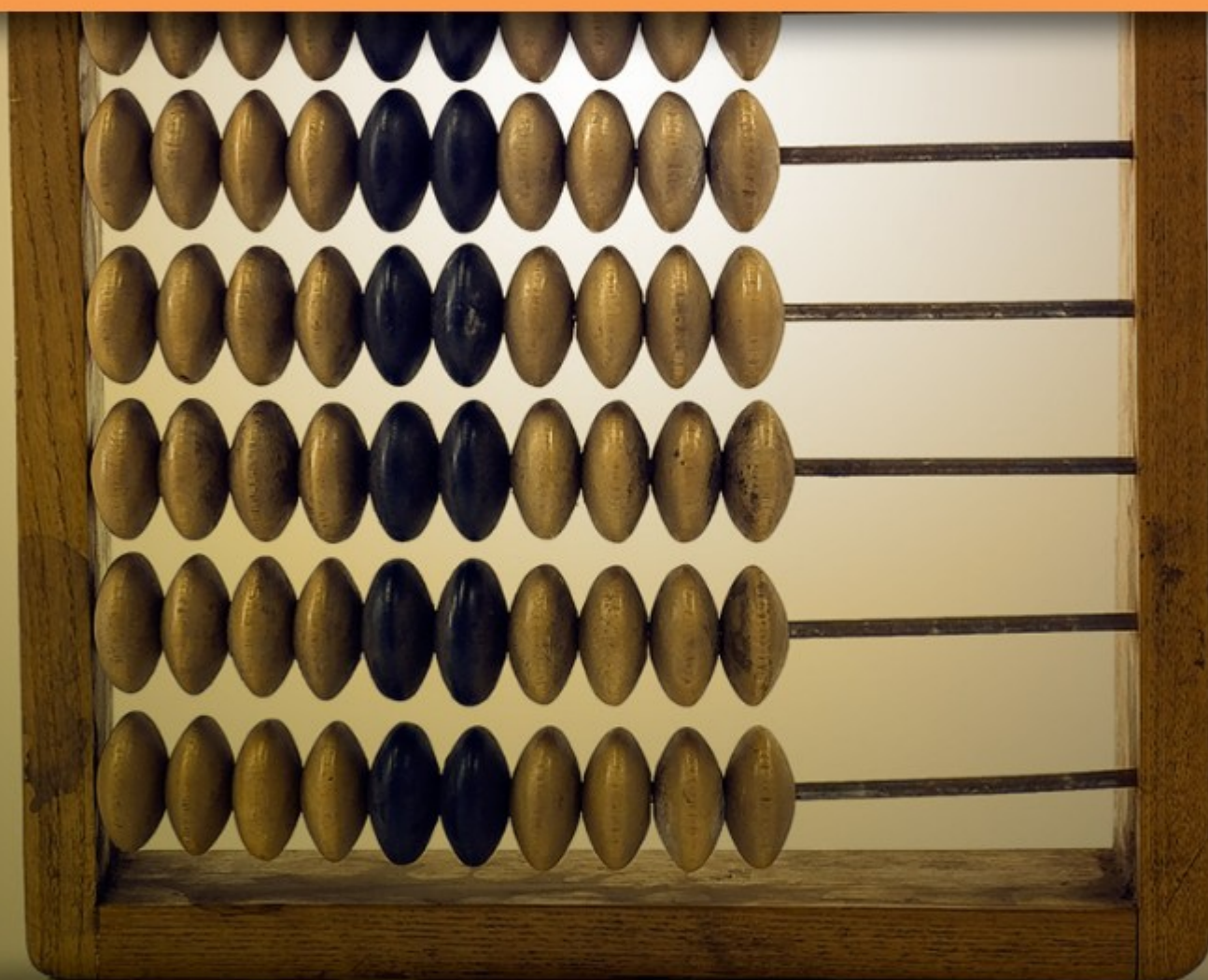


# Real Functions of Several Variables - Guidelines..

Leif Mejlbro



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# Real Functions of Several Variables

Guidelines for Solutions of Some Types of Problems

Calculus 2b

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Real Functions of Several Variables – Guidelines for Solutions of Some Types of  
Problems – Calculus 2b

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\* Figures taken from London Business School's Masters in Management 2010 employment report

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## Preface

The purpose of this volume is to give some guidelines for the student concerning the solution of problems in the theory of Functions in Several Variables.

The intension is *not* to write a textbook, but instead to give some hints of how to solve problems in this field. It therefore cannot replace any given textbook, but it may be used as a supplement to such a book on Functions in Several Variables.

The chapters have been kept as short as possible, each dealing with one particular type of standard problem, supplied with short descriptions of the possible methods of solutions which may be available, their advantages and also their shortcomings.

In Appendix A the reader will find a collection of formulæ which otherwise tacitly are assumed to be known from high school. It is highly recommended that the student learns these by heart during the course, because they form the backbone of the elementary part of Calculus, which should be mastered, before one may proceed to more advanced parts of Mathematics.

The text is a continuation of *Calculus 1a, Real Functions in One Variable*. It will be followed by a volume called *Calculus 2c*, in which selected examples from the theory of functions in several variables are considered.

The text is based on my experiences in my teaching of students in this course. I realized that there was absolutely a need for a practical description of how to solve explicit problems.

Leif Mejlbro

In the revision of these notes the publisher and I agreed that the title should be *Calculus* followed by a number (1–3) and a letter, where

- a stands for *compendia*,
- b stands for *procedures of solutions*,
- c stands for *examples*.

Therefore, *Calculus 2b* means that this note is the second one on Calculus, where *b* indicates that it is a collection of procedure of solutions.

Leif Mejlbro  
27th April 2008

# 1 Integration of trigonometric polynomials

**Problem 1.1** Calculate the integral

$$\int \sin^m x \cos^n x \, dx \quad \text{for } m, n \in \mathbb{N}_0.$$

**Notation:** By the *degree* of the product  $\sin^m x \cos^n x$  we shall understand the sum  $m + n$  of the exponents.

**Division of the problem:** There are two main cases, *odd* and *even* degree. Each of these is again divided into two *subcases*:

1)  $m + n$  *odd*.

a)  $m$  even and  $n$  odd, i.e.  $m = 2p$  and  $n = 2q + 1$ ,  $p, q \in \mathbb{N}_0$ ,

b)  $m$  odd and  $n$  even, i.e.  $m = 2p + 1$  and  $n = 2q$ ,  $p, q \in \mathbb{N}_0$ .

2)  $m + n$  *even*.

a)  $m$  and  $n$  are both odd, i.e.  $m = 2p + 1$  and  $n = 2q + 1$ ,  $p, q \in \mathbb{N}_0$ ,

b)  $m$  and  $n$  are both even, i.e.  $m = 2p$  and  $n = 2q$ ,  $p, q \in \mathbb{N}_0$ .

The most difficult case is 2b), where both  $m$  and  $n$  are even.

**Method of solution:**

1) a)  $m = 2p$  and  $n = 2q + 1$ .

Apply the substitution  $u = \sin x$  corresponding to  $m = 2p$  *even*:

$$\begin{aligned} \int \sin^{2p} x \cos^{2q+1} x \, dx &= \int \sin^{2p} x \cdot \cos^{2q} x \cdot \cos x \, dx \\ &= \int \sin^{2p} x \cdot (1 - \sin^2 x)^q \, d \sin x \\ &= \int_{u=\sin x} u^{2p} (1 - u^2)^q \, du. \end{aligned}$$

b)  $m = 2p + 1$  and  $n = 2q$ .

Apply the substitution  $u = \cos x$  corresponding to  $n = 2q$  *even*:

$$\begin{aligned} \int \sin^{2p+1} x \cdot \cos^{2q} x \, dx &= \int \sin^{2p} x \cdot \cos^{2q} x \cdot \sin x \, dx \\ &= \int (1 - \cos^2 x)^p \cos^{2q} x \cdot (-1) \, d \cos x \\ &= - \int_{u=\cos x} (1 - u^2)^p u^{2q} \, du. \end{aligned}$$

2) When the degree  $m + n$  is *even*, the trick is to change the problem to a similar one by doubling the angle, thereby halving the degree. Therefore, we use the formulæ

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \sin x \cos x = \frac{1}{2} \sin 2x.$$



a)  $m = 2p + 1$  and  $n = 2q + 1$  are both *odd*.

The integrand is transformed in the following way:

$$\begin{aligned}\sin^{2p+1} x \cdot \cos^{2q+1} x &= (\sin^2 x)^p \cdot (\cos^2 x)^q \cdot \sin x \cos x \\ &= \left\{ \frac{1}{2} (1 - \cos 2x) \right\}^p \left\{ \frac{1}{2} (1 + \cos 2x) \right\}^q \cdot \frac{1}{2} \sin 2x.\end{aligned}$$

Hence we get a special case of 1b), so by the substitution  $u = \cos 2x$  we get

$$\begin{aligned}\int \sin^{2p+1} x \cos^{2q+1} x dx &= \frac{1}{2^{p+q+1}} \int (1 - \cos 2x)^p (1 + \cos 2x)^q \sin 2x dx \\ &= \frac{1}{2^{p+q+1}} \int (1 - \cos 2x)^p (1 + \cos 2x)^q \cdot \left( -\frac{1}{2} \right) d \cos 2x \\ &= -\frac{1}{2^{p+q+2}} \int_{u=\cos 2x} (1 - u)^p (1 + u)^q du.\end{aligned}$$

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b)  $m = 2p$  and  $n = 2q$  are both *even*.

In this case there is no final formula, but there is a procedure by which we can reduce the problem to a sum of several problems of the types 1a) and 2b) of lower degree. The result is obtained after a finite number of steps.

The integrand is rewritten in the following way:

$$\sin^{2p} x \cos^{2q} x = \left\{ \frac{1}{2} (1 - \cos 2x) \right\}^p \left\{ \frac{1}{2} (1 + \cos 2x) \right\}^q.$$

The left hand side is of degree  $2p+2q$  in  $x$ . The right hand side is a trigonometric polynomial of degree  $p+q$  in the doubled angle  $2x$ . Each term of this polynomial must be handled separately, depending on whether the degree  $j(\leq p+q)$  is *odd* (case 1a)) or *even* (case 2b)).

**Remark 1.1** It is of course in principle possible to create a specific solution formula, but it will be more confusing than the description of the procedure given above.  $\diamond$

## 2 Complex decomposition of a fraction between two polynomials

**Problem 2.1** Write the quotient  $\frac{P(x)}{Q(x)}$  between two polynomials as a sum of elementary fractions.

**Remark 2.1** This problem occurs typically in connection with integration, and in courses on series also in telescopic summation. If the denominator has complex conjugated roots of at least order 2, a complex decomposition is usually the easiest method. If the order is 1, then real decomposition may be applied instead. We shall here show the method of *complex decomposition*.  $\diamond$

### Procedure.

- 1) If the degree of the numerator is  $\geq$  the degree of the denominator, we first perform a *division by the denominator*,

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{R(x)}{Q(x)},$$

where the residual polynomial  $R(x)$  (the new numerator) has degree smaller than the degree of  $Q(x)$ . We save the resulting polynomial  $P_1(x)$  for the last step.

- 2) The *denominator*  $Q(x)$  is then factorized into polynomials of degree one (with *complex* roots):

$$Q(x) = c \cdot (x - a_1)^{p_1} \cdots (x - a_k)^{p_k}.$$

Check that the sum  $p_1 + \cdots + p_k$  of all exponents is equal to the degree of  $Q(x)$ . If  $Q(x)$  is a real polynomial, check that the complex conjugated roots occur of the *same degree*.

- 3) Choose the simplest of the polynomials  $P(x)$  and  $R(x)$ . The following method gives the same result, whether  $P(x)$  or  $R(x)$  is used. Since it is here theoretically most correct to use  $R(x)$ , we shall use  $R(x)$  in the rest of this description, and it is left to the reader to write  $P(x)$  instead of  $R(x)$ , whenever this will give us simpler calculations.
- 4) The fraction is rewritten in the following way

$$\frac{R(x)}{Q(x)} = \frac{1}{c} \cdot \frac{R(x)}{(x - a_1)^{p_1} \cdots (x - a_k)^{p_k}}.$$

We get the coefficient of the special simple fraction

$$\frac{1}{(x - a_1)^{p_1}}$$

by “holding the hand over” the factor  $(x - a_1)^{p_1}$  in the denominator and then putting  $x = a_1$  in the remainder part:

$$b_{1,p_1} = \frac{1}{c} \cdot \left. \frac{R(x)}{(x - a_2)^{p_2} \cdots (x - a_k)^{p_k}} \right|_{x=a_1}.$$

We save the result

$$\frac{b_{1,p_1}}{(x - a_1)^{p_1}}$$

for the last step in this procedure.

5) Repeat 4) on any other of the factors

$$(x - a_2)^{p_2}, \quad \dots \quad (x - a_k)^{p_k},$$

in the denominator and save all the found special fractions.

6) Subtract all the found special fractions from  $\frac{R(x)}{Q(x)}$  and reduce:

$$\begin{aligned} \frac{1}{c} \cdot \frac{R(x)}{(x - a_1)^{p_1} \dots (x - a_k)^{p_k}} - \frac{b_{1,p_1}}{(x - a_1)^{p_1}} - \dots - \frac{b_{k,p_k}}{(x - a_k)^{p_k}} \\ = \frac{1}{d} \cdot \frac{R_1(x)}{(x - a_1)^{q_1}} \dots (x - a_k)^{q_k}. \end{aligned}$$

If the calculation is performed without errors, then

$$q_1 < p_1, \quad \dots \quad q_k < p_k.$$

Check this! (A weak test.)

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7) Repeat 4), 5) and 6) on the reduced fraction

$$\frac{1}{d} \cdot \frac{R_1(x)}{(x - a_1)^{q_1} \cdots (x - a_k)^{q_k}}.$$

**Remember** in each step to write down the elementary fractions which have been found. The process must necessarily stop after a finite number of steps, because the degree of the denominator is becoming smaller by each iteration.

8) Finally, collect all the found elementary fractions together with the polynomial from 1).

The description above is the standard procedure. My experience has shown me that one often can find shortcuts, which are impossible to systemize here. I shall therefore here only give one example of many possibilities.

**Example 2.1** Let us here try to decompose the fractional function

$$\frac{1}{x^4 - 1}.$$

1) *The standard procedure* as described above. The denominator has the simple roots 1,  $i$ ,  $-1$ ,  $-i$ , hence

$$\begin{aligned} \frac{1}{x^4 - 1} &= \frac{1}{(x-1)(x-i)(x+1)(x+i)} \\ &= \frac{1}{(1-i)(1+i)(1+i)} \cdot \frac{1}{x-1} + \frac{1}{(i-1)(i+1)(i+i)} \cdot \frac{1}{x-i} \\ &\quad + \frac{1}{(-1-1)(-1-i)(-1+i)} \cdot \frac{1}{x+1} + \frac{1}{(-i-1)(-i-i)(-i+1)} \cdot \frac{1}{x+i} \\ &= \frac{1}{4} \cdot \frac{1}{x-1} - \frac{1}{4i} \cdot \frac{1}{x-i} - \frac{1}{4} \cdot \frac{1}{x+1} + \frac{1}{4i} \cdot \frac{1}{x+i} \\ &= \frac{1}{4} \cdot \frac{1}{x-1} - \frac{1}{4} \cdot \frac{1}{x+1} - \frac{1}{4i} \left\{ \frac{1}{x-i} - \frac{1}{x+i} \right\} \\ &= \frac{1}{4} \cdot \frac{1}{x-1} - \frac{1}{4} \cdot \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x^2 + 1}. \end{aligned}$$

This is of course fairly tiresome, though it works.

2) *Alternatively* it is seen that

$$x^4 - 1 = (x^2)^2 - 1 = (x^2 + 1)(x^2 - 1),$$

so if we write  $u = x^2$ , and first decompose after  $u$  followed by a decomposition after  $x$ , we easily get in two simpler steps that

$$\begin{aligned} \frac{1}{x^4 - 1} &= \frac{1}{u^2 - 1} = \frac{1}{(u-1)(u+1)} = \frac{1}{2} \frac{1}{u-1} - \frac{1}{2} \frac{1}{u+1} \\ &= \frac{1}{2} \frac{1}{x^2 - 1} - \frac{1}{2} \frac{1}{x^2 + 1} = \frac{1}{2} \frac{1}{(x-1)(x+1)} - \frac{1}{2} \frac{1}{x^2 + 1} \\ &= \frac{1}{4} \frac{1}{x-1} - \frac{1}{4} \frac{1}{x+1} - \frac{1}{2} \frac{1}{x^2 + 1}. \quad \diamond \end{aligned}$$

### 3 Integration of a fraction between two polynomials

**Problem 3.1** Calculate  $\int \frac{P(x)}{Q(x)} dx$ , where  $P(x)$  and  $Q(x)$  are (real) polynomials.

**Remark 3.1** In some cases the pocket calculators TI-92 and TI-89 may for some unknown reasons give strange results, so one cannot always rely on them.  $\diamond$

**Procedure.**

- 1) Decompose  $\frac{P(x)}{Q(x)}$  as described in the previous chapter on *complex decomposition*.


Then  $\frac{P(x)}{Q(x)}$  is written as a sum of a polynomial  $P_1(x)$  and some elementary fractions of the type  $\frac{c}{(x-a)^p}$ .

- 2) The polynomial  $P_1(x)$  is integrated in the usual way.
- 3) The elementary fractions where  $p > 1$  are also integrated in the usual way


$$\int \frac{c}{(x-a)^p} dx = -\frac{c}{p-1} \cdot \frac{1}{(x-a)^{p-1}},$$

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no matter whether  $a$  is real or complex. If  $P(x)$  and  $Q(x)$  are real, then any complex fraction of the type  $\frac{c}{(x-a)^p}$  will be accompanied by its complex conjugated fraction  $\frac{\bar{c}}{(x-\bar{a})^p}$ . This means that the integration of such a pair of complex conjugated fractions can be reduced to

$$\begin{aligned} \int \left\{ \frac{c}{(x-a)^p} + \frac{\bar{c}}{(x-\bar{a})^p} \right\} dx &= -\frac{1}{p-1} \left\{ \frac{c}{(x-a)^{p-1}} + \frac{\bar{c}}{(x-\bar{a})^{p-1}} \right\} \\ &= -\frac{2}{p-1} \operatorname{Re} \left\{ \frac{c}{(x-a)^{p-1}} \cdot \frac{(x-\bar{a})^{p-1}}{(x-\bar{a})^{p-1}} \right\} \\ &= -\frac{2}{p-1} \cdot \frac{\operatorname{Re}\{c \cdot (x-\bar{a})^{p-1}\}}{\{x^2 - 2 \operatorname{Re} a \cdot x + |a|^2\}^{p-1}} \end{aligned}$$

4) If  $p = 1$ , and  $a$  is real, then of course

$$\int \frac{x}{x-a} dx = c \cdot \ln |x-a|.$$

5) If  $p = 1$ , and  $a$  is complex, then both  $\frac{c}{x-a}$  and  $\frac{\bar{c}}{x-\bar{a}}$  occur in the decomposition. A direct integration is not possible, unless one knows the theory of *Complex Functions*. Instead we add the two elementary fractions *before* the integration. (Notice that when  $p > 1$ , this is done *after* the integration, cf. 3)).

More precisely we put  $a = \alpha + i\beta$  and  $c = r + is$ . Then

$$\begin{aligned} \frac{c}{x-a} + \frac{\bar{c}}{x-\bar{a}} &= \frac{r+is}{x-\alpha-i\beta} + \frac{r-is}{x-\alpha+i\beta} \\ &= \frac{(r+is)(x-\alpha-i\beta) + (r-is)(x-\alpha+i\beta)}{(x-\alpha)^2 + \beta^2} \\ &= \frac{2r(x-\alpha)}{(x-\alpha)^2 + \beta^2} - \frac{2s\beta}{(x-\alpha)^2 + \beta^2}, \end{aligned}$$

whence

$$\begin{aligned} \int \left\{ \frac{c}{x-a} + \frac{\bar{c}}{x-\bar{a}} \right\} dx &= r \int \frac{2(x-\alpha)}{(x-\alpha)^2 + \beta^2} dx - 2s \int \frac{1}{1 + \left(\frac{x-\alpha}{\beta}\right)^2} \frac{1}{\beta} dx \\ &= r \cdot \ln \{(x-\alpha)^2 + \beta^2\} - 2s \operatorname{Arctan} \left( \frac{x-\alpha}{\beta} \right). \end{aligned}$$

6) The final result is obtained by gathering all the results from 2), 3), 4) and 5).

**Example 3.1** In Example 2.1 we found the decomposition

$$\frac{1}{x^4-1} = \frac{1}{4} \frac{1}{x-1} - \frac{1}{4} \frac{1}{x+1} - \frac{1}{2} \frac{1}{x^2+1},$$

from which we immediately get

$$\int \frac{1}{x^4-1} dx = \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \operatorname{Arctan} x, \quad x \neq \pm 1. \quad \diamond$$



## 4 The domain of a function

**Problem 4.1** Let the structure of a function  $f(x, y, \dots)$  be given. Find the maximum domain of this function, based on this structure.

**Procedure.**

- 1) Divide the function into subfunctions according to the signs  $+$  and  $-$ , i.e. write  $f = f_+ + f_-$ , where  $f_+ \geq 0$  and  $f_- \leq 0$  (and  $f_+ \cdot f_- = 0$ ).
- 2) Find the domain for each of the subfunctions (if possible, sketch a figure).
- 3) Then the domain of  $f$  is the intersection of the domains of subfunctions. (Sketch a figure).

If  $\mathbf{f}(x, y, \dots)$  is a *vector function*, then perform the above separately for each coordinate function. The domain is the intersection of all the domains of the coordinate functions.

One should in particular be aware of the following rules:

- 1) **Never** divide by 0.

Analyze the set of zeros for the denominator, if it exists.

- 2) In real analysis, **never** take the square root of a negative number.

Find the set of zeros of the radicand of the square root. Check the sign in the domains which are bounded by this set of zeros.

- 3) In real analysis **never** take the logarithm of a negative number or of 0.

Find the set of zeros of the expression which we are going to take the logarithm of. Check the sign in the domains which are bounded by this set of zeros.

**Remark 4.1** Experience tells that the square root is in particular difficult to handle. A professor once said to me that “if one can handle the square root, then one can handle anything in mathematics!”. Notice that pocket calculators does not like square roots either.  $\diamond$

## 5 The chain rule

**Problem 5.1** Let  $\mathbf{f} = (f_1, \dots, f_m)$  be a differentiable vector function in the  $k$  real variables  $x_1, \dots, x_k$ , and assume that these again are differentiable functions in the  $n$  variables  $u_1, \dots, u_n$ . Find the (partial) derivatives of  $(f_1, \dots, f_m)$  after  $u_1, \dots, u_n$ .

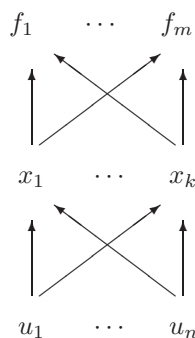


Figure 1: The general diagram of the chain rule.

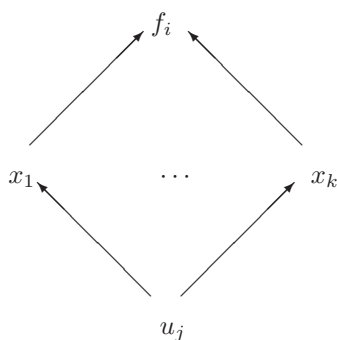


Figure 2: The reduced diagram of the chain rule.

### Procedure.

- 1) Sketch the general diagram as in figure 1, and reduce the  $i$ -th  $\mathbf{f}$ -coordinate and the  $j$ -th  $\mathbf{u}$ -coordinate as shown on figure 2.
- 2) “Pull the differentiation apart” in the following way with  $k$  specimens (i.e. the number of  $\mathbf{x}$ -coordinates) on the right hand side

$$\frac{\partial f_i}{\partial u_j} = \frac{\partial f_i}{\partial} \frac{\partial}{\partial u_j} + \dots + \frac{\partial f_i}{\partial} \frac{\partial}{\partial u_j}.$$

- 3) The empty places are then filled in with all the variables  $x_1, \dots, x_k$  from the layer in the middle,

$$\frac{\partial f_i}{\partial u_j} = \frac{\partial f_i}{\partial x_1} \frac{\partial x_1}{\partial u_j} + \dots + \frac{\partial f_i}{\partial x_k} \frac{\partial x_k}{\partial u_j}.$$

- 4) Repeat this process for every relevant  $i$  and  $j$ .

**Remark 5.1** If one of the layers of differentiations only contains one variable, then  $\partial$  is replaced by  $d$ , i.e. one writes  $\frac{d \dots}{d \dots}$  instead of  $\frac{\partial \dots}{\partial \dots}$ .  $\diamond$

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## 6 Calculation of the directional derivative

**Geometric interpretation:** Assume that  $f(\mathbf{x})$  is a differentiable function. Then the *directional derivative*

$$f'(\mathbf{x}; \mathbf{e})$$

of  $f$  at the point  $\mathbf{x}$  and in the direction  $\mathbf{e}$  indicates how much  $f(\mathbf{x})$  increases (decreases) per unit in the direction  $\mathbf{e}$ . By a *direction* we shall always understand a *unit vector*  $\mathbf{e}$ , i.e.  $\|\mathbf{e}\| = 1$ . In this case we have

$$f'(\mathbf{x}; \mathbf{e}) = \mathbf{e} \cdot \nabla f(\mathbf{x}).$$

Typical problems are:

**Problem 6.1** Let a unit vector  $\mathbf{e}$  be given.

Find the directional derivative  $f'(\mathbf{x}; \mathbf{e})$  of  $f$  in the direction  $\mathbf{e}$ .

**Procedure.**

- 1) Calculate the gradient  $\nabla f(\mathbf{x})$ .
- 2) Calculate the inner product  $f'(\mathbf{x}; \mathbf{e}) = \mathbf{e} \cdot \nabla f(\mathbf{x})$ .

**Problem 6.2** Given two points  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .

Find the directional derivative of  $f(\mathbf{x})$  at  $\mathbf{x}_1$  in the direction of  $\mathbf{x}_2$ .

**Procedure.**

- 1) Calculate the gradient  $\nabla f(\mathbf{x})$  i  $\mathbf{x}_1$ .
- 2) Find the directional vector  $\mathbf{e}$  from  $\mathbf{x}_1$  to  $\mathbf{x}_2$ :

$$\mathbf{e} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{\|\mathbf{x}_2 - \mathbf{x}_1\|},$$

(Do not forget to find the norm of the vector).

- 3) The directional derivative is given by

$$f'(\mathbf{x}_1; \mathbf{e}) = \mathbf{e} \cdot \nabla f(\mathbf{x}_1) = \frac{1}{\|\mathbf{x}_2 - \mathbf{x}_1\|} (\mathbf{x}_2 - \mathbf{x}_1) \cdot \nabla f(\mathbf{x}_1).$$

**Remark 6.1** The definition is extremely simple. Nevertheless it is causing the students a lot of trouble for some reason which is not understood by me. I have taken the consequence to include this chapter here. Notice that when we norm  $\nabla f(\mathbf{x})$ , we obtain the direction in which the function  $f(\mathbf{x})$  has its highest increase.  $\diamond$

## 7 Approximating polynomials

The most common case is given by the following problem. Other cases are obtained by suitable modifications of it.

**Problem 7.1** Find the approximating polynomial of at most second degree for a function  $f(x, y)$  in two variables from the point  $(x_0, y_0)$  in its open domain.

There are here several possible methods, which all have there advantages and disadvantages, so one cannot say that one particular method is always the easiest one to use. However, when the student encounters this problem for the first time, her or his preference will probably without doubt be the following

### A. Standard procedure

- 1) Start by explaining (*text*), why  $f$  is a  $C^2$ -function in the neighbourhood of  $(x_0, y_0)$ .
- 2) Calculate the following equations:

$$\begin{aligned} \text{order zero:} \quad & \{ f(x, y) = \cdots, \quad f(x_0, y_0) = \cdots, \\ \text{first order:} \quad & \begin{cases} f'_x(x, y) = \cdots, & f'_x(x_0, y_0) = \cdots, \\ f'_y(x, y) = \cdots, & f'_y(x_0, y_0) = \cdots, \end{cases} \\ \text{second order:} \quad & \begin{cases} f''_{xx}(x, y) = \cdots, & f''_{xx}(x_0, y_0) = \cdots, \\ f''_{xy}(x, y) = \cdots, & f''_{xy}(x_0, y_0) = \cdots, \\ f''_{yy}(x, y) = \cdots, & f''_{yy}(x_0, y_0) = \cdots. \end{cases} \end{aligned}$$

- 3) Insert the values of the column to the right into the formula

$$\begin{aligned} P_2(x, y) = & f(x_0, y_0) + \frac{1}{1!} \{ f'_x(x_0, y_0) \cdot (x - x_0) + f'_y(x_0, y_0) \cdot (y - y_0) \} \\ & + \frac{1}{2!} \{ f''_{xx}(x_0, y_0) \cdot (x - x_0)^2 + 2 f''_{xy}(x_0, y_0) \cdot (x - x_0)(y - y_0) + f''_{yy}(x_0, y_0) \cdot (y - y_0)^2 \}. \end{aligned}$$

**Remark 7.1** Since the approximating polynomial is the best description of  $f(x, y)$  in a neighbourhood of  $(x_0, y_0)$ , the right variables here are always  $x - x_0$  and  $y - y_0$ , and *not*  $x$  and  $y$ . Therefore, one shall *not* reduce the expression further to a polynomial in  $x$  and  $y$  alone, because we by doing this will obtain an expression with a higher numerical uncertainty!  $\diamond$

**B. Taylor expansions**

- 1) Explain (*text*) that  $f(x, y)$  is composed of *standard functions*, for which a Taylor expansion already is known from e.g. *Calculus 1a, Functions in one Variable*.
- 2) Reset the problem to zero, i.e. change the variables to

$$(h, k) = (x - x_0, y - y_0), \quad (x, y) = (x_0 + h, y_0 + k).$$

Then we get  $f(x, y) = F(h, k)$  in the new variables  $(h, k)$ .

- 3) Apply convenient *standard expansions* in  $F(h, k)$ . We list all the usual standard expansions for  $t$  small, which should be known from previous courses,

$$\frac{1}{1-t} = 1 + t + t^2 + \dots,$$

$$(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} t^2 + \dots,$$

$$\text{Arctan } t = t - \dots,$$

$$\sin t = t - \dots,$$

$$\sinh t = t + \dots,$$

$$\frac{1}{1+t} = 1 - t + t^2 - \dots,$$

$$\ln(1+t) = t - \frac{1}{2} t^2 + \dots,$$

$$e^t = 1 + t + \frac{1}{2} t^2 + \dots,$$

$$\cos t = 1 - \frac{1}{2} t^2 + \dots,$$

$$\cosh t = 1 + \frac{1}{2} t^2 + \dots,$$

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where the dots indicate terms of the type  $t^2\varepsilon(t)$ .

- 4) Calculate  $F(h, k)$ , where every term which contains *at least* three factors of the type  $h$ ,  $k$ , is symbolized by  $\cdots$  (of the type  $t^2\varepsilon(t)$ ).
- 5) One obtains the approximating polynomial by deleting the dots and then change variables back to  $(h, k) = (x - x_0, y - y_0)$ .

**Remark 7.2** One should always be very careful to rewrite to one of the ten standard functions above. We have for example (for  $\alpha = 1/2$ )

$$\begin{aligned}\sqrt{25+t} &= 5 \left(1 + \frac{t}{25}\right)^{\frac{1}{2}} = 5 \left\{1 + \frac{1}{2} \frac{t}{25} - \frac{1}{8} \frac{t^2}{625} + \cdots\right\} \\ &= 5 + \frac{1}{10} t - \frac{1}{1000} t^2 + \cdots, \quad t \text{ small.} \quad \diamond\end{aligned}$$

The following example shows that Taylor expansions may be easier to use than the standard procedure:

**Example 7.1** Find the approximating polynomial of at most second degree for the function

$$f(x, y) = \exp(x - y^2) \operatorname{Arctan}(x + 2y) \cos(x^2 + 4y)$$

from the point  $(x_0, y_0) = (2, -1)$ .

It is obvious that the standard procedure will give us a mess of calculations! Let us therefore turn to the method of using known Taylor expansions.

1. The function is a product of standard functions, where we know the Taylor expansions.
2. The change of variables is here

$$(h, k) = (x - 2, y + 1), \quad \text{i.e.} \quad (x, y) = (2 + h, -1 + k).$$

In particular we get for  $t = h + 2k$  that

$$\operatorname{Arctan}(x + 2y) = \operatorname{Arctan}(h + 2k) = (h + 2k) + \cdots,$$

so it suffices to expand the other factors of only *first degree*, since one degree is used in the factor  $\operatorname{Arctan}(h + 2k)$ .

**3. and 4.** Since

$$\begin{aligned}\exp(x - y^2) &= \exp(1 + h + 2k - k^2) = e \cdot \exp(h + k) \cdot \exp(-k^2) \\ &= e \cdot \{1 + (h + 2k) + \cdots\} \cdot \{1 + \cdots\} = e + e \cdot (h + 2k) + \cdots,\end{aligned}$$

and

$$\cos(x^2 + 4y) = \cos(4h + 4k + k^2) = 1 + \cdots,$$

we get

$$\begin{aligned}f(x, y) &= \exp(1 + h + 2k - k^2) \operatorname{Arctan}(h + 2k) \cos(4h + 4k + k^2) \\ &= \{e + e \cdot (h + 2k) + \cdots\} \cdot \{(h + 2k) + \cdots\} \cdot \{1 + \cdots\} \\ &= e \cdot (h + 2k) + e(h + 2k)^2 + \cdots.\end{aligned}$$



5. The approximating polynomial is obtained by deleting the dots and then use the inverse transformation of variables,

$$\begin{aligned}P_2(x, y) &= e(h + 2k) + e(h + 2k)^2 \\&= e\{(x-2) + 2(y+1) + (x-2)^2 + 4(x-2)(y+1) + 4(y+1)^2\}.\end{aligned}$$

It should be noticed that there also exist examples where the standard procedure is the easiest one, so it is impossible to say in advance that one method is always better than the other one.  $\diamond$

## 8 Gradient fields; antiderivatives

As the main case we consider vector fields in  $\mathbb{R}^3$ , i.e.

$$\mathbf{V}(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z)),$$

which is assumed to be of class  $C^1$  in an open domain  $A \subseteq \mathbb{R}^3$ . Whenever necessary we shall mention the modifications to  $\mathbb{R}^2$ .

**Problem 8.1** Check whether the vector field  $\mathbf{V} = (f, g, h)$  is a gradient field. When this is the case, find an antiderivative  $F$ . This means that the function  $F$  satisfies the equation

$$\nabla F = \mathbf{V}.$$

**Remark 8.1** The problem is tricky, because there exist so many solution methods that one may be confused the first time one is confronted with this situation. Furthermore, there exist *necessary* conditions which are not sufficient, and *sufficient* conditions which are not necessary. Finally, the standard procedure assumes some knowledge of line integrals, which is not always the case in every textbook, the first time this problem is encountered. It will, however, be known at the end of any course dealing with functions in several variables.  $\diamond$

**Procedure:**

**Existence.** This section is not necessary, if only one remembers to *test* the solution in the next section. The considerations of this section may, however, be useful in some particular situations.

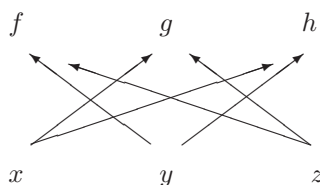


Figure 3: Diagram for “cross differentiation”.

- 1) Check that  $f, g, h \in C^2(A)$  satisfy

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

One may call these the “cross derivatives”. In case of 2 dimensions we only use the first equation.

- If these equations are *not* fulfilled, then  $\mathbf{V}(x, y, z)$  is *not* a gradient field, and the problem does not have an antiderivative.
- If the equations *are* satisfied, then  $\mathbf{V}(x, y, z)$  is indeed a gradient field in every *simply connected* region of  $A$ .

**Remark 8.2** Notice the extra condition that we only consider *simply connected* regions  $A$ . This is a sufficient condition, though not necessary.  $\diamond$

- 2) Suppose that the equations of 1) are satisfied. Check whether  $A$  is a *simply connected region*. If “yes”, then we have proved the existence. If “no”, construct a candidate by means of one of the methods in the next section and *test it*, i.e. check the equation

$$\nabla F = \mathbf{V}.$$

**Construction of a possible antiderivative.** We shall describe four methods, of which the former two have intrinsically built a test into them, while the latter two do *not* contain such a test! For that reason the latter two methods may be tricky, because their simple formulæ usually give some results, *even when no such antiderivative exists!* A reasonable strategy is therefore to skip the investigation in the section above and instead start by constructing a *candidate*  $F$  of an antiderivative and then as a rule *always* perform a *test*, i.e. check whether the *candidate* really satisfies the equation

$$\nabla F = \mathbf{V}.$$

1) **Indefinite integration.**

- a) Write down the differential form

$$\mathbf{V}(\mathbf{x}) \cdot d\mathbf{x} = f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz.$$

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b) Choose the simplest looking of the three terms in a), e.g.  $f(x, y, z) dx$ . Then calculate

$$F_1(x, y, z) = \int f(x, y, z) dx, \quad y, z \text{ are here considered as constants.}$$

c) **Check** the result, i.e. calculate

$$dF_1 = \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz,$$

and compare this with

$$\mathbf{V} \cdot d\mathbf{x} = f dx + g dy + h dz.$$

i) If  $\frac{\partial F_1}{\partial x} \neq f$ , then we have made an error in our calculations. There is only one thing to do: Start from the very beginning!

ii) If  $g_1 = g - \frac{\partial F_1}{\partial y}$ , or  $h_1 = h - \frac{\partial F_1}{\partial z}$  depends on  $x$ , then we have two possibilities: Either

(\*) we have made an error in our calculations above,

or

(\*\*)  $\mathbf{V}(x, y, z)$  is *not* a gradient field.

Notice that both possibilities may occur, so in this case one should check one's calculations an extra time.

iii) When neither  $g_1$  nor  $h_1$  depend on  $x$ , (which loosely speaking has been integrated in the first process, and therefore should have disappeared from the reduced problem), then we have

$$\mathbf{V} \cdot d\mathbf{x} = dF_1 + g_1(x, y) dy + h_1(y, z) dz.$$

In this case we repeat the process above on the *reduced* form

$$g_1(y, z) dy + h_1(y, z) dz.$$

d) After *at most* three repetitions of this process we either get

$\mathbf{V}(x, y, z)$  is *not* a gradient field (in which case the task is finished),

or

$$\mathbf{V}(\mathbf{x}) \cdot d\mathbf{x} = dF_1(x, y, z) + dF_2(y, z) + dF_3(z),$$

or something similar. The essential thing is that  $dF_1$  depends on all three variables, that  $dF_2$  only depends on two of them, and that  $dF_3$  only depends on one variable. Since all terms on the right hand side are “put under the  $d$ -sign”, it follows that  $\mathbf{V}(\mathbf{x})$  is a *gradient field*. One gets an *antiderivative* by removing the  $d$ -sign in all three terms,

$$F(x, y, z) = F_1(x, y, z) + F_2(y, z) + F_3(z).$$

Finally, we get *all* possible antiderivatives by adding an arbitrary constant.

## 2) The method of inspection.

In Danish this is called the “method of guessing”, but this is misleading, because it uses systematically the well-known rules of differentiation, read in the opposite direction of what one is used to from the reader’s previous education:

$$\text{Linearity:} \quad df + \alpha dg = d(f + \alpha g), \quad \alpha \text{ konstant,}$$

$$\text{Product:} \quad f df + g dg = d(f \cdot g),$$

$$\text{Quotient:} \quad g df - f dg = \begin{cases} g^2 d\left(\frac{f}{g}\right), & g \neq 0, \\ -f^2 d\left(\frac{g}{f}\right), & f \neq 0, \end{cases}$$

$$\text{Composition:} \quad F'(f) df = d(F \circ f).$$

These rules are all what we need, so learn them in this form!

a) Apply the rules of differentiation above to put as much as possible under the  $d$ -sign:

$$\mathbf{V}(\mathbf{x}) \cdot d\mathbf{x} = dF_1 + \mathbf{V}_1(\mathbf{x}) \cdot d\mathbf{x}.$$

b) If one by this process obtains that  $\mathbf{V}_1(\mathbf{x}) = \mathbf{0}$ , then  $\mathbf{V}(\mathbf{x})$  is indeed a *gradient field*,

$$\mathbf{V}(\mathbf{x}) \cdot d\mathbf{x} = dF_1 = \nabla F_1(\mathbf{x}) \cdot d\mathbf{x},$$

and  $F_1(\mathbf{x})$  is an *antiderivative*.

c) If one *cannot* obtain an equation of the form  $\mathbf{V}_1(\mathbf{x}) = \mathbf{0}$ , then either  $\mathbf{V}(\mathbf{x})$  is *not* a gradient field, or one has run out of ideas of further reductions. In this case one chooses another possible  $\mathbf{V}_1(\mathbf{x})$ , which is *simpler* than  $\mathbf{V}(\mathbf{x})$ , and uses one of the other methods on the reduced form  $\mathbf{V}_1(\mathbf{x}) \cdot d\mathbf{x}$ .

Even when  $\mathbf{V}$  is not a gradient field, it is often quite useful to remove a term of the form  $dF_1(\mathbf{x})$ , because later calculations of e.g. line integrals will be considerably easier to perform on the residual vector field. This technique may be useful in practical calculations in e.g. Thermodynamics.

## 3) Standard method; line integration along a curve consisting of axis parallel lines.

Once the *tangential line integral* has been introduced, and  $\mathbf{V}(\mathbf{x})$  is defined in  $\mathbb{R}^3$ , (or in some region which allows curves consisting of axis parallel lines as e.g. described in the following), it is easy to calculate a *candidate* to an antiderivative by *integration along such a curve* like e.g.

$$(0, 0, 0) \longrightarrow (x, 0, 0) \longrightarrow (x, y, 0) \longrightarrow (x, y, z).$$

a) Start by writing down the differential form

$$\mathbf{V}(\mathbf{x}) \cdot d\mathbf{x} = f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz.$$

b) Integrate this differential form along the curve mentioned above,

$$F_0(x, y, z) = \int_0^x f(t, 0, 0) dt + \int_0^y g(x, t, 0) dt + \int_0^z h(x, y, t) dt.$$

c) **Check the result!** This means that one should check the equation

$$\nabla F_0 = \mathbf{V}(\mathbf{x}).$$

If this is *not* fulfilled, then  $\mathbf{V}(\mathbf{x})$  is *not* a gradient field, *not even in the case where the candidate*  $F_0(x, y, z)$  *exists!* It is *not* an antiderivative in this case.

d) If on the other hand  $F_0(\mathbf{x})$  is an antiderivative, then we get all antiderivatives by adding an arbitrary constant.

#### 4) Radial integration.

In this case we integrate along the line

$$(0, 0, 0) \longrightarrow (x, y, z).$$

a) Be extremely careful when  $x$  is replaced by  $tx$ , and  $y$  by  $ty$ , and  $z$  by  $tz$  in  $\mathbf{V}(\mathbf{x})$ . By this process we get  $\mathbf{V}(tx, ty, tz)$ .

**Remark 8.3 Warning!** This seemingly simple process is far more difficult to perform than one would believe at the first sight!  $\diamond$

b) Calculate

$$F_0(x, y, z) = (x, y, z) \cdot \int_0^1 \mathbf{V}(tx, ty, tz) dt.$$

Notice that the dot product is used here.

c) **Check the result!** This means that one should check the equation

$$\nabla F_0(\mathbf{x}) = \mathbf{V}(\mathbf{x}).$$

**Remark 8.4** The method of *radial integration* is only mentioned here, because it may be found in some textbooks. I shall here strongly advise against the use of it, partly because the transform to  $\mathbf{x} \rightarrow t\mathbf{x}$  is far more difficult to perform than one would believe, and partly because the integral which is used in the calculation of  $F_0(x, y, z)$  in general is far more complicated than the analogous integral where we integrate along a simple curve consisting of straight lines parallel with one of the axis.  $\diamond$

## 9 An overview of integration in the plane and in the space

Consider the abstract integral  $\int_A f(\mathbf{x}) d\mu$ , where  $f : A \rightarrow \mathbb{R}$  always is assumed to be continuous. When one wants to classify the type of integration, one should go through the following flow diagram.

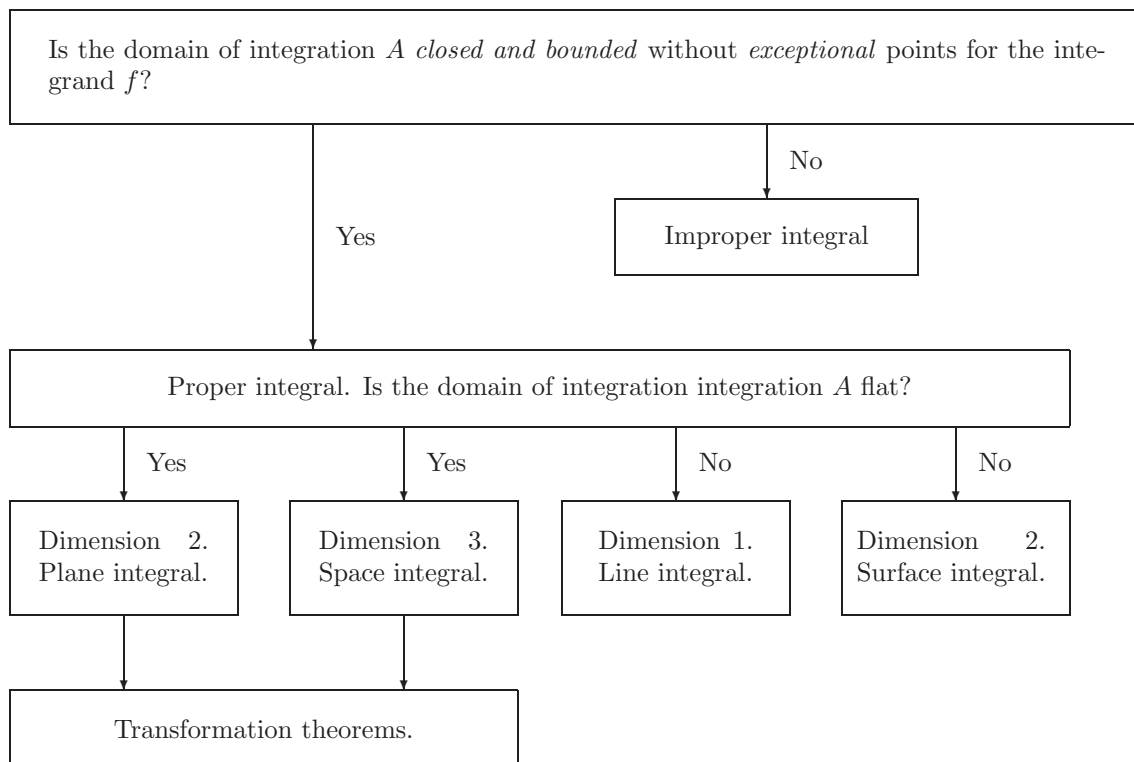


Figure 4: Flow diagram for types of integration.

These are ordered according to their difficulties in the following way:

- 1) **Plane integral** (rectangular, polar)
- 2) **Space integral** (rectangular, semi-polar, spherical)
- 3) **Line integral** (rectangular, polar; parametrical representation)
- 4) **Surface integral** (rectangular, semi-polar, spherical; parametrical representation)
- 5) **Transformation theorems** (plane, space)
- 6) **Improper integral** (bounded or unbounded domain).



## 10 Reduction of a plane integral; rectangular version

In this case we give a more thorough treatment, because it is the basic form which all other reductions are referring to.

The problems of understanding what is going on here is caused by the *historical* notation of an integral in the form

$$\int \cdots d\mu,$$

which says that the integral is actually written as an advanced form of a pair of brackets, where  $\int$  is the first bracket and  $d\mu$  is the second bracket. Other alternative notations are, however, possible but they have all some other disadvantages, so we shall stick to the familiar notation.

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### Geometry.

Whenever possible, one should always start by sketching a figure of the *parametric domain*. In its basic form (the rectangular version) the parametric domain is identical with the domain itself, so one does not distinguish between whether some of the coordinates are lying in the domain itself, while other coordinates at the same time are considered as lying in the parametric domain. Therefore, the same letter may occur both as a variable and as a parameter, depending on the actual situation. This may at a first glance look very confusing. In other words, one lacks a notation, by which one can distinguish between a variable and a parameter. Some experiments have been made by using different colours for the different aspects with some success. We shall, however, for technical reasons not use such a procedure here.

In the rectangular case there are two *basic geometrical forms*:

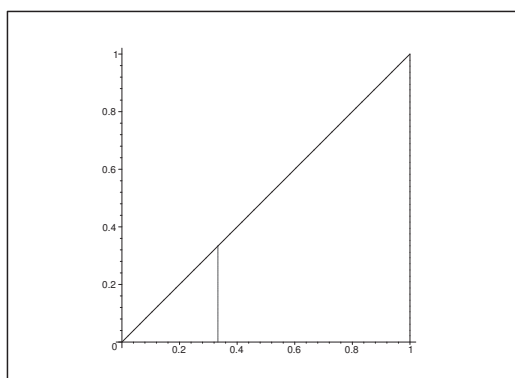


Figure 5: First version, where one in the inner (i.e. the first) integral integrates vertically. The bounds are here  $Y_1(x) = 0$  and  $Y_2(x) = x$ .

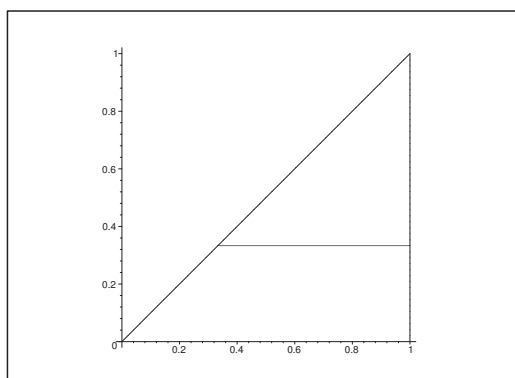


Figure 6: Second version, where one in the inner (i.e. the first) integral integrates horizontally. The bounds are here  $X_1(y) = y$  and  $X_2(y) = 1$ .

The actual figure is divided into sub-figures of one of these two types. Then each sub-figure is treated separately. We use the following principles:

**First version.** The variable  $x$  lies in an interval  $[a, b]$ , where  $a$  and  $b$  are constants. Notice that  $x$  is considered as a *parameter* in the first (i.e. the inner) integration. The variable  $y$  lies between the two curves of equations  $y = Y_1(x)$  and  $y = Y_2(x)$ .

**Second version.** The variable  $y$  lies in an interval  $[c, d]$ , where  $c$  and  $d$  are constants. Notice that  $y$  is considered as a *parameter* in the first (i.e. the inner) integration. The variable  $x$  lies between the two curved of equations  $x = X_1(y)$  and  $x = X_2(y)$ .

**Remark 10.1** Some figures, like e.g. axis-parallel rectangles, can be described equally well in the two geometrical versions. Then it depends on the structure of the *integrand* whether one should choose the first or the second version, because the integrations themselves do not be equally easy in the two versions. Whenever one is in trouble with the calculations in one of the two versions, one may try the other one instead. In some cases these integrations can in fact be calculated.  $\diamond$

**Problem 10.1** Calculate the abstract plane integral

$$\int_B f(x, y) dS,$$

by reduction in rectangular coordinates.

**Procedure:**

- 1) Sketch the domain of integration  $B$ . If necessary, divide  $B$  into sub-domains of type 1 or type 2, described above.
- 2) If  $B$  (or some sub-domain) is of type 1, then write

$$B = \{(x, y) \mid a \leq x \leq b, Y_1(x) \leq y \leq Y_2(x)\},$$

and set up the reduction formula

$$\int_B f(x, y) dS = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} f(x, y) dy \right\} dx.$$

According to the figure of the first version, one first keeps  $x$  fixed and integrates *vertically*. This can be interpreted as if we are collecting the “mass” of the vertical line above the point  $x$  in this point on the  $x$ -axis. Then the total “mass” is afterwards obtained by integrating the partial results from the inner integrations after  $x$ .

- 3) Perform a *separate* calculation of the inner integral:

$$\varphi(x) = \int_{Y_1(x)}^{Y_2(x)} f(x, y) dy,$$

where  $x$  is considered as a constant.

**Remark 10.2** When one has obtained the necessary training one can of course skip this point.  $\diamond$

- 4) Insert and calculate by the usual methods known from e.g. *Calculus 1a, Functions in One Variable*

$$\int_B f(x, y) dS = \int_a^b \varphi(x) dx.$$

- 5) If  $B$  (or a sub-domain of  $B$ ) is of type 2, then write instead

$$B = \{(x, y) \mid c \leq y \leq d, X_1(y) \leq x \leq X_2(y)\},$$

and continue with the reduction formula

$$\int_B f(x, y) dS = \int_c^d \left\{ \int_{X_1(y)}^{X_2(y)} f(x, y) dx \right\} dy.$$

According to the figure of the second version one starts by keeping  $y$  fixed and integrate *horizontally* after  $x$ . This can be interpreted as collecting the “mass” over den horizontal line in the corresponding point  $y$  on the  $y$ -axis. Afterwards the total “mass” is obtained by integrating the partial results after  $y$ .

- 6) Perform a *separate* calculation of the inner integral:

$$\psi(y) = \int_{X_1(y)}^{X_2(y)} f(x, y) dx,$$

where  $y$  is considered as a constant.

**Remark 10.3** When one has obtained the necessary training one can of course skip this point.  $\diamond$

- 7) Insert and calculate by the usual methods known from e.g. *Calculus 1a, Functions in One Variable*

$$\int_B f(x, y) dS = \int_c^d \psi(y) dy.$$

## 11 Reduction of a plane integral; polar version

### Geometry.

When the domain of integration  $B$  is bounded by radial half lines from  $(0,0)$  or circular arcs (with or without their centres in  $(0,0)$ ), an application of polar coordinates

$$x = \varrho \cos \varphi \quad \text{og} \quad y = \varrho \sin \varphi,$$

will often give simpler calculations. In this case the *parametric domain*  $\tilde{B}$  is *not* equal to the original domain. One pays for this by adding a *weight function* as a factor to the integrand, which is  $\varrho$  in the polar case. The gain is that one after this transformation can reduce the integral as if  $(\varrho, \varphi)$  were *rectangular coordinates*.

Formally this is expressed by that the *abstract* element of area  $dS$  is written abstractly (and rectangularly) as  $\varrho d\varrho d\varphi$ , i.e.

$$dS = \varrho d\varrho d\varphi,$$

or in the form of an integral,

$$\int_B f(x, y) dS = \int_{\tilde{B}} f(x, y) \text{ weight } d\varrho d\varphi = \int_{\tilde{B}} f(\varrho \cos \varphi, \varrho \sin \varphi) \varrho d\varrho d\varphi.$$

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Then we have reduced the problem to the original one of the rectangular case, and one might stop here. We shall not do this, because by continuing a little further we shall obtain some reductions, which one cannot see immediately from the above, and which would be quite convenient to have in the following.

As in the rectangular case there are also here two basic forms,

**Type 1:** The domain of integration  $B$  lies in an angular sector.

**Type 2:** The domain of integration  $B$  lies between two circular arcs of the same centre.

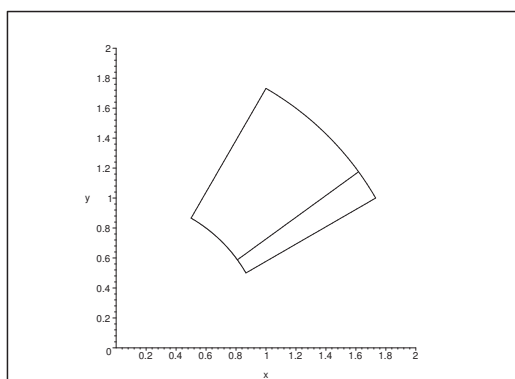


Figure 7: First version. First one integrates over  $B$  (the inner integral) radially after  $\varrho$ .

### Procedure.

- 1) Sketch  $B$ , and divide if necessary  $B$  into sub-domains of type 1 or type 2.
- 2) If  $B$  is of type 1 (i.e.  $B$  lies in some angular sector), then sketch if necessary the *parametrical domain*  $\tilde{B}$ , which we here write in the form

$$\tilde{B} = \{(\varphi, \varrho) \mid \alpha \leq \varphi \leq \beta, P_1(\varphi) \leq \varrho \leq P_2(\varphi)\}.$$

Notice that e.g.  $P_1$  is read as “capital rho 1”.

Then set up the reduction formula

$$\begin{aligned} \int_B f(x, y) dS &= \int_{\tilde{B}} f(\varrho \cos \varphi, \varrho \sin \varphi) \varphi d\varphi d\varrho \\ &= \int_{\alpha}^{\beta} \left\{ \int_{P_1(\varphi)}^{P_2(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi) \varrho d\varrho \right\} d\varphi. \end{aligned}$$

- 3) Perform a *separate* calculation (keep  $\varphi$  fixed and integrate after  $\varrho$ ):

$$F(\varphi) = \int_{P_1(\varphi)}^{P_2(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi) \varrho d\varrho.$$

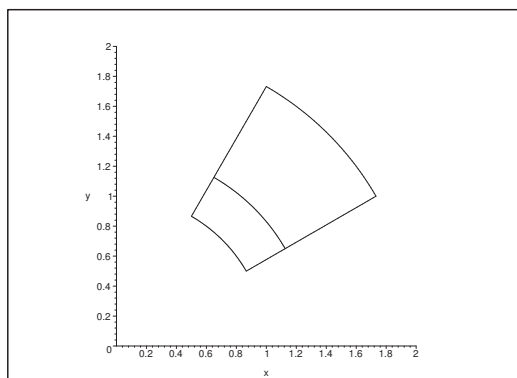


Figure 8: Second version. Integrate first over  $B$  (the inner integral) along the circular arc after  $\varphi$ .

4) Insert and calculate

$$\int_B f(x, y) dS = \int_{\alpha}^{\beta} F(\varphi) d\varphi.$$

5) If  $B$  is of type 2 (i.e.  $B$  lies between two circular arcs with their centre in  $(0, 0)$ ), we sketch if necessary the *parametric domain*  $\tilde{B}$ , which here is written

$$\tilde{B} = \{(\varphi, \varrho) \mid a \leq \varrho \leq b, \Phi_1(\varrho) \leq \varphi \leq \Phi_2(\varrho)\}.$$

Then the reduction formula can be written

$$\begin{aligned} \int_B f(x, y) dS &= \int_{\tilde{B}} f(\varrho \cos \varphi, \varrho \sin \varphi) \varrho d\varphi d\varrho \\ &= \int_a^b \left\{ \int_{\Phi_1(\varrho)}^{\Phi_2(\varrho)} f(\varrho \cos \varphi, \varrho \sin \varphi) d\varphi \right\} \varrho d\varrho. \end{aligned}$$

6) Perform a *separate* calculation (keep  $\varrho$  fixed and integrate circularly after  $\varphi$ ):

$$G(\varrho) = \int_{\Phi_1(\varrho)}^{\Phi_2(\varrho)} f(\varrho \cos \varphi, \varrho \sin \varphi) d\varphi.$$

7) Insert and calculate

$$\int_B f(x, y) dS = \int_a^b G(\varrho) \varrho d\varrho,$$

Notice that when the formula is written out of its normal context, this weight function does not seem natural. Therefore, be very careful about *not to forget* the weight function  $\varrho$  in the integrand.



## 12 Reduction of a space integral; rectangular version

The methods are close to the methods of Chapter 10. We get, however, a new complication, because the dimension 3 can be split into a sum of positive integers in three ways

**A. The “post method”:**  $3 = 2 + 1$ ,

**B. The “disc method”:**  $3 = 1 + 2$ ,

**C. The triple integral:**  $3 = 1 + 1 + 1$ .

These three cases are treated one by one.

### A. The “post method”.

In this case one of the variables, e.g.  $z$ , is lying between the graphs of two functions  $Z_1(x, y)$  and  $Z_2(x, y)$  in the other two variables  $(x, y)$ , where  $(x, y)$  lies in a domain  $B$  in the  $(x, y)$ -plane,  $(x, y) \in B$ . The graphs of the two functions are surfaces which cut the bigger domain  $A$  out of the cylinder over  $B$ .

#### Procedure.

- 1) Write the domain  $A$  in the form

$$A = \{(x, y, z) \mid (x, y) \in B, Z_1(x, y) \leq z \leq Z_2(x, y)\} \subseteq \mathbb{R}^3,$$

i.e. we get the set  $B \subseteq \mathbb{R}^2$  in the  $(x, y)$ -plane and the limiting functions  $Z_1(x, y)$  and  $Z_2(x, y)$ . Then write down the reduction formula

$$\int_A f(x, y, z) d\Omega = \int_B \left\{ \int_{Z_1(x, y)}^{Z_2(x, y)} f(x, y, z) dz \right\} dS.$$

- 2) Integrate for fixed  $(x, y) \in B$  the variable  $z$  along the *vertical “post”* between the limits  $Z_1(x, y)$  and  $Z_2(x, y)$ , i.e.

$$\varphi(x, y) = \int_{Z_1(x, y)}^{Z_2(x, y)} f(x, y, z) dz,$$

where the right hand side is an ordinary integral.

- 3) By insertion the *abstract space integral* is reduced to a simpler *abstract plane integral*,

$$\int_A f(x, y, z) d\Omega = \int_B \varphi(x, y) dS.$$

Finally, the right hand side is calculated by using one of the methods from Chapter 10 (rectangular case) or Chapter 11 (polar case).

### B. The “disc method”.

This is in particular used, when the integrand only depends on one of the three variables, e.g.  $z$ , though it of course is valid in general. The idea is that  $A$  is cut at the height  $z$  into a slice or disc  $B(z)$ , and then one first integrates the inner variables  $(x, y) \in B(z)$  and finally after  $z$ . Roughly speaking this means that we first find the “mass” of the disc  $B(z)$  and then afterwards collect all the “masses” by an integration after the “parameter”  $z$ .

**Procedure.**

- 1) Write the domain  $A$  in the form

$$A = \{(x, y, z) \mid a \leq z \leq b, (x, y) \in B(z)\},$$

i.e. we identify (by e.g. sketching a figure) the slice or disc  $B(z)$  for every relevant  $z$ .

Then write down the reduction formula


$$\int_A f(x, y, z) d\Omega = \int_a^b \left\{ \int_{B(z)} f(x, y, z) dS \right\} dz.$$

- 2) For fixed  $z \in [a, b]$  calculate the *abstract plane integral*

$$\psi(z) = \int_{B(z)} f(x, y, z) dS$$

applying one of the methods from Chapter 10 (rectangular coordinates) or Chapter 11 (polar coordinates).

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3) By insertion the *abstract space integral* is reduced to an ordinary integral,

$$\int_A f(x, y, z) d\Omega = \int_a^b \psi(z) dz.$$

**Remark 12.1** If the integrand only depends on  $z$ , then 2) above is reduced to

$$\psi(z) = f(z) \cdot \text{area } B(z),$$

and we can write 3) in the form

$$\int_A f(z) d\Omega = \int_a^b f(z) \cdot \text{area } B(z) dz,$$

where one often can find area  $B(z)$  by a simple geometrical consideration.  $\diamond$

### C. The triple integral.

This is a special case of the “*post*” *method*, because we assume that the domain  $B$  is bounded by graphs of functions in one of the variables.

#### Procedure.

1) Write the domain  $A$  in the form

$$A = \{(x, y, z) \mid a \leq x \leq b, Y_1(x) \leq y \leq Y_2(x), Z_1(x, y) \leq z \leq Z_2(x, y)\}.$$

Since this is the most difficult part of the process, one should always check whether the limiting functions  $Y_1(x)$ ,  $Y_2(x)$ ,  $Z_1(x, y)$ ,  $Z_2(x, y)$ , are correctly calculated. In practice this is the place where most of the errors occur in problems of this type.

2) Write down the reduction formula

$$\int_A f(x, y, z) d\Omega = \int_a^b \left\{ \int_{Y_1(x)}^{Y_2(x)} \left\{ \int_{Z_1(x, y)}^{Z_2(x, y)} f(x, y, z) dz \right\} dy \right\} dx.$$

3) For fixed  $(x, y)$ , calculate the inner integral,

$$f(x, y) = \int_{Z_1(x, y)}^{Z_2(x, y)} f(x, y, z) dz.$$

Check that the variable  $z$  has disappeared after this integration.

4) Then insert the result and calculate for fixed  $x$  the middle integral,

$$h(x) = \int_{Y_1(x)}^{Y_2(x)} g(x, y) dy.$$

Check that the corresponding  $y$  disappears by this integration.

5) Insert once again and calculate the outer integral,

$$\int_A f(x, y, z) d\Omega = \int_a^b h(x) dx.$$

### 13 Reduction of a space integral; semi-polar version

This method resembles the “*disc*” method. The difference is that the domain this time is cut like a *pie* to the  $z$ -axis.

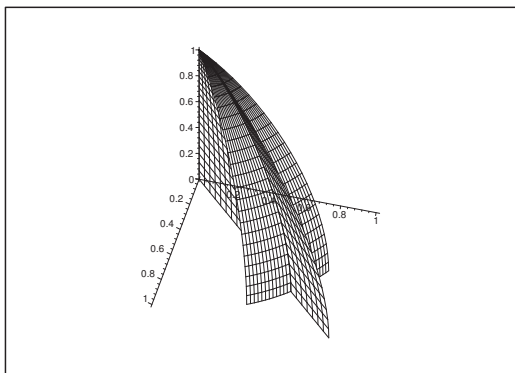


Figure 9: A fixed angle  $\varphi$  determines a half-plane through  $A$  with the  $z$ -axis as axis. This half-plane is then transferred into the meridian half-plane.

The formal reduction formula is

$$\int_A f(x, y, z) d\Omega = \int_{\alpha}^{\beta} \left\{ \int_{B(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi, z) \varrho d\varrho dz \right\} d\varphi,$$

where we have used *semi-polar* coordinates

$$x = \varrho \cos \varphi, \quad y = \varrho \sin \varphi, \quad z = z,$$

and the corresponding *volume element*

$$d\Omega = \varrho d\varrho dz d\varphi.$$

The pie cut is  $B(\varphi)$ , which represents for every fixed  $\varphi \in [\alpha, \beta]$  a domain in the meridian half-plane, i.e. in the  $(\varrho, z)$ -plane.

**Procedure:**

- 1) Sketch a figure (at least in the meridian half-plane).
- 2) Express the domain  $A$  in *semi-polar* coordinates

$$A = \{(\varrho, \varphi, z) \mid \alpha \leq \varphi \leq \beta, (\varrho, z) \in B(\varphi)\}.$$

Identify the *meridian cut*  $B(\varphi)$  for every fixed  $\varphi \in [\alpha, \beta]$ .

- 3) Keep  $\varphi \in [\alpha, \beta]$  fixed and calculate the abstract (inner) plane integral

$$\Phi(\varphi) = \int_{B(\varphi)} f(\varrho \cos \varphi, \varrho \sin \varphi, z) \varrho d\varrho dz,$$

by applying one of the methods from Chapter 10 or Chapter 11. Here one must *not* forget the weight function  $\varrho$  which should be put as an extra factor in the integrand.

4) Insert and calculate finally the *ordinary integral* on the right hand side,

$$\int_A f(x, y, z) d\Omega = \int_\alpha^\beta \Phi(\varphi) d\varphi.$$

**Remark 13.1** If  $\Omega$  is a *rotational domain*, then  $B(\varphi) = B$  is independent of  $\varphi$ . In this case we get a better procedure by interchanging the order of integration,

$$\int_A f(x, y, z) d\Omega = \int_B \left\{ \int_\alpha^\beta f(\varrho \cos \varphi, \varrho \sin \varphi, z) d\varphi \right\} \varrho d\varrho dz,$$

hence one starts in this case by calculating the inner *ordinary integral*.  $\diamond$

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## 14 Reduction of a space integral; spherical version

We use here the *spherical* coordinates

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta,$$

with the corresponding *volume element*

$$d\Omega = r^2 \sin \theta \, dr \, d\theta \, d\varphi,$$

i.e. the *weight function* is  $r^2 \sin \theta$ .

The reduction formula is not easy to comprehend,

$$\int_A f(x, y, z) \, d\Omega = \int_{\alpha}^{\beta} \left\{ \int_{B^*(\varphi)} f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta \, dr \, d\theta \right\} d\varphi.$$

### Procedure:

- 1) If  $A$  is a reasonable subset of a ball, then write in *spherical* coordinates

$$A = \{(r, \theta, \varphi) \mid \alpha \leq \varphi \leq \beta, (r, \theta) \in B^*(\varphi)\},$$

where  $B^*(\varphi)$  is the *meridian cut* for fixed  $\varphi \in [\alpha, \beta]$ , expressed in the *spherical* coordinates. Identify and sketch  $B^*(\varphi)$ .

**Remark 14.1** This is the most difficult part of this version.  $\diamond$

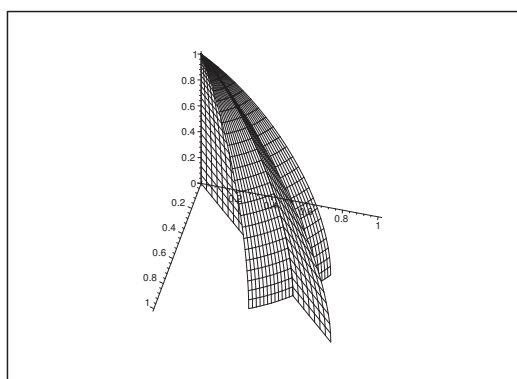


Figure 10: The meridian cut  $B^*(\varphi)$  for fixed  $\varphi$ .

- 2) Keep  $\varphi \in [\alpha, \beta]$  fixed and apply the methods from Chapter 10 or Chapter 11 in the calculation of the *abstract (inner) integral*,

$$H(\varphi) = \int_{B^*(\varphi)} f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta \, dr \, d\theta.$$

Do not forget the weight  $r^2 \sin \theta$  here as a factor of the integrand.

3) Insert the result and calculate the ordinary integral on the right hand side,

$$\int_A f(x, y, z) d\Omega = \int_\alpha^\beta H(\varphi) d\varphi.$$

**Remark 14.2** If  $\Omega$  is a *rotational domain*, then  $B^*(\varphi) = B^*$  is independent of  $\varphi$ . In this case one gets simpler calculations by interchanging the order of integration

$$\int_A f(x, y, z) d\Omega = \int_{B^*} \left\{ \int_\alpha^\beta f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) d\varphi \right\} r^2 \sin \theta dr d\theta.$$

Notice that the inner integral (which is calculated first)

$$\int_\alpha^\beta f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) d\varphi$$

becomes much simpler, because the weight  $r^2 \sin \theta$  only is added as a factor *after* the calculation of this integral.  $\diamond$

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## 15 Line integrals

This is a 1-dimensional case where the domain of integration is not an interval, but a curve in the plane or the space. The reduction formula becomes

$$\int_{\mathcal{K}} f(\mathbf{x}) ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt.$$

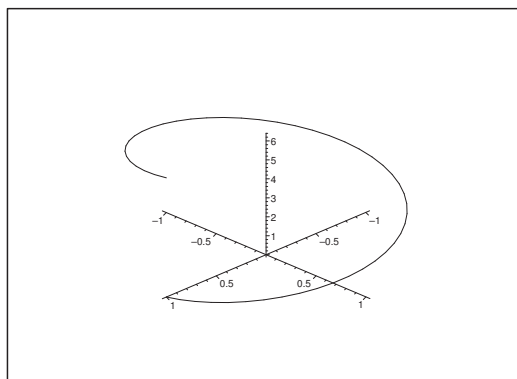


Figure 11: The curve of parameter representation  $\mathbf{r}(t) = (\cos t, \sin t, t)$ .

### Procedure:

- 1) Write down a *rectangular* parameter representation for the curve  $\mathcal{K}$ :

$$\text{In the plane: } (x, y) = \mathbf{r}(t), \quad t \in [a, b].$$

$$\text{In the space: } (x, y, z) = \mathbf{r}(t), \quad t \in [a, b].$$

- 2) Calculate the *curve element* (the weight is  $\|\mathbf{r}'(t)\|$ )

$$ds = \|\mathbf{r}'(t)\| dt.$$

**Mnemonic rule,** Pythagoras,

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}.$$

- 3) Insert the result and calculate the right hand side of

$$\int_{\mathcal{K}} f(\mathbf{x}) ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt.$$



**Special case:**1) *Curve length:*

$$\ell(\mathcal{K}) = \int_a^b \|\mathbf{r}'(t)\| dt = \int_a^b \sqrt{\left(\frac{dr_1}{dt}\right)^2 + \cdots + \left(\frac{dr_k}{dt}\right)^2} dt.$$

2) *Graph,  $y = Y(x)$ ,  $x \in [a, b]$ , rectangular.* The curve element is

$$ds = \sqrt{1 + \{Y'(x)\}^2} dx.$$

3) *Graph,  $\varrho = P(\varphi)$ ,  $\varphi \in [\alpha, \beta]$ , polar, first variant.* The curve element is

$$ds = \sqrt{\{P(\varphi)\}^2 + \{P'(\varphi)\}^2} d\varphi.$$

4) *Graph,  $\varphi = \Phi(\varrho)$ ,  $\varrho \in [a, b]$ , polar, second variant.* The curve element is

$$ds = \sqrt{1 + \{\varrho \Phi'(\varrho)\}^2} d\varrho.$$

**Remark 15.1** We see that the square root is quite natural here. Since even programmable pocket calculators are not too happy with square roots, they will usually stop, when they are not given some help from the user. Therefore, we cannot expect to get any final result by applying pocket calculators to problems of this type. Notice in particular that

$$\sqrt{g(t)^2} = |g(t)|.$$

(In other words: *Remember* the numerical signs!)  $\diamond$

## 16 Surface integrals

We consider 2-dimensional surfaces imbedded in  $\mathbb{R}^3$ . The idea is to pull the integration over the surface  $\mathcal{F}$  back to a *plane integral* over the *parameter domain*  $E$ , where we can use one of the methods from Chapter 10 or Chapter 11. This procedure has its price because we must add some *weight function* as a factor to the integrand.

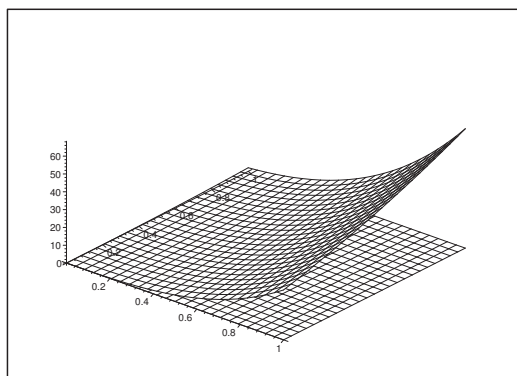


Figure 12: Example of a surface  $\mathcal{F}$  with a corresponding parameter domain  $E$  in the  $(x, y)$ -plane.

### Procedure:

- 1) Write down a *rectangular* parameter representation of the surface  $\mathcal{F}$ :

$$(x, y, z) = \mathbf{r}(u, v), \quad (u, v) \in E.$$

The *parameter domain*  $E \subseteq \mathbb{R}^2$  is then identified and sketched (a set lying in the plane).

**Remark 16.1** It is not always possible to sketch  $\mathcal{F}$  in the space, but this does not matter much, because the real calculations are taking place in the *parameter domain*  $E$ .  $\diamond$

- 2) Determine the *weight function*: Calculate the vectors

$$\mathbf{r}'_u(u, v) \quad \text{and} \quad \mathbf{r}'_v(u, v),$$

and the corresponding *normal vector* to the surface  $\mathcal{F}$  in this parameter representation,

$$\mathbf{N}(u, v) = \mathbf{r}'_u \times \mathbf{r}'_v = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

The wanted *weight function* is  $\|\mathbf{N}(u, v)\|$  (calculate it), and the *surface element* is

$$dS = \|\mathbf{N}(u, v)\| du dv.$$

- 3) Insert the parameter representation and the surface element, and calculate the right hand side by applying one of the methods from Chapter 10 or Chapter 11 in

$$\int_{\mathcal{F}} f(x, y, z) dS = \int_E f(\mathbf{r}(u, v)) \|\mathbf{N}(u, v)\| du dv.$$

If  $f(x, y, z) \equiv 1$ , the surface integral is interpreted as the *area of the surface*  $\mathcal{F}$ . In this case we get

**Theorem 16.1 Surface area:**

$$\text{area}(\mathcal{F}) = \int_{\mathcal{F}} dS = \int_E \|\mathbf{N}(u, v)\| du dv.$$

It is of course possible also to use some known area formulæ instead of calculating the cumbersome integral above. If for instance  $\mathcal{F}$  is the surface of a sphere of radius  $r$ , then it is well-known that

$$\text{area}(\mathcal{F}) = \text{area}(\partial B[0, r]) = 4\pi r^2.$$

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**Special cases:**

In the following special cases we reduce 2) in the procedure by inserting the given area element  $dS$ .

- 1) **Integral over a graph** for  $z = f(x, y)$ , *rectangular*:

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy.$$

(Compare with chapter 15 on the line integral, the case of a graph.)

- 2) **Integral over a cylindric surface**  $\mathbf{r}(t, z) = (X(t), Y(t), z)$ , *rectangular*:

$$dS = \sqrt{\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2} dt dz = ds dz,$$

where  $ds$  is the *curve element* for  $\tilde{\mathbf{r}}(t) = (X(t), Y(t))$  in the *plane*, cf. Chapter 15. Line integral.

- 3) **Integral over a rotational surface**

$$\mathbf{r}(t, \varphi) = (P(t) \cos \varphi, P(t) \sin \varphi, Z(t)),$$

*semi-polar*

$$dS = P(t) \sqrt{\left(\frac{dP}{dt}\right)^2 + \left(\frac{dZ}{dt}\right)^2} dt d\varphi = \varrho ds d\varphi.$$

The latter abstract form

$$dS = \varrho ds d\varphi,$$

can often be of some help when one is using the geometry (sketch a figure) when one sets up the reduced plane integral.

## 17 Transformation theorems

All the reduction formulæ in the Chapters 10–16 are special cases of more general formulæ. The presentations were using the classical coordinate systems: the *rectangular*, *polar*, *semi-polar*, and the *spherical* coordinate systems. When the coordinate system under consideration is *not* one of these we must use the general formulæ from the present chapter instead.

### A. Dimension 2.

- 1) Find a suitable parameter representation  $(x, y) = \mathbf{r}(u, v)$ ,  $(u, v) \in D$ . (Usually this will be given in advance in exercises).

Sketch the parametric domain  $D$  and argue *briefly* that  $\mathbf{r}(u, v)$  is injective, with the exception of e.g. a *finite* number of points. (More precisely one can neglect a so-called null set; which usually is not defined in elementary courses in Calculus).

Show also that the *range* is  $\mathbf{r}(D) = B$ .

- 2) Calculate the **Jacobian**

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Notice here that one is *forced* to find  $x$  and  $y$  as *functions of*  $(u, v)$  in 1) in order to calculate the Jacobian.

The *area element* is

$$dS = dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where the area element  $dx dy$  is lying in  $B$ , while the area element  $du dv$  is lying in the parametric domain  $D$ .

- 3) Insert the result and calculate the right hand side by using the methods from Chapter 10 or Chapter 11 in the expression

$$\int_B f(x, y) dx dy = \int_D f(\mathbf{r}(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

**Remark 17.1** Usually  $u$  and  $v$  are given as functions in  $x$  and  $y$  instead of the form we shall use:

$$(1) \quad u = U(x, y) \quad \text{and} \quad v = V(x, y).$$

Then one has to solve these equations with respect to  $x$  and  $y$ . If we in this way obtain a unique solution, then we have at the same time implicitly proved that the map is injective. Apply furthermore (1) to find the images of the boundary curves of  $B$ , thereby finding the boundary of  $D$ . Finally, the parametric domain  $D$  is identified.  $\diamond$

**B. Dimension 3.**

Formally the procedure is the same as in section A with obvious modifications due to the higher dimension.

- 1) Find a suitable parametric representation  $(x, y, z) = \mathbf{r}(u, v, w)$ ,  $(u, v, w) \in D$ . (This will usually be given in exercises, possibly in the form  $u = U(x, y, z)$ ,  $v = V(x, y, z)$ ,  $w = W(x, y, z)$ . If so, solve these equations with respect to  $x, y, z$ ).

Sketch the parametric domain  $D$  and argue (briefly) that the mapping  $\mathbf{r}(u, v, w)$  is injective almost everywhere.

Show that the *range* is  $\mathbf{r}(D) = B$ . Cf. also the remarks to section A.

- 2) Calculate the **Jacobian**

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Then the *volume element* is

$$d\Omega = dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw,$$

where one must be careful not to forget the numerical signs of the Jacobian.

- 3) Insert and calculate the right hand side by means of one of the methods from the chapter 12–14 in the formula

$$\int_B f(x, y, z) dx dy dz = \int_D f(\mathbf{r}(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw.$$

## 18 Improper integrals; bounded domain

**Problem 18.1** Calculate the integral

$$\int_A f(\mathbf{x}) dS,$$

where the domain  $A$  is bounded, and where either  $A$  is not closed, or the integrand  $f(\mathbf{x})$  is not defined for all  $\mathbf{x} \in A$ .

We choose for convenience  $A \subset \mathbb{R}^2$ .

**Idea:** Approximate  $\int_A f(\mathbf{x}) dS$  by  $\int_{A_n} f(\mathbf{x}) dS$ , where

- 1) The function  $f(\mathbf{x})$  is continuous for all  $x \in A_n$ .
- 2) All sets  $A_n \subset A$  are closed and bounded,  $n \in \mathbb{N}$ .
- 3)  $\text{area}(A \setminus A_n) \rightarrow 0$  for  $n \rightarrow +\infty$ .

**Remark 18.1** Even if we are approaching the solution by this idea, it is not enough. We shall below supply it with some sufficient conditions in the description of the procedure.  $\diamond$

**Procedure.**

- 1) Examine whether  $f(\mathbf{x})$  has a continuous extension  $F(\mathbf{x})$  to  $\overline{A}$ . If this is the case, then

$$\int_A f(\mathbf{x}) dS = \int_{\overline{A}} F(\mathbf{x}) dS,$$

where the right hand side is calculated by well-known methods, and the problem is solved.

- 2) If  $f(\mathbf{x})$  does *not* have a continuous extension to  $\overline{A}$ , we continue by an *analysis of the sign of  $f(\mathbf{x})$* . This step is extremely *important!* Divide  $A$  into two subsets

$$A^+ = \{(x, y) \in A \mid f(x, y) \geq 0\} \quad \text{and} \quad A^- = \{(x, y) \in A \mid f(x, y) \leq 0\}.$$

If  $f(x, y)$  is continuous where it is defined, then this division is most easily performed by finding the *null curves* i.e. the curves where  $f(x, y) = 0$ . These curves divide the domain into open sub-domains. Due to the continuity the sign of  $f$  is constant in each of these sub-domains, so in order to find the sign one just has to apply  $f$  to one point in each sub-domain. The curves where  $f(x, y) = 0$  can afterwards be included in any of the two sets  $A^+$  or  $A^-$ , and even in both of them, because since the integrand is 0 on these curves they will not contribute to the final result.

- 3) Let us first consider  $A^+$ . If  $A^+ = \emptyset$ , then go to 4) in the following. If on the other hand  $A^+ \neq \emptyset$ , we choose a convenient increasing sequence of *closed and bounded* subsets  $A_\varepsilon^+ \subset A^+$ , such that

$$\text{area}(A^+ \setminus A_\varepsilon^+) \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0+.$$

Calculate

$$I_\varepsilon^+ = \int_{A_\varepsilon^+} f(x, y) dS \quad (\geq 0).$$

If  $I_\varepsilon^+ \rightarrow +\infty$  for  $\varepsilon \rightarrow 0+$ , then the improper integral is *divergent*.

If on the other hand  $\lim_{\varepsilon \rightarrow 0} I_\varepsilon^+ < +\infty$  (there are only these two possibilities, because  $I_\varepsilon^+$  increases, when  $\varepsilon$  decreases towards 0), then

$$\int_{A^+} f(x, y) dS = \lim_{\varepsilon \rightarrow 0+} \int_{A_\varepsilon^+} f(x, y) dS \quad (\geq 0).$$

4) Then consider the other subset  $A^-$ . If  $A = \emptyset$ , go to 5).

If on the other hand,  $A^- \neq \emptyset$ , choose a convenient increasing sequence of *closed and bounded* subsets  $A_\varepsilon^- \subset A^-$ , such that

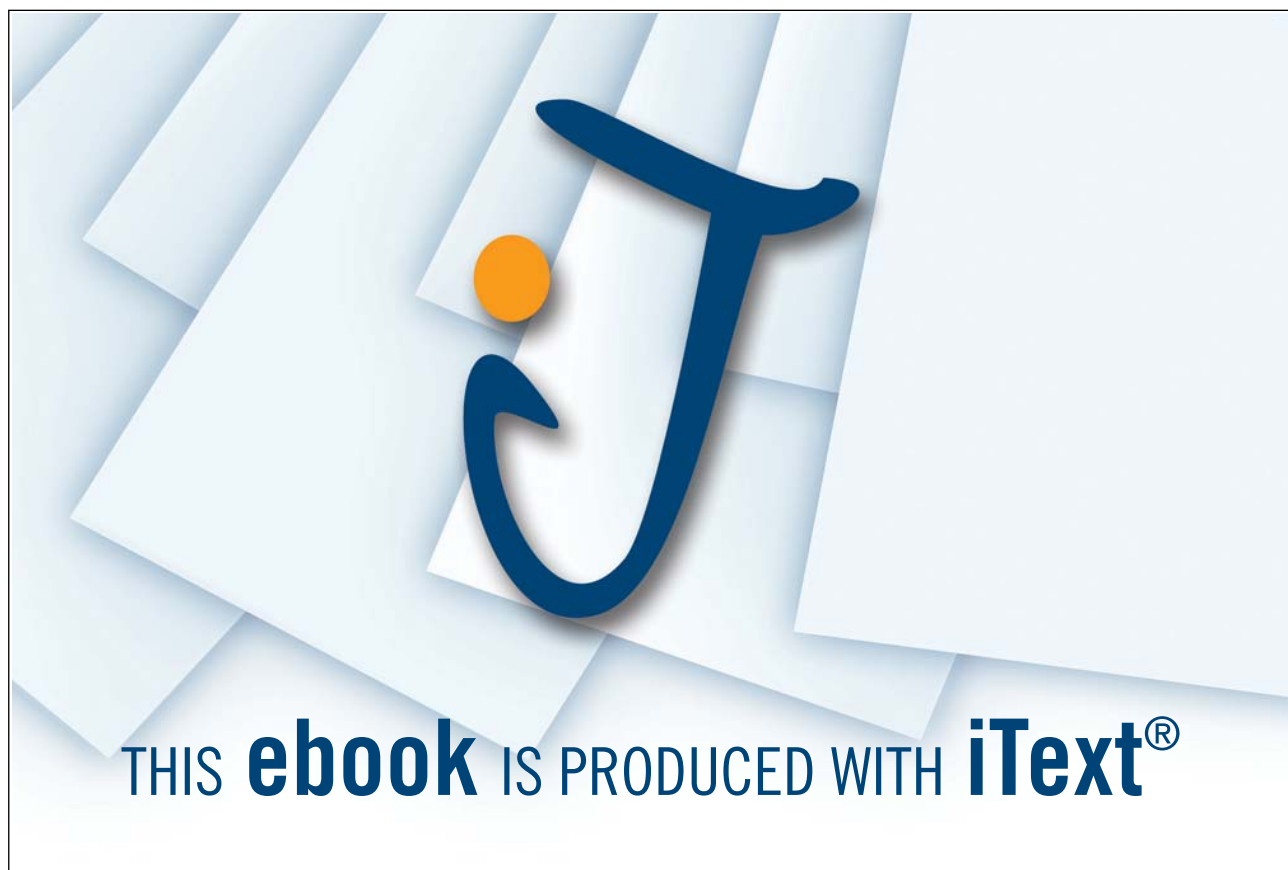
$$\text{area}(A^- \setminus A_\varepsilon^-) \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0+.$$

Calculate

$$I_\varepsilon^- = \int_{A_\varepsilon^-} f(x, y) dS \quad (\leq 0).$$

(To be continued)

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If  $I_\varepsilon^- \rightarrow -\infty$  for  $\varepsilon \rightarrow 0$ , then the improper integral is *divergent*.

If instead  $\lim_{\varepsilon \rightarrow 0} I_\varepsilon^- > -\infty$  (there are only these two possibilities), then

$$\int_{A^-} f(x, y) dS = \lim_{\varepsilon \rightarrow 0^+} \int_{A_\varepsilon^-} f(x, y) dS \quad (\leq 0).$$

- 5) It is only when we have obtained *convergence* in both 3) and 4) that we can conclude that the improper integral is *convergent* with the value

$$\int_A f(x, y) dS = \int_{A^+} f(x, y) dS + \int_{A^-} f(x, y) dS.$$

## 19 Improper integrals; unbounded domain

**Problem 19.1** Calculate the integral

$$\int_A f(x, y) dS,$$

where the domain of integration  $A \subseteq \mathbb{R}^2$  is unbounded.

**Procedure:**

- 1) Examine whether  $f$  has a continuous extension to  $\bar{A} \setminus \{\infty\}$ . If this is not the case, remove suitable bounded neighbourhoods of the “sick” points, and treat each of these neighbourhoods as in Chapter 18.

The residual set which we here denote by  $A$  is then treated in the following way.

- 2) The important *analysis of the sign of the integrand*. Divide  $A$  into the two sets

$$A^+ = \{(x, y) \in A \mid f(x, y) \geq 0\} \quad \text{and} \quad A^- = \{(x, y) \in A \mid f(x, y) \leq 0\}.$$

The sets  $A^+$  and  $A^-$  are found by analyzing the null curves, i.e. the set defined by the equation  $f(x, y) = 0$ .

- 3) Let us first consider  $A^+$ . If  $A^+ = \emptyset$ , then go to 4) in the following.

If  $A^+ \neq \emptyset$ , we choose one of the possible methods of trimming the set

$$\text{a) Rectangular: } A_n^+ = [-n, n]^2 \cap A^+, \quad \text{b) Polar: } A_R^+ = \overline{K}(\mathbf{0}; R) \cap A^+,$$

and we calculate

$$\text{a) Rectangular: } I_n^+ = \int_{A_n^+} f(x, y) dS, \quad \text{b) Polar: } I_R^+ = \int_{A_R^+} f(x, y) dS.$$

If  $I_n^+ \rightarrow +\infty$  for  $n \rightarrow +\infty$ , resp.  $I_R^+ \rightarrow +\infty$  for  $R \rightarrow +\infty$ , then the improper integral is *divergent*.

Otherwise,

$$\text{a) Rectangular: } \int_{A^+} f(x, y) dS = \lim_{n \rightarrow +\infty} \int_{A_n^+} f(x, y) dS.$$

$$\text{b) Polar: } \int_{A^+} f(x, y) dS = \lim_{R \rightarrow +\infty} \int_{A_R^+} f(x, y) dS.$$

Then go to 4).

- 4) Next turn to  $A^-$ . If  $A^- = \emptyset$ , then go to 5).

If  $A^- \neq \emptyset$ , choose one of the trimming possibilities

$$\text{a) Rectangular: } A_n^- = [-n, n]^2 \cap A^-, \quad \text{b) Polar: } A_R^- = \overline{K}(\mathbf{0}; R) \cap A^-,$$

and calculate

$$\text{a) Rectangular: } I_n^- = \int_{A_n^-} f(x, y) dS, \quad \text{b) Polar: } I_R^- = \int_{A_R^-} f(x, y) dS.$$

If  $I_n^- \rightarrow -\infty$  for  $n \rightarrow +\infty$ , resp.  $I_R^- \rightarrow -\infty$  for  $R \rightarrow +\infty$ , then the improper integral is *divergent*.

Otherwise,

a) Rectangular:  $\int_{A^-} f(x, y) dS = \lim_{n \rightarrow +\infty} \int_{A_n^-} f(x, y) dS.$

b) Polar:  $\int_{A^-} f(x, y) dS = \lim_{R \rightarrow +\infty} \int_{A_R^-} f(x, y) dS.$

5) If we obtain *convergence* in both 3) and 4), then the improper integral is *convergent* with the value

$$\int_A f(x, y) dS = \int_{A^+} f(x, y) dS + \int_{A^-} f(x, y) dS.$$

**Remark 19.1** If we put

$$A^0 = \{(x, y) \in A \mid f(x, y) = 0\},$$

it is seen that

$$\int_{A^0} f(x, y) dS = \int_{A^0} 0 dS = 0,$$

so it is of no importance whether we include  $A^0$  in our calculations, or we do not do it. The contribution from  $A^0$  is always 0.  $\diamond$

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\* Figures taken from London Business School's Masters in Management 2010 employment report

## 20 The setting up of a line, a plane, a surface or a space integral

When we set up an integral, we have two possible approaches:

- 1) A geometrical analysis.
- 2) Measure theory (i.e. concerning integration).

In elementary books on Calculus the *geometrical analysis* is dominating, although there may occur examples, where the *measure theory* plays a bigger role than the geometry. The latter is often the case when we apply the transformation theorems. It may also occur when we shall choose between semi-polar and spherical coordinates. We shall in the following not consider these exceptions, so usually we start with a *geometrical analysis* of the domain of integration. This analysis is depending on

**1a.** Dimension.

**1b.** Choice of coordinates.

The classical coordinates are er:

**Dimension 2:**

**Rectangular:**  $(x, y)$ , or  $(u, v)$ ,  $(\varrho, \varphi)$  or similarly in the parametric domain.

**Polar:**  $x = \varrho \cos \varphi$ ,  $y = \varrho \sin \varphi$ .

**General:**  $x = X(u, v)$ ,  $y = Y(u, v)$ , injective almost everywhere.

**Dimension 3:**

**Rectangular:**  $(x, y, z)$ , or  $(u, v, w)$ ,  $(\varrho, \varphi, z)$ ,  $(r, \theta, \varphi)$  or similarly in the *parametric domain*.

**Semi-polar:**  $x = \varrho \cos \varphi$ ,  $y = \varrho \sin \varphi$ ,  $z = z$ ,

**Spherical:**  $x = r \sin \theta \cos \varphi$ ,  $y = r \sin \theta \sin \varphi$ ,  $z = r \cos \theta$ .

**General:**  $x = X(u, v, w)$ ,  $y = Y(u, v, w)$ ,  $z = Z(u, v, w)$ , injective almost everywhere.

These are our building stones in the ongoing geometrical analysis.

**Remark 20.1** If the exercise does not contain any hint, then choose among the *rectangular*, *polar*, *semi-polar* or *spherical* coordinates. On the other hand, if more general coordinates are needed in an exercise, then these will always be given, usually in their *inverse* form:

$$\text{dimension 2 :} \quad u = U(x, y), \quad v = V(x, y),$$

$$\text{dimension 3 :} \quad u = U(x, y, z), \quad v = V(x, y, z), \quad w = W(x, y, z).$$

If they are given in their inverse form, we start by solving them with respect to  $x$ ,  $y$  (and  $z$ ), cf. Chapter 17.  $\diamond$

**Solution strategy:**

Choose the coordinates, such that

- a) the geometry of the *parametric domain* becomes simple,
- b) the *integrand* becomes simple.

This is an order of priority, so the *geometry* is most important in a), while the *measure theory* is dominating in b). Both cases can be found in elementary textbooks, though most of the examples are of the type given in a).

In the following we set up an overview guided by the dimension of the classical coordinates. In each case the structure will be given by:

- i) **Formula**, where the right hand side is the reduced expression.
- ii) **Geometry**, where the domain is compared with the parametric domain.
- iii) **Measure theory**, which briefly describes the weight function.
- iv) **Possible comments**.

It is seen that we are aiming at the reduction of a given abstract integral to a *rectangular* integration over a convenient *parametric domain*.

**Dimension 1.**

Characteristics: There is only one variable of integration  $t$ .

We have two cases, a) **An ordinary integral** and b) **A line integral**.

**a) The ordinary integral.**

Formula:	$\int_a^b f(t) dt.$
Geometry:	The domain of integration = the parametric interval.
Measure theory:	Weight = 1.
Comment:	Basic form known from high school and previous courses in Calculus.

**b) The line integral.**

Formula:	$\int_{\mathcal{K}} f(x, y, z) ds = \int_a^b f(\mathbf{r}(t)) \ \mathbf{r}'(t)\  dt,$ where $\mathcal{K} : (x, y, z) = \mathbf{r}(t).$
Geometry:	Curve $\neq$ parametric interval.
Measure theory:	Weight = $\ \mathbf{r}'(t)\ .$
Comments:	By the transform given above b) is transferred back to the basic form a). Notice that there occurs a <i>hidden square root</i> in the weight function, so pocket calculators cannot always be successfully applied. Notice that the curve is embedded in the <i>space</i> $\mathbb{R}^3$ .

**Dimension 2.**

Characteristics: There are *two* variables of integration, e.g.  $(u, v)$ .

Here we have four cases:

a) **Rectangular plane integral.**

b) **Polar plane integral.**

c) **General transform.**

d) **Surface integral.**

These cases are treated one by one in the following.

a) **Rectangular plane integral.**

Formula:  $\int_A f(x, y) dx dy.$

Geometry: domain of integration = parametric domain.

Measure theory: Weight = 1.

Comment: Basic form. Further reductions are obtained by means of Chapter 10.

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**b) Polar plane integral.**

Formula:  $\int_A f(x, y) dx dy = \int_B f(\varrho \cos \varphi, \varrho \sin \varphi) \varrho d\varrho d\varphi.$

Geometry: domain of integration  $\neq$  parametric domain.

Measure theory: Weight =  $\varrho$ .

Comment: This is a brief description of Chapter 11. Notice that b) is reduced to a).

**c) General.**

Formula:  $\int_A f(x, y) dx dy = \int_B f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$

Geometry: domain of integration  $\neq$  parametric domain.

Measure theory: Weight =  $\left| \frac{\partial(x, y)}{\partial(u, v)} \right|.$

Comment: This is a brief description of a part of Chapter 17.

**d) Surface integral.**

Formula:  $\int_A f(x, y, z) dS = \int_B f(\mathbf{r}(u, v)) \|\mathbf{N}(u, v)\| du dv,$

where the surface  $\mathcal{F}$  is given by  $(x, y, z) = \mathbf{r}(u, v).$

Geometry: Surface  $\neq$  parametric domain.

Measure theory: Weight =  $\|\mathbf{N}(u, v)\|.$

Comment: This is a brief description of Chapter 16, where several special cases are described. Notice that for *rotational surfaces* we usually apply the *semi-polar* coordinates with the weight function  $\varrho$ , where  $\varrho$  is a function of one parameter  $t$ .  
Notice also that the surface is embedded in  $\mathbb{R}^3$ .

**Dimension 3.**

Characteristics: There are *three* variables of integration, e.g.  $(u, v, w)$ .

Here we have five cases, of which only four usually are treated in elementary courses in Calculus.

**a) Rectangular space integral.**

Formula:  $\int_A f(x, y, z) dx dy dz.$

Geometry: domain of integration = parametric domain.

Measure theory: Weight = 1.

Comment: Basic form as described in Chapter 12.

**b) Semi-polar space integral.**

Formula:  $\int_A f(x, y, z) d\Omega = \int_B f(\varrho \cos \varphi, \varrho \sin \varphi, z) \varrho d\varrho d\varphi dz.$

Geometry: domain of integration  $\neq$  parametric domain.

Measure theory: Weight =  $\varrho$ .

Comment: Cf. Chapter 13 for further details. The method is in particular applied on rotational bodies.

**c) Spherical space integral.**

Formula:  $\int_A f(x, y, z) d\Omega = \int_B f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta dr d\theta d\varphi.$

Geometry: domain of integration  $\neq$  parametric domain.

Measure theory: Weight =  $r^2 \sin \theta$ .

Comment: Cf. also Chapter 14. The method is usually applied on *spherical shells*.

**d) General.**

Formula: 
$$\int_A f(x, y, z) d\Omega = \int_B f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw.$$

Geometry: domain of integration  $\neq$  parametric domain.

Measure theory: Weight =  $\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|.$

Comment: Cf. also Chapter 17.



**e) Curved space integral.**

One may come across integration over a curved 3-dimensional space in the Theory of Relativity (embedded in  $\mathbb{R}^4$ ) in Physics. This is of course analogous to the surface integral embedded in  $\mathbb{R}^3$ . Usually it does not occur in elementary textbooks of Calculus.

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## 21 Tangential line integral; gradient field

The **current** of a *vector field*  $\mathbf{V}$  along a *curve*  $\mathcal{K}$  of parametric representation  $\mathbf{r}(t)$  is defined by:

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t}, ds = \int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = \int_{\alpha}^{\beta} \mathbf{V}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt,$$

where we have identified

$$\mathbf{x} = \mathbf{r}(t) \quad \text{and} \quad d\mathbf{x} = \mathbf{r}'(t) dt.$$

It can in some cases be identified as an electric current along a wire, represented by the curve.

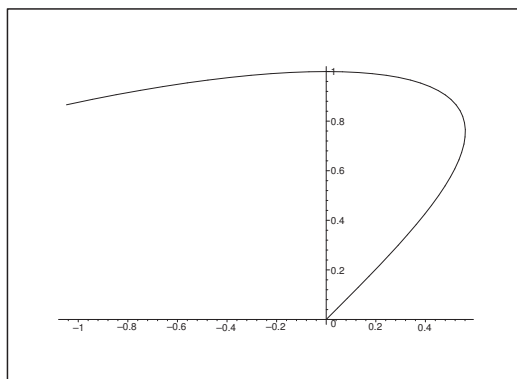


Figure 13: Example of a plane curve  $\mathcal{K}$  with initial point  $(0, 0)$ .

There are here two important special cases:

1) **The gradient integral theorem:**

$$\int_{\mathcal{K}} \nabla F(\mathbf{x}) \cdot d\mathbf{x} = F(\mathbf{b}) - F(\mathbf{a}),$$

no matter how the curve  $\mathcal{K}$  from  $\mathbf{a}$  to  $\mathbf{b}$  is chosen.

2) **Circulation**, i.e.  $\mathcal{K}$  is a *closed* curve.

Whenever the word “*circulation*” occurs in an exercise, always think of *Stokes’s theorem*,

$$\oint_{\partial \mathcal{F}} \mathbf{V} \cdot \mathbf{t} ds = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} dS,$$

and see if it applies, cf. Chapter 23.

We shall here only consider the **gradient integral theorem**, cf. Chapter 8, because the *circulation* will be treated separately later.

**A necessary condition** (which is not sufficient). The “cross derivatives” agree,

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i} \quad \text{for all } i, j = 1, \dots, k.$$

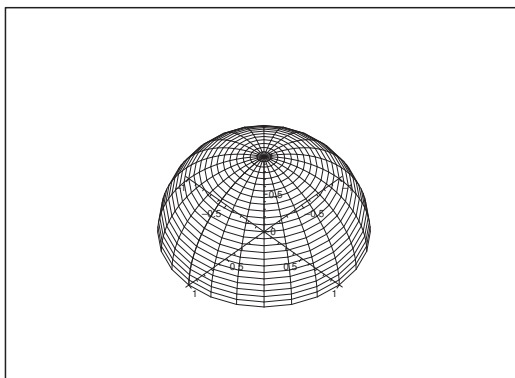


Figure 14: The half sphere  $\mathcal{F}$  gives a typical example, when we shall apply Stokes's theorem.

**A trap:** Even if the necessary conditions are all fulfilled, the field  $\mathbf{V}$  is *not* always a gradient field, although many students believe it.

**A sufficient condition** (which is not necessary). The “cross derivatives” agree:

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i} \quad \text{for all } i, j = 1, \dots, k,$$

and

the domain  $A$  is *star shaped*.

**Remark 21.1** Even when  $\mathbf{V}$  is a gradient field, the corresponding domain  $A$  does *not* have to be star shaped.  $\diamond$

Concerning the calculations in practice we refer to Chapter 8:

- 1) Indefinite integration,
- 2) Method of inspection,
- 3) Integration along a curve consisting of lines parallel with one of the axes,
- 4) Radial integration.

The *radial integration* cannot be recommended as a standard procedure.

In some cases a differential form can be simplified by removing a gradient field:

$$\mathbf{V} = \nabla F + \mathbf{U},$$

or more conveniently,

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= V_x dx + V_y dy + V_z dz \\ &= dF + U_x dx + U_y dy + U_z dz, \end{aligned}$$

where  $\mathbf{U}$  ought to be simpler than  $\mathbf{V}$ .

If so, then

$$\int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = F(\mathbf{b}) - F(\mathbf{a}) + \int_{\mathcal{K}} \mathbf{U} \cdot d\mathbf{x}.$$

This method is e.g. used in Thermodynamics, where the vector field usually is *not* a gradient field.

In these reductions one can take advantage of the well-known *rules of calculus for differentials*:

$$\alpha df + dg = d(\alpha f + g), \quad \alpha \text{ constant}$$

$$f dg + g df = d(fg),$$

$$f dg - g df = f^2 d\left(\frac{g}{f}\right), \quad f \neq 0,$$

$$F'(f) df = d(F \circ f).$$

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## 22 Flux and divergence of a vector field; Gauß's theorem

The **flux**  $\Phi$  of a vector field  $\mathbf{V}$  through a surface  $\mathcal{F}$  is given by a *surface integral* (cf. Chapter 16) in the following way:

If  $\mathbf{x} = \mathbf{r}(u, v)$ ,  $(u, v) \in E$  is a *parametric representation* of the surface  $\mathcal{F}$  with a given continuous unit normal vector field  $\mathbf{n}$ , then the flux is given by

$$\Phi_{\mathcal{F}}(\mathbf{V}) = \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS = \int_{\mathcal{F}} \mathbf{V}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, dS = \int_E \mathbf{V}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v) \, du \, dv.$$

It is the amount of the vector field which “flows through the surface in the direction of the normal vector” (e.g. per time unit).

Typically there are two different ways in which the flux can be calculated.

### Standard procedure.

In principle this can always be applied, but it is often very cumbersome.

- 1) Divide if necessary  $\mathcal{F}$  into convenient sub-surfaces  $\mathcal{F}_1, \dots, \mathcal{F}_k$  each having its own unit normal vector field  $\mathbf{n}_1, \dots, \mathbf{n}_k$ .
- 2) Check, whether  $\mathcal{F}$  (or  $\mathcal{F}_j$ ) is “flat”, and if it is not too difficult to calculate  $\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS$  as an ordinary *plane integral*, because  $\mathcal{F}$  is lying in a plane set.
- 3) If  $\mathcal{F}$  is not flat, we calculate the normal vector corresponding to the specific parametric representation in the variables  $(u, v)$ ,

$$\mathbf{N}(u, v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

Notice that  $\mathbf{N}(u, v)$  no longer is a *unit* normal vector field. Stated roughly, we build the weight function into the new normal vector field  $\mathbf{N}(u, v)$ .

- 4) Calculate the *plane integral* over the *parametric domain*  $E$ ,

$$\Phi_{\mathcal{F}}(\mathbf{V}) = \int_E \mathbf{V}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v) \, du \, dv.$$

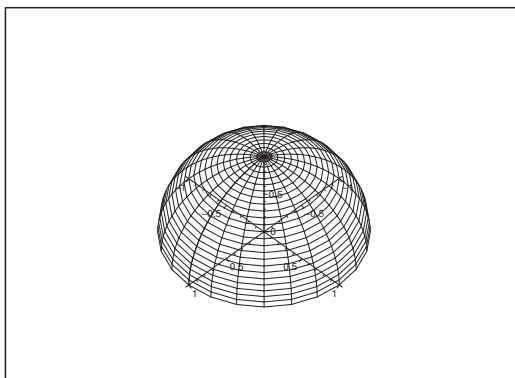


Figure 15: The surface of the unit half sphere is closed by adding the unit disc in the  $(x, y)$ -plane with a normal vector pointing downwards (this gives us the closed upper unit half sphere with normal vectors pointing outwards) and the unit disc in the  $(x, y)$ -plane with the normal vector pointing upwards.

### Application of Gauß's theorem.

The method can in principle always be applied when the surface is “closed”, i.e. one adds a surface with *two* numerically equal normal vector fields, which are pointing in the opposite directions,  $\mathbf{n}$  and  $-\mathbf{n}$ , such that one surrounds a 3-dimensional domain  $\Omega$  with outgoing normal, and an additional surface integral, which hopefully should be easy to calculate.

- 1) Check that  $\mathcal{F} = \partial\Omega$  is *closed*, i.e. this surface surrounds a 3-dimensional *body*  $\Omega$ .
- 2) Quote *Gauß's theorem* and reduce the surface integral to a *space integral*,

$$\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} dS = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega.$$

- 3) Calculate the space integral  $\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega$  by applying one of the methods from the Chapters 12–14.

**Remark 22.1** Usually one would not call it a reduction to go from 2 dimensions to 3 dimensions; but notice that the surface  $\mathcal{F}$  of dimension 2 may have a fairly complicated geometry, while we in principle always end up with rectangular coordinates in 3 dimensions, which here may be considered as a simpler situation.  $\diamond$

## 23 Rotation of a vector field; Stokes's theorem

Let us first mention the most practical way of calculating the *rotation* of a 3-dimensional  $C^1$  vector field by means of a formal determinant,

$$\mathbf{rot} \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}, \quad \mathbf{V} = (V_x, V_y, V_z).$$

Notice that this rule is only valid *when rectangular coordinates are used!*

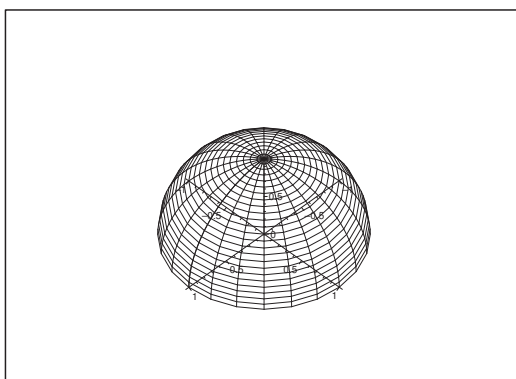


Figure 16: As an example we choose again the surface of the upper unit half sphere  $\mathcal{F}$ , where the boundary curve  $\delta\mathcal{F}$  is the unit circle in the  $(x, y)$ -plane.

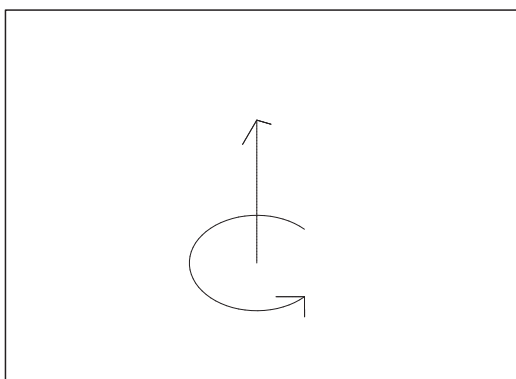


Figure 17: The connection between a vector and the corresponding whirl which fixes the orientation.

The fixing of the *orientation of a surface* (i.e. the *direction of the normal vector*), when the *direction of the run through of the boundary curve*, is given by:

- **The convention of orientation:** Let the *normal vector field* on the surface  $\mathcal{F}$  be supplied with a *whirl* around the foot of the normal vector. The *direction* of the whirl is by continuity fixed by the *direction* of the run through of the boundary curve. If you put your right hand along the normal vector with the four fingers in the direction of the whirl, then the thumb will point in the direction of the normal. Cf. also Figure 17.

Whenever *circulation of a vector field* is mentioned, one should think of an application of **Stokes's theorem**:

$$\oint_{\partial\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} \, dS,$$

where the left hand side is the circulation of  $\mathbf{V}$  (1 dimension), and the right hand side is a flux of the vector field  $\mathbf{rot} \, \mathbf{V}$  (2 dimensions).

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In exercises the student will often in an earlier question have calculated  $\mathbf{rot} \mathbf{V}$ , so the task is reduced to the choice of a convenient  $\mathcal{F}$  for the given closed (boundary) curve  $\delta\mathcal{F}$ . As long as the student is learning these ideas, the surface  $\mathcal{F}$  will typically either be flat or a part of a sphere.

**Remark 23.1** Stokes's theorem can also be applied from the right to the left. If e.g. one shall find the flux  $\int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{U} dS$  of a *rotational field*  $\mathbf{U} = \mathbf{rot} \mathbf{V}$ , and  $\mathbf{V}$  is *fairly easy to find*, then

$$\int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{U} dS = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} dS = \oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} ds. \quad \diamond$$

## 24 Green's theorems in the plane

When Gauß's and Stokes's theorems are “translated” to the 2-dimensional case, they are called **Green's theorems in the plane**. They are applicable at the most strange places in the technical sciences, so they ought to be more emphasized in the teaching.

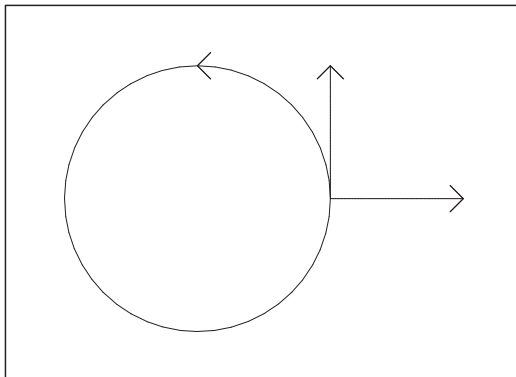


Figure 18: The unit circle  $\partial E$  and its unit tangent  $\mathbf{t}$  and unit normal  $\mathbf{n}$ .

*Conventions for Green's theorems in the plane:*

Let  $\partial E$  be a closed and plane curve without double points. The *orbital direction* for  $\partial E$  is always chosen such that the domain  $E$  to the *left* of  $\partial E$ , is *bounded*.

The *direction of the tangent* is determined by the orbital direction, so  $\mathbf{t}$  is unique. The *normal* is always *directed away* from  $E$ , so  $\mathbf{n}$  is also uniquely determined.

### Theorem 24.1 Green's theorems in the plane:

1) *Gauß-like:*

$$\oint_{\partial E} (V_x n_x + V_y n_y) ds = \int_E \left( \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) dS,$$

2) *Stokes-like:*

$$\oint_{\partial E} (V_x t_x + V_y t_y) ds = \int_E \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) dS.$$

Notice that

$$V_x n_x + V_y n_y = \mathbf{V} \cdot \mathbf{n} \quad \text{and} \quad V_x t_x + V_y t_y = \mathbf{V} \cdot \mathbf{t}.$$

## 25 Nabla calculus

In complicated cases it may be useful to apply the abstract theory of *nabla calculus* instead of the enormous calculations with coordinates.

The theory may look a little confusing the first time one sees it, until one realizes that there are three different products in the vector analysis in  $\mathbb{R}^3$ :

<b>scalar multiplication:</b>	no special notation	$\alpha \mathbf{V}$ ,	vector,
<b>inner product</b> ( <i>dot product</i> ):	dot	$\mathbf{U} \cdot \mathbf{V}$ ,	scalar,
<b>vector product</b> ( <i>cross product</i> ):	cross	$\mathbf{U} \times \mathbf{V}$ ,	vector.

These are transferred to a nabla notation,

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

by the relations

no special notation	<b>grad</b> $f = \nabla f$ ,
dot	$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V}$ ,
cross	<b>rot</b> $\mathbf{V} = \nabla \times \mathbf{V}$ .

When we apply the correspondence  $\nabla \sim \mathbf{n}$  we get **Gauß's theorem** in three versions:

$$\begin{aligned} \int_{\partial\Omega} \mathbf{n} f \, dS &= \int_{\Omega} \nabla f \, d\Omega = \int_{\Omega} \mathbf{grad} \, f \, d\Omega, && \text{no special notation} \\ \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} \, dS &= \int_{\Omega} \nabla \cdot \mathbf{V} \, d\Omega = \int_{\Omega} \operatorname{div} \mathbf{V} \, d\Omega, && \text{dot product, usual version} \\ \int_{\partial\Omega} \mathbf{n} \times \mathbf{V} \, dS &= \int_{\Omega} \nabla \times \mathbf{V} \, d\Omega = \int_{\Omega} \mathbf{rot} \, \mathbf{V} \, d\Omega, && \text{cross product.} \end{aligned}$$

When we apply the correspondence  $(\mathbf{n}, \nabla, \times) \sim \mathbf{t}$ , where  $\cdot$  and  $\times$  are always put in a meaningful connection, we obtain **Stokes's theorem** in three versions:

$$\begin{aligned} \oint_{\partial\mathcal{F}} \mathbf{t} f \, ds &= \int_{\mathcal{F}} \mathbf{n} \times \nabla f \, dS = \int_{\mathcal{F}} \mathbf{n} \times \mathbf{grad} \, f \, dS, && \text{no special notation} \\ \oint_{\partial\mathcal{F}} \mathbf{t} \cdot \mathbf{V} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot (\nabla \times \mathbf{V}) \, dS = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} \, dS, && \text{dot product, usual version} \\ \oint_{\partial\mathcal{F}} \mathbf{t} \times \mathbf{V} \, ds &= \int_{\mathcal{F}} (\mathbf{n} \times \nabla) \times \mathbf{V} \, dS, && \text{cross product.} \end{aligned}$$

## 26 Partial integration in higher dimensions

The simplest case is described by

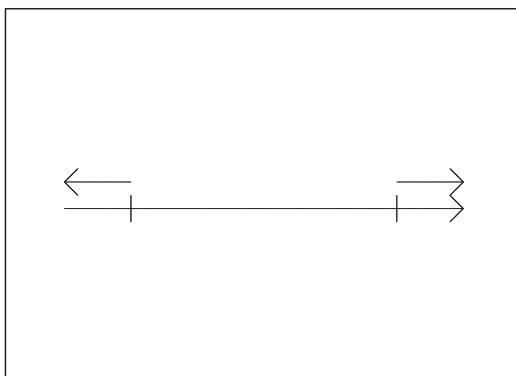


Figure 19: The interval  $[a, b]$ , where one goes from 1 dimension to 0 dimensions.

1) **Main theorem of the differential and integral calculus**, 1 dimension:

$$\int_a^b \frac{df}{dx} dx = [f(x)]_a^b = f(b) - f(a),$$

i.e. a 1-dimensional integral is transformed into a 0-dimensional “boundary integral”. Note that in the right end point, where the outer normal is equal to the direction of the axis, we use the sign  $+$ , and in the left end point, where the outer normal is opposite to the direction of the axis, we use the sign  $-$ .

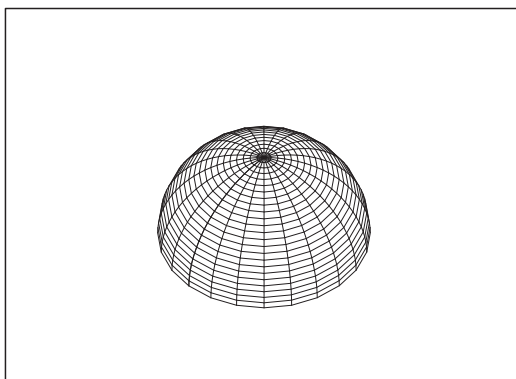


Figure 20: A surface  $\mathcal{F}$  with its limiting curve  $\delta\mathcal{F}$ .

2) **Stokes’s theorem**, 2 dimensions:

$$\int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} dS = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{B} ds,$$

i.e. a 2-dimensional surface integral is transferred into a 1-dimensional “boundary integral”.

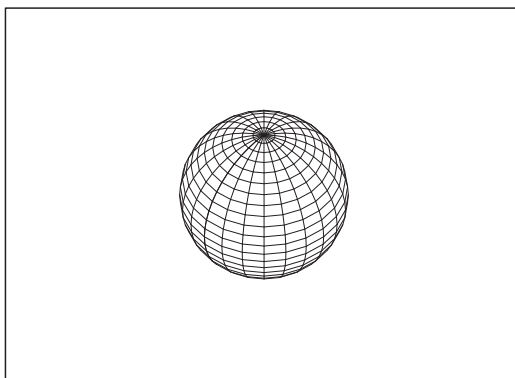


Figure 21: A domain in space  $\Omega$  with surface  $\partial\Omega$ . The unit normal vector field is everywhere directed away from  $\Omega$ .

3) **Gauß's theorem**, 3 dimensions:

$$\int_{\Omega} \operatorname{div} \mathbf{V} \, d\Omega = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} \, dS,$$

i.e. a 3-dimensional space integral is transferred into a 2-dimensional “boundary integral”.

The more advanced applications use the nabla calculus, cf. Chapter 25. Here we have several special cases, because there in e.g. in  $\mathbb{R}^3$  exist three types of product (scalar multiplication, dot and cross product) and three types of differentiation (gradient, divergence, rotation). The basic idea is that if the integrand is of the form  $\Phi \otimes \mathcal{D}\Psi$ , where  $\otimes$  stands for any of the possible products, and  $\mathcal{D}$  stands for any of the possible differentiations, then we get a new integrand (by a partial integration) of a simpler form  $-\tilde{\mathcal{D}}\Phi \otimes \Psi$ , where  $(\tilde{\otimes}, \tilde{\mathcal{D}})$  does not have to be equal to  $(\otimes, \mathcal{D})$ . The price for this is that we get an additional boundary integral.

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## 27 Vector potential

If  $\mathbf{V}$  satisfies  $\operatorname{div} \mathbf{V} = 0$  in a *star shaped domain*  $\Omega$  in  $\mathbb{R}^3$ , then  $\mathbf{V}$  has a *vector potential*  $\mathbf{W}$ , i.e.

$$\mathbf{V} = \nabla \times \mathbf{W}.$$

Furthermore, if  $\mathbf{0}$  is a “star point”, we even have a solution formula,

$$\mathbf{W}_0(\mathbf{x}) = \left\{ \int_0^1 \tau \mathbf{V}(\tau \mathbf{x}) d\tau \right\} \times \mathbf{x}.$$

However, be careful here, because this formula is not easy to apply! Since it may be used in practice, we add the following

### Procedure:

- 1) Check, whether  $\nabla \times \mathbf{W} = \mathbf{V}$  has been calculated at some earlier state of the exercise. If “yes”, then the problem is solved. If “no”, continue with
- 2) Check, whether  $\operatorname{div} \mathbf{V} = 0$ . If “no”, then  $\mathbf{V}$  does not have a vector potential. If “yes”, continue with
- 3) Calculate

$$\mathbf{T}(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ V_x & V_y & V_z \\ x & y & z \end{vmatrix}.$$

- 4) Then write  $\mathbf{T}(\xi, \eta, \zeta)$  instead of  $\mathbf{T}(x, y, z)$ , and substitute

$$\xi = \tau x, \quad \eta = \tau y, \quad \zeta = \tau z.$$

In this way we obtain  $\mathbf{T}(\tau \mathbf{x})$ .

**Remark 27.1** Most of the errors in exercises of this type are made by the substitution above, so be extremely careful at this step.  $\diamond$

- 5) Calculate the *candidate*

$$\mathbf{W}_0(\mathbf{x}) = \int_0^1 \mathbf{T}(\tau \mathbf{x}) d\tau.$$

- 6) If  $\Omega$  is *star shaped*, the problem is solved.
- 7) If  $\Omega$  is *not* star shaped, just check that

$$\nabla \times \mathbf{W}_0 = \mathbf{V}.$$

## 28 Global maximum and minimum

Start always by quoting:

**Theorem 28.1 The second main theorem for continuous functions.** *If  $A$  is a closed and bounded set (some call it “compact”), and  $f : A \rightarrow \mathbb{R}$  is a continuous map, then  $f$  has both a global maximum and a global minimum on  $A$ .*

If  $f : A \rightarrow \mathbb{R}$  is continuous on the *closed and bounded* set  $A$ , then the *global maximum* and the *global minimum* should be found in one of the following point sets:

- 1) **The exceptional points**, i.e.  $u \in A^\circ$  is an interior point of  $A$  and  $\nabla f$  does *not* exist.
- 2) **The stationary points**, i.e.  $u \in A^\circ$  is an interior point of  $A$ , and  $\nabla f = \mathbf{0}$ .
- 3) **The boundary points**, i.e. a point set of lower dimension (at least in the elementary courses of Calculus).

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**Procedure:** Assume that  $A$  is closed and bounded, and that  $f : A \rightarrow \mathbb{R}$  is continuous. Then:

- 1) *Sketch* the set  $A$ ; quote and apply the second main theorem for continuous functions.
- 2) Find the **exceptional points** (if any) with their corresponding values of the function.
- 3) Find the **stationary points** (if any) with their corresponding values of the function.
- 4) Continue with an investigation of the same type on the boundary which is of at least 1 lower dimension, i.e. find the maximum and minimum on the boundary. If the boundary is 1-dimensional, then this is task is known from high school; if the dimension of the boundary is  $> 1$ , then iterate the procedure above.
- 5) Finally **compare numerically** the values found in 2)–4) above.

**Procedure**, when  $A$  is *bounded*, but *not* closed.

- A. If  $f$  has a *continuous extension*,  $\bar{f} : \bar{A} \rightarrow \mathbb{R}$ , then find the (global) maximum and minimum for  $\bar{f}$  by using the methods above. Then compare with  $f : A \rightarrow \mathbb{R}$ , which is defined in fewer points.
- B. If  $f$  does *not* have a continuous extension to  $\bar{A}$ , then add separate investigation of what happens in the neighbourhood of the *exceptional points*.

**Procedure**, if  $A = \mathbb{R}^k$  (no boundary).

- i) Possible maxima and minima can only occur among the *exceptional* and *stationary points*.
- ii) If we can *find* some restriction of  $f$ , such that the values of the restriction tend to  $+\infty$  (or to  $-\infty$ ), then  $f$  has *no global maximum* (or *no global minimum*).
- iii) If  $f(\mathbf{x}) \rightarrow L$  for  $\mathbf{x} \rightarrow \infty$ , then compare  $L$  with the already found values in the *exceptional points* and the *stationary points*.



## 29 Extrema; local investigation

An extremum  $\mathbf{u}$  is either

- an **exceptional point**, i.e.  $\nabla f$  is not defined in  $\mathbf{u} \in A^\circ$ ,

or

- a **stationary point**, i.e.  $\nabla f(\mathbf{u}) = \mathbf{0}$ .

When we have an *extremum*, then the next task is to decide whether it is a local maximum or a local minimum.

Assume that  $f \in C^2$  and that the point  $\mathbf{u}$  is stationary. Then we have two possible methods:

- 1) *Approximating polynomial* of at most second degree, cf. Chapter 7.

Find this approximating polynomial developed from the point under consideration and check whether there is a (local) maximum or minimum at this point.

This method can in principle be applied in any dimension.

- 2) *The so-called  $(r, s, t)$ -method.*

This method can only in the form below be used in the *Euclidean plane*, where the dimension is 2.

It is not possible in  $\mathbb{R}^2$ , where we have two methods available, to give any rule of thumb of which one is the easiest one to apply. In particular one should notice that the  $(r, s, t)$ -method is far from being that universal standard solution as many students may think. Since the students in general make more errors in the application of the  $(r, s, t)$ -method than by constructing the approximating polynomial, I shall here only recommend the method of the approximating polynomial.

Whenever the dimension is  $> 2$ , we have only the first method of approximating polynomial at hand, unless the exercise is so special that some unusual solution looks more tempting due to some special structure of the problem.

## A Formulæ

Some of the following formulæ can be assumed to be known from high school. Others are introduced in *Ventus: Calculus 1a*. It is highly recommended that one *learns most of these formulæ in this appendix by heart*.

### A.1 Squares etc.

The following simple formulæ occurs very frequently in the most different situations.

$$\begin{array}{ll} (a+b)^2 = a^2 + b^2 + 2ab, & a^2 + b^2 + 2ab = (a+b)^2, \\ (a-b)^2 = a^2 + b^2 - 2ab, & a^2 + b^2 - 2ab = (a-b)^2, \\ (a+b)(a-b) = a^2 - b^2, & a^2 - b^2 = (a+b)(a-b), \\ (a+b)^2 = (a-b)^2 + 4ab, & (a-b)^2 = (a+b)^2 - 4ab. \end{array}$$

### A.2 Powers etc.

**Logarithm:**

$$\begin{aligned} \ln |xy| &= \ln |x| + \ln |y|, & x, y &\neq 0, \\ \ln \left| \frac{x}{y} \right| &= \ln |x| - \ln |y|, & x, y &\neq 0, \\ \ln |x^r| &= r \ln |x|, & x &\neq 0. \end{aligned}$$

**Power function, fixed exponent:**

$$\begin{aligned} (xy)^r &= x^r \cdot y^r, x, y > 0 & (\text{extensions for some } r), \\ \left( \frac{x}{y} \right)^r &= \frac{x^r}{y^r}, x, y > 0 & (\text{extensions for some } r). \end{aligned}$$

**Exponential, fixed base:**

$$\begin{aligned} a^x \cdot a^y &= a^{x+y}, & a > 0 & \quad (\text{extensions for some } x, y), \\ (a^x)^y &= a^{xy}, & a > 0 & \quad (\text{extensions for some } x, y), \\ a^{-x} &= \frac{1}{a^x}, & a > 0, & \quad (\text{extensions for some } x), \\ \sqrt[n]{a} &= a^{1/n}, & a \geq 0, & \quad n \in \mathbb{N}. \end{aligned}$$

**Square root:**

$$\sqrt{x^2} = |x|, \quad x \in \mathbb{R}.$$

**Remark A.1** It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value*!  $\diamond$

### A.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x) \left\{ \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \right\},$$

where the latter rearrangement presupposes that  $f(x) \neq 0$  and  $g(x) \neq 0$ .

If  $g(x) \neq 0$ , we get the usual formula known from high school

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{d}{dx} \left\{ f(x) \cdot \frac{1}{g(x)} \right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)} \left\{ \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that  $f(x) \neq 0$  and  $g(x) \neq 0$ . Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}, \quad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

### A.4 Special derivatives.

**Power like:**

$$\frac{d}{dx} (x^\alpha) = \alpha \cdot x^{\alpha-1}, \quad \text{for } x > 0, \text{ (extensions for some } \alpha).$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad \text{for } x \neq 0.$$

**Exponential like:**

$$\frac{d}{dx} \exp x = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} (a^x) = \ln a \cdot a^x, \quad \text{for } x \in \mathbb{R} \text{ og } a > 0.$$

**Trigonometric:**

$$\frac{d}{dx} \sin x = \cos x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \cos x = -\sin x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx} \cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \quad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

**Hyperbolic:**

$$\frac{d}{dx} \sinh x = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \cosh x = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \quad \text{for } x \neq 0.$$

**Inverse trigonometric:**

$$\frac{d}{dx} \operatorname{Arcsin} x = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } x \in ]-1, 1[,$$

$$\frac{d}{dx} \operatorname{Arccos} x = -\frac{1}{\sqrt{1-x^2}}, \quad \text{for } x \in ]-1, 1[,$$

$$\frac{d}{dx} \operatorname{Arctan} x = \frac{1}{1+x^2}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \operatorname{Arccot} x = \frac{1}{1+x^2}, \quad \text{for } x \in \mathbb{R}.$$

**Inverse hyperbolic:**

$$\frac{d}{dx} \operatorname{Arsinh} x = \frac{1}{\sqrt{x^2+1}}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \operatorname{Arcosh} x = \frac{1}{\sqrt{x^2-1}}, \quad \text{for } x \in ]1, +\infty[,$$

$$\frac{d}{dx} \operatorname{Artanh} x = \frac{1}{1-x^2}, \quad \text{for } |x| < 1,$$

$$\frac{d}{dx} \operatorname{Arcoth} x = \frac{1}{1-x^2}, \quad \text{for } |x| > 1.$$

**Remark A.2** The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class.  $\diamond$

## A.5 Integration

The most obvious rules are about linearity

$$\int \{f(x) + \lambda g(x)\} dx = \int f(x) dx + \lambda \int g(x) dx, \quad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and about that differentiation and integration are “inverses to each other”, i.e. modulo some arbitrary constant  $c \in \mathbb{R}$ , which often tacitly is missing,

$$\int f'(x) dx = f(x).$$

If we in the latter formula replace  $f(x)$  by the product  $f(x)g(x)$ , we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}' dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx.$$

Hence, by a rearrangement

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**The rule of partial integration:**

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term  $f(x)g(x)$ .

**Remark A.3** This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself.  $\diamond$

**Remark A.4** This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. See also Chapter 4.  $\diamond$

**Integration by substitution:**

If the integrand has the special structure  $f(\varphi(x)) \cdot \varphi'(x)$ , then one can change the variable to  $y = \varphi(x)$ :

$$\int f(\varphi(x)) \cdot \varphi'(x) dx = \int f(\varphi(x)) d\varphi(x) = \int_{y=\varphi(x)} f(y) dy.$$

**Integration by a monotonous substitution:**

If  $\varphi(y)$  is a *monotonous* function, which maps the  $y$ -interval *one-to-one* onto the  $x$ -interval, then

$$\int f(x) dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y)) \varphi'(y) dy.$$

**Remark A.5** This rule is usually used when we have some “ugly” term in the integrand  $f(x)$ . The idea is to put this ugly term equal to  $y = \varphi^{-1}(x)$ . When e.g.  $x$  occurs in  $f(x)$  in the form  $\sqrt{x}$ , we put  $y = \varphi^{-1}(x) = \sqrt{x}$ , hence  $x = \varphi(y) = y^2$  og  $\varphi'(y) = 2y$ .  $\diamond$

**A.6 Special antiderivatives**

**Power like:**

$$\int \frac{1}{x} dx = \ln |x|, \quad \text{for } x \neq 0. \quad (\text{Do not forget the numerical value!})$$

$$\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1}, \quad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2} dx = \text{Arctan } x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, \quad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2} dx = \text{Artanh } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2} dx = \text{Arcoth } x, \quad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \text{Arcsin } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = -\text{Arccos } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \text{Arsinh } x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \ln(x + \sqrt{x^2+1}), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \sqrt{x^2-1}, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \text{Arcosh } x, \quad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \ln |x + \sqrt{x^2-1}|, \quad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that  $\sqrt{x^2-1} < |x|$  so if  $x < -1$ , then  $x + \sqrt{x^2-1} < 0$ . Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

**Exponential like:**

$$\int \exp x \, dx = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \quad \text{for } x \in \mathbb{R}, \text{ og } a > 0, a \neq 1.$$

**Trigonometric:**

$$\int \sin x \, dx = -\cos x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln |\cos x|, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln |\sin x|, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left( \frac{1 + \sin x}{1 - \sin x} \right), \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left( \frac{1 - \cos x}{1 + \cos x} \right), \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

**Hyperbolic:**

$$\int \sinh x \, dx = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln |\sinh x|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln \left( \frac{\cosh x - 1}{\cosh x + 1} \right), \quad \text{for } x \neq 0,$$



$$\int \frac{1}{\sinh x} dx = \ln \left| \frac{e^x - 1}{e^x + 1} \right|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x} dx = \tanh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x} dx = -\coth x, \quad \text{for } x \neq 0.$$

## A.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus  $(\cos u, \sin u)$  are the coordinates of a point  $P$  on the unit circle corresponding to the angle  $u$ , cf. figure A.1. This geometrical interpretation is used from time to time.

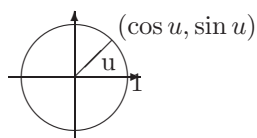


Figure 22: The unit circle and the trigonometric functions.

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**The fundamental trigonometric relation:**

$$\cos^2 u + \sin^2 u = 1, \quad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point  $P$  with the coordinates  $(\cos u, \sin u)$  always has distance 1 from the origo  $(0, 0)$ , i.e. it is lying on the boundary of the circle of centre  $(0, 0)$  and radius  $\sqrt{1} = 1$ .

**Connection to the complex exponential function:**

The *complex exponential* is for imaginary arguments defined by

$$\exp(iu) := \cos u + i \sin u.$$

It can be checked that the usual functional equation for  $\exp$  is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for  $\exp(iu)$  and  $\exp(-iu)$  it is easily seen that

$$\cos u = \frac{1}{2}(\exp(iu) + \exp(-iu)),$$

$$\sin u = \frac{1}{2i}(\exp(iu) - \exp(-iu)),$$

.

**Moivre's formula:** By expressing  $\exp(inu)$  in two different ways we get:

$$\exp(inu) = \cos nu + i \sin nu = (\cos u + i \sin u)^n.$$

**Example A.1** If we e.g. put  $n = 3$  into Moivre's formula, we obtain the following typical application,

$$\begin{aligned} \cos(3u) + i \sin(3u) &= (\cos u + i \sin u)^3 \\ &= \cos^3 u + 3i \cos^2 u \cdot \sin u + 3i^2 \cos u \cdot \sin^2 u + i^3 \sin^3 u \\ &= \{\cos^3 u - 3 \cos u \cdot \sin^2 u\} + i\{3 \cos^2 u \cdot \sin u - \sin^3 u\} \\ &= \{4 \cos^3 u - 3 \cos u\} + i\{3 \sin u - 4 \sin^3 u\} \end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4 \cos^3 u - 3 \cos u, \quad \sin 3u = 3 \sin u - 4 \sin^3 u. \quad \diamond$$

**Addition formulæ:**

$$\sin(u + v) = \sin u \cos v + \cos u \sin v,$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v,$$

$$\cos(u + v) = \cos u \cos v - \sin u \sin v,$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v.$$

**Products of trigonometric functions to a sum:**

$$\sin u \cos v = \frac{1}{2} \sin(u + v) + \frac{1}{2} \sin(u - v),$$

$$\cos u \sin v = \frac{1}{2} \sin(u + v) - \frac{1}{2} \sin(u - v),$$

$$\sin u \sin v = \frac{1}{2} \cos(u - v) - \frac{1}{2} \cos(u + v),$$

$$\cos u \cos v = \frac{1}{2} \cos(u - v) + \frac{1}{2} \cos(u + v).$$

**Sums of trigonometric functions to a product:**

$$\sin u + \sin v = 2 \sin \left( \frac{u + v}{2} \right) \cos \left( \frac{u - v}{2} \right),$$

$$\sin u - \sin v = 2 \cos \left( \frac{u + v}{2} \right) \sin \left( \frac{u - v}{2} \right),$$

$$\cos u + \cos v = 2 \cos \left( \frac{u + v}{2} \right) \cos \left( \frac{u - v}{2} \right),$$

$$\cos u - \cos v = -2 \sin \left( \frac{u + v}{2} \right) \sin \left( \frac{u - v}{2} \right).$$

**Formulæ of halving and doubling the angle:**

$$\sin 2u = 2 \sin u \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2 \cos^2 u - 1 = 1 - 2 \sin^2 u,$$

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

## A.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

**The fundamental relation:**

$$\cosh^2 x - \sinh^2 x = 1.$$

**Definitions:**

$$\cosh x = \frac{1}{2} (\exp(x) + \exp(-x)), \quad \sinh x = \frac{1}{2} (\exp(x) - \exp(-x)).$$

**“Moivre’s formula”:**

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

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**Addition formulæ:**

$$\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y),$$

$$\sinh(x - y) = \sinh(x) \cosh(y) - \cosh(x) \sinh(y),$$

$$\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y),$$

$$\cosh(x - y) = \cosh(x) \cosh(y) - \sinh(x) \sinh(y).$$

**Formulæ of halving and doubling the argument:**

$$\sinh(2x) = 2 \sinh(x) \cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2 \cosh^2(x) - 1 = 2 \sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm \sqrt{\frac{\cosh(x) - 1}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x) + 1}{2}}.$$

**Inverse hyperbolic functions:**

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right), \quad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right), \quad |x| > 1.$$

**A.9 Complex transformation formulæ**

$$\cos(ix) = \cosh(x), \quad \cosh(ix) = \cos(x),$$

$$\sin(ix) = i \sinh(x), \quad \sinh(ix) = i \sin x.$$

**A.10 Taylor expansions**

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1 \cdot 2 \cdots n},$$

with  $n$  factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e.  $= 1$  for the standard series) and *exponential like* (the radius of convergency is infinite).

Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1,$$


$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j, \quad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1,$$

$$\text{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

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**Exponential like:**

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R}.$$

### A.11 Magnitudes of functions

We often have to compare functions for  $x \rightarrow 0+$ , or for  $x \rightarrow \infty$ . The simplest type of functions are therefore arranged in an hierarchy:

- 1) logarithms,
- 2) power functions,
- 3) exponential functions,
- 4) faculty functions.

When  $x \rightarrow \infty$ , a function from a higher class will always dominate a function from a lower class. More precisely:

**A)** A *power function* dominates a *logarithm* for  $x \rightarrow \infty$ :

$$\frac{(\ln x)^\beta}{x^\alpha} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, \beta > 0.$$

**B)** An *exponential* dominates a *power function* for  $x \rightarrow \infty$ :

$$\frac{x^\alpha}{a^x} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, a > 1.$$

**C)** The *faculty function* dominates an *exponential* for  $n \rightarrow \infty$ :

$$\frac{a^n}{n!} \rightarrow 0, \quad n \rightarrow \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

**D)** When  $x \rightarrow 0+$  we also have that a *power function* dominates the *logarithm*:

$$x^\alpha \ln x \rightarrow 0-, \quad \text{for } x \rightarrow 0+, \quad \alpha > 0.$$