)} = \frac{|x|}{\sqrt[n]{n} \cdot \sqrt[n]{n^2+1}} = \frac{|x|}{(\sqrt[n]{n})^3 \sqrt[n]{1+\dfrac{1}{n^2}}} \to |x|$$

for $n \to \infty$, idet $\sqrt[n]{n} \to 1$ og $\sqrt[n]{1+\dfrac{1}{n^2}} \to 1$ for $n \to \infty$.

The condition of convergence becomes $|x| < 1$, so $\varrho = 1$ and $I =]-1,1[$.

2) For $x = \pm 1$ we get the estimate

$$\left| \sum_{n=1}^{\infty} \frac{x^n}{n(n^2+1)} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n(n^2+1)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

It follows from the **criterion of comparison** that the power series is convergent at both endpoints of the interval of convergence.

A **variant** is to note that $\dfrac{1}{n(n^2+1)} \sim \dfrac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ is convergent, the convergence at the endpoints follows from the **criterion of equivalence**.

3) If $x \in [-1,1]$, then we get as in (2) that

$$\left| \sum_{n=1}^{\infty} \frac{x^n}{n(n^2+1)} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The power series has a convergent majoring series in the interval $[-1,1]$, hence it is uniformly convergent.

Example 1.27 *Consider the power series*

$$\sum_{n=1}^{\infty} \frac{n+1}{n} x^n.$$

1) Find the radius of convergence.

2) Does the series converge at the endpoints of the interval of convergence?

1) The radius of convergence is 1, which is proved in the following in four different ways:

a) The **criterion of quotients**. If $x \neq 0$, then we get for $n \to \infty$ that

$$\left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \frac{(n+1)+1}{n+1} |x|^{n+1} \cdot \frac{n}{n+1} \frac{1}{|x|^n} = \frac{n(n+2)}{n+1)^2} |x| \to |x|.$$

We conclude from the **criterion of quotients** that we have convergence for $|x| < 1$ and divergence for $|x| > 1$, hence the radius of convergence is 1.

b) The **criterion of roots**. It follows from

$$1 \le \sqrt[n]{\frac{n+1}{n}} = \sqrt[n]{1+\frac{1}{n}} \le \sqrt[n]{2} \to 1 \quad \text{for } n \to \infty,$$

that

$$\sqrt[n]{\frac{n+1}{n}|x|^n} = \sqrt[n]{1+\frac{1}{n}} \cdot |x| \to |x| \quad \text{for } n \to \infty.$$

We conclude from the **criterion of roots** that we have convergence for $|x| < 1$ and divergence for $|x| > 1$, and the radius of convergence must be 1.

c) **Criterion of comparison**. It follows from

$$\sum_{n=1}^{\infty} |x|^n \le \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right) |x|^n \le 2 \sum_{n=1}^{\infty} |x|^n,$$

that the series $\sum x^n$ and $\sum (n+1)/n \cdot x^n$ have the same radius of convergence, namely 1 (known for $\sum x^n$).

d) **Known series**. If $|x| < 1$, then

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \quad \text{og} \quad \sum_{n=1}^{\infty} \frac{1}{n}\, x^n = \ln \frac{1}{1-x}.$$

By addition we get (at least) convergence for $|x| < 1$ and the sum is

$$\sum_{n=1}^{\infty} \frac{n+1}{n}\, x^n = \sum_{n=1}^{\infty} x^n + \sum_{n=1}^{\infty} \frac{1}{n}\, x^n = \frac{x}{1-x} + \ln \frac{1}{1-x}.$$

Both terms on the right hand side tend towards $+\infty$, when $x \to 1-$, so we conclude that the radius of convergence is 1.

2) We have at the endpoints ± 1 that

$$\frac{n+1}{n} |(\pm 1)^n| = \frac{n+1}{n} \to 1 \ne 0 \quad \text{for } n \to \infty,$$

thus the necessary condition for convergence is not fulfilled. Hence we have divergence at both endpoints.

Example 1.28 *Given the power series*

$$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(n-1)n}\, x^n.$$

1) Find the interval of convergence for the power series.

2) Prove that in the interval of convergence,

$$(1+x)^2 \cdot f''(x) = -x-3.$$

1) We can find the interval of convergence in several different ways:

a) We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{\frac{2n-1}{(n-1)n}}\,|x| \to |x| \qquad \text{for } n \to \infty.$$

The condition of convergence $|x| < 1$ shows that the radius of convergence is $\varrho = 1$m so the interval of convergence is $]-1,1[$.

b) If we instead apply the **criterion of quotients**, we get for $x \neq 0$ that

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{(2n+1)|x|^{n+1}}{(n+1)(n+2)} \cdot \frac{(n-1)n}{(2n-1)|x|^n} \to |x| \quad \text{for } n \to \infty.$$

The condition of convergence is $|x| < 1$, so the radius of convergence is $\varrho = 1$, and the interval of convergence is $]-1,1[$.

2) If $|x| < 1$, we can find the sum function in the following way,

$$\begin{aligned}
f(x) &= \sum_{n=2}^{\infty}(-1)^{n+1} \cdot \frac{2n-1}{(n-1)n}\, x^n = \sum_{n=2}^{\infty}(-1)^{n+1}\left(\frac{1}{n-1}+\frac{1}{n}\right)x^n \\
&= \sum_{n=2}^{\infty}(-1)^{n+1} \cdot \frac{1}{n-1}\, x^n + \sum_{n=2}^{\infty}(-1)^{n+1} \cdot \frac{1}{n}\, x^n \quad \text{(NB. Both series are convergent for } |x|<1) \\
&= \sum_{n=1}^{\infty}(-1)^{n} \cdot \frac{1}{n}\, x^{n+1} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\, x^n - x \\
&= -x\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\, x^n + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\, x^n - x \\
&= (1-x)\ln(1+x) - x, \qquad \text{for } x \in\,]-1,1[,
\end{aligned}$$

where we recognize the logarithmic series. Hence

$$f'(x) = -\ln(1+x) + \frac{1-x}{1+x} - 1 = -\ln(1+x) + \frac{2}{1+x} - 2$$

and

$$f''(x) = -\frac{1}{1+x} - \frac{2}{(1+x)^2} = -\frac{3+x}{(1+x)^2},$$

and we finally get

$$(1+x)^2 f''(x) = -x-3 \qquad \text{for } x \in\,]-1,1[.$$

3) **Alternatively** it follows by differentiation of each term before the summation of the series,

$$f(x) = \sum_{n=2}^{\infty}\frac{(-1)^{n+1}(2n-1)}{(n-1)n}\, x^n, \qquad |x|<1,$$

that

$$\begin{aligned}
f''(x) &= \sum_{n=2}^{\infty}(-1)^{n+1}(2n-1)x^{n-2} = \sum_{n=0}^{\infty}(-1)^{n+1}(2n+3)x^n \\
&= 2\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n + \sum_{n=0}^{\infty}(-1)^{n+1}x^n \\
&= -2\sum_{n=0}^{\infty}(n+1)(-x)^n - \sum_{n=0}^{\infty}(-x)^n \\
&= -\frac{2}{(1+x)^2} - \frac{1}{1+x},
\end{aligned}$$

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$$(1+x)^2 f''(x) = -2 - (1+x) = -x - 3 \qquad \text{for } |x| < 1.$$

2 Power series expansions of functions

Example 2.1 *Find the power series of the function*

$$f(x) = \cos^2 x,$$

and its radius of convergence. Check, if the series is convergent or divergent at the endpoints of the interval of convergence.

Since $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, we get for every $x \in \mathbb{R}$ that

$$f(x) = \frac{1}{2} + \frac{1}{2}\cos 2x = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n-1} \cdot x^{2n}.$$

The interval of convergence is $\mathbb{R}$, and the radius of convergence is $\varrho = \infty$.
In this case we do not have an endpoint. Notice that one should always check, if the endpoints exist or not.

Example 2.2 *Find the power series of the function*

$$f(x) = \sin^2 x,$$

and its radius of convergence. Check if the series is convergent or divergent at the endpoints of the interval of convergence.

Since $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, we get for every $x \in \mathbb{R}$ that

$$f(x) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \cdot 2^{2n-1} \cdot x^{2n}.$$

The interval of convergence is again $\mathbb{R}$, hence the radius of convergence is $\varrho = \infty$.
Again we have no endpoints of the interval of convergence.

Example 2.3 *Find the power series of the function*

$$f(x) = \sin x \cdot \cos x,$$

and its radius of convergence. Check if the series is convergent or divergent at the endpoints of the interval of convergence.

By a small trigonometric reformulation we get for ever y $x \in \mathbb{R}$ that

$$f(x) = \sin x \cdot \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot 4^n \cdot x^{2n+1}.$$

The interval of convergence is $\mathbb{R}$, and the radius of convergence is $\varrho = \infty$.
We have no endpoint, so the last question does not make sense.

Example 2.4 *Find the power series of the function*

$$f(x) = \ln \sqrt{\frac{1+x}{1-x}},$$

ant its radius of convergence. Check if the series is convergent or divergent at the endpoints of the interval of convergence.

Since $\dfrac{1+x}{1-x} > 0$ for $-1 < x < 1$, and $f(0) = 0$, we get in this interval

$$\begin{aligned}
f(x) &= \ln \sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left\{\ln(1+x) - \ln(1-x)\right\} \\
&= \frac{1}{2}\int_0^x \left\{\frac{d}{dt}\ln(1+t) - \frac{d}{dt}\ln(1-t)\right\} dt \\
&= \frac{1}{2}\int_0^x \left\{\frac{1}{1+t} + \frac{1}{1-t}\right\} dt = \int_0^x \frac{dt}{1-t^2} = \sum_{n=0}^{\infty} \int_0^x t^{2n}\, dt \\
&= \sum_{n=0}^{\infty} \frac{1}{2n+1}\, x^{2n+1}.
\end{aligned}$$

Obviously, the series $\sum_{n=0}^{\infty} \dfrac{1}{2n+1}\, x^{2n+1}$ has the radius of convergence $\varrho = 1$, and the series is *divergent* for $x = \pm 1$, i.e. at the endpoints of the interval of convergence.

Example 2.5 *Find the power series of the function*

$$f(x) = \frac{1}{2-x},$$

and its radius of convergence. Check if the series is convergent or divergent at the endpoints of the interval of convergence.

Whenever we are considering an expression consisting of two terms, the general strategy is to norm it, such that the dominating term is adjusted to 1. This is here done in the following way:
If $|x| < 2$, then $\left|\dfrac{x}{2}\right| < 1$, hence

$$f(x) = \frac{1}{2-x} = \frac{1}{2}\cdot\frac{1}{1-\dfrac{x}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n.$$

It follows from the above that $\varrho = 2$.

We get at the endpoints of the interval of convergence $x = \pm 2$ that $|a_n(x)| = \dfrac{1}{2}$. Since this does not tend towards 0, the series is coarsely divergent at the endpoint of the interval of convergence.

Remark 2.1 If instead $|x| > 2$, then x becomes the dominating term in the denominator. Then we get formally

$$f(x) = \frac{1}{2-x} = -\frac{1}{x}\cdot\frac{1}{1-\dfrac{2}{x}} = -\frac{1}{x}\sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n = -\sum_{n=1}^{\infty}\frac{2^{n-1}}{x^n}.$$

This is, however, not a *power series*, because the exponents of x are negative. Such series are called *Laurent series*. They are very important in *Complex Function Theory*.

Example 2.6 *Find the power series of the function*

$$f(x) = \frac{x}{1+x-2x^2},$$

and its radius of convergence. Check if the series is convergent or divergent at the endpoints of the interval of convergence.

Since

$$1+x-2x^2 = (1-x)(1+2x),$$

we get by a decomposition for $|x| < \frac{1}{2}$,

$$\begin{aligned} f(x) &= \frac{x}{1+x-2x^2} = \frac{x}{(1-x)(1+2x)} = \frac{1}{3}\cdot\frac{1}{1-x} - \frac{1}{3}\cdot\frac{1}{1+2x} \\ &= \frac{1}{3}\sum_{n=0}^{\infty} x^n - \frac{1}{3}\sum_{n=0}^{\infty}(-2)^n x^n = \sum_{n=1}^{\infty}\frac{1}{3}\left\{1-(-2)^n\right\}x^n. \end{aligned}$$

The radius of convergence is $\varrho = \frac{1}{2}$.
The series is coarsely divergent at the endpoints of the interval of convergence, because

$$\left|\frac{1}{3}\left\{1-(-2)^n\right\}\left(\pm\frac{1}{2}\right)^n\right| = \frac{1}{3}\left\{1-\left(\mp\frac{1}{2}\right)^n\right\} \to \frac{1}{3} \neq 0 \quad \text{for } n \to \infty.$$

Example 2.7 *Find the power series of the function*

$$f(x) = \left(1+x^2\right)\ln(1+x).$$

Find its radius of convergence. Check if the series is convergent or divergent at the endpoints of the interval of convergence.

The logarithmic series (for $\ln(1+x)$) is convergent for $x \in]-1,1[$, hence we have in this interval

$$\begin{aligned} f(x) &= \left(1+x^2\right)\ln(1+x) = \left(1+x^2\right)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n \\ &= \sum_{n-1}^{\infty}\frac{(-1)^{n-1}}{n}x^n + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{n+2} \quad \text{(multiply by } 1+x^2\text{)} \\ &= x - \frac{1}{2}x^2 + \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n}x^n + \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n-2}x^n \quad \text{(removal of some terms and a change of index)} \\ &= x - \frac{1}{2}x^2 + \sum_{n=3}^{\infty}(-1)^{n-1}\left\{\frac{1}{n}+\frac{1}{n-2}\right\}x^n \quad \text{(collecting the series)} \\ &= x - \frac{1}{2}x^2 + \sum_{n=3}^{\infty}(-1)^{n-1}\cdot\frac{2(n-1)}{n(n-2)}\,x^n. \end{aligned}$$

Clearly, the radius of convergence is $\varrho = 1$.

We get at the *endpoint* $x = -1$ the *divergent* series

$$-1 - \frac{1}{2} - \sum_{n=3}^{\infty} \left\{ \frac{1}{n} + \frac{1}{n-2} \right\}.$$

We get at the *endpoint* $x = 1$ the alternating series

$$1 - \frac{1}{2} + \sum_{n=3}^{\infty} (-1)^{n-1} \left\{ \frac{1}{n} + \frac{1}{n-2} \right\}.$$

Since $\dfrac{1}{n} + \dfrac{1}{n-2} \to 0$ is *decreasing* for $n \to \infty$, it is convergent according to **Leibniz's criterion**. It is, however, not absolutely convergent, so it must be *conditionally convergent*.

Remark 2.2 We obtain by applying *Abel's theorem*,

$$1 - \frac{1}{2} + \sum_{n=3}^{\infty} (-1)^{n-1} \left\{ \frac{1}{n} + \frac{1}{n-2} \right\} = \lim_{x \to 1-} f(x) = 2 \cdot \ln 2.$$

Example 2.8 *Find the power series of the function*

$$f(x) = \text{ Arctan } x + \ln \sqrt{1+x^2}.$$

Find its radius of convergence. Check if the series is convergent or divergent at the endpoints of the interval of convergence.

Since

$$\frac{d}{dx} \text{ Arctan } x = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{for } |x| < 1,$$

we get by integrating each term in the same interval,

$$\text{Arctan } x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \, x^{2n+1} \qquad \text{for } |x| < 1.$$

Since

$$\ln \sqrt{1+x^2} = \frac{1}{2} \ln \left(1+x^2\right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \, x^{2n} \qquad \text{for } |x| < 1,$$

we get by adding the two series in the common domain of convergence that

$$\text{Arctan } x + \ln \sqrt{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \, x^{2n+1} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} \, x^{2n} \text{ for } |x| < 1.$$

the radius of convergence is of course $\varrho = 1$.

Since both series by **Leibniz's criterion** are (conditionally) convergent at the *endpoints* of the interval of convergence, the power series for $f(x)$ is also convergent for $x = \pm 1$.
It is possible by a small consideration to conclude that the convergence at $x = \pm 1$ is conditional.

Example 2.9 *Find the power series for the function*

$$f(x) = \ln\left(x + \sqrt{1+x^2}\right)$$

by applying the formula

$$\ln\left(x + \sqrt{1+x^2}\right) = \int_0^x \frac{1}{\sqrt{1+t^2}}, dt, \qquad x \in \mathbb{R}.$$

Find the radius of convergence of the series.

We have

$$\frac{1}{\sqrt{1+t^2}} = \left(1+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} t^{2n} \qquad \text{for } |t|<1,$$

i.e. $\varrho = 1$, where

$$\begin{aligned}\binom{-1/2}{n} &= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}+1-n\right)}{n!} = \frac{(-1)^n}{2^n}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \\ &= \frac{(-1)^n}{2^n\cdot 2^n}\cdot\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)\cdot 2n}{n!\,n!} = \frac{(-1)^n}{4^n}\cdot\frac{(2n)!}{n!\,n!} = \frac{(-1)^n}{4^n}\binom{2n}{n}.\end{aligned}$$

By integrating each term and then add them all, we get

$$\ln\left(x + \sqrt{1+x^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\cdot\frac{1}{4^n}\binom{2n}{n} x^{2n+1} \quad \text{for } x \in]-1,1[.$$

The radius of convergence does not change by an integration, hence $\varrho = 1$.

Example 2.10 *Find the power series for the function*

$$f(x) = \text{ Arcsin } x$$

by using the formula

$$\text{Arcsin } x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt, \qquad x \in]-1,1[.$$

Find the radius of convergence of the series.

Now,

$$\frac{1}{\sqrt{1-t^2}} = \left(1-t^2\right)^{-1/2} = \sum_{n=0}^{\infty}(-1)^n \binom{-1/2}{n} t^n \quad \text{for } |t|<1,$$

so $\varrho = 1$, where

$$\begin{aligned}(-1)^n\binom{-1/2}{n} &= (-1)^n\cdot\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} = \frac{1}{2^n}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \\ &= \frac{1}{2^n\cdot 2^n}\cdot\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)2n}{n!\,n!} = \frac{1}{4^n}\cdot\frac{(2n)!}{n!\,n!} = \frac{1}{4^n}\binom{2n}{n}.\end{aligned}$$

We get by integration of each term that

$$\operatorname{Arcsin} x = \sum_{n=0}^{\infty} \frac{1}{2n+1} \cdot \frac{1}{4^n} \binom{2n}{n} x^{2n+1} \quad \text{for } x \in]-1,1[.$$

The radius of convergence does not change by an integration, so $\varrho = 1$.

Example 2.11 *1) Prove that*

$$(x+1)\ln(1+x) = x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n(n+1)}, \qquad x \in]-1,1[.$$

2) Given $a_1 = 1$ *and the recursion formula*

(1) $a_{n+1} = a_n + (-1)^n (n+1)^2, \qquad n \in \mathbb{N},$

which produces the sequence

$$a_n = 1^2 - 2^2 + \cdots + (-1)^{n-1} n^2, \qquad n \in \mathbb{N}.$$

Show by testing in (1) that a_n *can also be written*

(2) $a_n = \dfrac{(-1)^{n-1} n(n+1)}{2}, \qquad n \in \mathbb{N}.$

3) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{1^2 - 2^2 + \cdots + (-1)^{n-1} n^2}$$

is convergent and find its sum.
Hint: Exploit (2) and possibly also the result of (1).

1) It follows from a known power series expansion that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \qquad \text{for } x \in]-1,1[,$$

that

$$\begin{aligned}
(x+1)\ln(1+x) &= x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{n+1} + x + \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n+1} x^{n+1} \\
&= x + \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ \frac{1}{n} - \frac{1}{n+1} \right\} x^{n+1} = x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n(n+1)}.
\end{aligned}$$

These calculations are correct for $x \in]-1,1[$, and it must be noted that the series is also absolutely convergent at the endpoints of the interval, because the denominator is $n(n+1) \sim n^2$.

2) By insertion of $n = 1$ into (2) we get $a_1 = \dfrac{1}{2}\{(-1)^{1-1} \cdot 1 \cdot (1+1)\} = 1$ as required.

Assume that (2) is true for some $n \in \mathbb{N}$. Then

$$\begin{aligned} a_{n+1} &= \frac{(-1)^{n-1}n(n+1)}{2} + (-1)^n(n+1) \\ &= \frac{(-1)^n(n+1)}{2}\{-n+2n+2\} = \frac{(-1)^n(n+1)(n+2)}{2}. \end{aligned}$$

This is the same as the result we obtain by replacing n by $n+1$ in (2):

$$a_{n+1} = \frac{(-1)^{(n+1)-1}(n+1)((n+1)+1)}{2} = \frac{(-1)^n(n+1)(n+2)}{2}.$$

Hence, if the formula holds for some $n \in \mathbb{N}$, then it also holds for $n+1$. Since the formula is valid for $n = 1$, we conclude that (2) holds in general by induction.

3) When we insert (2) we formally get

$$\sum_{n=1}^{\infty} \frac{1}{1^2 - 2^2 + \cdots + (-1)^{n-1} n^2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}.$$

This series is, however, absolutely convergent:

$$\left| 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} \right| \le 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \le 2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3},$$

proving the first question. The last question is now proved in two different ways:

a) **Sum by means of (1)**. If we apply Abel's theorem for $x = 1$ on the series of (1), then

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 1^{n+1}}{n(n+1)} = (1+1)\ln(1+1) - 1 = 2\ln 2 - 1,$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{1^2 - 2^2 + \cdots + (-1)^{n-1} n^2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} = 4\ln 2 - 2.$$

b) **Sum by means of the sequence of sections**. We get by a decomposition that

$$\begin{aligned}
s_N &= \sum_{n=1}^{N} (-1)^{n-1} \cdot \frac{2}{n(n+1)} = 2 \sum_{n=1}^{N} (-1)^{n-1} \left\{ \frac{1}{n} - \frac{1}{n+1} \right\} \\
&= 2 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} + 2 \sum_{n=1}^{N} \frac{(-1)^{n}}{n+1} = 2 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} + 2 \sum_{n=2}^{N+1} \frac{(-1)^{n+1}}{n} \\
&= 4 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} - 2 + 2 \frac{(-1)^{N+1}}{N} \quad \to \quad 4\ln 2 - 2, \qquad \text{for } N \to \infty,
\end{aligned}$$

hence the series is convergent with the sum

$$\sum_{n=1}^{\infty} \frac{1}{1^2 - 2^2 + \cdots + (-1)^{n-1} n^2} = 4\ln 2 - 2.$$

Example 2.12 *1) Find, by using power series for elementary functions, the power series for the functions* $\sin(x^2)$ *and* $\ln(1+2x)$*, and find the intervals of convergence of the series.*

2) Prove that one has

$$\sin x - x\cos x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!(2n+1)!}\, x^{2n+1}, \qquad x \in \mathbb{R}.$$

1) a) Since

$$\sin u = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, u^{2n+1} \qquad \text{for } u \in \mathbb{R},$$

it follows by the substitution $u = x^2$ that

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{4n+2} \qquad \text{for } x \in \mathbb{R}.$$

b) Since

$$\ln(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, u^n \qquad \text{for } -1 < u < 1,$$

it follows by the substitution $u = 2x \in]-1,1[$, i.e. $x \in \left]-\frac{1}{2}, \frac{1}{2}\right[$, that

$$\ln(1+2x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, 2^n x^n \qquad \text{for } x \in \left]-\frac{1}{2}, \frac{1}{2}\right[.$$

2) The interval of convergence is $\mathbb{R}$ for both of the series for $\sin x$ and $\cos x$ (and the radius of convergence is ∞). Hence, we get by legal operations of calculations that we have for $x \in \mathbb{R}$

$$\begin{aligned} \sin x - x\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\{1-(2n+1)\}x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,(-2n)x^{2n+1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot 2n}{(2n-1)!2n(2n+1)}\, x^{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!(2n+1)}\, x^{2n+1}. \end{aligned}$$

3 Cauchy multiplication

Example 3.1 *Prove by using Cauchy multiplication that for any* $x \in]-1,1[$,

$$(\text{Arctan}\, x)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)x^{2n}.$$

Since

$$\text{Arctan}\, x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1} \qquad \text{for } |x|<1,$$

we get in this interval that

$$(\text{Arctan}\, x)^2 = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\, x^{2j+1} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k}\cdot x^{2(j+k)+2}}{(2j+1)(2k+1)}$$

$$= \sum_{n=0}^{\infty}\sum_{j+k=n} \frac{(-1)^{j+k}x^{2(j+k)+2}}{(2j+1)(2k+1)} = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{(-1)^n\, x^{2n+2}}{(2j+1)(2n-2j+1)} \qquad (\text{sæt } k = n-j)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{n} \sum_{j=0}^{n-1} \frac{n}{(2j+1)(2n-2j-1)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left\{\frac{1}{2}\sum_{j=0}^{n-1}\left(\frac{1}{2j+1}+\frac{1}{2n-2j-1}\right)\right\}x^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left\{\sum_{j=1}^{n}\frac{1}{2j-1}\right\}x^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)x^{2n}.$$

Example 3.2 *Find the first five terms of the power series of*

$$f(x) = e^x \sin x.$$

First method. If we interpret “the first five terms” as the terms up to a_5x^5, then we get by a simple multiplication of known power series that

$$\begin{aligned} e^x \sin x &= \left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots\right)\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5+\cdots\right) \\ &= x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5+\cdots-\frac{1}{6}x^3-\frac{1}{6}x^4-\frac{1}{12}x^5+\cdots+\frac{1}{120}x^5+\cdots \\ &= x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5+\cdots . \end{aligned}$$

Second method. Calculation of the *Taylor coefficients*. In this calculation we have

$$\begin{array}{lcllcr} f(x) &=& e^x \sin x, & f(0) &=& 0, \\ f'(x) &=& e^x(\sin x+\cos x), & f'(0) &=& 1, \\ f''(x) &=& 2e^x\cos x, & f''(0) &=& 2, \\ f^{(3)}(x) &=& 2e^x(\cos x-\sin x), & f^{(3)}(0) &=& 2, \\ f^{(4)}(x) &=& -4e^x\sin x = -4f(x), & f^{(4)}(0) &=& 0, \\ f^{(5)}(x) &=& -4f'(x) = -4e^x(\sin x+\cos x), & f^{(5)}(0) &=& -4, \end{array}$$

hence

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5+\cdots .$$

Third method. Complex calculations:

$$\begin{aligned}
e^x\sin x = {} & \operatorname{Im}\{e^x(\cos x+i\sin x)\} = \operatorname{Im}\left\{e^x e^{ix}\right\} = \operatorname{Im}\{\exp((1+i)x)\}\\
&= \operatorname{Im}\left\{1+(1+i)x+\frac{(1+i)^2x^2}{2!}+\frac{(1+i)^3x^3}{3!}+\frac{(1+i)^4x^4}{4!}+\frac{(1+i)^5x^5}{5!}+\cdots\right\}\\
&= \operatorname{Im}\left\{1+(1+i)x+ix^2+\frac{-1+i}{3}x^3-\frac{1}{6}x^4-\frac{1+i}{30}x^5+\cdots\right\}\\
&= x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5+\cdots .
\end{aligned}$$

Example 3.3 *Find the first five terms of the power series of the function*

$$f(x) = e^x \cos x.$$

First method. If we interpret "the first five terms" as the terms up to $a_5 x^5$, then we get by a simple multiplication of known power series,

$$\begin{aligned}
e^x \cos x &= \left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\cdots\right)\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\cdots\right)\\
&= 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\cdots-\frac{1}{2}x^2-\frac{1}{2}x^3-\frac{1}{4}x^4-\frac{1}{12}x^5+\cdots+\frac{1}{24}x^4+\frac{1}{24}x^5+\cdots\\
&= 1+x-\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{30}x^5+\cdots .
\end{aligned}$$

Second method. Calculation of the *Taylor coefficients*. We get

$$\begin{array}{lcllcr}
f(x) & = & e^x \cos x, & f(0) & = & 1,\\
f'(x) & = & e^x(\cos x - \sin x), & f'(0) & = & 1,\\
f''(x) & = & -2e^x \sin x, & f''(0) & = & 0,\\
f^{(3)}(x) & = & -2e^x(\sin x + \cos x), & f^{(3)}(0) & = & -2,\\
f^{(4)}(x) & = & -4e^x \cos x = -4f(x), & f^{(4)}(0) & = & -4,\\
f^{(5)}(x) & = & -4f'(x) = -4e^x(\cos x - \sin x), & f^{(5)}(0) & = & -4,
\end{array}$$

hence

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = 1+x-\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{30}x^5+\cdots .$$

Third method. Complex calculations:

$$\begin{aligned}
e^x \cos x &= \operatorname{Re}\{e^x(\cos x + i\sin x)\} = \operatorname{Re}\left\{e^x e^{ix}\right\} = \operatorname{Re}\{\exp((1+i)x)\}\\
&= \operatorname{Re}\left\{1+(1+i)x+\frac{(1+i)^2x^2}{2!}+\frac{(1+i)^3x^3}{3!}+\frac{(1+i)^4x^4}{4!}+\frac{(1+i)^5x^5}{5!}+\cdots\right\}\\
&= \operatorname{Re}\left\{1+(1+i)x+ix^2+\frac{-1+i}{3}x^3-\frac{1}{6}x^4-\frac{1+i}{30}x^5+\cdots\right\}\\
&= 1+x-\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{30}x^5+\cdots .
\end{aligned}$$

4 Integrals described by series

Example 4.1 *Find (expressed as a sum of an infinite series) the value of the integral*

$$\int_0^1 \frac{\sin x}{x}\, dx.$$

We have

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n}, \quad \text{for } x \in \mathbb{R} \setminus \{0\},$$

which is supplied by the value 1 for $x = 0$. The series is uniformly convergent in $[0,1]$) (because $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} = \sinh 1$ is a convergent majoring series). Hence, we get by integrating each term before summation that

$$\int_0^1 \frac{\sin x}{x}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}.$$

Example 4.2 *Find (expressed by the sum of an infinite series) the value of the integral*

$$\int_0^{1/2} \frac{1}{1+x^4}\, dx.$$

We have

$$\frac{1}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n x^{4n}, \qquad \text{for } |x| < 1,$$

which is *uniformly* convergent in $\left[0, \frac{1}{2}\right]$, because it has the convergent majoring series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{4n}$.
Hence, by integrating each term,

$$\int_0^{1/2} \frac{dx}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n \int_0^{1/2} x^{4n}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} \cdot \frac{1}{2^{4n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} \cdot \frac{1}{16^n}.$$

Remark 4.1 One *can* in fact directly find the value of the integral. However, this is not so easy. We show below how it is done:
First we get by a smart decomposition

$$\begin{aligned}
\frac{1}{1+x^4} &= \frac{1}{1+2x^2+x^4-2x^2} && (2x^2 \text{ is added and subtracted}) \\
&= \frac{1}{\left(x^2+1\right)^2 - (\sqrt{2}x)^2} && (\text{difference of two squares}) \\
&= \frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} && (a^2-b^2=(a+b)(a-b)) \\
&= \frac{1}{2\sqrt{2}} \left\{ \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1} \right\} && (\text{decomposition}).
\end{aligned}$$

When this expression is integrated, we get (where some of the details have been left out)

$$\begin{aligned}
\int_0^{1/2} \frac{dx}{1+x^4} &= \frac{1}{4\sqrt{2}}\left[\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| + 2\left\{\text{Arctan}(\sqrt{2}x+1)+\ \text{Arctan}(\sqrt{2}x-1)\right\}\right]_0^{1/2} \\
&= \frac{1}{4\sqrt{2}}\ln\left(\frac{5+2\sqrt{2}}{5-2\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\left\{\text{Arctan}\left(1+\frac{\sqrt{2}}{2}\right)-\ \text{Arctan}\left(1-\frac{\sqrt{2}}{2}\right)\right\} \\
&= \frac{1}{4\sqrt{2}}\ln\left(\frac{5+2\sqrt{2}}{5-2\sqrt{2}}\right) \\
&\quad +\frac{1}{2\sqrt{2}}\text{Arctan}\left[\tan\left(\text{Arctan}\left(1+\frac{\sqrt{2}}{2}\right)\right)-\tan\left(\text{Arctan}\left(1-\frac{\sqrt{2}}{2}\right)\right)\right] \\
&= \frac{1}{4\sqrt{2}}\ln\left(\frac{5+2\sqrt{2}}{5-2\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\text{Arctan}\left(\frac{\left(1+\frac{\sqrt{2}}{2}\right)-\left(1-\frac{\sqrt{2}}{2}\right)}{1+\left(1+\frac{\sqrt{2}}{2}\right)\left(1-\frac{\sqrt{2}}{2}\right)}\right) \\
&= \frac{1}{4\sqrt{2}}\ln\left(\frac{5+2\sqrt{2}}{5-2\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\text{Arctan}\left(\frac{2\sqrt{2}}{3}\right).
\end{aligned}$$

Example 4.3 *Find (expressed by the sum of an infinite series the value of the integral,*

$$\int_0^1 \cos\sqrt{x}\,dx.$$

Here

$$\cos\sqrt{x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\sqrt{x})^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n, \qquad \text{for } x \geq 0,$$

is *uniformly convergent* in $[0,1]$, hence by integrating each term before summation

$$\int_0^1 \cos(\sqrt{x})\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\int_0^1 x^n\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(2n)!}.$$

Remark 4.2 The integral can be given an exact value by the substitution $u=\sqrt{x}$, i.e. $x=u^2$ and $dx = 2udu$, thus

$$\begin{aligned}
\int_0^1 \cos\sqrt{x}\,dx &= \int_0^1 \cos u\cdot 2u\,du = [2u\sin u]_0^1 - \int_0^1 2\sin u\,du \\
&= [2u\sin u + 2\cos u]_0^1 = 2(\sin 1+\cos 1-1).
\end{aligned}$$

Example 4.4 *Find the value of the integral below expressed by the sum of an infinite series*

$$\int_0^{1/2}\frac{x-\ \text{Arctan}\,x}{x^3}\,dx.$$

When $0 < x < 1$, we get from

$$\text{Arctan}\, x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad \text{for } |x| < 1,$$

that

$$\frac{x - \text{Arctan}\, x}{x^3} = \frac{1}{x^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}\, x^{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}\, x^{2n-2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+3}\, x^{2n}.$$

We see immediately that the series has $\sum_{n=0}^{\infty} 4^{-n}$ as a convergent majoring series in the interval $\left[0, \frac{1}{2}\right]$, hence the series is *uniformly convergent* in this interval. By integrating each term before summing we get

$$\int_0^{1/2} \frac{x - \text{Arctan}\, x}{x^3}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \cdot \frac{1}{2^{2n+1}}.$$

Remark 4.3 We can also here find the exact value of the integral. If $x \neq 0$, then by a partial integration,

$$\begin{aligned}
\int \frac{x - \text{Arctan}\, x}{x^3}\, dx &= \int \frac{dx}{x^2} - \int \frac{\text{Arctan}\, x}{x^3}\, dx = -\frac{1}{x} + \frac{1}{2} \cdot \frac{\text{Arctan}\, x}{x^2} - \frac{1}{2} \int \frac{dx}{x^2(1+x^2)} \\
&= -\frac{1}{x} + \frac{1}{2} \cdot \frac{\text{Arctan}\, x}{x^2} - \frac{1}{2} \cdot \int \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) dx = -\frac{1}{x} + \frac{1}{2} \cdot \frac{\text{Arctan}\, x}{x^2} + \frac{1}{2} \cdot \frac{1}{x} + \frac{1}{2} \cdot \text{Arctan}\, x \\
&= \frac{1}{2} \cdot \frac{\text{Arctan}\, x - x}{x^2} + \frac{1}{2} \cdot \text{Arctan}\, x.
\end{aligned}$$

Det ses ved rækkeudvikling, at singulariteten i $x = 0$ er hævelig (værdien er her 0), så

$$\int_0^{1/2} \frac{x - \text{Arctan}\, x}{x^3}\, dx = \frac{1}{2} \cdot 4 \left\{ \text{Arctan}\, \frac{1}{2} - \frac{1}{2} \right\} + \frac{1}{2} \text{Arctan}\, \frac{1}{2} = \frac{5}{2} \text{Arctan}\, \frac{1}{2} - 1.$$

5 Sums of series

Example 5.1 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty}(-1)^n x^{2n},$$

and find (inside the interval of convergence) an explicit expression for the function which is defined by the series.

The series is a quotient series of quotient $-x^2$, thus $\varrho = 1$, and

$$f(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

for $|x| < 1$.

Example 5.2 *Find the radius of convergence for*

$$\sum_{n=0}^{\infty} nx^n.$$

Find inside the interval of convergence an explicit expression for the function defined by the series.

We put t $a_n(x) = n|x|^n \geq 0$, which is > 0 for $x \neq 0$, $n \geq 1$.

Criterion of roots.

$$\sqrt[n]{a_n(x)} = \sqrt[n]{n|x|^n} = \sqrt[n]{n} \cdot |x| \to |x| \qquad \text{for } n \to \infty.$$

The condition of convergence is $|x| < 1$, thus the interval of convergence is $I =]-1, 1[$.

Criterion of quotients. We have for $x \neq 0$ and $n \geq 1$ that

$$\frac{a_{n+1}(x)}{a_n(x)} = \frac{(n+1)|x|^{n+1}}{n|x|^n} = \left(1 + \frac{1}{n}\right)|x| \to |x| \text{ for } n \to \infty.$$

The condition of convergence is $|x| < 1$, thus $I =]-1, 1[$.

Alternatively, if $|x| \geq 1$ then $a_n(x) = n|x|^n \to \infty$, and the series is coarsely divergent, thus we conclude that $\varrho \leq 1$.
On the other hand, if $|x| < 1$, then $n(\sqrt{|x|})^n \to 0$ for $n \to \infty$ by the laws of magnitudes. In particular $n(\sqrt{|x|})^n \leq c(x)$ for every n, and we get the estimate

$$\sum_{n=0}^{\infty} n|x|^n = \sum_{n=0}^{\infty} n(\sqrt{|x|})^n(\sqrt{|x|})^n \leq c(x) \sum_{n=0}^{\infty} (\sqrt{|x|})^n < \infty$$

(quotient series of quotient $\sqrt{|x|} < 1$). Consequently we have absolute convergence for $|x| < 1$, hence $\varrho \geq 1$. Putting the things together we get $\varrho = 1$, and the interval of convergence is $I =]-1, 1[$.

Sum function. The series looks like the standard series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1.$$

When this series is differentiated, we get

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \qquad \text{for } |x| < 1.$$

It is seen that we are only missing a factor x in order to obtain the wanted result, so

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} nx^n \qquad \text{for } |x| < 1,$$

where we have added 0 corresponding to $n = 0$ in the series.

Example 5.3 *Find the radius of convergence for the power series*

$$\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{(n-2)n}\, x^n.$$

Find (inside the interval of convergence) an explicit expression for the function which is defined by the series.

The coefficient $\dfrac{1}{(n-2)n}$ is a rational function, hence $\varrho = 1$, because we have e.g.

$$\sqrt[n]{(n-2)n} \to 1 \qquad \text{for } n \to \infty.$$

We get by a decomposition,

$$\frac{1}{(n-2)n} = \frac{1}{2} \cdot \frac{1}{n-2} - \frac{1}{2} \cdot \frac{1}{n}.$$

Here n occurs in the denominator, so we are aiming at a logarithmic series. We get for $|x| < 1$,

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{(n-2)n}\, x^n &= \frac{1}{2}\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{n-2}\, x^n - \frac{1}{2}\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{n}\, x^n \\ &= \frac{1}{2}\, x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, x^n - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, x^n + \frac{1}{2}\left\{x - \frac{1}{2}x^2\right\} \\ &= \frac{1}{2}\, x - \frac{1}{4}\, x^2 + \frac{1}{2}(x^2-1)\ln(1+x).\end{aligned}$$

Remark 5.1 Since $\sum_{n=3}^{\infty} \dfrac{1}{(n-2)n}$ is equivalent with the convergent series $\sum_{n=3}^{\infty} \dfrac{1}{n^2}$, the given series is absolutely convergent at the endpoints of the interval of convergence, hence by *Abel's theorem*,

$$\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{(n-2)n}\, 1^n = \lim_{x \to 1-} \left\{\frac{1}{2}\, x - \frac{1}{4}\, x^2 + \frac{1}{2}(x^2-1)\ln(1+x)\right\} = \frac{1}{4},$$

and

$$\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{(n-2)n}\, (-1)^n = \lim_{x \to -1+} \left\{\frac{1}{2}\, x - \frac{1}{4}\, x^2 + \frac{1}{2}(x^2-1)\ln(1+x)\right\} = -\frac{3}{4},$$

because we get by the laws of magnitudes $(x^2-1)\ln(1+x) = (x-1)\{(1+x)\ln(1+x)\} \to 0$ for $1+x \to 0+$. $\Diamond$

Example 5.4 *Find the radius of convergence for the power series*

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\, x^n.$$

Find (inside the interval of convergence) an explicit expression for the function defined by the series.

Here we immediately recognize the structure of the logarithmic series. If we put $y = 2x$, then

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\, x^n = \sum_{n=1}^{\infty} \frac{1}{n}\, y^n = -\ln(1-y) = -\ln(1-2x),$$

which holds for $|y| = |2x| < 1$, hence for $|x| < \dfrac{1}{2}$, so $\varrho = \dfrac{1}{2}$.

Example 5.5 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} (-1)^n (n+1) x^n.$$

Find (inside the interval of convergence) an explicit expression for the function defined by the series.

The coefficient $n+1$ is a polynomial, hence $\varrho = 1$. One may here use that $\sqrt[n]{n+1} \to 1$ for $n \to \infty$ and the **criterion of roots**.

Sum function. It is well-known that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \text{for } |x| < 1.$$

When this equation is differentiated, we get

$$-\frac{1}{(1+x)^2} = \frac{d}{dx}\left\{\sum_{n=0}^{\infty} (-1)^n x^n\right\} = \sum_{n=1}^{\infty} (-1)^n x^{n-1} = -\sum_{n=0}^{\infty} (-1)^n (n+1) x^n,$$

hence

$$\sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \frac{1}{(1+x)^2}.$$

Alternatively we put

$$f(x) = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n,$$

hence by termwise integration for $|x| < 1$,

$$F(x) = \int_0^x f(t)\, dt = \sum_{n=0}^{\infty} (-1)^n x^{n+1} = x \sum_{n=0}^{\infty} (-x)^n = \frac{x}{1+x} = 1 - \frac{1}{1+x},$$

and thus

$$f(x) = F'(x) = \frac{d}{dx}\left\{1 - \frac{1}{1+x}\right\} = \frac{1}{(1+x)^2}.$$

Example 5.6 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty}(-2)^n \, \frac{n+2}{n+1}\, x^n.$$

Find (inside the interval of convergence) an explicit expression for the function which is defined by the series.

The condition of convergence is by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{2^n \cdot \frac{n+2}{n+1} \cdot |x|^n} = \sqrt[n]{\frac{n+2}{n+1}} \cdot 2|x| \to 2|x| < 1, \ \text{for } n \to \infty,$$

so $|x| < \varrho = \dfrac{1}{2}$.

If $x = 0$, then the sum is $(-2)^0 \cdot \dfrac{0+2}{0+1} \cdot 1 = 2$.

Now, $\dfrac{n+2}{n+1} = 1 + \dfrac{1}{n+1}$, so we get for $0 < |x| < \dfrac{1}{2}$ that

$$\sum_{n=0}^{\infty}(-2)^n \cdot \frac{n+2}{n+1}\,x^n = \sum_{n=0}^{\infty}(-2x)^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\,(2x)^n = \sum_{n=0}^{\infty}(-2x)^n + \frac{1}{2x}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,(2x)^n$$
$$= \frac{1}{1+2x} + \frac{\ln(1+2x)}{2x}.$$

Summing up we get the sum function

$$f(x) = \begin{cases} \dfrac{1}{1+2x} + \dfrac{\ln(1+2x)}{2x} & \text{for } 0 < |x| < \dfrac{1}{2} \\[2mm] 2 & \text{for } x = 0 \end{cases}$$

Example 5.7 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty}\frac{x^n}{(n+3)!}.$$

Find (inside the interval of convergence) an explicit expression for the function defined by the series.

We put $a_n(x) = \dfrac{|x|^n}{(n+3)!}$. Then we get by the **criterion of quotients** for $x \neq 0$,

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{|x|^{n+1}}{(n+4)!} \cdot \frac{(n+3)!}{|x|^n} = \frac{|x|}{n+4} \to 0 < 1 \text{ for } n \to \infty.$$

The series is convergent for every $x \in \mathbb{R}$, thus $\varrho = \infty$.

Since we have a faculty in the denominator, we aim at an exponential function.

If $x = 0$, then $f(0) = \dfrac{1}{3!} = \dfrac{1}{6}$.

If $x \neq 0$, we get by changing indices,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{x^n}{(n+3)!} = \sum_{n=3}^{\infty}\frac{x^{n-3}}{n!} = \frac{1}{x^3}\sum_{n=3}^{\infty}\frac{x^n}{n!} = \frac{1}{x^3}\left\{\sum_{n=0}^{\infty}\frac{x^n}{n!} - 1 - x - \frac{x^2}{2}\right\} \\ &= \frac{1}{x^3}\left(e^x - 1 - x - \frac{x^2}{2}\right). \end{aligned}$$

Summing up we get the sum function

$$f(x) = \begin{cases} \dfrac{1}{x^3}\left(e^x - 1 - x - \dfrac{x^2}{2}\right), & \text{for } x \neq 0, \\[2mm] \dfrac{1}{6}, & \text{for } x = 0. \end{cases}$$

Example 5.8 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n+1)}\, x^{2n}.$$

Find (inside the interval of convergence) an explicit expression for the function which is defined by the series.

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{1}{\sqrt[n]{2n+1}} \cdot \left|\frac{x}{2}\right|^2 \to \left|\frac{x}{2}\right|^2 \qquad \text{for } n \to \infty.$$

From the condition $\left|\dfrac{x}{2}\right| < 1$ we get the radius of convergence $\varrho = 2$.

If $x = 0$, then $f(0) = 1$.
If $0 < |x| < 2$, the structure $\dfrac{(-1)^n}{2n+1}$ indicates that we should think of Arctan. With that function in our mind we easily get

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n+1)}\, x^{2n} = \frac{2}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{x}{2}\right)^{2n+1} = \frac{2}{x}\, \text{Arctan}\left(\frac{x}{2}\right).$$

Summing up we get the sum function

$$f(x) = \begin{cases} \dfrac{2}{x}\, \text{Arctan}\left(\dfrac{x}{2}\right), & \text{for } 0 < |x| < 2, \\ 1, & \text{for } x = 0. \end{cases}$$

Example 5.9 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}.$$

Find inside the interval of convergence an explicit expression for the function which is defined by the series.

It follows from the rearrangement

$$\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

that the series is a quotient series of quotient $\dfrac{x}{3}$. This is convergent for $\left|\dfrac{x}{3}\right| < 1$, thus for $x \in]-3, 3[$, and the radius of convergence is $\varrho = 3$.

Inside the interval of convergence the sum function is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1 - \dfrac{x}{3}} = \frac{1}{3 - x}, \quad \text{for } |x| < 3.$$

Example 5.10 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, x^{3n}.$$

Find inside the interval of convergence an explicit expression for the function which is defined by the series.

The faculty in the denominator indicates that we should think of an exponential function. One should immediately recognize

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, x^{3n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-x^3\right)^n = \exp\left(-x^3\right),$$

which is true for every $x \in \mathbb{R}$, så $\varrho = \infty$.

Example 5.11 *Find the radius of convergence for the power series*

$$\sum_{n=3}^{\infty} \frac{x^n}{(n-2)(n-1)n}.$$

Find inside the interval of convergence an explicit expression for the function, given by the series.

It follows from

$$\sqrt[n]{|a_n(x)|} = \frac{|x|}{\sqrt[n]{n-2}\cdot\sqrt[n]{n-1}\cdot\sqrt[n]{n}} \to |x| \qquad \text{for } n \to \infty,$$

that $\varrho = 1$.
The sum function can be found in several ways.

First method. If we define

$$f(x) = \sum_{n=3}^{\infty} \frac{x^n}{(n-2)(n-1)n} \qquad \text{for } |x| < 1,$$

we get by successive differentiations

$$f'(x) = \sum_{n=3}^{\infty} \frac{x^{n-1}}{(n-2)(n-1)} = \sum_{n=2}^{\infty} \frac{x^n}{(n-1)n}, \quad |x| < 1, \quad f'(0) = 0,$$

$$f''(x) = \sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n}\, x^n = -\ln(1-x) \quad \text{for } |x| < 1,$$

hence by successive integrations with $f'(0) = f(0) = 0$,

$$\begin{aligned} f'(x) &= \int_0^x (-1)\cdot\ln(1-t)\,dt = [-(t-1)\ln(1-t)]_0^x + \int_0^x \frac{t-1}{t-1}\,dt \\ &= -(x-1)\ln(1-x) + x, \end{aligned}$$

and

$$
\begin{aligned}
f(x) &= \int_0^x f'(t)\,dt = \int_0^x \{-(t-1)\ln(1-t)+t\}\,dt \\
&= \left[-\frac{(t-1)^2}{2}\ln(1-t)+\frac{1}{2}\,t^2\right]_0^x + \int_0^x \frac{(t-1)^2}{2}\cdot\frac{1}{t-1}\,dt \\
&= -\frac{1}{2}(x-1)^2\ln(1-x)+\frac{1}{2}\,x^2+\frac{1}{4}\left\{(x-1)^2-1\right\} \\
&= -\frac{1}{2}(x-1)^2\ln(1-x)+\frac{3}{4}\,x^2-\frac{1}{2}x.
\end{aligned}
$$

Second method. We get by a *decomposition* a simpler variant. In fact, it follows from

$$\frac{1}{(n-2)(n-1)n} = \frac{1}{2}\cdot\frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{2}\cdot\frac{1}{n}$$

that whenever $|x|<1$, then

$$\begin{aligned}
f(x) &= \frac{1}{2}\sum_{n=3}^{\infty}\frac{x^n}{n-2} - \sum_{n=3}^{\infty}\frac{x^n}{n-1} + \frac{1}{2}\sum_{n=3}^{\infty}\frac{1}{n}\,x^n \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,x^{n+2} - \sum_{n=2}^{\infty}\frac{1}{n}\,x^{n+1} + \frac{1}{2}\sum_{n=3}^{\infty}\frac{1}{n}\,x^n \\
&= \frac{x^2}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,x^n - x\left\{\sum_{n=1}^{\infty}\frac{1}{n}\,x^n - x\right\} + \frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\,x^n - x - \frac{x^2}{2}\right\} \\
&= \frac{1}{2}(x^2-2x+1)\sum_{n=1}^{\infty}\frac{1}{n}\,x^n + x^2 - \frac{1}{2}\,x - \frac{1}{4}\,x^2 \\
&= -\frac{1}{2}(x-1)^2\ln(1-x) + \frac{3}{4}\,x^2 - \frac{1}{2}\,x.
\end{aligned}$$

Remark 5.2 Since $\dfrac{1}{(n-2)(n-1)n} \sim \dfrac{1}{n^3}$, and $\sum_{n=1}^{\infty}\dfrac{1}{n^3}$ is convergent, the series is absolutely convergent at the endpoints of the interval of convergence. Then by *Abel's theorem*,

$$\sum_{n=3}^{\infty}\frac{1}{(n-2)(n-1)n} = \lim_{x\to 1-} f(x) = \frac{1}{4},$$

and

$$\sum_{n=3}^{\infty}\frac{(-1)^n}{(n-2)(n-1)n} = \lim_{x\to -1+} f(x) = -2\ln 2 + \frac{5}{4}.$$

Example 5.12 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty}\frac{n+1}{n!}\,x^{2n}.$$

Find inside the interval of convergence an explicit expression for the function given by the series.

By the **criterion of quotients** we get for $x \neq 0$,

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{n+2}{(n+1)!}\cdot x^{2n+2}\cdot\frac{n!}{n+1}\cdot\frac{1}{x^{2n}} = \frac{n+2}{(n+1)^2}\cdot x^2 \to 0 \text{ for } n\to\infty,$$

so the series is convergent for every $x \in \mathbb{R}$, and $\varrho = \infty$.

The sum function is found by a comparison with the exponential series,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{n+1}{n!}\,x^{2n} = \sum_{n=0}^{\infty} \frac{n}{n!}\,x^{2n} + \sum_{n=0}^{\infty} \frac{1}{n!}\,x^{2n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\,x^{2n} = \exp(x^2) + \sum_{n=0}^{\infty} \frac{1}{n!}\,x^{2n+2} \\ &= \exp(x^2) + x^2 \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n = (1+x^2)\exp(x^2). \end{aligned}$$

Example 5.13 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty} \frac{3(-1)^n}{(n+2)(n+1)(2n+1)}\,x^{2n+2},$$

and find its sum in the interval of convergence. Check, if the power series is convergent for $x = \varrho$ or $x = -\varrho$.

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{\sqrt[n]{3}}{\sqrt[n]{n+2}\cdot\sqrt[n]{n+1}\cdot\sqrt[n]{2n+1}}\,x^2\cdot\sqrt[n]{x^2} \to x^2 \quad \text{for } n\to\infty.$$

The condition of convergence is $|x|^2 < 1$, hence the radius of convergence is $\varrho = 1$.
The *sum function* is found in various ways.

First method. We put

$$f(x) = \sum_{n=0}^{\infty} \frac{3(-1)^n}{(n+2)(n+1)(2n+1)}\,x^{2n+2}, \text{ for } |x|<1, \qquad f(0)=0.$$

Then we get in the interval of convergence by termwise differentiation,

$$f'(x) = \sum_{n=0}^{\infty} \frac{2(n+1)\cdot 3(-1)^n}{(n+2)(n+1)(2n+1)}\,x^{2n+1} = 6\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)(2n+1)}\,x^{2n+1},$$

and $f'(0) = 0$. By another differentiation we get

$$f''(x) = 6\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}\,x^{2n} = 6\sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,x^{2n-4}.$$

If $x = 0$, then $f''(0) = 6\cdot\frac{1}{2}\cdot 1 = 3$, and if $0 < |x| < 1$, we get by a comparison with the *logarithmic series*,

$$f''(x) = \frac{6}{x^4}\left\{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\,(x^2)^n + x^2\right\} = 6\cdot\frac{x^2 - \ln(1+x^2)}{x^4},$$

where we have changed the lower bound to $n=1$ and added x^2.

We get by integration $f'(x)$ as a convergent improper integral,

$$\begin{aligned}
f'(x) &= f'(0)+6\cdot\int_{0+}^{x}\left\{\frac{t^2-\ln(1+t^2)}{t^4}\right\}dt \\
&= 0+6\left[-\frac{1}{3}\,\frac{t^2-\ln(1+t^2)}{t^3}\right]_{0+}^{x}+\frac{6}{3}\int_{0+}^{x}\frac{1}{t^3}\left(2t-\frac{2t}{1+t^2}\right)dt \\
&= -2\,\frac{x^2-\ln(1+x^2)}{x^3}+4\int_{0+}^{x}\frac{1}{t^2}\cdot\frac{(1+t^2)-1}{1+t^2}\,dt \\
&= -2\,\frac{x^2-\ln(1+x^2)}{x^3}+4\text{Arctan}\,x.
\end{aligned}$$

By another integration we find $f(x)$ for $0<|x|<1$ as an improper integral,

$$\begin{aligned}
f(x) &= f(0)+4\int_0^x \text{Arctan}\,t\,dt-2\int_{0+}^{x}\frac{t^2-\ln(1+t^2)}{t^3}\,dt \\
&= [4t\cdot\text{Arctan}\,t]_0^x-4\int_0^x\frac{t}{1+t^2}\,dt+\left[\frac{t^2-\ln(1+t^2)}{t^2}\right]_{0+}^{x}-\int_{0+}^{x}\frac{1}{t^2}\left(2t-\frac{2t}{1+t^2}\right)dt \\
&= 4x\cdot\text{Arctan}\,x-2\left[\ln(1+t^2)\right]_0^x+\frac{x^2-\ln(1+x^2)}{x^2}-\int_{0+}^{x}\frac{2t}{1+t^2}\,dt \\
&= 4x\cdot\text{Arctan}\,x-3\ln(1+x^2)+1-\frac{\ln(1+x^2)}{x^2},
\end{aligned}$$

supplied by $f(0)=0$.

Second method. We get by a decomposition,

$$\frac{3}{(n+2)(n+1)(2n+1)}=\frac{1}{n+2}-\frac{3}{n+1}+\frac{4}{2n+1}.$$

We have $f(0)=0$ as before. For $0<|x|<1$ we get by the decomposition

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\frac{3(-1)^n}{(n+2)(n+1)(2n+1)}\,x^{2n+2} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}\,x^{2n+2}-\sum_{n=0}^{\infty}\frac{3(-1)^n}{n+1}\,x^{2n+2}+\sum_{n=0}^{\infty}\frac{4(-1)^n}{2n+1}\,x^{2n+2} \\
&= \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\,x^{2n-2}-3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,(x^2)^n+4x\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\,x^{2n+1} \\
&= \frac{1}{x^2}\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,(x^2)^n+x^2\right\}-3\ln(1+x^2)+4x\,\text{Arctan}\,x \\
&= 1-\frac{\ln(1+x^2)}{x^2}-3\ln(1+x^2)+4x\,\text{Arctan}\,x.
\end{aligned}$$

Summing up we get

$$f(x)=\begin{cases} 1-\left(3+\dfrac{1}{x^2}\right)\ln(1+x^2)+4x\,\text{Arctan}\,x, & \text{for } 0<|x|<1, \\ 0, & \text{for } x=0. \end{cases}$$

Endpoints. Since

$$\frac{3}{(n+2)(n+1)(2n+1)} \sim \frac{3}{2} \cdot \frac{1}{n^3},$$

and since $n = 3 > 1$ secures that the equivalent series is convergent, the original series is convergent at the endpoints of the interval of convergence.

The sum function is *even* (only even exponents occur in the series), hence it follows by *Abel's theorem* that the value for $x = \pm 1$ is

$$\lim_{x \to 1-} f(x) = 1 - (3+1)\ln(1+1) + 4 \cdot 1 \cdot \text{Arctan } 1 = 1 - 4\ln 2 + \pi.$$

Example 5.14 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\right\}\cdot\frac{x^{2n-1}}{4^n}.$$

Find the sum function of the power series in the interval of convergence. Check if the power series is convergent for $x=\varrho$ or for $x=-\varrho$.

By the **criterion of roots** it follows from

$$\begin{aligned}\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1} &= \frac{1}{n(n+1)(2n+1)}\{(n+1)(2n+1)+n(2n+1)-4n(n+1)\}\\ &= \frac{1}{n(n+1)(2n+1)}\{(2n+1)^2-4n^2-4n\}=\frac{1}{n(n+1)(2n+1)}\sim\frac{1}{2n^3},\end{aligned}$$

for $x\neq 0$ that

$$\sqrt[n]{|a_n(x)|}=\frac{1}{\sqrt[n]{n}\cdot\sqrt[n]{n+1}\cdot\sqrt[n]{2n+1}}\,\frac{x^2}{4}\cdot\frac{1}{\sqrt[n]{|x|}}\to\frac{x^2}{4}=\left(\frac{|x|}{2}\right)^2$$

for $n\to\infty$. The condition of convergence is $\left(\dfrac{|x|}{2}\right)^2<1$, hence $|x|<2$, and $\varrho=2$.

Endpoints 1. Since the equivalent series $\sum_{n=1}^{\infty}\dfrac{1}{2n^3}$ is convergent, we conclude that the series is convergent at the endpoints of the interval of convergence $x=\pm 2$ and that the sum can be found by using *Abel's theorem*, if only the sum function is found.

Sum function. If $x=0$, then $f(0)=0$. if $0<|x|<2$, then we get the sum function by the following splitting (note that all series are convergent for $|x|<2$),

$$\begin{aligned}f(x) &= \sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\right\}\frac{x^{2n-1}}{4^n}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cdot\frac{1}{x}\left(\frac{x^2}{4}\right)^n+\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}\cdot\frac{1}{x}\left(\frac{x^2}{4}\right)^n-4\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{x^{2n-1}}{2^{2n}}\\ &= -\frac{1}{x}\ln\left(1+\frac{x^2}{4}\right)+\frac{4}{x^3}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{x^2}{4}\right)^n-\frac{8}{x^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{x}{2}\right)^{2n+1}\\ &= -\frac{1}{x}\ln\left(1+\frac{x^2}{4}\right)+\frac{4}{x^3}\ln\left(1+\frac{x^2}{4}\right)-\frac{4}{x^3}\cdot\frac{x^2}{4}-\frac{8}{x^2}\operatorname{Arctan}\left(\frac{x}{2}\right)+\frac{8}{x^2}\cdot\frac{x}{2}\\ &= \left(\frac{4}{x^3}-\frac{1}{x}\right)\ln\left(1+\frac{x^2}{4}\right)-\frac{8}{x^2}\operatorname{Arctan}\left(\frac{x}{2}\right)+\frac{3}{x}.\end{aligned}$$

Summing up we have found the sum function

$$f(x)=\begin{cases}\left(\dfrac{4}{x^3}-\dfrac{1}{x}\right)\ln\left(1+\dfrac{x^2}{4}\right)-\dfrac{8}{x^2}\operatorname{Arctan}\left(\dfrac{x}{2}\right)+\dfrac{3}{x} & \text{for } 0<|x|<2,\\ 0 & \text{for } x=0.\end{cases}$$

Endpoints 2. As mentioned earlier the series is convergent at the endpoints of the interval of convergence. By using *Abel's theorem* we get the sum for $x = 2$,

$$\lim_{x \to 2-} f(x) = \left(\frac{4}{8} - \frac{1}{2}\right) \ln\left(1 + \frac{4}{4}\right) - \frac{8}{4}\,\text{Arctan}\,1 + \frac{3}{2} = \frac{3}{2} - \frac{\pi}{2}.$$

Since the power series only contains *odd* exponents, the sum function is *odd*, and we get by *Abel's theorem* the sum for $x = -2$,

$$\lim_{x \to -2+} f(x) = -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}.$$

Example 5.15 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty} \frac{2^n - n}{n!}\, x^n,$$

and find its sum function in the interval of convergence.

By the rules of calculation,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2^n - n}{n!}\, x^n &= \sum_{n=0}^{\infty} \frac{2^n}{n!}\, x^n - \sum_{n=0}^{\infty} \frac{n}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{1}{n!}(2x)^n - \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\, x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(2x)^n - x \sum_{m=0}^{\infty} \frac{1}{m!}\, x^m, \end{aligned}$$

$[m = n - 1$, i.e. $n = m + 1]$, in the *common domain of convergence for the series on the right hand side.*

By inspection of the standard series it follows that

$$\sum_{n=0}^{\infty} \frac{1}{n!}(2x)^n = \exp(2x) \quad \text{for all } x \in \mathbb{R}, \quad [\varrho = \infty],$$

and

$$\sum_{m=0}^{\infty} \frac{1}{m!}\, x^m = e^x \quad \text{for alle } x \in \mathbb{R}, \quad [\varrho = \infty].$$

We conclude that $\varrho = \min\{\infty, \infty\} = \infty$, and the sum function is

$$\sum_{n=0}^{\infty} \frac{2^n - n}{n!}\, x^n = e^{2x} - x e^x \quad \text{for all } x \in \mathbb{R}, \quad \varrho = \infty.$$

The question of convergence at the interval of convergence does not give sense because $\pm\infty \notin \mathbb{R}$.

Example 5.16 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty}(2n+1)x^n,$$

and find its sum function in the interval of convergence. Check if the power series is convergent for $x=\varrho$ or for $x=-\varrho$.

We put $a_n(x)=(2n+1)|x|^n \geq 0$. Then

$$\sqrt[n]{a_n(x)} = \sqrt[n]{2n+1}\cdot|x| \to |x| \qquad \text{for } n\to\infty.$$

By the **criterion of roots** the series is convergent for $|x|<1$, thus $\varrho=1$.

Since $\varrho=1$, we can split the series into two series which both have $\varrho=1$,

$$\sum_{n=0}^{\infty}(2n+1)x^n = 2\sum_{n=0}^{\infty}nx^n+\sum_{n=0}^{\infty}x^n, \qquad |x|<1.$$

Here, $\sum_{n=0}^{\infty}x^n=\dfrac{1}{1-x}$, $|x|<1$, is the well-knows quotient series.

Then we get by termwise differentiation,

$$\sum_{n=1}^{\infty}nx^{n-1}=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2}, \qquad \text{for } |x|<1.$$

This looks very much like the first series on the right hand side. When we multiply by $2x$ and add some zero terms, we get

$$\frac{2x}{(1-x)^2}=2\sum_{n=1}^{\infty}nx^n=2\sum_{n=0}^{\infty}nx^n \qquad \text{for } |x|<1.$$

We get by insertion for $|x|<1$,

$$\sum_{n=0}^{\infty}(2n+1)x^n=\frac{2x}{(1-x)^2}+\frac{1}{1-x}=\frac{1+x}{(1-x)^2}=f(x).$$

Since $(2n+1)|\pm 1|^n \to \infty$ for $n\to\infty$, the series is **coarsely divergent** at the endpoints of the interval of convergence.

The underhand dealing here is that the *sum function* can be extended continuously to $x=-1$; and the series is not convergent here.

Example 5.17 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{4n^2-1},$$

and find its sum in the interval of convergence. Check, if the power series is convergent for $x = \varrho$ or for $x = -\varrho$.

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{x^2}{\sqrt[n]{4n^2-1}} \to x^2 \qquad \text{for } n \to \infty,$$

so the condition of convergence $x^2 < 1$ gives $|x| < 1$, thus $\varrho = 1$.

For $x = 0$ we get the sum $f(0) = 0$. It follows by a decomposition that

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right),$$

hence for $0 < |x| < 1$,

$$f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{4n^2-1} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{x^{2n}}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{x^{2n}}{2n+1} = \frac{1}{2}\left(x-\frac{1}{x}\right)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}+\frac{1}{2}.$$

Here,

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} = \int_0^x \sum_{n=0}^{\infty} t^{2n}\,dt = \int_0^x \frac{dt}{1-t^2} = \frac{1}{2}\int_0^x \left\{\frac{1}{1-t}+\frac{1}{1+t}\right\}dt = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$

which by insertion gives the *sum function*

$$f(x) = \begin{cases} \dfrac{1}{4}\left(x-\dfrac{1}{x}\right)\ln\left(\dfrac{1+x}{1-x}\right)+\dfrac{1}{2} & \text{for } 0<|x|<1, \\ 0 & \text{for } x=0. \end{cases}$$

The sum at the *endpoints* is here directly obtained by a decomposition without any reference to *Abel's theorem:*

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right) = \frac{1}{2}\lim_{N\to\infty}\left(1-\frac{1}{2N+1}\right) = \frac{1}{2}.$$

Alternatively, $\dfrac{1}{4n^2-1} \sim \dfrac{1}{4n^2}$, and since $\sum_{n=1}^{\infty}\dfrac{1}{4n^2}$ is convergent, the series is convergent at the *endpoints*. Then we get by *Abel's theorem* and the **laws of magnitude** that since the value is the same at ± 1, we have

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \lim_{x\to 1-} f(x) = \frac{1}{2} + \lim_{x\to 1-}\frac{1}{4}(x+1)(x-1)\{\ln(1+x)-\ln(1-x)\} = \frac{1}{2}.$$

Example 5.18 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=1}^{\infty}\left(1+\frac{(-1)^n}{n}\right)x^{2n},$$

and find its sum function in the interval of convergence. Check if the power series is convergent for $x=\varrho$ or for $x=-\varrho$.

We can here find the *radius of convergence* more or less elegantly (there are several variants). Here is one of them. If $|x| \geq 1$, then we have for the n-th term that

$$\left|\left(1+\frac{(-1)^n}{n}\right)x^{2n}\right| \to \infty \quad \text{for } n\to\infty, \quad \text{hence coarsely divergens,}$$

so $\varrho \leq 1$. If on the other hand, $|x| < 1$, then we have the estimate

$$\sum_{n=1}^{\infty}\left|\left(1+\frac{(-1)^n}{n}\right)x^{2n}\right| \leq 2\sum_{n=1}^{\infty}(x^2)^n, \qquad \text{konvergens,}$$

(quotient series with the quotient $x^2 < 1$). It follows that $\varrho \geq 1$. Summarizing we get $\varrho = 1$.

It follows from the first argument that the series is **coarsely divergent** for $x = \pm 1 = \pm\varrho$.

Sum function. For $|x| < 1$ we get according to standard series,

$$\sum_{n=1}^{\infty}\left(1+\frac{(-1)^n}{n}\right)x^{2n} = \sum_{n=1}^{\infty}x^{2n} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}(x^2)^n = \frac{x^2}{1-x^2} - \ln(1+x^2), \quad \text{for } |x|<1,$$

because the two series in the splitting both have $\varrho = 1$, so the splitting is legal.

Note that the right hand side is not defined for $x = \pm 1$.

Example 5.19 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=1}^{\infty}\left\{n+(-1)^n\right\}x^{2n}.$$

Find the sum function of the power series in the interval of convergence. Check if the power series is convergent for $x = \varrho$ or for $x = -\varrho$.

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{n+(-1)^n}\cdot x^2 \to x^2 \qquad \text{for } n \to \infty.$$

In fact, for $n = 2m$ even we get $\sqrt[n]{n+(-1)^n} = \sqrt[2m]{2m+1} \to 1$ for $n = 2m \to \infty$ through even indices, and for $n = 2m+1$ odd we get $\sqrt[n]{n+(-1)^n} = \sqrt[2m+1]{(2m+1)-1} = \sqrt[2m+1]{2m} \to 1$ for $n = 2m+1 \to \infty$ through odd indices.
The condition of convergence $x^2 < 1$ gives $|x| < 1$, hence $\varrho = 1$.

At the endpoints of the interval of convergence we get $|a_n(x)| = n + (-1)^n \to \infty$ for $n \to \infty$, so the **necessary condition** for convergence is *not* fulfilled, and the series is (coarsely) divergent for $x = \pm 1$.

The **sum function** is for $|x| < 1$ given by

$$f(x) = \sum_{n=1}^{\infty}\left\{n+(-1)^n\right\}x^{2n} = \sum_{n=1}^{\infty}nx^{2n} + \sum_{n=1}^{\infty}(-x^2)^n = x^2\sum_{n=1}^{\infty}n(x^2)^{n-1} + \frac{-x^2}{1+x^2}.$$

From

$$\frac{1}{1-y} = \sum_{n=0}^{\infty}y^n \quad \text{og} \quad \frac{d}{dy}\left(\frac{1}{1-y}\right) = \frac{1}{(1-y)^2} = \sum_{n=1}^{\infty}ny^{n-1}, \quad |y|<1,$$

follows by inserting $y = x^2$ that

$$f(x) = \frac{x^2}{(1-x^2)^2} - \frac{x^2}{1+x^2}.$$

Example 5.20 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)(2n+2)}.$$

Find its sum function in the interval of convergence. Check if the power series is convergent for $x = \varrho$ or for $x = -\varrho$.

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{1}{\sqrt[n]{2n+1} \cdot \sqrt[n]{2n+2}}\, x^2 \to x^2 \qquad \text{for } n \to \infty,$$

so the condition of convergence gives $x^2 < 1$, hence $|x| < 1$, and thus $\varrho = 1$.

Sum function. We get for $x = 0$ that $f(0) = \dfrac{1}{2}$. Then by a decomposition,

$$\frac{1}{(2n+1)(2n+2)} = \frac{1}{2n+1} - \frac{1}{2n+2},$$

hence we get for $0 < |x| < 1$ the sum function

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)(2n+2)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+2} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} - \frac{1}{2x^2} \sum_{n=1}^{\infty} \frac{(x^2)^n}{n} \\
&= \frac{1}{x} \int_0^x \sum_{n=0}^{\infty} t^{2n}\, dt + \frac{1}{2x^2} \ln(1-x^2) = \frac{1}{x} \int_0^x \frac{dt}{1-t^2} + \frac{1}{2x^2} \ln(1-x^2) \\
&= \frac{1}{2x} \ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2x^2} \ln(1-x^2).
\end{aligned}$$

As conclusion we get

$$f(x) = \begin{cases} \dfrac{1}{2x} \ln\left(\dfrac{1+x}{1-x}\right) + \dfrac{1}{2x^2} \ln(1-x^2), & \text{for } 0 < |x| < 1, \\ \dfrac{1}{2} & \text{for } x = 0. \end{cases}$$

Since $\dfrac{1}{(2n+1)(2n+2)} \sim \dfrac{1}{4n^2}$, and $\sum_{n=1}^{\infty} \dfrac{1}{4n^2}$ is convergent, the series is convergent at the endpoints of the interval of convergence. Since $(\pm 1)^{2n} = 1$, we find the sum at the endpoints according to *Abel's theorem*,

$$\begin{aligned}
\lim_{x \to 1-} f(x) &= \lim_{x \to 1-} \frac{x \ln(1+x) - x \ln(1-x) + \ln(1-x) + \ln(1+x)}{2x^2} \\
&= \lim_{x \to 1-} \frac{1}{2x^2} \{(x+1)\ln(1+x) - (x-1)\ln(1-x)\} = \ln 2.
\end{aligned}$$

Alternatively,

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \lim_{N \to \infty} \sum_{n=0}^{N} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right) = \lim_{N \to \infty} \sum_{n=1}^{2N+2} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2.$$

Example 5.21 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{4n^2-1}\,x^{2n}.$$

Find its sum function in the interval of convergence. Check if the power series is convergent for $x = \varrho$ or for $x = -\varrho$.

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{\sqrt[n]{n}}{\sqrt[n]{4n^2-1}}\,x^2 \to x^2 \qquad \text{for } n \to \infty.$$

The condition of convergence $x^2 < 1$ implies that $|x| < 1$, hence $\varrho = 1$.

Convergence at the endpoints. If $x = \pm 1$, then we get the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n}{4n^2-1}.$$

Then by a decomposition,

$$\frac{n}{4n^2-1} = \frac{1}{4}\left(\frac{1}{2n-1}+\frac{1}{2n+1}\right) \to 0 \quad \textit{decreasingly} \text{ for } n \to \infty.$$

By **Leibniz's criterion** the series is convergent at the endpoints of the interval of convergence (same value which is found below by means of *Abel's theorem*, once the sum function is found).

Sum function. If $x = 0$, we get the sum $f(0) = 0$. If $0 < |x| < 1$, we get by the decomposition above that

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{4n^2-1}\,x^{2n} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\,x^{2n} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}\,x^{2n} \\ &= \frac{1}{4}\,x\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\,x^{2n+1} - \frac{1}{4x}\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\,x^{2n+1} + \frac{1}{4} = \frac{1}{4}\left(x-\frac{1}{x}\right)\operatorname{Arctan} x + \frac{1}{4}. \end{aligned}$$

Summing up we get

$$f(x) = \begin{cases} \dfrac{1}{4}\left(x-\dfrac{1}{x}\right)\operatorname{Arctan} x + \dfrac{1}{4} & \text{for } 0 < |x| < 1, \\ 0 & \text{for } x = 0. \end{cases}$$

Value at the endpoints. Since the series is convergent at the endpoints of the interval of convergence, we can apply **Abel's theorem:**

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n}{4n^2-1} = \lim_{x\to 1-} f(x) = \frac{1}{4}.$$

Example 5.22 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)(n+3)}\,x^{2n+4}.$$

Find its sum function in the interval of convergence. Check if the power series is convergent for $x = \varrho$ or for $x = -\varrho$.

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{1}{2}\cdot\frac{1}{\sqrt[n]{n+1}\cdot\sqrt[n]{n+3}}\cdot x^2\cdot\sqrt[n]{x^4} \to \frac{x^2}{2} \qquad \text{for } n\to\infty.$$

It follows from the condition of convergence $\dfrac{x^2}{2} < 1$ that $|x| < \sqrt{2}$, thus $\varrho = \sqrt{2}$.

Sum function. If $x = 0$, then $f(0) = 0$. If $0 < |x| < \sqrt{2}$, then we exploit the decomposition

$$\frac{1}{(n+1)(n+3)} = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

when we find the sum function

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{x^{2n+4}}{2^n(n+1)(n+3)} = \sum_{n=0}^{\infty}\frac{1}{n+1}\,\frac{x^{2(n+1)}}{2^{n+1}}\cdot x^2 - \frac{4}{x^2}\sum_{n=0}^{\infty}\frac{1}{n+3}\cdot\frac{x^{2(n+3)}}{2^{n+3}} \\ &= x^2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^2}{2}\right)^n - \frac{4}{x^2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^2}{2}\right)^n - \frac{x^2}{2} - \frac{1}{2}\left(\frac{x^2}{x}\right)^2\right\} \\ &= -x^2\ln\left(1-\frac{x^2}{2}\right) + \frac{4}{x^2}\ln\left(1-\frac{x^2}{2}\right) + 2 + \frac{x^2}{2}. \end{aligned}$$

Summing up we get the sum function

$$f(x)=\begin{cases} 2\left(\dfrac{2}{x^2}-\dfrac{x^2}{2}\right)\ln\left(1-\dfrac{x^2}{2}\right)+2+\dfrac{x^2}{2} & \text{for } 0<|x|<\sqrt{2}, \\ \\ 0 & \text{for } x=0. \end{cases}$$

We get at the **endpoints** $x=\pm\sqrt{2}$ by using the sequence of segments,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\sqrt{2})^4}{(n+1)(n+3)} &= \frac{4}{2}\sum_{n=0}^{\infty}\left\{\frac{1}{n+1}-\frac{1}{n+3}\right\}=2\lim_{N\to\infty}\left\{\sum_{n=0}^{N}\frac{1}{n+1}-\sum_{n=0}^{N}\frac{1}{n+3}\right\} \\ &= 2\lim_{N\to\infty}\left\{1+\frac{1}{2}-\frac{1}{N+2}-\frac{1}{N+3}\right\}=3.\end{aligned}$$

We can **alternatively** show the latter by **Abel's theorem**, because the series is convergent, using that

$$\frac{1}{(n+1)(n+3)}\sim\frac{1}{n^2},$$

hence

$$\lim_{x\to\pm\sqrt{2}}f(x)=2\cdot 0+2+\frac{2}{2}=3,$$

where we have applied that

$$\left(\frac{2}{x^2}-\frac{x^2}{2}\right)\ln\left(1-\frac{x^2}{2}\right)=\left(1+\frac{2}{x^2}\right)\left(1-\frac{x^2}{2}\right)\ln\left(1-\frac{x^2}{2}\right)\to 0 \qquad \text{for } \frac{x^2}{2}\to 1-\,.$$

Example 5.23 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{4^n(n+1)},$$

and find for each $x\in\,]-\varrho,\varrho[$ the sum function $f(x)$ of the series.
(Apply e.g. some suitable substitution).

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n|}=\frac{x^2}{4}\cdot\frac{\sqrt[n]{|x|}}{\sqrt[n]{n+1}}\to\frac{x^2}{4}=\left(\frac{|x|}{2}\right)^2 \qquad \text{for } n\to\infty.$$

The condition of convergence gives $\left(\dfrac{|x|}{2}\right)^2<1$, hence $|x|<2$, så $\varrho=2$.

Sum function. If $x=0$ then $f(0)=0$.

If $|x| \in]0,2[$, we get the sum function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{4^n(n+1)} = \frac{4}{x}\sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{x^2}{4}\right)^{n+1} = \frac{4}{x}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{4}{x}\right)^n = -\frac{4}{x}\ln\left(1-\frac{x^2}{4}\right).$$

Summing up we get

$$f(x) = \begin{cases} -\dfrac{4}{x}\ln\left(1-\dfrac{x^2}{4}\right) & \text{for } 0<|x|<2, \\ 0 & \text{for } x=0. \end{cases}$$

Remark 5.3 Obviously, the series is divergent at the endpoints of the interval of convergence, so we cannot apply Abel's theorem.

Example 5.24 *Let $F :]-\varrho,\varrho[\mapsto \mathbb{R}$ be the integral of*

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{4^n(n+1)},$$

for which $F(0)=0$. Find the power series for $F(x)$ and prove that it is convergent for $x=-\varrho$ and $x=\varrho$.
By means of the given formula, $\sum_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$, one shall find the value of the integral $\int_0^{\varrho} f(x)\,dx$.

Background. It is easily seen that $\varrho = 2$ and

$$f(x) = \begin{cases} -\dfrac{4}{x}\ln\left(1-\dfrac{x^2}{4}\right) & \text{for } 0<|x|<2, \\ 0 & \text{for } x=0. \end{cases}$$

Direct integration of $f(x)$ is not possible in practice, because we get by $t = \dfrac{x^2}{4}$,

$$\int f(x)\,dx = -\int \frac{4}{x^2}\ln\left(1-\frac{x^4}{4}\right)\cdot x\,dx = -2\int \frac{\ln(1-t)}{t}\,dt = -2\ln t\cdot\ln(1-t) + 2\int\frac{\ln t}{t-1}\,dt,$$

thus an integral of the same structure where one cannot proceed further.

We use instead for $|x|<2$ **termwise integration**

$$F(x) = \int_0^x \sum_{n=0}^{\infty}\frac{t^{2n+1}}{4^n(n+1)}\,dt = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{4^n}\,\frac{x^{2n+2}}{(n+1)^2} = 2\sum_{n=0}^{\infty}\frac{1}{n^2}\left(\frac{x}{2}\right)^{2n}.$$

This series is clearly absolutely convergent for $x=\varrho=2$, and

$$F(2) = \int_0^2 f(t)\,dt = 2\sum_{n=1}^{\infty}\frac{1}{n^2} = 2\cdot\frac{\pi^2}{6} = \frac{\pi^2}{3}.$$

Example 5.25 *Find the radius of convergence ϱ for the power series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n(3n+2)}{(n+1)(2n+1)}\, x^{2n+1}.$$

Find its sum function in the interval of convergence.
Prove that the series is conditionally convergent at the endpoints of the interval of convergence, and find the sum function for $x = \varrho$.

1) **Radius of convergence**. We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{\sqrt[n]{3n+2}}{\sqrt[n]{n+1}\cdot\sqrt[n]{2n+1}} \cdot x^2 \cdot \sqrt[n]{|x|} \to x^2 \qquad \text{for } n \to \infty.$$

The condition of convergence becomes $x^2 < 1$, thus $|x| < 1$, and $\varrho = 1$.

2) **Sum function**. If $x = 0$, then $f(0) = 0$.

Then by a decomposition,

$$\frac{3n+2}{(n+1)(2n+1)} = \frac{1}{n+1} + \frac{1}{2n+1},$$

thus if $0 < |x| < 1$, then we get the sum function

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n(3n+2)}{(n+1)(2n+1)} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\, x^{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1} \\ &= \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, (x^2)^n + \text{ Arctan } x = \frac{1}{x} \ln(1+x^2) + \text{ Arctan } x, \end{aligned}$$

i.e. by summing up,

$$f(x) = \begin{cases} \dfrac{1}{x} \ln(1+x^2) + \text{ Arctan } x & \text{for } 0 < |x| < 1, \\ \\ 0 & \text{for } x = 0. \end{cases}$$

3) **Conditional convergence at the endpoints**. Since

$$\frac{3n+2}{(n+1)(2n+1)} \geq \frac{1}{n+1},$$

and since $\sum_{n=0}^{\infty} \dfrac{1}{n+1}$ is divergent, it follows that the series is *not* absolutely convergent.

Now, $x^{2n+1} = x$ for $x = \pm 1$, and $\sum_{n=0}^{\infty} \dfrac{(-1)^n(3n+2)}{(n+1)(2n+1)}$ is *alternating* with

$$\frac{3n+2}{(n+1)(2n+1)} = \frac{1}{n+1} + \frac{1}{2n+1} \to 0 \qquad \textit{decreasingly.}$$

Hence it follows from **Leibniz's criterion** that the series is convergent and thus *conditionally convergent* at the endpoints of the interval of convergence.

4) **Value at the endpoints**. The series is convergent for $x = 1$, so it follows from **Abel's theorem** that the value is

$$\sum_{n=0}^{\infty} \frac{(-1)^n(3n+2)}{(n+1)(2n+1)} = \lim_{x \to 1-} f(x) = \frac{1}{1} \ln(1+1) + \text{ Arctan } 1 = \ln 2 + \frac{\pi}{4}.$$

Example 5.26 *1) Find the radius of convergence ϱ for the power series*

$$\sum_{n=1}^{\infty} \frac{3n+4}{n(n+1)(n+2)}\, x^n.$$

2) Prove that the series is absolutely convergent at the endpoints of the interval of convergence and find the sum of the series

$$\sum_{n=1}^{\infty} \frac{3n+4}{n(n+1)(n+2)}\, \varrho^n.$$

3) Prove that the sum of the power series is

$$f(x) = \begin{cases} \dfrac{1+x-2x^2}{x^2}\ln(1-x) + \dfrac{3x+2}{2x}, & \text{for } |x| < \varrho \text{ og } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

1) We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{\sqrt[n]{3n+4}}{\sqrt[n]{n}\cdot\sqrt[n]{n+1}\cdot\sqrt[n]{n+2}}\,|x| \to |x| \quad \text{for } n \to \infty.$$

The condition of convergence is here $|x| < 1$, so $\varrho = 1$.

2) Since $\dfrac{3n+4}{n(n+1)(n+2)} \sim \dfrac{3}{n^2}$, and since $\sum_{n=1}^{\infty} \dfrac{3}{n^2}$ is convergent, we conclude that series is absolutely convergent at the endpoints of the interval of convergence, hence for $x = \pm 1$.

Then by a *decomposition*,

$$\frac{3n+4}{n(n+1)(n+2)} = \frac{2}{n} - \frac{1}{n+1} - \frac{1}{n+2} = 2\left\{\frac{1}{n} - \frac{1}{n+1}\right\} + \left\{\frac{1}{n+1} - \frac{1}{n+2}\right\}.$$

This gives us the segmental sequence

$$\begin{aligned} s_N &= \sum_{n=1}^{N} \frac{3n+4}{n(n+1)(n+2)} = 2\sum_{n=1}^{N}\left\{\frac{1}{n} - \frac{1}{n+1}\right\} + \sum_{n=1}^{N}\left\{\frac{1}{n+1} - \frac{1}{n+2}\right\} \\ &= 2\left(1 - \frac{1}{N+1}\right) + \left(\frac{1}{2} - \frac{1}{N+2}\right) = \frac{5}{2} - \frac{2}{N+1} - \frac{1}{N+2} \to \frac{5}{2} \qquad \text{for } n \to \infty. \end{aligned}$$

We conclude that the sum of the series is

$$\sum_{n=1}^{\infty} \frac{3n+4}{n(n+1)(n+2)}\, \varrho^n = \lim_{N\to\infty} s_N = \frac{5}{2}.$$

3) If $x = 0$, then $f(0) = 0$.

If $0 < |x| < 1$, then it follows by the decomposition in (2) that

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{3n+4}{n(n+1)(n+2)}\,x^n = 2\sum_{n=1}^{\infty}\frac{1}{n}\,x^n - \sum_{n=1}^{\infty}\frac{1}{n+1}\,x^n - \sum_{n=1}^{\infty}\frac{1}{n+2}\,x^n \\
&= 2\sum_{n=1}^{\infty}\frac{1}{n}\,x^n - \frac{1}{x}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\,x^n - x\right\} - \frac{1}{x^2}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\,x^n - x - \frac{x^2}{2}\right\} \\
&= \left(2-\frac{1}{x}-\frac{1}{x^2}\right)\sum_{n=1}^{\infty}\frac{1}{n}\,x^n + 1 + \frac{1}{x} + \frac{1}{2} = \frac{1+x-2x^2}{x^2}\ln(1-x) + \frac{3x+2}{2x},
\end{aligned}$$

and the claim is proved.

Remark 5.4 Alternatively it is possible in (3) to expand the given function

$$\frac{1+x-2x^2}{x^2}\ln(1-x) + \frac{3x+2}{2x}$$

by known power series and then compare with the series in (1).

Remark 5.5 Since $(1-x)\ln(1-x) \to 0$ for $x \to 1-$ by the laws of magnitudes, we get (cf. Abel's theorem) that

$$\begin{aligned}\lim_{x\to 1-} f(x) &= \lim_{x\to 1-}\left\{\frac{1+x-2x^2}{x^2}\ln(1-x)+\frac{3x+2}{2x}\right\} = \frac{5}{2}+\lim_{x\to 1-}\frac{(1-x)(1+2x)}{x^2}\ln(1-x)\\ &= \frac{5}{2}+3\lim_{x\to 1-}(1-x)\ln(1-x)=\frac{5}{2},\end{aligned}$$

in accordance with the value found in (2).

Example 5.27 *Find the interval of convergence for the power series*

$$\sum_{n=1}^{\infty}\frac{6n^2x^{n+4}}{(n+1)(n+2)(n+3)(n+4)}.$$

Prove that the power series is absolutely convergent at the endpoints of the interval of convergence.

Since

$$\frac{6n^2x^{n+4}}{(n+1)(n+2)(n+3)(n+4)} \sim \frac{6n^2}{n^4}\cdot x^{n+4} = \frac{6}{n^2}\,x^{n+4},$$

and since $\sum_{n=1}^{\infty}\dfrac{6}{n^{n+4}}$ has the radius of convergence $\varrho = 1$, the same holds by the **criterion of equivalence** for the given series.

Since $\sum_{n=1}^{\infty}\dfrac{6}{n^2}$ is convergent, we conclude that both series are *absolutely* convergent at the endpoints of the interval of convergence.

Comment. One can actually find the sum function. First we get by a decomposition

$$\frac{6n^2}{(n+1)(n+2)(n+3)(n+4)} = \frac{1}{n+1}-\frac{12}{n+2}+\frac{27}{n+3}-\frac{16}{n+4}.$$

Hence for $|x|<1$,

$$\begin{aligned}f(x) &= \sum_{n=1}^{\infty}\frac{6n^2x^{n+4}}{(n+1)(n+2)(n+3)(n+4)}\\ &= \sum_{n=1}^{\infty}\frac{1}{n+1}\,x^{n+4}-12\sum_{n=1}^{\infty}\frac{1}{n+2}\,x^{n+4}+27\sum_{n=1}^{\infty}\frac{1}{n+3}\,x^{n+4}-16\sum_{n=1}^{\infty}\frac{1}{n+4}\,x^{n+4}\\ &= x^3\left\{\sum_{n=1}^{\infty}\frac{x^n}{n}-x\right\}-12x^2\left\{\sum_{n=1}^{\infty}\frac{x^n}{n}-x-\frac{x^2}{2}\right\}+27x\left\{\sum_{n=1}^{\infty}\frac{x^n}{n}-x-\frac{x^2}{2}-\frac{x^3}{3}\right\}\\ &\qquad -16\left\{\sum_{n=1}^{\infty}\frac{x^n}{n}-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}\right\}\\ &= -\left(x^3-12x^2+27x-16\right)\ln(1-x)+\frac{23}{6}\,x^3-19x^2+16x\\ &= (x^2-11x+16)\cdot(1-x)\ln(1-x)+\frac{23}{6}\,x^3-19x^2+16x.\end{aligned}$$

We get in particular (cf. *Abel's theorem*),

$$\lim_{x\to 1-} f(x) = \frac{23}{6} - 19 + 16 = \frac{5}{6}$$

and

$$\lim_{x\to -1+} f(x) = 28\cdot 2\cdot \ln 2 - \frac{23}{6} - 19 - 16 = 56\ln 2 - \frac{233}{6}.$$

Example 5.28 *Find the radius of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(n+1)(2n+3)}\, x^{2n+3} = \sum_{n=0}^{\infty}(-1)^n \left\{\frac{1}{n+1} - \frac{1}{2n+3}\right\} x^{2n+2}.$$

Find its sum in the interval of convergence.

Obviously, $\dfrac{n+2}{(n+1)(2n+3)} = \dfrac{1}{n+1} - \dfrac{1}{2n+3}$, hence we have equality.

Then we get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{\sqrt[n]{n+2}}{\sqrt[n]{n+1}\cdot\sqrt[n]{2n+3}}\cdot |x|^2 \cdot \sqrt[n]{x^2} \to |x|^2 \qquad \text{for } n\to\infty.$$

The condition of convergence $|x|^2 < 1$ implies that $|x| < 1$, hence $\varrho = 1$.

Sum function. If $x = 0$, then $f(0) = 0$. If $0 < |x| < 1$, then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}(-1)^n\left\{\frac{1}{n+2} - \frac{1}{2n+3}\right\} x^{2n+2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\,(x^2)^{n+1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3}\,x^{2(n+1)} \\ &= \ln(1+x^2) + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\,x^{2n} = \ln(1+x^2) + \frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\,x^{2n+1} - 1 \\ &= \ln(1+x^2) + \frac{1}{x}\,\text{Arctan}\, x - 1. \end{aligned}$$

Summing up we get

$$f(x) = \begin{cases} \ln(1+x^2) + \dfrac{1}{x}\,\text{Arctan}\, x - 1, & \text{for } 0 < |x| < 1, \\ 0 & \text{for } x = 0. \end{cases}$$

Remark 5.6 We get at the endpoints the *alternating* series

$$\sum_{n=1}^{\infty}(-1)^n\,\frac{n+2}{(n+1)(2n+3)} = \sum_{n=1}^{\infty}\left\{\frac{1}{n+1} - \frac{1}{2n+3}\right\}.$$

Since

$$\frac{1}{n+1} - \frac{1}{2n+3} \to 0 \qquad \textit{decreasingly},$$

the series is convergent according to **Leibniz's criterion**.

Now $\dfrac{n+2}{(n+1)(2n+3)} \sim \dfrac{1}{2n}$, and $\sum_{n=1}^{\infty} \dfrac{1}{2n}$ is divergent, so the series is not absolutely convergent, hence it is *conditionally* convergent.

Finally, we get by **Abel's theorem**,

$$\sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{n+1} - \frac{1}{2n+3} \right\} = \lim_{x \to 1-} f(x) = \ln 2 + \frac{\pi}{4} - 1.$$