Examples of Applications of The Power Series...

Series Method By Solut

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Examples of Applications of The Power Series Method By Solution of Differential Equations with Polynomial Coefficients

Calculus 3c-4

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Introduction

Here follows a collection of examples of how one can solve linear differential equations with polynomial coefficients by the *method of power series*. The reader is also referred to *Calculus 3b*, to *Calculus 3c-3*, and to *Complex Functions*.

It should no longer be necessary rigourously to use the ADIC-model, described in *Calculus 1c* and *Calculus 2c*, because we now assume that the reader can do this himself.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

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1 Solution of differential equations by the power series method

Example 1.1 *1) Find the radius of convergence ϱ for the power series*

$$(1)\quad \sum_{p=1}^{\infty} p \cdot x^{2p-1}.$$

2) Find the sum function $f(x)$ for (1), when $x \in]-\varrho, \varrho[$, e.g. by termwise integration of (1).

3) Prove that if $y = \sum_{n=0}^{\infty} a_n x^n$ is a power series solution of the differential equation

$$(2)\quad (x^2-1)\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 4y = 0,$$

then we have the recursion formula

$$a_{n+2} = \frac{n+4}{n+2}\, a_n, \qquad n \in \mathbb{N}_0.$$

4) Find the solution $y = \varphi(x)$, $x \in I$, of (2), for which $\varphi(0) = 1$ and $\varphi'(0) = 0$.

1) We get by the **criterion of roots** $x \neq 0$ that

$$\sqrt[p]{|a_p(x)|} = \sqrt[p]{p} \cdot x^2 \cdot \frac{1}{\sqrt[p]{|x|}} \to x^2 \qquad \text{for } p \to \infty.$$

From the condition of convergence $x^2 < 1$ follows that $\varrho = 1$.

2) If we put

$$f(x) = \sum_{p=1}^{\infty} p x^{2p-1}, \qquad \text{for } |x| < 1,$$

then

$$F(x) = \int_0^x f(t)\,dt = \frac{1}{2}\sum_{p=1}^{\infty} x^{2p} = \frac{1}{2} \cdot \frac{x^2}{1-x^2} = \frac{1}{2} \cdot \frac{1}{1-x^2} - \frac{1}{2},$$

hence by differentiation,

$$f(x) = F'(x) = \frac{1}{2} \cdot \frac{2x}{(1-x^2)^2} = \frac{x}{(1-x^2)^2}, \qquad \text{for } |x| < 1.$$

3) If we put

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

(formal standard series), then we get by insertion into the differential equation,

$$\begin{aligned}
0 &= (x^2-1)y''+6xy'+4y \\
&= \sum_{\substack{n=2\\(n=0)}}^{\infty} n(n-1)a_nx^n - \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{\substack{n=2\\(n=0)}}^{\infty} 6na_nx^n + \sum_{n=0}^{\infty} 4a_nx^n \\
&= \sum_{n=0}^{\infty}(n^2-n+6n+4)a_nx^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \\
&= \sum_{n=0}^{\infty}(n+1)(n+4)a_nx^n - \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n \\
&= \sum_{n=0}^{\infty}(n+1)\left\{(n+4)a_n-(n+2)a_{n+2}\right\}x^n.
\end{aligned}$$

It follows from the **identity theorem** that

$$(n+1)\{(n+4)a_n-(n+2)a_{n+2}\}=0 \qquad \text{for } n\in\mathbb{N}_0,$$

thus for every n in the summation domain. Since $n+1\neq 0$ for $n\in\mathbb{N}_0$, we get

$$(n+4)a_n=(n+2)a_{n+2} \qquad \text{for } n\in\mathbb{N}_0,$$

which is fulfilled if and only if just one of the following two formulæ is satisfied

$$a_{n+2}=\frac{n+4}{n+2}\,a_n,\quad \text{or}\quad \frac{a_{n+2}}{n+4}=\frac{a_n}{n+2},\qquad \text{for } n\in\mathbb{N}_0.$$

Remark 1.1 In the text, only the former one is required. Note that the latter one is easier to treat. In fact, if we put $b_n=\dfrac{a_n}{n+2}$, then it is written

$$b_{n+2}=b_n,\quad n\in\mathbb{N}_0,\qquad \text{where } b_n=\frac{a_n}{n+2}.\qquad \Diamond$$

4) If $\varphi(0)=1$ and $\varphi'(0)=0$, then $a_0=1$ and $a_1=0$, hence $b_0=\dfrac{1}{2}$ and $b_1=0$. It follows from the recursion formula for b_n that $b_{2n+1}=0$, and thus $a_{2n+1}=0$ for every $n\in\mathbb{N}_0$. Furthermore, $b_{2n}=b_{2n-2}=\cdots=b_0=\dfrac{1}{2}$, hence

$$a_{2n}=(2n+2)b_{2n}=n+1,\qquad \text{for } n\in\mathbb{N}_0.$$

If $x\neq 0$, then we get by (1) that the power series solution (where $\varrho=1$) is given by

$$\varphi(x)=\sum_{n=0}^{\infty}a_{2n}x^{2n}=\sum_{n=0}^{\infty}(n+1)x^{2n}=\sum_{p=1}^{\infty}px^{2p-2}=\frac{1}{x}\cdot\frac{x}{(1-x^2)^2}=\frac{1}{(1-x^2)^2}.$$

By insertion into the differential equation we see that this is a solution

$$\varphi(x)=\frac{1}{(1-x^2)^2}\qquad \text{for } x\in\,]-1,1[.$$

Remark 1.2 It is often worth the trouble to inspect the equation instead of immediately to start on the method of inserting a power series into the differential equation. In the present case we e.g. get by using the rules of differentiation for $|x|<1$ that

$$\begin{aligned}
0 &= (x^2-1)\frac{d^2y}{dx^2}+6x\frac{dy}{dx}+4y=\left\{(x^2-1)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}\right\}+4\left\{x\frac{dy}{dx}+y\right\}\\
&= \left\{(x^2-1)\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{d}{dx}(x^2-1)\cdot\frac{dy}{dx}\right\}+4\left\{x\frac{dy}{dx}+\frac{dx}{dx}\cdot y\right\}=\frac{d}{dx}\left\{(x^2-1)\frac{dy}{dx}+4xy\right\}\\
&= \frac{d}{dx}\left\{\frac{1}{x^2-1}\left[(x^2-1)^2\frac{dy}{dx}+2\cdot 2x(x^2-1)y\right]\right\}=\frac{d}{dx}\left\{\frac{1}{x^2-1}\frac{d}{dx}\left[(x^2-1)^2y\right]\right\}.
\end{aligned}$$

Then by an integration,

$$\frac{1}{x^2-1}\frac{d}{dx}\left[(x^2-1)^2y\right]=c,$$

thus

$$\frac{d}{dx}[(x^2-1)^2y]=c\cdot(x^2-1).$$

Putting $c=3c_2$ we get by another integration,

$$(x^2-1)^2y=c_1+c\left(\frac{x^3}{3}-x\right)=c_1+c_2(x^3-3x),$$

and the complete solution of the differential equation becomes

$$y = \frac{c_1}{(x^2-1)^2} + c_2 \cdot \frac{x^3-3x}{(x^2-1)^2}, \quad |x|<1, \quad c_1, c_2 \in \mathbb{R} \text{ arbitrære.}$$

Since

$$\frac{dy}{dx} = c_1 \cdot x \cdot \{\cdots\} + c_2 \cdot \left\{ \frac{3x^2-3}{(x^2-1)^2} + x(\cdots) \right\},$$

we get

$$y(0) = c_1 = 1 \qquad \text{and} \qquad y'(0) = -3c_2 = 0,$$

hence $c_1 = 0$ and $c_2 = 0$, and the specific solution is

$$\varphi(x) = \frac{1}{(1-x^2)^2} \qquad \text{for } x \in]-1,1[.$$

Example 1.2 *Prove that the differential equation*

$$x\,\frac{d^2y}{dx^2} + (3-2x^2)\frac{dy}{dx} - 4xy = 0, \qquad x \in \mathbb{R},$$

has a simple infinity of solutions which can be written as power series from $x = 0$, i.e. on the form

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad x \in]-\varrho, \varrho[,$$

where $a_0 \in \mathbb{R}$ is an arbitrary constant.

Find the radius of convergence ϱ and the sum function $f(x)$ for $a_0 = 1$.

1) The equation is linear of second order with polynomial coefficients. The coefficient x of $\frac{d^2y}{dx^2}$ is only 0 for $x = 0$, so the formal power series solutions either have radius of convergence $\varrho = 0$ or $\varrho = \infty$.

2) By insertion of

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

we get for $|x| < \varrho$ by adding some convenient zero terms,

$$\begin{aligned}
0 &= \sum_{\substack{n=2\\(n=1)}}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 3na_n x^{n-1} (\text{ same group } a_n x^{n-1}, \text{ same domain } n=1,2,\dots) \\
&\quad - \sum_{\substack{n=1\\(n=0)}}^{\infty} 2na_n x^{n+1} - \sum_{n=0}^{\infty} 4a_n x^{n+1} \qquad (\text{same group } a_n x^{n+1}, \text{ same domain } n=0,1,\dots) \\
&= \sum_{n=1}^{\infty} n(n+2)a_n x^{n-1} - \sum_{n=0}^{\infty} 2(n+2)a_n x^{n+1} \qquad (\text{collecting each group}) \\
&= \sum_{n=1}^{\infty} n(n+2)a_n x^{n-1} - \sum_{n=2}^{\infty} 2na_{n-2} x^{n-1} \qquad (\text{adjust according to the exponent } n-1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{i.e. } m-1=n+1) \\
&= 2a_1 + \sum_{n=2}^{\infty} n(n+2)a_n x^{n-1} - \sum_{n=2}^{\infty} 2na_{n-2} x^{n-1},
\end{aligned}$$

where we have removed terms such that we get the same domain in both places. Summing up we get by collecting the two series,

$$0 = 2a_1 + \sum_{n=2}^{\infty} n\{(n+2)a_n - 2a_{n-2}\}x^{n-1}.$$

3) It follows from the **identity theorem** that every coefficient is 0, hence

$$2a_1 = 0, \quad \text{og} \quad n\{(n+2)a_n - 2a_{n-2}\} = 0 \quad \text{for } n \geq 2,$$

because the summation domain is given by $n \geq 2$.

Since $n \neq 0$ for $n \geq 2$, this is reduced to $a_1 = 0$, and to the **recursion formula**

(3) $(n+2)a_n = 2a_{n-2}$ eller $a_n = \dfrac{2}{n+2}\, a_{n-2}$ for $n \geq 2$.

4) **Solution of the recursion formula**.

Since we have a leap of 2 in the indices of (3), we have to consider separately the two cases, where n is odd or even.

a) It follows from $a_1 = 0$ and (3) that $a_3 = 0$, $a_5 = 0$, …, hence by induction, $a_{2n+1} = 0$ for every odd index $2n+1$, $n \in \mathbb{N}_0$.

b) For even indices we start by replacing n by $2n$ in (3), thus

$$a_{2n} = \frac{2}{2n+2}\, a_{2n-2} = \frac{1}{n+1}\, a_{2(n-1)}, \qquad n \in \mathbb{N}.$$

We define the auxiliary sequence $b_n = a_{2n}$ and then get the two variants of the recursion formula (3),

(4) $b_n = \dfrac{1}{n+1}\, b_{n-1}$ or $(n+1)b_n = b_{n-1}$, $n \in \mathbb{N}$.

Here we have three methods of solution:

i) **Induction**. Since b_0 is "free", the first coefficients are

$$b_1 = \frac{1}{2}b_0, \quad b_2 = \frac{1}{3}b_1 = \frac{1}{3\cdot 2}b_0, \quad b_3 = \frac{1}{4}b_2 = \frac{1}{4\cdot 3\cdot 2}b_0.$$

Set up the **hypothesis**

$$b_n = \frac{1}{(n+1)!}\,b_0, \quad n \in \mathbb{N}, \quad (\text{rigtig for } n = 1, 2, 3).$$

It follows from the recursion formula that the successor becomes

$$b_{n+1} = \frac{1}{(n+1)+1}\,b_{(n+1)-1} = \frac{1}{n+2}\,b_n = \frac{1}{(n+2)!}\,b_n,$$

which is the same as the hypothesis only with n replaced by $n+1$. Then the hypothesis follows by induction.

ii) **Recursion**. By iteration of the recursion formula we get (note that the difference between the denominator and the index is constantly equal to 2),

$$\begin{aligned} b_n &= \frac{1}{n+1}\,b_{n-1} = \frac{1}{n+1}\cdot\frac{1}{n}\,b_{n-2} = \cdots \\ &= \frac{1}{n+1}\cdot\frac{1}{n}\cdots\frac{1}{2}\,b_0 = \frac{1}{(n+1)!}\,b_0. \end{aligned}$$

iii) **Multiplication by** $n! \neq 0$ (an integrating factor) gives

$$(n+1)!b_n = n!b_{n-1} = \cdots = 1!b_0, \text{ dvs. } b_n = \frac{1}{(n+1)!}b_0.$$

5) **Insertion into to formal power series**. We get in all three cases that

$$a_{2n+1} = 0 \quad \text{and} \quad a_{2n} = b_n = \frac{1}{(n+1)!}a_0 \quad \text{for } n \in \mathbb{N}_0,$$

hence

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{a_0}{(n+1)!}\, x^{2n}.$$

6) **Radius of convergence**. If we put $b_n(x) = |a_0| \cdot \dfrac{1}{(n+1)!}\, x^{2n} \geq 0$, we get for $a_0 \neq 0$ and $x \neq 0$ that $b_n(x) > 0$, hence

$$\frac{b_{n+1}(x)}{b_n(x)} = \frac{|a_0| x^{2(n+1)}}{(n+2)!} \cdot \frac{(n+1)!}{|a_0| x^{2n}} = \frac{x^2}{n+2} \to 0 < 1$$

for $n \to \infty$. It follows from the **criterion of quotients** that the series is convergent for every $x \in \mathbb{R}$, thus $\varrho = \infty$, and the interval of convergence is $\mathbb{R}$.

7) **Sum function**. The coefficient $\dfrac{1}{(n+1)!}$ indicates that we should think of an exponential function. When $a_0 = 1$ and $x \neq 0$ we get by the change of indices $n \mapsto n-1$ that

$$y = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^{2n} = \sum_{n=1}^{\infty} \frac{1}{n!} x^{2(n-1)} = \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{n!} x^{2n}.$$

A comparison with the exponential series

$$\exp(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\, t^n, \qquad t \in \mathbb{R},$$

shows that it would be a good idea to put $t = x^2$,

$$\exp(x^2) = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\, x^{2n}.$$

By a rearrangement,

$$\sum_{n=1}^{\infty} \frac{1}{n!}\, x^{2n} = \exp(x^2) - 1,$$

hence for $x \neq 0$,

$$y = \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{n!}\, x^{2n} = \frac{\exp(x^2) - 1}{x^2}.$$

If $x = 0$, then $y(0) = a_0 = 1$, and the sum function is

$$y = f(x) = \begin{cases} \dfrac{\exp(x^2) - 1}{x^2} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

8) **Alternative solution** (without using the power series method). With some deftness we get

$$\begin{aligned}
0 &= x\,\frac{d^2y}{dx^2}+(3-2x^2)\,\frac{dy}{dx}-4xy\\
&= \left\{x\,\frac{d}{dx}\left(\frac{dy}{dx}\right)+1\cdot\frac{dy}{dx}\right\}+2\,\frac{dy}{dx}-\left\{2x^2\,\frac{dy}{dx}+4x\cdot y\right\} \qquad \left(\text{split} \quad 3\,\frac{dy}{dx}=1\cdot\frac{dy}{dx}+2\cdot\frac{dy}{dx}\right)\\
&= \frac{d}{dx}\left\{x\,\frac{dy}{dx}\right\}+\frac{d}{dx}\{2y\}-\frac{d}{dx}\{2x^2y\} \qquad \text{(rule of differentiation of a product)}\\
&= \frac{d}{dx}\left\{x\,\frac{dy}{dx}+2(1-x^2)y\right\}.
\end{aligned}$$

Then by integration and adding an arbitrary constant c_2',

$$x\,\frac{dy}{dx}+2(1-x^2)y=c_2',$$

thus

$$\frac{dy}{dx}+2\left(\frac{1}{x}-x\right)y=c_2'\cdot\frac{1}{x} \qquad \text{for } x\neq 0.$$

Then we get by the usual solution formula for linear differential equations of first order that the complete solution is

$$\begin{aligned} y &= c_1 \cdot \frac{\exp(x^2)}{x^2} + c_2' \cdot \frac{\exp(x^2)}{x^2} \int \exp(-x^2) x \, dx = c_1 \cdot \frac{\exp(x^2)}{x^2} - \frac{c_2'}{2} \cdot \frac{\exp(x^2)}{x^2} \cdot \exp(-x^2) \\ &= c_1 \cdot \frac{\exp(x^2)}{x^2} - \frac{c_2'}{2} \cdot \frac{1}{x^2} = c_1 \cdot \frac{\exp(x^2) - 1}{x^2} + c_2 \cdot \frac{1}{x^2}, \end{aligned}$$

where we have put $c_2 = c_1 - \dfrac{c_2'}{2}$. This is due to the fact that

$$\frac{\exp(x^2) - 1}{x^2} = \sum_{n=1}^{\infty} \frac{1}{n!} x^{2(n-1)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^{2n},$$

is a convergent power series with $\varrho = \infty$.

Example 1.3 *1) Prove that the differential equation*

$$(5) \quad x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + (3 - x^4) y = 0, \qquad x \in \mathbb{R},$$

has precisely one power series solution $y = \sum_{n=0}^{\infty} a_n x^n$, *for which* $(a_0, a_1, a_2, a_3) = \left(0, 0, 0, \dfrac{1}{2}\right)$, *and find this solution.*

Find the interval of convergence and sum function of the series.

2) Find another solution on $\mathbb{R}$ *of (5), which is linearly independent of the first one, and which at* $(x_0, y_0) = (0, 0)$ *fulfils* $(y_0', y_0'', y_0''') = (1, 0, 0)$.

3) Find the complete solution of (5) on $\mathbb{R}_+$.

The problem is strictly speaking over-determined, because the singularity at $x = 0$, where the coefficient of y'' is 0 also creates some conditions.

1) If we insert the formal power series

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

into the differential equation we get by adding some convenient zero terms that

$$\begin{aligned}
0 &= x^2\,\frac{d^2y}{dx^2} - 3x\,\frac{dy}{dx} + (3-x^4)y \\
&= \sum_{\substack{n=2\\(n=0)}}^{\infty} n(n-1)a_n x^n - \sum_{\substack{n=1\\(n=0)}}^{\infty} 3na_n x^n \\
&\quad \text{(here we add some zero terms, cf. the lower bound)} \\
&\quad + \sum_{n=0}^{\infty} 3a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+4} \\
&= \sum_{n=0}^{\infty}(n^2-4n+3)a_n x^n - \sum_{n=4}^{\infty} a_{n-4}x^n \\
&\quad \text{(collect according to the group } a_n x^n\text{, and change of index)} \\
&= \sum_{n=0}^{\infty}(n-1)(n-3)a_n x^n - \sum_{n=4}^{\infty} a_{n-4}x^n \\
&\quad \text{(splitting into factors)} \\
&= \sum_{n=0}^{3}(n-1)(n-3)a_n x^n + \sum_{n=4}^{\infty}\{(n-1)(n-3)a_n - a_{n-4}\}x^n \\
&\quad \text{(removal of terms and collecting the series).}
\end{aligned}$$

Hence it follows form the **identity theorem** that we have for the removed terms (the first finite sum),

$$\begin{array}{lll}
n=0: & 3a_0 = 0, & \text{i.e. } a_0 = 0, \\
n=1: & 0 = 0, & \text{i.e. } a_1 \text{ arbitrary}, \\
n=2: & -a_2 = 0, & \text{i.e. } a_2 = 0, \\
n=3: & 0 = 0, & \text{i.e. } a_3 \text{ arbitrary},
\end{array}$$

and the recursion formula for $n \geq 4$ (from the infinite series)

$$(n-1)(n-3)a_n = a_{n-4}, \qquad \text{for } n \geq 4.$$

It follows in particular, since $a_0 = a_2 = 0$, and since there is a leap of 4 in the indices, by induction that $a_{2n} = 0$ for every $n \in \mathbb{N}_0$.

For odd indices $n = 2m+1$ the recursion formula is written

$$2m \cdot 2(m-1)a_{2m+1} = a_{2m-3} = a_{2(m-2)+1}, \qquad m \geq 2.$$

If we put $b_m = a_{2m+1}$, then

$$2^2 m(m-1)b_m = b_{m-2}, \qquad m \geq 2,$$

hence by a multiplication by $2^{m-2}(m-2)! \neq 0$ for $m \geq 2$,

$$(6) \quad 2^m m! b_m = 2^{m-2}(m-2)! b_{m-2}, \qquad m \geq 2.$$

If we put

$$c_m = 2^m m! b_m = 2^m m! a_{2m+1},$$

then $c_m = c_{m-2}$ by (6), thus $c_{2p+1} = \cdots = c_1$ and $c_{2p} = \cdots = c_0$, hence

$$c_{2p+1} = 2^{2p+1}(2p+1!\, a_{4p+3} = c_1 = 2 \cdot 1!\, a_3 = 2a_3,$$

and

$$c_{2p} = 2^{2p}(2p)!\, a_{4p+1} = c_0 = a_1,$$

so

$$a_{4p+1} = a_1 \cdot \frac{1}{(2p)!} \cdot \frac{1}{2^{2p}} \quad \text{og} \quad a_{4p+3} = 2a_3 \cdot \frac{1}{(2p+1)!} \cdot \frac{1}{2^{2p+1}}.$$

2) We get by insertion that all the power series solutions are given by

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = a_1 \sum_{p=0}^{\infty} \frac{1}{(2p)!} \left(\frac{x^2}{2}\right)^{2p} \cdot x + 2a_3 \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \left(\frac{x^2}{2}\right)^{2p+1} \cdot x \\ &= a_1 x \cosh\left(\frac{x^2}{2}\right) + 2a_3 x \sinh\left(\frac{x^2}{2}\right), \end{aligned}$$

where we have recognized the series for cosh and sinh with $\varrho = \infty$.

If $(a_0, a_1, a_2, a_3) = \left(0, 0, 0, \dfrac{1}{2}\right)$, we get the solution

$$y = x \sinh\left(\frac{x^2}{2}\right).$$

Notice that we are here forced to put $a_0 = a_2 = 0$.

3) If $y = x \cosh\left(\dfrac{x^2}{2}\right)$, then

$$\begin{aligned} y' &= \cosh\left(\frac{x^2}{2}\right) + x^2 \sinh\left(\frac{x^2}{2}\right), & y'(0) &= 1, \\ y'' &= x^3 \cosh\left(\frac{x^2}{2}\right) + 3x \sinh\left(\frac{x^2}{2}\right), & y''(0) &= 0, \\ y''' &= 6x^2 \cosh\left(\frac{x^2}{2}\right) + (x^4+3) \sinh\left(\frac{x^2}{2}\right), & y'''(0) &= 0. \end{aligned}$$

Remark 1.3 It is actually possible directly to solve the equation by using some “dirty tricks”. The idea is to divide by x^5 for $x \neq 0$ and then split in a clever way and of course apply the rules of differentiation of a product in the opposite way of the usual one:

$$\begin{aligned}
0 &= \frac{1}{x^3}\frac{d^2y}{dx^2}-\frac{3}{x^4}\frac{dy}{dx}+\frac{3}{x^5}\,y-\frac{y}{x}\\
&= \frac{1}{x^2}\left\{\frac{1}{x}\frac{d^2y}{dx^2}-\frac{1}{x^2}\frac{dy}{dx}\right\}-\frac{1}{x^2}\left\{\frac{1}{x^2}\frac{dy}{dx}-\frac{2}{x^3}\,y\right\}-\frac{1}{x^3}\left\{\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}\,y\right\}-\frac{y}{x}\\
&= \frac{1}{x^2}\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right)-\frac{1}{x^2}\frac{d}{dx}\left(\frac{1}{x^2}\,y\right)-\frac{1}{x^3}\frac{d}{dx}\left(\frac{y}{x}\right)-\frac{y}{x}\\
&= \frac{1}{x^2}\frac{d}{dx}\left\{\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}\,y\right\}-\frac{1}{x^3}\frac{d}{dx}\left(\frac{y}{x}\right)-\frac{y}{x}\\
&= \frac{1}{x^2}\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{y}{x}\right)\right\}-\frac{1}{x^3}\frac{d}{dx}\left(\frac{y}{x}\right)-\frac{y}{x}\\
&= \frac{1}{x}\left[\frac{1}{x}\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{y}{x}\right)\right\}-\frac{1}{x^2}\frac{d}{dx}\left(\frac{y}{x}\right)\right]-\frac{y}{x}\\
&= \frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left(\frac{y}{x}\right)\right\}-\frac{y}{x}.
\end{aligned}$$

By putting $A=\frac{1}{x}\frac{d}{dx}$, we see that the equation can be written $A^2\left(\frac{y}{x}\right)=\frac{y}{x}$, so we only need to find a more handy variable than x. It is tempting to put

$$\text{“}A=\frac{1}{x}\frac{d}{dx}=\frac{d}{d(x^2/2)}\text{”}.$$

More precisely we apply on $\mathbb{R}_+$ the monotone substitution $t = x^2/2$, $x = \sqrt{2t}$. Then $\dfrac{dt}{dx} = x$, hence by the simplest form of the chain rule,

$$\frac{d}{dt} = \frac{dx}{dt}\,\frac{d}{dx} = \frac{1}{x}\,\frac{d}{dx}.$$

Then the equation is reduce to

$$\frac{d^2}{dt^2}\left(\frac{y}{\sqrt{2t}}\right) - \left(\frac{y}{\sqrt{2t}}\right) = 0,$$

the complete solution of which is

$$\frac{t}{\sqrt{2t}} = c_1 \cosh t + c_2 \sinh t,$$

so

$$y = c_1\sqrt{2t}\cosh t + c_2\sqrt{2t}\sinh t = c_1 x \cosh\left(\frac{x^2}{2}\right) + c_2 x \sinh\left(\frac{x^2}{2}\right). \qquad \Diamond$$

Example 1.4 *Given the power series*

$$\sum_{n=1}^{\infty} \frac{x^{n+2}}{\begin{pmatrix} n+2 \\ n \end{pmatrix}}.$$

Find the radius of convergence ϱ.

Prove that the sum function $y = f(x)$ of the power series in the interval of convergence $-\varrho < x < \varrho$, satisfies the differential equation

$$(1-x)\,\frac{d^2y}{dx^2} = 2x,$$

and find an explicit expression of $f(x)$.

Prove that the power series is convergent for $x = \varrho$, and find the sum of the series for $x = \varrho$.

1) **Radius of convergence**. It follows from

$$\begin{pmatrix} n+2 \\ n \end{pmatrix} = \begin{pmatrix} n+2 \\ 2 \end{pmatrix} = \frac{(n+2)(n+1)}{1\cdot 2},$$

that

$$\sum_{n=1}^{\infty} \frac{x^{n+2}}{\begin{pmatrix} n+2 \\ n \end{pmatrix}} = \sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)}\,x^{n+2},$$

hence by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{\sqrt[n]{2}\cdot\sqrt[n]{x^2}}{\sqrt[n]{n+1}\cdot\sqrt[n]{n+2}}\cdot|x| \to |x| \quad \text{for } n \to \infty.$$

The condition of convergence becomes $|x| < 1$, so $\varrho = 1$.

2) **Differential equation**. If we put

$$y = f(x) = \sum_{n=1}^{\infty} \frac{x^{n+2}}{\binom{n+2}{n}} = \sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)}\, x^{n+2}, \quad |x| < 1,$$

with $f(0) = 0$, then

$$y' = f'(x) = \sum_{n=1}^{\infty} \frac{2}{n+1}\, x^{n+1}, \qquad |x| < 1, \qquad f'(0) = 0,$$

and

$$y'' = f''(x) = \sum_{n=1}^{\infty} 2x^n = 2x \sum_{n=0}^{\infty} x^n = \frac{2x}{1-x}, \qquad |x| < 1.$$

It follows that

$$(1-x)\,\frac{d^2y}{dx^2} = 2x.$$

3) **Determination of** $f(x)$. Since $f(0) = f'(0) = 0$, we get for $|x| < 1$ by integration of

$$\frac{d^2y}{dx^2} = \frac{2x}{1-x} = \frac{2}{1-x} - 2,$$

that

$$\frac{dy}{dx} = \int_0^x \frac{2}{1-t}\, dt - 2x = -2\ln(1-x) - 2x,$$

and

$$\begin{aligned} y &= f(x) = -2\int_0^x \ln(1-t)\, dt - \int_0^x 2t\, dt = 2[(1-t)\ln(1-t) + t]_0^x - x^2 \\ &= 2(1-x)\ln(1-x) + 2x - x^2 = 1 - (1-x)^2 + 2(1-x)\ln(1-x), \end{aligned}$$

hence

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{(n+1)(n+2)}\, x^{n+2} = 1 - (1-x)^2 + 2(1-x)\ln(1-x)$$

for $|x| < 1$.

4) **Convergence for** $x = \varrho = 1$**.** Since $\dfrac{2}{(n+1)(n+2)} \sim \dfrac{2}{n^2}$, and since $\sum_{n=1}^{\infty} \dfrac{2}{n^2}$ is convergent, the series is even absolutely convergent for $x = \pm 1$.

The sum is traditionally found in the following way,

$$\sum_{n=1}^{\infty} \frac{2}{(n+2)(n+1)} = 2 \lim_{N\to\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{n+1} - \frac{1}{n+2} \right\} = 2 \lim_{N\to\infty} \left\{ \frac{1}{2} - \frac{1}{N+1} \right\} = 1.$$

Alternatively, it follows by *Abel's theorem* that

$$\sum_{n=1}^{\infty} \frac{2}{(n+2)(n+1)} = \lim_{x\to 1-} f(x) = \lim_{x\to 1-} \left\{1-(1-x)^2+2(1-x)\ln(1-x)\right\} = 1-0+0=1$$

according to the laws of magnitudes.

Example 1.5 *Given the differential equation*

(7) $(1+x^2)\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} + 6y = 0, \qquad x \in \mathbb{R}.$

Find, expressed by a power series, a solution of (7) where

$$(x_0, y(x_0), y'(x_0)) = (0,1,0),$$

and find the sum function of the series.

By insertion of the formal power series

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

into (7), and adding some zero terms we get

$$\begin{aligned}
0 &= (1+x^2)\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{\substack{n=2\\(n=0)}}^{\infty} n(n-1)a_n x^n + \sum_{\substack{n=1\\(n=0)}}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty}(n^2+5n+6)a_n x^n \quad \text{(grouping according to } a_{n+2}x^n \text{ and } a_n x^n\text{)} \\
&= \sum_{n=0}^{\infty}\{(n+1)(n+2)a_{n+2}+(n+2)(n+3)a_n\}\,x^n \qquad \text{(factorize and collect the series)} \\
&= \sum_{n=0}^{\infty}(n+2)\{(n+1)a_{n+2}+(n+3)a_n\}\,x^n \qquad \text{(remove the common factor).}
\end{aligned}$$

It follows from the **identity theorem** for $n \in \mathbb{N}_0$ (the summation domain) that

$$(n+2)\{(n+1)a_{n+2}+(n+3)a_n\} = 0, \qquad n \in \mathbb{N}_0.$$

Since $n+2 \neq 0$ for every $n \in \mathbb{N}_0$, this equation is reduced to the recursion formula,

$$(n+1)a_{n+2} + (n+3)a_n = 0, \qquad \text{for } n \in \mathbb{N}_0.$$

Since also $n+1 \neq 0$ and $n+3 \neq 0$, this can more conveniently be written

$$\frac{a_{n+2}}{n+3} = -\frac{a_n}{n+1}, \qquad \text{for } n \in \mathbb{N}_0.$$

It follows from the line element that $a_0 = 1$ and $a_1 = 0$, hence the recursion formula gives by induction that $a_{2n+1} = 0$ for every $n \in \mathbb{N}_0$. We get for even indices $(n \mapsto 2n-2)$

$$\frac{a_{2n}}{2n+1} = -\frac{a_{2(n-1)}}{2(n-1)+1} = \cdots = (-1)^n, \quad \text{i.e. } a_{2n} = (-1)^n(2n+1).$$

The formal power series solution is

$$y = \sum_{n=0}^{\infty} a_{2n}x^{2n} = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n}.$$

Putting $a_n(x) = (-1)^n(2n+1)x^{2n}$, we get by the **criterion of roots** th at

$$\sqrt[n]{|a_n(x)|} = \sqrt[n]{2n+1}\cdot x^2 \to |x|^2 \qquad \text{for } n \to \infty.$$

The condition of convergence is $|x|^2 < 1$, thus $|x| < 1$ and $\varrho = 1$.

Sum function. If $|x|<1$, we obtain by known series expansions,

$$\begin{aligned} y &= \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n} = 2\sum_{n=0}^{\infty} n(-x^2)^n + \sum_{n=0}^{\infty}(-x^2)^n = -2x^2\sum_{n=1}^{\infty} n(-x^2)^{n-1} + \frac{1}{1+x^2} \\ &= -2x^2\cdot\frac{1}{(1+x^2)^2} + \frac{1}{1+x^2} = \frac{1-x^2}{(1+x^2)^2}. \end{aligned}$$

Alternatively,

$$\begin{aligned} y &= \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n} = \frac{d}{dx}\int_0^x \sum_{n=0}^{\infty}(-1)^n(2n+1)t^{2n}\,dt = \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^{2n+1} \\ &= \frac{d}{dx}\left\{x\sum_{n=0}^{\infty}(-x^2)^n\right\} = \frac{d}{dx}\left\{\frac{x}{1+x^2}\right\} = \frac{1\cdot(1+x^2)-x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}. \end{aligned}$$

Alternatively the equation can directly be solved in the following way for $x \neq 0$ by a multiplication by x, i.e. the integrating factor. This gives

$$\begin{aligned} 0 &= x\left\{(1+x^2)\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y\right\} \\ &= \left\{(x+x^3)\frac{d^2y}{dx^2} + (1+3x^2)\frac{dy}{dx}\right\} - (1+3x^2)\frac{dy}{dx} + 6x^2\frac{dy}{dx} + 6xy \\ &= \frac{d}{dx}\left\{(x+x^3)\frac{dy}{dx}\right\} + \left\{(3x^2-1)\frac{dy}{dx} + 6x\cdot y\right\} \\ &= \frac{d}{dx}\left\{(x+x^3)\frac{dy}{dx} + (3x^2-1)y\right\} \\ &= \frac{d}{dx}\left\{\frac{x^2}{1+x^2}\left[\frac{(1+x^2)^2}{x}\frac{dy}{dx} + \frac{(3x^2-1)(1+x^2)}{x^2}y\right]\right\} \\ &= \frac{d}{dx}\left\{\frac{x^2}{1+x^2}\frac{d}{dx}\left(\frac{(1+x^2)^2}{x}y\right)\right\}. \end{aligned}$$

For $x \neq 0$ we get by an integration,

$$\frac{x^2}{1+x^2}\frac{d}{dx}\left\{\frac{(1+x^2)^2}{x}y\right\} = -c_1, \qquad c_1 \in \mathbb{R},$$

hence by a rearrangement,

$$\frac{d}{dx}\left\{\frac{(1+x^2)^2}{x}y\right\} = -c_1\left(\frac{1+x^2}{x^2}\right) = -c_1\left(1+\frac{1}{x^2}\right).$$

Then by another integration,

$$\frac{(1+x^2)^2}{x}y = -c_1\left(x-\frac{1}{x}\right) + c_2 = c_1\cdot\frac{1-x^2}{x} + c_2.$$

We finally obtain the complete solution for $x \neq 0$,

$$y = c_1\cdot\frac{1-x^2}{(1+x^2)^2} + c_2\cdot\frac{x}{(1+x^2)^2}, \qquad c_1, c_2 \text{ arbitrary.}$$

By insertion into the differential equation it is easily seen that this is in fact the complete solution in all of $\mathbb{R}$.

Remark 1.4 This example is an excellent illustration of the limitations of the power series method: We only obtain the solution in the *interval of convergence* $]-1,1[$, and we have to insert into the differential equation that the solution is valid in all of $\mathbb{R}$.

The reason of this strange phenomenon can be found in the concept of a "singular point of the differential equation". By this we understand a zero of the coefficient of the highest order term $\dfrac{d^2y}{dx^2}$, here $1+x^2$. The singular points are here *complex*, $\pm i$, and they cannot be seen in the *real* analysis. They nevertheless influence the radius of convergence, because the numerical values of all (complex) singular points are the *candidates* of the radius of convergence. Here $|\pm i| = 1$, in accordance with $\varrho = 1$.

Example 1.6 *Prove that the differential equation*

$$x^2\,\frac{d^2y}{dx^2} + 4x\,\frac{dy}{dx} + \left(2 - \frac{1}{4}\,x^2\right) y = 1$$

has precisely one power series solution $y = \sum_{n=0}^{\infty} a_n x^n$*, and find this solution.*

Find the radius of convergence and the sum function of the series.

By insertion of the formal series

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

into the differential equation and adding some zero terms we get

$$\begin{aligned}
1 &= \sum_{\substack{n=2\\(n=0)}}^{\infty} n(n-1)a_n x^n + \sum_{\substack{n+1\\(n=0)}}^{\infty} 4na_n x^n \qquad \text{(add some trivial zero terms)}\\
&\qquad + \sum_{n=0}^{\infty} 2a_n x^n - \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+2}\\
&= \sum_{n=0}^{\infty}(n^2 - n + 4n + 2)a_n x^n - \frac{1}{4}\sum_{n=2}^{\infty} a_{n-2}x^n\\
&= 2a_0 + 6a_1 x + \sum_{n=2}^{\infty}\left\{(n+1)(n+2)a_n - \frac{1}{4}a_{n-2}\right\} x^n \qquad \text{(remove some terms).}
\end{aligned}$$

It follows from the **identity theorem** that $a_0 = \dfrac{1}{2}$, $a_1 = 0$ and we also get the recursion formula (over the summation domain)

$$2^2(n+2)(n+1)a_n = a_{n-2} \qquad \text{for } n \geq 2.$$

When this is multiplied by $2^{n-2}n! \neq 0$, we get

$$b_n := 2^n(n+2)!a_n = 2^{n-2}n!a_{n-2} = b_{n-2} \qquad \text{for } n \geq 2.$$

Then by recursion, $b_{2n+1} = \cdots = b_1 = 2 \cdot 3! a_1 = 0$, hence $a_{2n+1} = 0$ for every $n \in \mathbb{N}_0$, and

$$b_{2n} = 2^{2n}(2n+2)! a_{2n} = \cdots = b_0 = 2 \cdot a_0 = 2 \cdot \frac{1}{2} = 1,$$

so

$$a_{2n} = \frac{1}{(2n+2)!} \cdot \frac{1}{2^{2n}}, \qquad n \geq 0.$$

By insertion we get the formal power series solution

$$y = \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{2^{2n}}\, x^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} \left(\frac{x}{2}\right)^{2n}.$$

The faculty in the denominator assures that $\varrho = \infty$. (Apply e.g. the **criterion of quotients** with $a_n(x) = \dfrac{1}{(2n+2)!} \left(\dfrac{x}{2}\right)^{2n}$).

Sum function. If $x = 0$, then er $f(0) = \dfrac{1}{2}$. If $x \neq 0$, the structure is vary similar to that of cosh, so we try the following rearrangement,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} \left(\frac{x}{2}\right)^{2n} = \frac{4}{x^2} \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left(\frac{x}{2}\right)^{2n} = \frac{4}{x^2} \left\{ \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{x}{2}\right)^{2n} - 1 \right\} \\ &= \frac{4}{x^2} \left\{ \cosh\left(\frac{x}{2}\right) - 1 \right\}, \end{aligned}$$

hence

$$f(x) = \begin{cases} \dfrac{4}{x^2} \left\{ \cosh\left(\dfrac{x}{2}\right) - 1 \right\} & \text{for } x \neq 0, \\[2ex] 1/2 & \text{for } x = 0. \end{cases}$$

Alternatively, the equation can be completely solved by inspection for $x \neq 0$. In fact, we get by some small reformulations

$$\begin{aligned} 1 &= x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + \left(2 - \frac{1}{4}\, x^2\right) y = \left\{ x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} \right\} + \left\{ 2x \frac{dy}{dx} + 2y \right\} - \frac{1}{4}\, x^2 y \\ &= \frac{d}{dx} \left\{ x^2 \frac{dy}{dx} \right\} + \frac{d}{dx}\{2xy\} - \frac{1}{4}\, x^2 y = \frac{d}{dx} \left\{ x^2 \frac{dy}{dy} + 2x \cdot y \right\} - \frac{1}{4}\, x^2 y = \frac{d^2}{dx^2} \left\{ x^2 y \right\} - \frac{1}{4}\, x^2 y. \end{aligned}$$

We have above assumed that $x \neq 0$, so we can put $z = x^2 y$, and thus $y = \dfrac{z}{x^2}$. Then we get the equation

$$\frac{d^2z}{dx^2} - \frac{1}{4}\, z = 1,$$

the complete solution of which is

$$z = c_1 \cosh\left(\frac{x}{2}\right) + c_2 \sinh\left(\frac{x}{2}\right) - 4,$$

thus

$$y = -\frac{4}{x^2} + c_1\,\frac{1}{x^2}\cosh\left(\frac{x}{2}\right) + c_2\,\frac{1}{x^2}\sinh\left(\frac{x}{2}\right), \qquad x \neq 0.$$

This has only a continuation to $x = 0$, if $c_1 = 4$ and $c_2 = 0$ in accordance with the solution above.

Example 1.7 *Given the differential equation*

$$(8)\quad (1-x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + 64y = 0, \qquad x \in\,]-1,1[.$$

1) Prove that if $y = \sum_{n=0}^{\infty} a_n x^n$ is a power series solution of (8), then we have the recursion formula

$$a_{n+2} = \frac{n^2-64}{(n+1)(n+2)}\,a_n \qquad \textit{for } n \geq 0.$$

2) Prove that the power series solution $\varphi(x)$ of (8), which satisfies $\varphi(0) = 1$ and $\varphi'(0) = 0$, is a polynomial, and find all its coefficients.

1) By insertion of the formal series

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

into the differential equation we get by adding some zero terms that

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{\substack{n=1\\(n=0)}}^{\infty} n(n-1)a_n x^n - \sum_{\substack{n=1\\(n=0)}}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 64 a_n x^n \\ &\qquad \text{(we have added zero terms in the second and third series)} \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n^2-64)a_n x^n \quad \text{(change of index; grouping with } a_n x^n\text{)} \\ &= \sum_{n=0}^{\infty}\left\{(n+2)(n+1)a_{n+2} - (n^2-64)a_n\right\} x^n, \qquad \text{(collect the series after } x^n\text{)}. \end{aligned}$$

We get from the **identity theorem** for $n \in \mathbb{N}_0$ (the summation domain) that

$$(n+2)(n+1)a_{n+2} - (n^2-64)a_n = 0 \qquad \text{for } n \in \mathbb{N}_0.$$

Since $(n+2)(n+1) \neq 0$ for $n \in \mathbb{N}_0$, this can be rewritten as

$$a_{n+2} = \frac{n^2-64}{(n+1)(n+2)}\,a_n = \frac{(n+8)(n-8)}{(n+1)(n+2)}\,a_n, \qquad n \in \mathbb{N}_0.$$

2) If $\varphi(0) = 1$ and $\varphi'(0) = 0$, then $a_0 = 1$ and $a_1 = 0$. The leap of the recursion formula is 2, so it follows by induction that $a_{2n+1} = 0$ for $n \in \mathbb{N}_0$.

For even indices we rewrite the recursion formula $(n \mapsto 2n)$

$$a_{2(n+1)} = \frac{4(n+4)(n-4)}{(2n+1)(2n+2)}\, a_{2n} = \frac{2(n+4)(n-4)}{(2n+1)(n+1)}\, a_{2n}, \qquad n \in \mathbb{N}_0.$$

For $n = 4$ we get $a_{10} = 0$, and then it follows from the recursion formula that $a_{2n} = 0$ for $n \geq 5$. Hence

$$\varphi(x) = a_8 x^8 + a_6 x^6 + a_4 x^4 + a_2 x^2 + a_0.$$

We now have the following two methods.

a) From $a_0 = 1$ follows from the recursion formula

$$\begin{aligned}
a_2 &= \frac{2\cdot 4(-4)}{1\cdot 1} a_0 = -32,\\
a_4 &= \frac{2\cdot 5(-3)}{3\cdot 2}\, a_2 = (-5)(-32) = 160,\\
a_6 &= \frac{2\cdot 6(-2)}{5\cdot 3}\, a_5 = -\frac{8}{5}\cdot 160 = -256,\\
a_8 &= \frac{2\cdot 7(-1)}{7\cdot 4}\, a_6 = -\frac{1}{2}(-256) = 128,
\end{aligned}$$

thus

$$\begin{aligned}\varphi(x) &= 128x^8-256x^6+160x^4-32x^2+1\\ &= 32(x^2-1)(2x^2-1)^2x^2+1.\end{aligned}$$

b) **Alternatively** we have found the structure

$$\varphi(x)=a_8x^8+a_6x^6+a_4x^4+a_2x^2+1, \qquad a_0=1,$$

so

$$\varphi'(x)=8a_8x^7+6a_6x^5+4a_4x^3+2a_2x,$$

$$\varphi''(x)=56a_8x^6+30a_6x^4+12a_4x^2+2a_2,$$

hence by insertion,

$$\begin{aligned}0 &= (1-x^2)\varphi''(x)-x\varphi'(x)+64\varphi(x)\\ &= 64+64a_2x^2+64a_4x^4+64a_6x^6+64a_8x^8\\ &\qquad -2a_2x^2-4a_4x^4-6a_6x^6-8x^8\\ &\qquad -2a_2x^2-12a_4x^4-30a_6x^6-56a_8x^8\\ &\qquad +2a_2+12a_4x^2+30a_6x^4+56a_8x^6\\ &= (64+2a_2)+(60a_2+12a_4)x^2+(48a_4+30a_6)x^4+(28a_6+56a_8)x^6.\end{aligned}$$

Since the coefficients of this equation are 0, we must have

$$a_2=-32, \qquad a_4=-\frac{60}{12}\cdot a_2=160, \qquad a_6=-\frac{48}{30}\cdot a_4=-256, \qquad a_8=-\frac{1}{2}\,a_6=128,$$

hence by insertion,

$$\varphi(x)=128x^8-256x^6+160x^4-32x^2+1.$$

Example 1.8 *Find, expressed by means of a power series, a solution through the line element* $(0,0,1)$ *of the differential equation*

$$x\,\frac{d^2y}{dx^2}+x\,\frac{dy}{dx}+y=0.$$

Find the radius of convergence and sum function of the series.

1) The coefficient x of the highest order term $\dfrac{d^2y}{dx^2}$ is only 0 for $x=0$ in the complex plane. Hence, any formal power series solution of the differential equation can only have its radius of convergence $\varrho\in\{0,\infty\}$. We shall hope for $\varrho=\infty$.

2) **Insert the formal series into the differential equation**. Thus we assume that we have the series expansions kkefremstillinger

$$y=\sum_{n=0}^{\infty}a_nx^n, \quad \frac{dy}{dx}=\sum_{n=1}^{\infty}na_nx^{n-1}, \quad \frac{d^2y}{dx^2}=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}.$$

When these are inserted into the differential equation, we get

$$\begin{aligned}
0 &= x\,\frac{d^2y}{dx^2}+x\,\frac{dy}{dx}+y = x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+x\sum_{n=1}^{\infty} na_n x^{n-1}+\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{\substack{n=2\\(n=1)}}^{\infty} n(n-1)a_n x^{n-1}+\sum_{\substack{n=1\\(n=0)}}^{\infty} na_n x^n+\sum_{n=0}^{\infty} a_n x^n \qquad \text{(reduction and addition of zero terms)} \\
&= \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n+\sum_{n=0}^{\infty}(n+1)a_n x^n \qquad \text{(change of index } n\mapsto n+1 \text{ in the first series)} \\
&= \sum_{n=0}^{\infty}\{(n+1)na_{n+1}a_{n+1}+(n+1)a_n\}x^n \qquad \text{(collecting the series).}
\end{aligned}$$

3) **Identity theorem**. We have here a power series for 0. This power series is unique with all its coefficients equal to 0. Hence, every index which is included in the summation must be zero, hence for $n \in \mathbb{N}_0$. We thus obtain the **recursion formula**

$$0=(n+1)na_{n+1}+(n+1)a_n=(n+1)\{na_{n+1}+a_n\} \quad \text{for } n\in\mathbb{N}_0.$$

Since $n+1 \neq 0$ for $n \in \mathbb{N}_0$, we can remove this factor, so we get the simpler recursion formula

(9) $na_{n+1}+a_n=0 \qquad \text{for } n \in \mathbb{N}_0.$

Remark 1.5 If a common factor of a recursion formula has zeros in the domain of validity, then these zeros must be excepted in the further investigator, before we can remove the factor. There is no problem here, because $n+1 \neq 0$. $\Diamond$

4) **Solution of the recursion formula (9)**.

a) **Standard method**. Express a_{n+1} by a_n. (Warning: One must *never* divide by 0.) Calculate the first coefficients. Set up an hypothesis of induction and prove it by induction (the bootstrap principle).

i) If $n=0$, then $0\cdot a_1+a_0=a_0=0$ (in agreement with the line element).
Then we get by the line element that $a_1=f'(0)=1$.
If $n\geq 1$, then (9) is rewritten as

(10) $a_{n+1}=-\dfrac{1}{n}\,a_n \qquad \text{for } n\in\mathbb{N}.$

ii) Since $a_1=1$, it follows from (10) that

$$\begin{aligned}
a_2 &= -\frac{1}{1}\,a_1=-1, && n=1,\\
a_3 &= -\frac{1}{2}\,a_2=+\frac{1}{2}, && n=2,\\
a_4 &= -\frac{1}{3}\,a_3=-\frac{1}{3\cdot 2\cdot 1}, && n=3.
\end{aligned}$$

Based on these values, we set up the *hypothesis*

(11) $a_n = (-1)^{n-1} \cdot \dfrac{1}{(n-1)!}, \qquad n \in \mathbb{N}.$

Check: It is true for $n = 1, 2, 3, 4$.

iii) **Induction**. *Assume* that the hypothesis is true for some $n \in \mathbb{N}$. Then by the recursion formula (10) the successor is

$$a_{n+1} = -\frac{1}{n}\, a_n = (-1)^{(n+1)-1} \cdot \frac{1}{((n+1)-1)!},$$

which is precisely the hypothesis where n has been replaced by $n+1$.
It follows by the bootstrap principle that (11) holds for every $n \in \mathbb{N}$.

b) **Recursion** We get as in (a) that $a_0 = 0$ and $a_1 = 1$, and (10), hence

$$a_{n+1} = -\frac{1}{n}\, a_n, \qquad n \in \mathbb{N}.$$

If we replace n by $n-1$, we get

$$a_n = -\frac{1}{n-1}\, a_{n-1}, \qquad n \geq 2,$$

hence by insertion of repetition of the process,

$$\begin{aligned} a_{n+1} &= -\frac{1}{n}\, a_n = \left(-\frac{1}{n}\right)\left(-\frac{1}{n-1}\right) a_{n-1} = \cdots = \left(-\frac{1}{n}\right)\left(-\frac{1}{n-1}\right)\cdots\left(-\frac{1}{1}\right) a_1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(we get } n \text{ factors by counting)} \\ &= (-1)^n \cdot \frac{1}{n!}\, a_1 = \frac{(-1)^n}{n!}, \qquad n \in \mathbb{N}_0. \end{aligned}$$

We get by the change of index $n \mapsto n-1$,

$$a_n = (-1)^{n-1} \cdot \frac{1}{(n-1)!}, \qquad n \in \mathbb{N}.$$

c) **Integrating factor**. Write (9) as

$$n a_{n+1} = -a_n \quad \text{for } n \in \mathbb{N} \text{ and } a_0 = 0, a_1 = 1.$$

Multiply this equation by $(-1)^n(n-1)! \neq 0$ for $n \in \mathbb{N}$,

$$(-1)^n n! a_{n+1} = (-1)^{n-1}(n-1)! a_n \quad \text{for } n \in \mathbb{N}.$$

If we put $b_{n+1} = (-1)^n n! a_{n+1}$, then

$$b_{n+1} = b_n = \cdots = b_1 = (-1)^0 0! a_1 = 1,$$

hence

$$b_n = (-1)^{n-1}(n-1)! a_n = 1, \text{ dvs. } a_n = \frac{(-1)^{n-1}}{(n-1)!},\ n \in \mathbb{N}.$$

We get in all three cases that

$$a_0 = 0 \quad \text{and} \quad a_n = \frac{(-1)^{n-1}}{(n-1)!} \quad \text{for } n \in \mathbb{N}.$$

Of course, only one of the methods above is necessary.

5) **The setup of a formal power series**. This is

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!}\, x^n.$$

6) **Determination of the radius of convergence!** (The task is without any significance without this step).

   a) **The secure method** (not necessary, cf. (7) below). Put

   $$c_n(x) = \left| \frac{(-1)^{n-1}}{(n-1)!}\, x^n \right| = \frac{|x|^n}{(n-1)!} \geq 0 \qquad \text{for } n \in \mathbb{N}.$$

   If $x \neq 0$, then $c_n(x) > 0$, and

   $$\frac{c_{n+1}(x)}{c_n(x)} = \frac{|x|^{n+1}}{n!} \cdot \frac{(n-1)!}{|x|^n} = \frac{|x|}{n} \to 0 < 1 \quad \text{for } n \to \infty.$$

   It follows from the **criterion of quotients** that the formal series is convergent for every $x \in \mathbb{R}$, so $\varrho = \infty$, and the formal solution is indeed a solution!

   b) We know that $\varrho = \lim_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$, if the limit exists. It follows from (10) (i.e. the recursion formula)

   $$a_{n+1} = -\frac{1}{n}\, a_n$$

   that by a rearrangement

   $$\varrho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} n = \infty.$$

7) **Recognize the power series** by a comparison with standard series:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!}\, x^n &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, x^{n+1} && \text{(change of index } n \mapsto n+1) \\
&= x \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n && \text{(reformulation)} \\
&= x \exp(-x) && \text{for } x \in \mathbb{R},
\end{aligned}$$

where we have recognized the exponential series, the radius of convergence of which is $\varrho = \infty$, and the investigation in (6) is superfluous.

Summing up we get the solution via the power series method,

$$y = xe^{-x} \qquad \text{for } x \in \mathbb{R}.$$

Alternative solution method. The power series method is rather cumbersome. In the most elementary examples they can actually be solved alternatively by a *trick* follows by applications of some rules of calculus. The not so obvious trick here is to add

$$1 \cdot \frac{dy}{dx} - \frac{dy}{dx} = 0$$

to the equation. Then we get

$$\begin{aligned}
0 &= x\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + y = \left\{ x\,\frac{d^2y}{dx^2} + 1 \cdot \frac{dy}{dx} \right\} - \frac{dy}{dx} + \left\{ x\,\frac{dy}{dx} + 1 \cdot y \right\} = \frac{d}{dx}\left\{ x\,\frac{dy}{dx} - y + xy \right\} \\
&= \frac{d}{dx}\left\{ x\,\frac{dy}{dx} + (x-1)y \right\} = \frac{d}{dx}\left\{ x^2 e^{-x} \left[\frac{e^x}{x}\,\frac{dy}{dx} + \frac{e^x(x-1)}{x^2} \cdot y \right] \right\} = \frac{d}{dx}\left\{ x^2 e^{-x}\, \frac{d}{dx}\left(\frac{e^x y}{x} \right) \right\}.
\end{aligned}$$

We get by integration of the equation

$$\frac{d}{dx}\left\{x^2 e^{-x}\,\frac{d}{dx}\left(\frac{e^x}{x}\right)\right\} = 0$$

that

$$x^2 e^{-x}\,\frac{d}{dx}\left(\frac{e^x y}{x}\right) = c_2,$$

hence

$$\frac{d}{dx}\left(\frac{e^x y}{x}\right) = c_2\,\frac{e^x}{x^2}.$$

Then by another integration,

$$\frac{e^x y}{x} = c_1 + c_2 \int \frac{e^x}{x^2}\,dx,$$

and the complete solution is

$$y = c_1\,x\,e^{-x} + c_2\,x\,e^{-x} \int \frac{e^x}{x^2}\,dx.$$

For $(c_1, c_2) = (1, 0)$ we get the wanted solution, i.e.

$$y_0 = x\,e^{-x}.$$

It can be proved that

$$\int \frac{e^x}{x^2}\,dx = \int \left\{ \frac{1}{x^2} + \frac{1}{x} + \sum_{n=2}^{\infty} \frac{1}{n!}\,x^{n-2} \right\} dx = -\frac{1}{x} + \ln|x| + \sum_{n=1}^{\infty} \frac{1}{(n+1)!\cdot n}\,x^n, \qquad x \neq 0,$$

cannot be expressed by known elementary functions.

Example 1.9 *Given the differential equation*

(12) $x\,\dfrac{d^2y}{dx^2} - (x+2)\,\dfrac{dy}{dx} + 2y = 0.$

1) Find a power series solution $\varphi_1(x)$ of (12), for which

$$\varphi_1(0) = \varphi_1'(0) = \varphi_1''(0) = \varphi_1'''(0) = 1.$$

Find the radius of convergence and the sum function of the series.

2) Find a power series solution $\varphi_2(x)$ of (12), for which

$$\varphi_2(0) = \varphi_2'(0) = \varphi_2''(0) = 2 \quad og \quad \varphi_2'''(0) = 0.$$

Find the radius of convergence and the sum function of this series-

3) Find for each of the intervals $]-\infty, 0[$ and $]0, \infty[$ the complete solution of (12).

We see that both (1) and (2) are over-determined problems, because we in both cases have four equations in two unknowns.

1) By insertion of the formal power series

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad \frac{dy}{dx}=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y}{dx^2}=n(n-1)a_n x^{n-2}$$

into the differential equation (12), we get by adding some zero terms in the first two series that

$$\begin{aligned} 0 &= \sum_{\substack{n=2\\(n=1)}}^{\infty} n(n-1)a_n x^{n-1} - \sum_{\substack{n=1\\(n=0)}}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=1}^{\infty} n(n-3)a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left\{(n+1)(n-2)a_{n+1} - (n-2)a_n\right\} x^n. \end{aligned}$$

Identity theorem. This gives for $n \in \mathbb{N}_0$ (the summation domain)

$$(n-2)\left\{(n+1)a_{n+1} - a_n\right\} = 0 \qquad \text{for } n \geq 0,$$

hence by a multiplication by $n! \neq 0$,

(13) $(n-2)\left\{(n+1)!a_{n+1} - n!a_n\right\} = 0, \qquad \text{for } n \geq 0,$

which can also be applied in (2).

If $n = 2$, the (13) trivially, no matter the values of a_2 and a_3.

If $n \neq 2$, the recursion formula is reduced to

(14) $(n+1)!\, a_{n+1} = n!\, a_n \qquad \text{for } n = 0, 1 \text{ og } n \geq 3.$

It follows from $\varphi_1(0) = \varphi_1'''(0) = 1$ that $a_0 = 1$ and $3!\, a_3 = 1$, hence $a_3 = \dfrac{1}{6}$. Notice that

$$\varphi_1'(0) = 1!\, a_1 = 0!\, a_0 = 1 \quad \text{og} \quad \varphi_1''(0) = 2!\, a_2 = 0!\, a_0 = 1,$$

thus the conditions $\varphi_1'(0) = \varphi_1''(0) = 1$ are automatically satisfied. They should not have been assumed, because they can be derived.

Since

$$(n+1)!\, a_{n+1} = n!\, a_n = \cdots = 3!\, a_3 = 1 \quad \text{for } n \geq 3,$$

we have $a_n = \dfrac{1}{n!}$ (also for $n = 2$, 1, 0), so

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} = e^x \qquad \text{med } \varrho = \infty.$$

2) If $\varphi_1'''(0) = 3!\, a_3 = 0$, then $a_n = 0$ for $n \geq 3$ by (13), and

$$\varphi_2(x) = 2 + \frac{1}{1!}\, x + \frac{2}{2!}\, x^2 = x^2 + 2x + 2,$$

because $\varphi_2(0) = 2$. In particular, $\varphi_2'(0) = \varphi_2''(0) = 2$, so these conditions are automatically fulfilled, i.e. they are superfluous.

3) The complete solution for $x \neq 0$ is then by the **existence and uniqueness theorem** given by

$$y = c_1\varphi_1(x) + c_2\varphi_2(x) = c_1 e^x + c_2(x^2 + 2x + 2),$$

where c_1, $c_2 \in \mathbb{R}$ are arbitrary constants.

We have some additional variants of solution, in which one does not apply the power series method.

First alternative. The trick for $x \neq 0$ is to divide by x^3. Then we get by some reformulations,

$$\begin{aligned} 0 &= \frac{1}{x^2}\frac{d^2y}{dx^2} - \frac{2}{x^3}\frac{dy}{dx} - \frac{1}{x^2}\frac{dy}{dx} + \frac{2}{x^3}\, y = \frac{d}{dx}\left\{\frac{1}{x^2}\frac{dy}{dx}\right\} - \frac{d}{dx}\left\{\frac{y}{x^2}\right\} = \frac{d}{dx}\left\{\frac{1}{x^2}\frac{dy}{dx} - \frac{y}{x^2}\right\} \\ &= \frac{d}{dx}\left\{\frac{e^x}{x^2}\left[e^{-x}\frac{dy}{dx} - e^{-x}y\right]\right\} = \frac{d}{dx}\left\{\frac{e^x}{x^2}\frac{d}{dx}\left(e^{-x}y\right)\right\}. \end{aligned}$$

This equation is immediately integrated:

$$\frac{e^x}{x^2}\frac{d}{dx}\left(e^{-x}y\right) = -c_2, \quad \text{dvs.} \quad \frac{d}{dx}\left(e^{-x}y\right) = -c_2 x^2 e^{-x},$$

hence by another integration

$$e^{-x}y = c_1 - c_2 \int x^2 e^{-x}\,dx = c_1 + c_2 e^{-x}(x^2+2x+2).$$

The complete solution for $x \neq 0$ is

$$y = c_1 e^x + c_2(x^2+2x+2).$$

Second alternative. By inspection of the equation we see that the sum of the coefficients is 0,

$$x - (x+2) + 2 = 0.$$

Now $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} = y$ for $y = e^x$, so $y = e^x$ is a solution of the homogeneous equation. Then we get for $x \neq 0$ by norming the equation,

$$\frac{d^2y}{dx^2} - \left(1+\frac{2}{x}\right)\frac{dy}{dx} + \frac{2}{x}\,y = 0.$$

Then a linearly independent solution is given by

$$\begin{aligned}\varphi_2(x) &= \varphi_1(x)\int \frac{1}{\varphi_1(x)^2}\exp\left(\int\left(1+\frac{2}{x}\right)dx\right)dx = e^x\int e^{-2x}\cdot e^x\cdot x^2\,dx = e^x\int e^{-x}x^2\,dx \\ &= e^x\left\{-e^{-x}\left(x^2+2x+2\right)\right\} = -\left(x^2+2x+2\right),\end{aligned}$$

and the complete solution is er

$$y = c_1 e^x + c_2(x^2+2x+2), \qquad c_1, c_2 \text{ arbitrære konstanter.}$$

Example 1.10 *Prove that the function*

$$f(x) = \sqrt{1-x^2}\cdot \mathit{Arcsin}\ x, \qquad x \in\,]-1,1=B4[,$$

fulfils the differential equation

$$(1-x^2)\frac{dy}{dx} + xy = 1 - x^2.$$

Then find a power series expansion of $f(x)$.

When we insert

$$f(x) = \sqrt{1-x^2}\ \text{Arcsin}\ x \text{ og } f'(x) = -\frac{x}{\sqrt{1-x^2}}\ \text{Arcsin}\ x + 1$$

into the left hand side of the differential equation, we get

$$(1-x^2)f'(x) + x\,f(x) = -x\sqrt{1-x^2}\,\text{Arcsin}\ x + (1-x^2) + x\sqrt{1-x^2}\,\text{Arcsin}\ x = 1-x^2,$$

which shows that $f(x) = \sqrt{1-x^2}\,\text{Arcsin}\ x$ fulfils the differential equation.

Since $f(0) = 0$, we must have $a_0 = 0$ in any power series expansion of $f(x)$, if it exists, thus

$$f(x) = \sum_{n=1}^{\infty} a_n x^n, \quad \text{where} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

By insertion of these formal series into the differential equation we get

$$\begin{aligned}
1 - x^2 &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=1}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{\substack{n=1\\(n=2)}}^{\infty} (n-1) a_n x^{n+1} \\
&= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{4}^{\infty} (n-3) a_{n-2} x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \sum_{n=4}^{\infty} \{n a_n - (n-3) a_{n-2}\} x^{n-1}.
\end{aligned}$$

It follows from the identity theorem that

$$a_1 = 1, \qquad a_2 = 0, \qquad a_3 = -\frac{1}{3},$$

and

$$n a_n - (n-3) a_{n-2} = 0 \qquad \text{for } n \geq 4.$$

By the change of index $n \mapsto n+2$ and a rearrangement we get

(15) $a_{n+2} = \dfrac{n-1}{n+2}\,a_n \qquad \text{for } n \geq 2.$

Now $a_2 = 0$, so (15) implies by induction that $a_{2n} = 0$ for every n. For odd indices (i.e. $n \mapsto 2n-1$ in (15)) we get by recursion

$$\begin{aligned} a_{2n+1} &= \frac{2n-2}{2n+1}\,a_{2n-1} = \frac{2n-2}{2n+1}\cdot\frac{2n-4}{2n-1}\cdots\frac{4}{7}\cdot\frac{2}{5}\cdot\left(-\frac{1}{3}\right) = -\frac{2n(2n-2)^2(2n-4)^2\cdots 4^2\cdot 2^2}{(2n+1)!} \\ &= -\frac{2^{2n-1}n!(n-1)!}{(2n+1)!} \end{aligned}$$

for $n \in \mathbb{N}$. As $a_1 = 1$, we find the formal series expansion

$$f(x) = x - \sum_{n=1}^{\infty} \frac{n!(n-1)!}{(2n+1)!}\cdot 2^{2n-1}\cdot x^{2n+1} = x - \frac{1}{4}\sum_{n=1}^{\infty}\frac{n!(n-1)!}{(2n+1)!}(2x)^{2n+1}.$$

When $x \neq 0$ the general term of the series is

$$a_n(x) = -\frac{1}{4}\,\frac{n!(n-1)!}{(2n+1)!}(2x)^{2n+1} \neq 0 \qquad \text{for } n \in \mathbb{N}.$$

Hence, by the **criterion of quotients**

$$\begin{aligned} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| &= \frac{(n+1)!n!}{(2n+3)!}\,|2x|^{2n+3}\cdot\frac{(2n+1)!}{n!(n-1)!}\cdot\frac{1}{|2x|^{2n+1}} \\ &= \frac{(n+1)n}{(2n+3)(2n+2)}\cdot|2x|^2 = \frac{n+1}{n+\frac{3}{2}}\cdot x^2 \to x^2 \text{ for } n \to \infty. \end{aligned}$$

The convergence condition is $x^2 < 1$, hence $|x| < 1$, and thus $\varrho = 1$. We have proved that

$$f(x) = \sqrt{1-x^2}\,\operatorname{Arcsin} x = x - \frac{1}{4}\sum_{n=1}^{\infty}\frac{n!(n-1)!}{(2n+1)!}(2x)^{2n+1} \quad \text{for } |x| < 1.$$

Example 1.11 *Find the power series solution of the differential equation*

(16) $(x-x^2)\dfrac{d^2y}{dx^2} - 3x\,\dfrac{dy}{dx} - y = 0,$

which goes through the line element $(0,0,1)$*. Find the radius of convergence and sum function of the series. Finally find the complete solution of (16) in the interval* $]-1,1[$.

1) The equation is a linear homogeneous differential equation of second order. The coefficient $x-x^2 = x(1-x)$ of the highest order term $\dfrac{d^2y}{dx^2}$ er 0 for either $x=0$ or $x=1$. Therefore, we can expect that the power series solutions have the radius of convergence $\varrho \in \{0, 1, \infty\}$.

2) When we insert the formal series

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad \frac{dy}{dx}=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y}{dx^2}=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}$$

into (16), we get for $|x| < \varrho$ by adding some zero terms in the first three series,

$$\begin{aligned}
0 &= \sum_{\substack{n=2\\(n=1)}}^{\infty} n(n-1)a_n x^{n-1} - \sum_{\substack{n=2\\(n=0)}}^{\infty} n(n-1)a_n x^n - \sum_{\substack{n=1\\(n=0)}}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=0}^{\infty}(n^2-n+3n+1)a_n x^n \qquad \text{(the latter three series are joined)} \\
&= \sum_{n=0}^{\infty} n(n+1)a_{n+1} x^n - \sum_{n=0}^{\infty}(n+1)^2 a_n x^n \qquad \text{(adjust the exponent of } x^n\text{)} \\
&= \sum_{n=0}^{\infty} \left\{n(n+1)a_{n+1} - (n+1)^2 a_n\right\} x^n \qquad \text{(join the series)} \\
&= \sum_{n=0}^{\infty} (n+1)\{n a_{n+1} - (n+1)a_n\} x^n \qquad \text{(remove the common factor).}
\end{aligned}$$

3) An application of the **identity theorem** gives the recursion formula

$$(n+1)\{n a_{n+1} - (n+1)a_n\} = 0 \qquad \text{for } n \in \mathbb{N}_0,$$

where $\mathbb{N}_0$ is the summation domain. Now, $n+1 \neq 0$ for every $n \in \mathbb{N}_0$, so this can be reduced to

(17) $n a_{n+1} = (n+1)a_n, \qquad \text{for } n \geq 0.$

4) The **line element** $(0,0,1)$ implies that

$$y(0) = a_0 = 0, \quad \text{og} \quad y'(0) = 1!\,a_1 = 1, \quad \text{hence } a_1 = 1.$$

Notice that if we put $n = 0$ into (17), we also get $a_0 = 0$, so the setup is strictly speaking over-determined.

5) The **recursion formula** (17) has now been reduced to

$$n a_{n+1} = (n+1)a_n \quad \text{or} \quad a_{n+1} = \frac{n+1}{n}\,a_n \quad \text{for } n \geq 1,$$

because $a_0 = 0$ and $a_1 = 1$.

Solution of the recursion formula.

a) **Induction**. It follows from $a_1 = 1$ that

$$a_2 = \frac{2}{1}\cdot a_1 = 2, \qquad a_3 = \frac{3}{2}\cdot a_2 = 3.$$

Induction hypothesis:

$$a_n = n \qquad \text{for } n \in \mathbb{N}, \qquad (\text{rigtig for } n = 1, 2, 3)$$

Assume that the hypothesis is true for some $n \in \mathbb{N}$. Then we get for the successor that

$$a_{n+1} = \frac{n+1}{n} \cdot a_n = \frac{n+1}{n} \cdot n = n+1,$$

which has the same structure as the hypothesis, only with n replaced by $n+1$.

The hypothesis then follows by *induction*. (The bootstrap principle).

b) **Recursion**. From $a_1 = 1$ and $a_n = \dfrac{n}{n-1}\, a_{n-1}$ for $n \geq 2$ it follows recursively that

$$a_n = \frac{n}{n-1}\, a_{n-1} = \frac{n}{n-1} \left\{ \frac{n-1}{n-2}\, a_{n-2} \right\} = \cdots = \frac{n}{n-1}\, \frac{n-1}{n-2} \cdots \frac{2}{1}\, a_1 = n.$$

Since also $a_1 = 1$, we have in general $a_n = n$ for $n \in \mathbb{N}$.

c) The **divine inspiration**. If we divide the recursion formula by $(n+1)n \neq 0$, then

$$\frac{a_{n+1}}{n+1} = \frac{a_n}{n}, \qquad n \in \mathbb{N}.$$

By putting $b_n = \dfrac{a_n}{n}$, this is also written

$$b_{n+1} = b_n = \cdots = b_1 = \frac{a_1}{1} = 1, \quad \text{dvs. } b_n = \frac{a_n}{n} = 1,$$

hence

$$a_n = n \qquad \text{for } n \in \mathbb{N}.$$

6) The *formal* power series solution is

$$y = \sum_{n=1}^{\infty} n x^n.$$

The radius of convergence may e.g. be found by the **criterion of roots**. In fact, put $c_n(x) = n|x|^n \geq 0$. Then

$$\sqrt[n]{c_n(x)} = \sqrt[n]{n} \cdot |x| \to 1 \cdot |x| \qquad \text{for } n \to \infty.$$

The condition of convergence is that $|x| < 1 = \varrho$, hence the radius of convergence is $\varrho = 1$, and the series is convergent for $x \in]-1, 1[$.

Once the interval of convergence has been found, we know that we have found a true power series solution

$$y = \sum_{n=1}^{\infty} n x^n, \qquad \text{for } x \in]-1, 1[.$$

7) **Sum function**. When $x \in]-1,1[$, we are allowed to perform the following reformulations,

$$y=\sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1} = x \frac{d}{dx}\left\{ \sum_{\substack{n=1 \\ (n=0)}}^{\infty} x^n \right\} = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}.$$

8) The **complete solution** in $]-1,1[$. We have proved that

$$y_1 = \frac{x}{(1-x)^2}, \qquad \text{for } x \in]-1,1[,$$

is a solution of (16). When $x \neq 0$ we *norm* (16),

$$\frac{d^2y}{dx^2} - \frac{3}{1-x}\,\frac{dy}{dx} - \frac{1}{x(1-x)}\,y = 0 \qquad \text{for } 0 < |x| < 1,$$

where

$$f_1(x) = -\frac{3}{1-x}.$$

Then

$$\exp\left(-\int f_1(x)\,dx\right) = \exp\left(+3\int \frac{dx}{1-x}\right) = \exp(-3\ln|1-x|) = \frac{1}{(1-x)^3},$$

hence for $0 < |x| < 1$

$$\begin{aligned}
y_2(x) &= y_1(x)\int \frac{1}{y_1(x)^2}\,\exp\left(-\int f_1(x)\,dx\right)dx \\
&= \frac{x}{(1-x)^2}\int \frac{(1-x)^4}{x^2}\cdot\frac{1}{(1-x)^3}\,dx = \frac{x}{(1-x)^2}\int \frac{1-x}{x^2}\,dx \\
&= \frac{x}{(1-x)^2}\int \left\{\frac{1}{x^2}-\frac{1}{x}\right\}dx = \frac{x}{(1-x)^2}\left\{-\frac{1}{x}-\ln|x|\right\} = -\frac{x\ln|x|+1}{(1-x)^2}.
\end{aligned}$$

By changing sign, $y_2(x) \mapsto -y_2(x)$, we get the complete solution

$$y(x) = c_1 y_1(x) - c_2 y_2(x) = c_1\cdot\frac{x}{(1-x)^2} + c_2\cdot\frac{x\ln|x|+1}{(1-x)^2},$$

for $0 < |x| < 1$ with arbitrary constants c_1 and $c_2 \in \mathbb{R}$.

9) **Extension to** $x = 0$. (This is here fairly difficult.) Due to the laws of magnitude we have $x\cdot\ln|x| \to 0$ for $x \to 0$. Hence, $y_2(x)$ can be extended *continuously* to $x = 0$ by taking the limit,

$$\lim_{x\to 0}\left(-y_2(x)\right) = \lim_{x\to 0}\frac{x\ln|x|+1}{(1-x)^2} = \frac{0+1}{(1-0)^2} = 1,$$

hence we have the continuous extension

$$-y_2(x) = \begin{cases} \dfrac{x\ln|x|+1}{(1-x)^2} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

We note here that $-y_2(x)$ is **not** continuously differentiable at $x = 0$. We have e.g.

$$\frac{d}{dx}(x\ln|x|) = 1 + \ln|x| \to -\infty \qquad \text{for } x \to 0,$$

and $y_2(x)$ does not belong to the class C^2 i $x = 0$.

It is possible to interpret the solution, if we use the concept of *weak differentiation*.

10) **Alternative solution** for $0 < |x| < 1$ without using series. This variant is very hard, so it is only given here without comments. It shall only illustrate that it is also a possible method in this case. We rewrite the equation in the following way:

$$\begin{aligned}
0 &= (x-x^2)\frac{d^2y}{dx^2} - 3x\,\frac{dy}{dx} - y \\
&= \left\{(x-x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right) + (1-2x)\frac{dy}{dx}\right\} - \frac{dy}{dx} - \left\{x\,\frac{dy}{dx} + 1\cdot y\right\} \\
&= \frac{d}{dx}\left\{(x-x^2)\frac{dy}{dx}\right\} - \frac{dy}{dx} - \frac{d}{dx}\{x\cdot y\} \\
&= \frac{d}{dx}\left\{x(1-x)\frac{dy}{dx} - (1+x)y\right\} \qquad \text{(can be integrated)} \\
&= \frac{d}{dx}\left\{\frac{x^2}{1-x}\left[\frac{(1-x)^2}{x}\,\frac{dy}{dx} - \frac{1-x^2}{x^2}\,y\right]\right\} = \frac{d}{dx}\left\{\frac{x^2}{1-x}\,\frac{d}{dx}\left[\frac{(1-x)^2}{x}\,y\right]\right\}.
\end{aligned}$$

Integration of

$$\frac{d}{dx}\left\{\frac{x^2}{1-x}\,\frac{d}{dx}\left[\frac{(1-x)^2}{x}\,y\right]\right\}=0$$

gives with an arbitrary constant $-c_2$ (notice the sign)

$$\frac{x^2}{1-x}\,\frac{d}{dx}\left[\frac{(1-x)^2}{x}\,y\right]=-c_2,$$

from which

$$\frac{d}{dx}\left[\frac{(1-x)^2}{x}\,y\right]=-c_2\cdot\frac{1-x}{x^2}=c_2\left(\frac{1}{x}-\frac{1}{x^2}\right).$$

Then by another integration,

$$\frac{(1-x)^2}{x}\,y=c_1+c_2\left(\ln|x|+\frac{1}{x}\right),$$

and thus

$$y=c_1\cdot\frac{x}{(1-x)^2}+c_2\cdot\frac{x\ln|x|+1}{(1-x)^2}\qquad\text{for } 0<|x|<1.$$

Then proceed as in (9).

Example 1.12 *Find a power series solution of the differential equation*

$$(1-x^2)\frac{d^2y}{dx^2}-2x\,\frac{dy}{dx}+2y=0,$$

through the line element $(0,1,0)$. *Find the interval of convergence of the series and check, if the series is convergent at the endpoints of the interval. Finally, find the sum function of the series.*

When we insert the formal series

$$y=\sum_{n=0}^{\infty}a_nx^n,\quad \frac{dy}{dx}=\sum_{n=1}^{\infty}na_nx^{n-1},\quad \frac{d^2y}{dx^2}=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

into the differential equation, we get by adding some zero terms in the second series that

$$\begin{aligned}
0 &= (1-x^2)\frac{d^2y}{dx^2}-2x\,\frac{dy}{dx}+2y\\
&= \sum_{n=2}^{\infty}n(n-1)a_n-\sum_{\substack{n=2\\(n=0)}}^{\infty}n(n-1)a_nx^n-\sum_{\substack{n=1\\(n=0)}}^{\infty}2na_nx^n+\sum_{n=0}^{\infty}2a_nx^n\\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n^2-n+2n-2)a_nx^n\\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n^2+n-2)a_nx^n\\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+2)(n-1)a_nx^n\\
&= \sum_{n=0}^{\infty}(n+2)\{(n+1)a_{n+2}-(n-1)a_n\}x^n.
\end{aligned}$$

It follows from the **identity theorem** that if $n \in \mathbb{N}_0$ (the summation domain), then

$$(n+2)\{(n+1)a_{n+2} - (n-1)a_n\} = 0 \qquad \text{for } n \in \mathbb{N}_0.$$

Now, $n+2 \neq 0$ for every $n \in \mathbb{N}_0$, so this can also be written

$$(n+1)a_{n+2} = (n-1)a_n \qquad \text{for } n \in \mathbb{N}_0.$$

There is a leap of 2 in the indices, and for $n = 1$,

$$2 \cdot a_3 = 0 \cdot a_1 = 0,$$

hence by induction, $a_{2n+1} = 0$ for $n \geq 1$, while a_1 is arbitrary.

In particular, $y = x$ is a solution which can also be seen immediately.

We get for even indices

$$(2n-1)a_{2n} = \cdots = (0-1) \cdot a_0 = -a_0, \text{ dvs. } a_{2n} = -\frac{1}{2n-1}\, a_0.$$

It follows from the line element that $a_0 = 1$ and $a_1 = 0$, hence the formal power series solution becomes

$$y = 1 - \sum_{n=1}^{\infty} \frac{1}{2n-1}\, x^{2n}.$$

We get by the **criterion of roots**,

$$\sqrt[n]{|a_n(x)|} = \frac{1}{\sqrt[n]{2n-1}}\, x^2 \to x^2 \qquad \text{for } n \to \infty.$$

The condition of convergence $x^2 < 1$ implies that $\varrho = 1$.

Clearly, $\sum_{n=1}^{\infty} \dfrac{1}{2n-1}$ is divergent, so the series is divergent at both of the endpoints of the interval of convergence.

Sum function. If $|x| < 1$, then

$$\begin{aligned} y &= 1 - \sum_{n=1}^{\infty} \frac{1}{2n-1}\, x^{2n} = 1 - x\sum_{n=1}^{\infty} \frac{1}{2n-1}\, x^{2n-1} = 1 - x\sum_{n=1}^{\infty} \int_0^x t^{2n-2}\, dt \\ &= 1 - x\int_0^x \sum_{n=0}^{\infty} t^{2n}\, dt = 1 - x\int_0^x \frac{1}{1-t^2}\, dt = 1 - \frac{x}{2}\int_0^x \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = 1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right). \end{aligned}$$

Alternatively we get the solution $y = x$ by inspection. It is therefore reasonable for $x \neq 0$ to find a differential equation in $z = \dfrac{y}{x}$ instead. By insertion of $y = xz$ we get

$$\begin{aligned} 0 &= (1-x^2)\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + 2y = (1-x^2)\frac{d^2}{dx^2}(xz) - 2x\,\frac{d}{dx}(xz) + 2xz \\ &= (1-x^2)\frac{d}{dx}\left(x\,\frac{dz}{dx} + z\right) - 2x^2\,\frac{dz}{dx} - 2xz + 2xz = (1-x^2)\left\{x\,\frac{d^2z}{dx^2} + 2\,\frac{dz}{dx}\right\} - 2x^2\,\frac{dz}{dx} \\ &= x(1-x^2)\frac{d^2z}{dx^2} + 2(1-2x^2)\frac{dz}{dx}. \end{aligned}$$

This is a linear differential equation of first order in $u = \dfrac{dz}{dx}$, and it can therefore be solved by the usual methods.

Alternatively the equation is multiplied by $x \neq 0$ (our assumption), hence

$$0 = (x^2 - x^4)\frac{d^2 z}{dx^2} + (2x - 4x^3)\frac{dz}{dx} = (x^2 - x^4)\frac{d}{dx}\left(\frac{dz}{dx}\right) + \frac{d}{dx}(x^2 - x^4)\cdot\frac{dz}{dx} = \frac{d}{dx}\left\{x^2(1-x^2)\frac{dz}{dx}\right\}.$$

Then by an integration,

$$x^2(1-x^2)\frac{dz}{dx} = c,$$

thus

$$\frac{dz}{dx} = \frac{c}{x^2(1-x^2)} = c\left\{\frac{1}{x^2} + \frac{1}{2}\,\frac{1}{x+1} - \frac{1}{2}\,\frac{1}{x-1}\right\}.$$

By another integration we get

$$z = c_1 + c\left\{-\frac{1}{x} + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|\right\}.$$

Since $y = xz$ and $|x| < 1$, and if we put $c = -c_2$, we get

$$y = c_1 x + c_2 \left\{ 1 - \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right)\right\} \qquad \text{for } |x| < 1.$$

Here we have allowed $x = 0$, because it is seen that the solution can be extended continuously to this value.

Example 1.13 *1) Prove that every power series solution of the differential equation*

$$(x - x^2)\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0,$$

is of the simple form $\varphi(x) = kx^2$, where k is an arbitrary constant.

2) Then find a power series solution $\varphi(x)$ with $\varphi''(0) = 0$ of the differential equation

$$(x - x^2)\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 3x^2,$$

and find the radius of convergence and sum function of this power series solution.

3) Finally, find the complete solution in the interval $]0, 1[$ of the two differential equations.

1) By insertion of the formal series

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad \frac{d^2y}{dx^2} \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

into the left hand side of the differential equation we get by adding some zero terms to the first two series,

$$\begin{aligned}(x - x^2)\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y &= \sum_{\substack{n=2\\(n=1)}}^{\infty} n(n-1)a_n x^{n-1} - \sum_{\substack{n=2\\(n=0)}}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=1}^{\infty} n(n-2)a_n x^{n-1} - \sum_{n=0}^{\infty} (n^2 - n - 2)a_n x^n \\ &= \sum_{n=0}^{\infty} (n-1)(n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} (n-2)(n+1)a_n x^n \\ &= \sum_{n=0}^{\infty} (n+1)\{(n-1)a_{n+1} - (n-2)a_n\} x^n.\end{aligned}$$

The right hand side of the differential equation is 0, and $n + 1 \neq 0$ for every $n \in \mathbb{N}_0$ (summation domain), hence we get by the identity theorem that

$$(n-1)a_{n+1} = (n-2)a_n, \qquad \text{for } n \in \mathbb{N}_0.$$

If $n = 2$, then $a_3 = 0$, hence $a_n = 0$ for every $n \geq 3$.

If $n = 1$, then $a_1 = 0$, hence $a_0 = 0$.

Summing up we get that $y = a_2 x^2$ are the only power series solutions of the homogeneous equation.

2) Since the left hand side of the inhomogeneous equation already has been calculated above as a series, we immediately get

$$3x^2 = \sum_{n=0}^{\infty}(n+1)\{(n-1)a_{n+1}-(n-2)a_n\}x^n.$$

Since $n+1 \neq 0$, we get by the identity theorem for $n=2$ that

$$3=(2+1)\{(2-1)a_{2+1}-(2-2)a_2\}=3a_3,\ \text{thus}\ a_3=1,$$

and

$$(n-1)a_{n+1}=(n-2)a_n \qquad \text{for } n\in\mathbb{N}_0\setminus\{2\}.$$

We get from $\varphi''(0)=0$ that $a_2=0$. If $n=1$ then $0=-a_1$, so $a_1=0$.
If $n=0$, then $-a_1=0=-2a_0$, hence $a_0=0$.
If $n\geq 3$, then

$$(n-1)a_{n+1}=(n-2)a_2=\cdots=(3-2)a_3=1,\ \text{dvs.}\ a_n=\frac{1}{n-2}.$$

The formal power series solution is

$$y=\sum_{n=3}^{\infty}\frac{1}{n-2}\,x^n=x^2\sum_{n=1}^{\infty}\frac{1}{n}\,x^n=-x^2\ln(1-x)\ \text{for}\ |x|<1,$$

where we have recognized the logarithmic series with $\varrho=1$.

3) By *norming* we get the homogeneous equation

$$\frac{d^2}{dx^2}-\frac{1}{x(1-x)}\,\frac{dy}{dx}+\frac{2}{x(1-x)}\,y=0, \qquad \text{for } x\in\,]0,1[.$$

Now, $\varphi_1(x)=x^2$ is a solution, so a linearly independent solution is given by

$$\begin{aligned}
\varphi_2(x) &= x^2\int\frac{1}{x^4}\,\exp\left(\int\frac{dx}{x(1-x)}\right)dx = x^2\int\frac{1}{x^4}\,\exp\left(\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx\right)dx\\
&= x^2\int\frac{1}{x^2}\cdot\frac{x}{1-x}\,dx = x^2\int\frac{1}{x^3(1-x)}\,dx\\
&= x^2\int\left\{\frac{1}{1-x}+\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}\right\}dx \quad \text{(decomposition)}\\
&= x^2\left\{\ln\left(\frac{x}{1-x}\right)-\frac{1}{2x^2}-\frac{1}{x}\right\}=x^2\ln\left(\frac{x}{1-x}\right)-\frac{1}{2}-x.
\end{aligned}$$

If $x\in\,]0,1[$ the complete solution of (1) is given by (the arbitrary constants are c_1, c_2),

$$y=c_1x^2+c_2\left\{x^2\ln\left(\frac{x}{1-x}\right)-\frac{1}{2}-x\right\},$$

and of (2),

$$y=-x^2\ln(1-x)+c_1x^2+c_2\left\{x^2\ln\left(\frac{x}{1-x}\right)-\frac{1}{2}-x\right\}.$$

The equations can also be solved in a different way:

First alternative. Since x^2 is a solution of the homogeneous equation, we insert $y = x^2 z$ into the inhomogeneous equation and set up a differential equation in z. Then

$$\begin{aligned}
3x^2 &= (x-x^2)\frac{d^2}{dx^2}(x^2z) - \frac{d}{dx}(x^2z) + 2x^2z = x(1-x)\frac{d}{dx}\left\{x^2\frac{dz}{dx}+2xz\right\} - x^2\frac{dz}{dx} - 2xz + 2x^2z \\
&= x(1-x)\left\{x^2\frac{d^2z}{dx^2}+4x\frac{dz}{dx}+2z\right\} - x^2\frac{dz}{dx} - 2xz(1-x) = x^3(1-x)\frac{d^2z}{dx^2} + x^2(4-4x-1)\frac{dz}{dx} \\
&= (x^3-x^4)\frac{d^2z}{dx^2} + (3x^2-4x^3)\frac{dz}{dx} \qquad \left(\text{can now be solved as a first order equation in } \frac{dz}{dx}\right) \\
&= (x^3-x^4)\frac{d}{dx}\left(\frac{dz}{dx}\right) + \frac{d}{dx}(x^3-x^4)\cdot\frac{dz}{dx} = \frac{d}{dx}\left\{x^3(1-x)\frac{dz}{dx}\right\} \\
&= \frac{d}{dx}\left\{x^3(1-x)\frac{d}{dx}\left(\frac{y}{x^2}\right)\right\}.
\end{aligned}$$

When we integrate this equation in the interval $]0,1[$, we get

$$x^3(1-x)\frac{d}{dx}\left(\frac{y}{x^2}\right) = x^3 + c_2,$$

hence

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{1-x} + \frac{c_2}{x^3(1-x)} = \frac{1}{1-x} + c_2\left(\frac{1}{1-x}+\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}\right).$$

The complete solution of the inhomogeneous equation is obtained by another integration followed by a multiplication by x^2,

$$\begin{aligned}
y &= -x^2\ln(1-x) + c_1x^2 + c_2x^2\left\{\ln\left(\frac{x}{1-x}\right) - \frac{1}{2x^2} - \frac{1}{x}\right\} \\
&= -x^2\ln(1-x) + c_1x^2 + c_2\left\{x^2\ln\left(\frac{x}{1-x}\right) - \frac{1}{2} - x\right\}, \quad x \in]0,1[.
\end{aligned}$$

Second alternative. It is possible directly to obtain the differential equation in the first alternative in the interval $]0,1[$ by using the following rearrangements

$$\begin{aligned}
3x^2 &= (x-x^2)\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y \qquad \text{(add something and subtract it again)} \\
&= \left\{(x-x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right) + (1-2x)\frac{dy}{dx}\right\} - 2\frac{dy}{dx} + 2\left\{x\frac{dy}{dx} + 1\cdot y\right\} \\
&= \frac{d}{dx}\left\{x(1-x)\frac{dy}{dx} - 2(1-x)y\right\} = \frac{d}{dx}\left\{x^3(1-x)\left[\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y\right]\right\} \\
&= \frac{d}{dx}\left\{x^3(1-x)\frac{d}{dx}\left(\frac{y}{x^2}\right)\right\}.
\end{aligned}$$

Then continue as in the first alternative.

2 Larger examples of the power series method

Example 2.1 *Find a power series solution* $y = \sum_{n=0}^{\infty} a_n y^n$ *where* $y(0) = y'(0) = 0$ *of the differential equation*

$$(x + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 2x.$$

Find the radius of convergence and the sum function of this power series solution.

1) Assume that the solution has the formal power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

We get from the initial conditions,

$$y(0) = a_0 = 0 \qquad \text{og} \qquad y'(0) = 1 \cdot a_1 = a_1 = 0,$$

so the formal series are

$$y=\sum_{n=2}^{\infty} a_n x^n, \quad y'=\sum_{n=2}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

with the same lower bound $n=2$.

2) The coefficient of the term of highest order is $x+x^2=x(x+1)=0$ for $x=0$ and $x=-1$ (the singular points). Hence, the radius of convergence satisfies $\varrho\in\{0,1,\infty\}$.

3) By insertion of the series into the differential equation we obtain by reading from the right towards the left,

$$\begin{aligned}
2x &= (x+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+x\sum_{n=2}^{\infty} n a_n x^{n-1}-\sum_{n=2}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^n+\sum_{n=2}^{\infty} a_n x^{n-1} a_n x^{n-1}+\sum_{n=2}^{\infty} n a_n x^n-\sum_{n=2}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty}\{n^2-n+n-1\}a_n x^n+\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n \\
&= 2a_2 x+\sum_{n=2}^{\infty}(n+1)\{(n-1)a_n+n a_{n+1}\}x^n.
\end{aligned}$$

4) Then it follows from the identity theorem,

$$2a_2=2, \qquad \text{thus} \quad a_2=1,$$

and the recursion formula [NB: The factor $n+1>0$ can be removed]

$$(n-1)a_n+n a_{n+1}=0 \qquad \text{for } n\ge 2,$$

i.e. when n goes through the summation domain.

5) **Solution of the recursion formula**. We write this formula in one of the following ways

$$a_{n+1}=-\frac{n-1}{n}a_n, \quad n\ge 2, \text{ eller } n a_{n+1}=-(n-1)a_n, \quad n\ge 2.$$

a) **Induction**. Since $a_2=1$, we get

$$\begin{aligned}
a_3 &= -\frac{2-1}{2}a_2=-\frac{1}{2}, && n=2,\\
a_4 &= -\frac{3-1}{3}a_3=\left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right)=\frac{1}{3}, && n=3,\\
a_5 &= -\frac{4-1}{4}a_4=\left(-\frac{3}{4}\right)\cdot\frac{1}{3}=-\frac{1}{4}, && n=4.
\end{aligned}$$

A reasonable induction hypothesis is

$$(18)\quad a_n=\frac{(-1)^n}{n-1} \qquad \text{for } n\ge 2.$$

At least this is fulfilled for $n = 2, 3, 4$. Assume that (18) holds for some $n \geq 2$. Then we get for the successor a_{n+1} by using the recursion formula,

$$a_{n+1} = -\frac{n-1}{n} \cdot \frac{(-1)^n}{n-1} = \frac{(-1)^{n+1}}{n} = \frac{(-1)^{n+1}}{\{n+1\}-1},$$

which has the same structure as (18), only with n replaced by $n+1$. We conclude by induction (the bootstrap principle) that (18) holds in general.

b) **Recursion**. When we repeat the recursion formula *downwards*, we get instead

$$\begin{aligned} a_{n+1} &= -\frac{n-1}{n} a_n = \left(-\frac{n-1}{n}\right)\left(-\frac{n-2}{n-1}\right) a_{n-1} = \cdots \\ &= \left(-\frac{n-1}{n}\right)\left(-\frac{n-2}{n-1}\right)\cdots\left(-\frac{1}{2}\right) a_2 \qquad (n-1 \text{ factors}) \\ &= (-1)^{n-1} \cdot \frac{1}{n}, \end{aligned}$$

hence by the change of index, $n+1 \mapsto n$,

$$a_n = \frac{(-1)^n}{n-1} \qquad \text{for } n \geq 2.$$

c) **Inspection**. If we put $b_n = (n-1)a_n$, then

$$b_n = -b_{n-1} = (-1)^2 b_{n-2} = \cdots = (-1)^{n-2} b_{n-(n-2)} = (-1)^n b_2,$$

so

$$b_n = (n-1)a_n = (-1)^n b_2 = (-1)^n (2-1) a_2 = (-1)^n,$$

and

$$a_n = \frac{(-1)^n}{n-1} \qquad \text{for } n \geq 2.$$

We obtain by all three methods

$$a_0 = a_1 = 0 \qquad \text{and} \qquad a_n = \frac{(-1)^n}{n-1} \quad \text{for } n \geq 2.$$

6) The formal series is

$$y = \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} x^n = x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

Here we recognize the logarithmic series with $\varrho = 1$, and the sum of the series is for $|x| < 1$,

$$y = x \ln(1+x), \qquad x \in]-1, 1[.$$

7) (No details). This equation can also be rewritten in a convenient way:

$$\frac{d}{dx}\left\{x^2(x+1)\frac{d}{dx}\left(\frac{y}{x}\right)\right\} = 2x \quad \text{for } x \neq 0, -1.$$

The complete solution is then obtained by two integrations and a reduction,

$$y = x\ln|1+x| + c_1 x + c_2\left\{x\ln\left|\frac{x+1}{x}\right| - 1\right\} \qquad \text{for } x \neq 0, -1.$$

Every solution can by continuity be extended to $x = 0$.

8) It should be mentioned that it is also possible just to guess the solution $y = x$ of the homogeneous equation, and the the equation can be solved by some known solution formula.

Example 2.2 *Find the radius of convergence and sum function of the power series*

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}.$$

Prove that a power series solution of the differential equation

(19) $y'' - 2xy' - 4y = 0,$

which fulfils $y(0) = 0$, is of the form $k\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{n!}$, where k is an arbitrary constant. Check if there exist other power series solutions of the differential equation (19).

1) We immediately get the sum function by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} = x\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = x \cdot \exp(x^2),$$

and the radius of convergence is $\varrho = \infty$.

2) When we insert the formal power series

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

into the differential equation, we get by adding some zero terms,

$$\begin{aligned} 0 &= y'' - 2xy' - 4y \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{\substack{n=1 \\ (n=0)}}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 2(n+2)a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)\{(n+1)a_{n+2} - 2a_n\}x^n. \end{aligned}$$

Now, $n + 2 \neq 0$ for $n \in \mathbb{N}_0$, so we conclude from the identity theorem that

$$(n+1)a_{n+2} = 2a_n, \qquad \text{i.e. } a_{n+2} = \frac{2}{n+1}\,a_n, \quad n \in \mathbb{N}_0,$$

with a leap of 2 in the indices.

If $y(0) = 0$, then $a_0 = 0$, so $a_{2n} = 0$ by induction.

If $a_1 = y'(0) = k$, then

$$a_{2n+1} = \frac{2}{2n}\, a_{2n-1} = \frac{1}{n}\, a_{2n-1}.$$

If we multiply by $n! \neq 0$, then we get by recursion,

$$n! a_{2n+1} = (n-1)! a_{2(n-1)+1} = \cdots = 1! a_1 = k,$$

hence

$$a_{2n+1} = \frac{k}{n!}, \qquad n \in \mathbb{N}_0.$$

By (1) the power series solution is

$$k \sum_{n=0}^{\infty} \frac{1}{n!}\, x^{2n+1} = kx \cdot \exp(x^2), \qquad x \in \mathbb{R}.$$

3) Then let $y(0) = a_0 = k \neq 0$ and $y'(0) = 0$. We get $a_{2n+1} = 0$, and

$$a_{2n} = \frac{2}{2n-1} \cdot a_{2(n-1)} = \frac{4n}{2n(2n-1)} \cdot a_{2(n-1)}.$$

When we multiply by $\dfrac{(2n)!}{4^n n!}$ this equation is transferred into

$$\frac{(2n)!}{4^n n!}\, a_{2n} = \frac{(2(n-1))!}{4^{n-1}(n-1)!}\, a_{2(n-1)} = \cdots = \frac{0!}{4^0 \cdot 0!}\, a_0 = k,$$

hence

$$a_{2n} = \frac{4^n n!}{(2n)!} \cdot k, \qquad n \in \mathbb{N}_0.$$

The *formal* series is

$$k \sum_{n=0}^{\infty} \frac{4^n n!}{(2n)!}\, x^{2n}.$$

For $x \neq 0$ we get by the **criterion of quotients** that

$$\left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \frac{k \cdot 4^{n+1}(n+1)! x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{k \cdot 4^n n! x^{2n}} = \frac{4(n+1)x^2}{(2n+2)(2n+1)} = \frac{2x^2}{2n+1} \to 0 \text{ for } n \to \infty.$$

This proves that the formal series is convergent for every $x \in \mathbb{R}$.

It follows from the **existence and uniqueness theorem** that the complete solution of the differential equation is given by

$$y = c_1\, x \exp(x^2) + c_2 \sum_{n=0}^{\infty} \frac{4^n n!}{(2n)!}\, x^{2n} \quad \text{for } x \in \mathbb{R},$$

where c_1 and c_2 are arbitrary constants.

Example 2.3 *Given the initial value problem*

$$(20)\quad \begin{cases} 4x\,\dfrac{d^2y}{dx^2}+2\,\dfrac{dy}{dx}-y=0, & x\geq 0,\\[2mm] y(0)=1, \qquad y'(0)=\dfrac{1}{2}. \end{cases}$$

1) Assume that

$$y=\sum_{n=0}^{\infty}a_nx^n$$

is a power series solution of (20). Find a recursion formula for the coefficients a_n.

2) Prove that the recursion formula is fulfilled for

$$a_n=\frac{1}{(2n)!},\qquad n\in\mathbb{N}_0.$$

3) Find the interval of convergence of the power series solution.

4) Find the sum function of the power series solution for $x\geq 0$.
(Hint: Replace x by $\sqrt{x}$ in a known power series).

1) When we insert the formal power series

$$y=\sum_{n=0}^{\infty}a_nx^n,\quad \frac{dy}{dx}=\sum_{n=1}^{\infty}na_nx^{n-1},\quad \frac{d^2y}{dx^2}=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

into the differential equation and add some zero terms, we get

$$\begin{aligned} 0 &= 4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-y=\sum_{\substack{n=2\\(n=1)}}^{\infty}4n(n-1)a_nx^{n-1}+\sum_{n=1}^{\infty}2na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^n\\ &= \sum_{n=1}^{\infty}2n(2n-1)a_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^n=\sum_{n=1}^{\infty}2n(2n-1)a_nx^{n-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n-1}\\ &= \sum_{n=1}^{\infty}\{2n(2n-1)a_n-a_{n-1}\}x^{n-1}. \end{aligned}$$

We derive from the **identity theorem** the recursion formula

$$2n(2n-1)a_n=a_{n-1}\qquad \text{for } n\in\mathbb{N}.$$

2) Then we get from the initial values

$$a_0=y(0)=1\qquad\text{and}\qquad a_1=y'(0)=\frac{1}{2}.$$

If we put $n=1$ into the recursion formula, we see that

$$2\cdot(2-1)a_1=2a_1=a_0,$$

which is in accordance with the given values.

If we multiply the recursion formula by $(2n-2)!$, we get

$$(2n)!a_n = (2(n-1))!a_{n-1} = \cdots = 0!\, a_0 = a_0 = 1,$$

hence

$$a_n = \frac{1}{(2n)!}, \qquad \text{in particular } a_0 = 1 \text{ and } a_1 = \frac{1}{2}.$$

Alternatively we assume that $a_{n-1} = \dfrac{1}{(2(n-1))!}$ for some $n \in \mathbb{N}$. This is true for $n = 1$ and $n = 2$. Then we get from the recursion formula

$$a_n = \frac{1}{2n(2n-1)}\, a_{n-1} = \frac{1}{2n(2n-1)(2n-2)!} = \frac{1}{(2n)!},$$

and the claim follows by induction.

3) The formal power series solution is

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^n.$$

If we put $c_n = \dfrac{|x|^n}{(2n)!}$, then $c_n > 0$ for $x \neq 0$, and

$$\frac{c_{n+1}}{c_n} = \frac{|x|^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^n} = \frac{|x|}{(2n+2)(2n+1)} \to 0 \qquad \text{for } n \to \infty,$$

It follows from the **criterion of quotients** that the interval of convergence is $\mathbb{R}$.

Alternatively,

$$\left| \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^n \right| \leq \sum_{n=0}^{\infty} \frac{1}{n!} |x|^n = e^{|x|} \qquad \text{for alle } x \in \mathbb{R},$$

and it follows from the **criterion of comparison** that the interval of convergence is $\mathbb{R}$

4) If $x \geq 0$, then

$$y = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (\sqrt{x})^{2n} = \cosh(\sqrt{x}).$$

Remark 2.1 If $y_1(t) = \cosh(\sqrt{x})$ is known for $x > 0$, then the complete solution can be found for $x > 0$ by a known solution formula. By norming we get

$$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} - \frac{1}{4x}\, y = 0, \qquad x > 0.$$

Then

$$
\begin{aligned}
y_2(x) &= y_1(x)\int \frac{1}{y_1(x)^2}\exp\left(-\int f_1(x)\,dx\right)dx = \cosh(\sqrt{x})\int \frac{1}{\cosh^2(\sqrt{x})}\exp\left(-\int \frac{1}{2x}\,dx\right)dx \\
&= \cosh(\sqrt{x})\int \frac{1}{\cosh^2(\sqrt{x})}\cdot\frac{1}{\sqrt{x}}\,dx = \cosh(\sqrt{x})\cdot 2\int_{u=\sqrt{x}} \frac{du}{\cosh^2 u} \\
&= 2\cosh(\sqrt{x})\cdot\tanh(\sqrt{x}) = 2\sinh(\sqrt{x}).
\end{aligned}
$$

The complete solution for $x \in \mathbb{R}_+$ is not surprisingly

$$y = c_1\cosh(\sqrt{x}) + c_2\sinh(\sqrt{x}), \qquad c_1,\ c_2 \text{ arbitrary}.$$

Notice that

$$\sinh(\sqrt{x}) = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(\sqrt{x})^{2n+1} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,x^{n+(1/2)}$$

formally is *not* a power series solution, because every exponent contains one half.

Remark 2.2 If $x < 0$, we get analogously [because $x = -|x|$]

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, |x|^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{|x|})^{2n} = \cos(\sqrt{|x|}). \end{aligned}$$

By repeating the calculations of Remark 2.1 we get the complete solution for $x < 0$,

$$y = c_1 \cos(\sqrt{|x|}) + c_2 \sin(\sqrt{|x|}), \qquad c_1,\, c_2 \text{ arbitrære.}$$

Example 2.4 *Given the differential equation*

$$(21)\quad x\frac{d^2y}{dx^2} - (2x^2+1)\frac{dy}{dx} - 4xy = 0, \qquad x \in \mathbb{R}.$$

1) Prove that if the power series of radius of convergence $\varrho > 0$,

$$(22)\quad \sum_{n=0}^{\infty} a_n x^n, \qquad x \in]-\varrho, \varrho[,$$

is a solution of (21), then

$$-a_1 - 4a_0 x + \sum_{n=3}^{\infty} n\{(n-2)a_n - 2a_{n-2}\}x^{n-1} = 0, \quad x \in]-\varrho, \varrho[.$$

2) Prove that (21) has a solution $y = \varphi(x)$ of the form (22), satisfying the conditions $\varphi(0) = 0$, $\varphi'(0) = 0$, $\varphi''(0) = 2$.

3) Find the sum function of the power series in (2).

1) The coefficient of the highest order term $\dfrac{d^2y}{dx^2}$ is 0 for $x = 0$. Hence, we can expect that $\varrho \in \{0, \infty\}$.

We get by insertion of the formal series (22) into (21) (where we add some zero terms) that

$$\begin{aligned} 0 &= \sum_{\substack{n=2\\(n=1)}}^{\infty} n(n-1)a_n x^{n-1} - \sum_{\substack{n=1\\(n=0)}}^{\infty} 2na_n x^{n+1} - \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} 4a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} 2(n+2)a_n x^{n+1} = \sum_{n=1}^{\infty} n(n-2)a_n x^{n-1} - \sum_{n=2}^{\infty} 2na_{n-2} x^{n-1} \\ &= -a_1 - 4a_0 x + \sum_{n=3}^{\infty} n\{(n-2)a_n - 2a_{n-2}\}x^{n-1}. \end{aligned}$$

2) Assume that $\varphi(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution of (21) with

$$\varphi(0) = a_0 = 0, \quad \varphi'(0) = 1 \cdot a_1 = 0, \quad \varphi''(0) = 2a_2 = 2.$$

Then we must have

$$a_0 = 0, \qquad a_1 = 0 \qquad \text{og} \qquad a_2 = 1.$$

By these assumptions the expression of (1) is reduced to

$$0 = \sum_{n=3}^{\infty} n\{(n-2)a_n - 2a_{n-2}\}x^{n-1}.$$

It follows from the **identity theorem** that we get the recursion formula

$$n\{(n-2)a_n - 2a_{n-2}\} = 0, \qquad \text{for } n \geq 3.$$

Since $n \neq 0$, this equation is reduced to

$$(n-2)a_n = 2a_{n-2} \qquad \text{for } n \geq 3.$$

There is a leap of 2 in the indices, so we must consider the cases of n odd or even separately.

a) If $n = 2p+1$ is odd, then the recursion formula becomes

$$(2p-1)a_{2p+1} = 2a_{2p-1}, \qquad p \geq 1.$$

Now $a_1 = 0$, so $a_3 = 0$, etc., and it follows by **induction** that all $a_{2p+1} = 0$.

b) If $n = 2p\,(\geq 3)$, hence $p \geq 2$, we get instead

$$(2p-2)a_{2p} = 2a_{2p-2}, \qquad p \geq 2,$$

which is reduced to

$$(p-1)a_{2p} = a_{2(p-1)}, \qquad p \geq 2.$$

Here there are more possible solutions:

i) **The elegant one**. Multiply by $(p-2)! \neq 0$. It follows immediately that

$$(p-1)!a_{2p} = (p-2)!a_{2(p-1)} = \cdots = 1!a_2 = 1,$$

hence

$$a_{2p} = \frac{1}{(p-1)!} \quad \text{for } p \geq 2 \ \text{ (even for } p \geq 1).$$

ii) **Recursion**. By iteration of

$$a_{2p} = \frac{1}{p-1}\, a_{2(p-1)}, \qquad p \geq 2,$$

(notice how $p-1$ occurs on the right hand side) we get

$$a_{2p} = \frac{1}{p-1} \cdot \frac{1}{p-2} \cdots \frac{1}{1} \cdot a_{2 \cdot 1} = \frac{1}{(p-1)!}, \quad p \geq 2, \ \ (p \geq 1).$$

iii) **Induction**. It follows from $a_2 = 1$ that

$$p = 2: \quad a_4 = 1; \qquad p = 3: \quad a_6 = \frac{1}{3-1}\, a_4 = \frac{1}{2},$$

$$p = 4: \quad a_8 = \frac{1}{4-1}\, a_6 = \frac{1}{3 \cdot 2} = \frac{1}{3!}.$$

Then we set up the **hypothesis**

$$a_{2p} = \frac{1}{(p-1)!} \qquad \text{(True for } p = 1, 2, 3, 4\text{)}.$$

By the recursion formula ($p \mapsto p+1$),

$$a_{2(p+1)} = \frac{1}{p}\, a_{2p} = \frac{1}{p} \cdot \frac{1}{(p-1)!} = \frac{1}{p!},$$

and the hypothesis follows in general by induction.

Summing up we have proved that we necessarily must have that

$$a_0 = 0, \quad a_{2n-1} = 0, \quad a_{2n} = \frac{1}{(n-1)!} \quad \text{for } n \in \mathbb{N}.$$

Then we get **formally**,

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{2n} x^{2n} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\, x^{2n}.$$

It remains to be proved that the radius of convergence is $\varrho > 0$. This can either be done by comparing with a standard series of known radius of convergence $\varrho > 0$ (this is here sufficient; we shall, however, postpone this method to (3)) or by a direct proof. (This is actually superfluous here, if we start with (3), though we shall nevertheless go through the argument here.)

Let $b_n(x) = \left| \dfrac{1}{(n-1)!}\, x^{2n} \right| = \dfrac{1}{(n-1)!}\, x^{2n}$. Then $b_n(x) > 0$, if $x \neq 0$. If so, we have

$$\frac{b_{n+1}(x)}{b_n(x)} = \frac{x^{2(n+1)}}{n!} \cdot \frac{(n-1)!}{x^{2n}} = \frac{x^2}{n} \to 0 < 1 \quad \text{for } n \to \infty,$$

for every fixed $x \in \mathbb{R}$, and the series is convergent for every $x \in \mathbb{R}$, and the radius of convergence is $\varrho = \infty$.

The solution is

$$\varphi(x) = \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\, x^{2n} \qquad \text{for } x \in \mathbb{R}.$$

3) The faculty in the denominator indicates that we must have "something including the exponential function". We get by the change of index $n \mapsto n+1$ that

$$\begin{aligned} \varphi(x) &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\, x^{2n} = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^{2(n+1)} = x^2 \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n \\ &= x^2 \exp(x^2) \qquad \text{for } x \in \mathbb{R}. \end{aligned}$$

ADDITION. This equation can also be solved **alternatively** by "a dirty trick". Since differentiation lowers the degree by 1, and multiplication by x increases the degree by 1, we get for the general term of the series,

$$
\begin{array}{ll}
x\dfrac{d^2y}{dx^2} & \text{has degree } n-2+1=n-1,\\[2ex]
2x^2\dfrac{dy}{dx} & \text{has degree } n-1+2=n+1,\\[2ex]
\dfrac{dy}{dx} & \text{has degree } n-1=n-1,\\[2ex]
4xy & \text{has degree } n+1=n+1.
\end{array}
$$

We see that only the degrees $n+1$ and $n-1$ with the leap of 2 occur, so the possible power series solutions must necessarily have the structure

$$y = c_1 f(x^2) + c_2 x g(x^2) = \varphi(x),$$

where f and g are functions of $t = x^2$.

We have $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\varphi''(0) = 2$, so we guess on a solution of the form

$$y = \varphi(x) = f(x^2) \qquad \text{which is even.}$$

For $x \neq 0$ we apply the change of variable $t = x^2$, (monotonous in each of the intervals $x < 0$ and $x > 0$), hence

$$\frac{dy}{dx} = \frac{dt}{dx}\,\frac{dy}{dt} = 2x\,\frac{dy}{dt},$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{2x\,\frac{dy}{dt}\right\} = 2\,\frac{dy}{dt} + 4x^2\,\frac{d^2}{dt^2} = 4t\,\frac{d^2y}{dt^2} + 2\frac{dy}{dt}.$$

By insertion into (21) we get

$$\begin{aligned} 0 &= x\frac{d^2y}{dx^2} - (2x^2+1)\frac{dy}{dx} - 4xy = x\left\{4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right\} - (2t+1)\cdot 2x\frac{dy}{dt} - 4xy \\ &= x\left\{4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 4t\frac{dy}{dt} - 2\frac{dy}{dt} - 4y\right\} = 4x\left\{t\frac{d^2y}{dt^2} - t\frac{dy}{dt} - y\right\}. \end{aligned}$$

Since $x \neq 0$, we reduce this equation in the following way

$$0 = t\frac{d^2y}{dt^2} - t\frac{dy}{dt} - y = \left\{t\frac{d^2y}{dt^2} + 1\cdot\frac{dy}{dt}\right\} - \frac{dy}{dt} - \frac{d}{dt}(ty) = \frac{d}{dt}\left\{t\cdot\frac{dy}{dt} - (t+1)y\right\}.$$

We get by an integration,

$$t\,\frac{dy}{dt} - (t+1)y = c_2,$$

hence, because $x \neq 0$ implies that $t > 0$,

$$\frac{dy}{dt} - \left(1 + \frac{1}{t}\right)y = \frac{c_2}{t}.$$

Now $\int\left(1 + \dfrac{1}{t}\right)dt = t + \ln t$ for $t > 0$, so the complete solution for $t > 0$ is

$$y = c_1 t e^t + c_2 t e^t \int \frac{dt}{t^2 e^t}.$$

It can be proved that if $c_2 \neq 0$, this integral cannot be written as an elementary function, so we can only integrate the corresponding series term by term (a transcendental function).

If $c_2 = 0$, then we get for $t = x^2 > 0$ that

$$y = c_1 t e^t = c_1 x^2 \exp(x^2) = c_1 \sum_{n=0}^{\infty} \frac{1}{n!}\,x^{2n+2} \quad \text{for } x \neq 0.$$

The remaining now follows by testing the equation $\varphi''(0) = 2$.

Example 2.5 *Given the differential equation*

$$(23)\quad x\frac{d^2y}{dx^2}-2\frac{dy}{dx}-xy=0, \qquad x\in\mathbb{R}.$$

1) Prove that if the power series of radius of convergence $\varrho>0$,

$$(24)\quad \sum_{n=0}^{\infty} a_n x^n, \qquad x\in\,]-\varrho,\varrho[,$$

is a solution of (23), then

$$-2a_1-(2a_2+a_0)x-a_1x^2+\sum_{n=4}^{\infty}\{n(n-3)a_n-a_{n-2}\}x^{n-1}=0, \quad x\in\,]-\varrho,\varrho[.$$

2) Prove by using the result of (1) that (23) has a solution $y=\varphi(x)$ of the form (24), which satisfies the conditions $\varphi(0)=1$, $\varphi'(0)=0$, $\varphi'''(0)=0$, and that this solution is given by

$$y=\varphi(x)=1-\sum_{p=1}^{\infty}\frac{2p-1}{(2p)!}\,x^{2p}, \qquad x\in\mathbb{R}.$$

3) Find the sum function of the power series of (2).
(Hint: Notice that

$$\left.-\frac{2p-1}{(2p)!}=\frac{1}{(2p)!}-\frac{1}{(2p-1)!}, \qquad p\in\mathbb{N}\right).$$

1) We get by insertion of (24) into (23) and addition of some zero terms that

$$\begin{aligned}
0 &= \sum_{\substack{n=2\\(n=1)}}^{\infty} n(n-1)a_nx^{n-1}-\sum_{n=1}^{\infty}2na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^{n+1}\\
&= \sum_{n=1}^{\infty}n(n-3)a_nx^{n-1}-\sum_{n=2}^{\infty}a_{n-2}x^{n-1}=-2a_1+\sum_{n=2}^{\infty}\{n(n-3)a_n-a_{n-2}\}x^{n-1}\\
&= -2a_1-(2a_2+a_0)x-a_1x^2+\sum_{n=4}^{\infty}\{n(n-3)a_n-a_{n-2}\}x^{n-1}.
\end{aligned}$$

2) It follows by the identity theorem from the structure of (1) that

$$a_1=0,\quad 2a_2+a_0=0,\quad a_1=0,\quad n(n-3)a_n-a_{n-2}=0 \text{ for } n\ge 4.$$

Then by the initial conditions,

$$\varphi(0)=a_0=1,\quad \varphi'(0)=a_1=0,\quad \varphi'''(0)=3\cdot 2\cdot 1\cdot a_3=0,$$

hence, summing up,

$$a_0=1,\quad a_1=0,\quad a_2=-\frac{1}{2},\quad a_3=0,$$

and the recursion formula

$$n(n-3)a_n = a_{n-2} \qquad \text{for } n \geq 4.$$

In particular, $a_{4p-1} = 0$ for $p \in \mathbb{N}$.
By choosing $n = 2p$ even the recursion formula is reduced to

$$2p(2p-3)a_{2p} = a_{2(p-1)} \quad \text{for } p \geq 2 \quad (\text{even for } p \geq 1).$$

When $a_0 = 1$ and $a_2 = a_{2\cdot 1} = -\dfrac{1}{2}$ the solution is unique and it suffices just to check $a_{2p} = -\dfrac{2p-1}{(2p)!}$.
Obviously, $a_0 = 1$ and $a_1 = -\dfrac{2-1}{2!} = -\dfrac{1}{2}$. Finally,

$$2p(2p-3)a_{2p} = 2p(2p-3)\cdot\left\{-\frac{2p-1}{(2p)!}\right\} = -\frac{2p-3}{2(p-1)!} = a_{2(p-1)}, \qquad p \geq 2,$$

and the recursion formula holds.

Then it follows from the recursion formula,

$$\varrho^2 = \lim_{p\to\infty} \frac{a_{2(p-1)}}{a_{2p}} = \lim_{p\to\infty} 2p(2p-3) = \infty,$$

so the radius of convergence is $\varrho = \infty$.

Thus, we have proved that the solution is given by

$$y = \varphi(x) = 1 - \sum_{p=1}^{\infty} \frac{2p-1}{(2p)!}\, x^{2p}, \qquad x \in \mathbb{R}.$$

3) By means of the hint we get

$$\begin{aligned} y &= \varphi(x) = 1 - \sum_{p=1}^{\infty} \frac{2p-1}{(2p)!}\, x^{2p} = 1 + \sum_{p=1}^{\infty}\left(\frac{1}{(2p)!} - \frac{1}{(2p-1)!}\right) x^{2p} \\ &= \sum_{p=0}^{\infty} \frac{1}{(2p)!}\, x^{2p} - \sum_{p=0}^{\infty} \frac{1}{(2p+1)!}\, x^{2p+2} = \cosh x - x\sinh x. \end{aligned}$$

Example 2.6 *Given the differential equation*

$$(25)\quad (2x^2+1)\frac{d^2y}{dx^2}+8x\,\frac{dy}{dx}+4y=0, \qquad x\in\mathbb{R}.$$

1) Prove that if the power series of radius of convergence $\varrho>0$*,*

$$(26)\quad \sum_{n=0}^{\infty} a_n x^n, \qquad x\in\,]-\varrho,\varrho[,$$

is a solution of (25), then

$$a_{n+2}+2a_n=0, \qquad n\in\mathbb{N}_0.$$

2) Prove that (25) has a solution $y=\varphi(x)$ *of the form (26), satisfying the conditions*

$$\varphi(0)=0, \qquad \varphi'(0)=1.$$

3) Find the sum function of the power series of (2).

4) Find a solution of (25) through the line element $(0,0,2)$*.*

The example can be treated in several ways. The main variant is of course the **power series method**.

1) By insertion of

$$y=\sum_{n=0}^{\infty}a_nx^n, \quad y'=\sum_{n=1}^{\infty}na_nx^{n-1}, \quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

and addition of some zero terms we get

$$\begin{aligned}
0 &= (2x^2+1)\frac{d^2y}{dx^2}+8x\,\frac{dy}{dx}+4y=(2x^2+1)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+8x\sum_{n=1}^{\infty}na_nx^{n-1}+4\sum_{n=0}^{\infty}a_nx^n\\
&= \sum_{\substack{n=2\\(n=0)}}^{\infty}2n(n-1)a_nx^n+\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{\substack{n=1\\(n=0)}}^{\infty}8a_nx^n+\sum_{n=0}^{\infty}4a_nx^n\\
&= \sum_{n=0}^{\infty}\{2n^2-2n+8n+4\}a_nx^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n\\
&= \sum_{n=0}^{\infty}\{2(n^2+3n+2)a_n+(n+2)(n+1)a_{n+2}\}x^n=\sum_{n=0}^{\infty}(n+2)(n+1)\{2a_n+a_{n+2}\}x^n.
\end{aligned}$$

Since $(n+2)(n+1)\neq 0$ for $n\in\mathbb{N}_0$, it follows from the **identity theorem** that we have the recursion formula

$$a_{n+2}+2a_n=0 \qquad \text{for } n\in\mathbb{N}_0.$$

2) Then we get from $\varphi(0) = 0$ that $a_0 = 0$, and from $\varphi'(0) = 1$ that $a_1 = 1$.
   The recursion formula has a leap of 2 in the indices, so we conclude by induction that $a_{2n} = 0$ for every $n \in \mathbb{N}_0$.

   For $n = 2p+1$, $p \in \mathbb{N}_0$, *odd* the recursion formula becomes

   $$a_{2(p+1)+1} = -2a_{2p+1}.$$

   The first terms are

   $$a_1 = 1, \qquad a_3 = -2, \qquad a_5 = (-2)^2,$$

   so we set up the *hypothesis* $a_{2p+1} = (-2)^p$.
   Assume this hypothesis. Then by the recursion formula,

   $$a_{2(p+1)+1} = -2a_{2p+1} = -2 \cdot (-2)^p = (-2)^{p+1},$$

   which is the hypothesis with p replaced by $p+1$. Then the hypothesis follows by induction, i.e. the bootstrap method.

Alternatively it follows by recursion that

$$a_{2p+1} = -2a_{2(p-1)+1} = (-2)^2 a_{2(p-2)+1} = \cdots = (-2)^p a_1 = (-2)^p.$$

The formal series becomes

$$\varphi(x) = \sum_{p=0}^{\infty} a_{2p+1} x^{2p+1} = \sum_{p=0}^{\infty} (-2)^p x^{2p+1} = x \sum_{p=0}^{\infty} (-2x^2)^p.$$

This is a **quotient series** of quotient $q = -2x^2$, so the condition of convergence is $|-2x^2| < 1$, i.e. $|x| < \dfrac{1}{\sqrt{2}}$, and we get $\varrho = \dfrac{1}{\sqrt{2}}$.

3) The quotient is $q = -2x^2$, hence the sum function is in $]-\varrho, \varrho[$ given by

$$\varphi(x) = x \sum_{p=0}^{\infty} q^p = \frac{x}{1-q} = \frac{x}{1+2x^2}, \qquad \text{for } |x| < \frac{1}{\sqrt{2}}.$$

4) The solution through the line element $(0,0,2)$ satisfies $\psi(0) = 0$ and $\psi'(0) = 2 = 2 \cdot \varphi'(0)$. The equation is linear, and since $\psi(0) = 2 \cdot \varphi(0)$, the solution is

$$\psi(x) = 2\varphi(x) = \frac{2x}{1+2x^2}, \qquad |x| < \frac{1}{\sqrt{2}}.$$

Remark 2.3 We have so far only found the solution $\varphi(x) = \dfrac{x}{1+2x^2}$ in the interval $|x| < \dfrac{1}{\sqrt{2}}$. It is, however, immediately seen that the function is defined for every $x \in \mathbb{R}$. By a simple test it is also a solution for $x \in \mathbb{R}$. We first calculate

$$\varphi'(x) = \frac{1}{1+2x^2} - \frac{4x^2+(2-2)}{(1+2x^2)^2} = \frac{1}{1+2x^2} - \frac{2}{1+2x^2} + \frac{2}{(1+2x^2)^2} = -\frac{1}{1+2x^2} + \frac{2}{(1+2x^2)^2}$$

and

$$\varphi''(x) = \frac{4x}{(1+2x^2)^2} - \frac{16x}{(1+2x^2)^3}.$$

We get by insertion

$$(2x^2+1)\varphi''(x) + 8x\varphi'(x) + 4\varphi(x) = \frac{4x}{1+2x^2} - \frac{16x}{(1+2x^2)^2} - \frac{8x}{1+2x^2} + \frac{16x}{(1+2x^2)^2} + \frac{4x}{1+2x^2} = 0.$$

It follows from the test that $\varphi(x) = \dfrac{x}{1+2x^2}$ is a solution of (25) in $\mathbb{R}$.

Here we have produced the full argument. One can, however, obtain this result without any calculation at all. In fact, according to the previously shown results φ fulfils the differential equation in the interval $|x| < \dfrac{1}{\sqrt{2}}$. Then note that since $\varphi(x)$ is a rational fractional function, defined in $\mathbb{R}$, the same is true for $\varphi'(x)$ and $\varphi''(x)$. When these fractional functions are inserted into (25), these inserted functions "cannot distinguish between if we are inside or outside the interval $|x| < \dfrac{1}{\sqrt{2}}$". Since the equation is satisfied inside the interval, it must also be fulfilled outside the interval, and we conclude that $\varphi(x)$ is a solution in $\mathbb{R}$.

Remark 2.4 Again we can solve the equation directly by an **inspection**. In fact, we get by a small reformulation of (25)

$$\begin{aligned} 0 &= (2x^2+1)\frac{d^2y}{dx^2}+8x\,\frac{dy}{dx}+4y=\left\{(2x^2+1)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}\right\}+\left\{4x\frac{dy}{dx}+4y\right\}\\ &= \left\{(2x^2+1)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{d}{dx}(2x^2+1)\cdot\frac{dy}{dx}\right\}+\left\{4x\cdot\frac{dy}{dx}+\frac{d}{dx}(4x)\cdot y\right\}\\ &= \frac{d}{dx}\left\{(2x^2+1)\frac{dy}{dx}+4xy\right\}=\frac{d}{dx}\left\{(2x^2+1)\frac{dy}{dx}+\frac{d}{dx}(2x^2+1)\cdot y\right\}\\ &= \frac{d^2}{dx^2}\{(2x^2+1)y\}. \end{aligned}$$

Then by two successive integrations

$$(2x^2+1)y=c_1+c_2x,$$

and the complete solution is

$$y=c_1\cdot\frac{1}{2x^2+1}+c_2\cdot\frac{x}{2x^2+1},\qquad x\in\mathbb{R}.$$

The rest is easy.

Example 2.7 *Given the differential equation*

$$(27)\quad x\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}+2y=0, \qquad x\in\mathbb{R}.$$

1) Prove that if the power series of radius of convergence $\varrho>0$,

$$(28)\quad \sum_{n=0}^{\infty} a_n x^n, \qquad x\in\,]-\varrho,\varrho[,$$

is a solution of (27), then

$$2a_0+a_1+\sum_{n=1}^{\infty}[(n+1)^2a_{n+1}+(n+2)a_n]x^n=0, \quad x\in\,]-\varrho,\varrho[.$$

(In some of the variants we get instead

$$\sum_{n=0}^{\infty}[(n+1)^2a_{n+1}+(n+2)a_n]x^n=0, \qquad x\in\,]-\varrho,\varrho[.\Big]$$

2) Prove that (27) has a solution $y=\varphi(x)$ of the form (28), satisfying the conditions

$$\varphi(0)=1, \qquad \varphi'(0)=-2.$$

3) Find the sum function of the power series of (2).

1) We insert the power series (28) into (27) and add some zero terms. Hereby,

$$\begin{aligned}
0 &= \sum_{\substack{n=2\\(n=1)}}^{\infty} n(n-1)a_nx^{n-1}+\sum_{\substack{n=1\\(n=0)}}^{\infty} na_nx^n+\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}2a_nx^n\\
&= \sum_{n=1}^{\infty}n^2a_nx^{n-1}+\sum_{n=0}^{\infty}(n+2)a_nx^n=\sum_{n=0}^{\infty}(n+1)^2a_{n+1}x^n+\sum_{n=0}^{\infty}(n+2)a_nx^n\\
&= \sum_{n=0}^{\infty}\{(n+1)^2a_{n+1}+(n+2)a_n\}x^n.
\end{aligned}$$

2) We get from the **identity theorem** the recursion formula

$$(n+1)^2a_{n+1}+(n+2)a_n=0, \qquad n\in\mathbb{N}_0,$$

which is equivalent to

$$n^2a_n+(n+1)a_{n-1}=0, \qquad n\in\mathbb{N}.$$

From the latter expression we get by recursion

$$\begin{aligned}
a_n &= -\frac{n+1}{n^2}\,a_{n-1}=(-1)^2\,\frac{n+1}{n^2}\cdot\frac{n}{(n-1)^2}\,a_{n-2}=\cdots=(-1)^n\,\frac{n+1}{n^2}\cdot\frac{n}{(n-1)^2}\cdots\frac{3}{2^2}\cdot\frac{2}{1^2}\,a_0\\
&= (-1)^n\cdot\frac{n+1}{n!}\,a_0, \qquad n\in\mathbb{N}.
\end{aligned}$$

We see that this formula also holds for $n = 0$.

We get from the initial conditions that

$$\varphi(0) = a_0 = 1 \quad \text{and} \quad \varphi'(0) = a_1 = (-1)^1 \, \frac{1+1}{1!} \, a_0 = -2a_0 = -2.$$

The *formal* power series solutions is

$$y = \sum_{n=0}^{\infty} (-1)^n \, \frac{n+1}{n!} \, x^n = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} + \sum_{n=1}^{\infty} \frac{(-x)^n}{(n-1)!} = (1-x) \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = (1-x)e^{-x},$$

where we have recognized the exponential series with $\varrho = \infty$.

Alternatively we use the quotient criterion for $x \neq 0$:

$$\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \left| -\frac{(n+2)x}{(n+1)^2} \right| \to 0 \qquad \text{for } n \to \infty$$

for every fixed $x \in \mathbb{R}$, so $\varrho = \infty$.

Remark 2.5 This equation can also be solved **alternatively** by some manipulation. It is, however, more tricky here. By some trial and error we see that the *right idea* must be to multiply the equation by $1-x$, then add something and immediately subtract it again. We introduce in this way a singularity at $x = 1$, which ought to be discussed as well. We shall, however, decline from doing this here.

We get by using the sketch above,

$$\begin{aligned}
0 &= x(1-x) \frac{d^2y}{dx^2} + (1-x^2) \frac{dy}{dx} + (2-2x)y \\
&= x(1-x) \frac{d^2y}{dx^2} + (x-x^2) \frac{dy}{dx} - x \cdot y + \left\{ (1-x) \frac{dy}{dx} + (2-x)y \right\} \\
&= x \left\{ (1-x) \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx} - y \right\} + \left\{ (1-x) \frac{dy}{dx} + (2-x)y \right\}.
\end{aligned}$$

Then notice that

$$\frac{d}{dx} \left\{ (1-x) \frac{dy}{dx} + (2-x)y \right\} = (1-x) \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx} - y,$$

so the equation can be rewritten as

$$\begin{aligned}
0 &= x \cdot \frac{d}{dx} \left\{ (1-x) \frac{dy}{dx} + (2-x) \frac{dy}{dx} + (2-x)y \right\} + 1 \cdot \left\{ (1-x) \frac{dy}{dx} + (2-x)y \right\} \\
&= \frac{d}{dx} \left\{ x \left[(1-x) \frac{dy}{dx} + (2-x)y \right] \right\} = \frac{d}{dx} \left\{ x(1-x)^2 e^{-x} \left[\frac{e^x}{1-x} \frac{dy}{dx} + \frac{2-x}{(1-x)^2} \, e^x y \right] \right\}.
\end{aligned}$$

Since $\dfrac{d}{dx} \left(\dfrac{e^x}{1-x} \right) = \dfrac{2-x}{(1-x)^2} \, e^x$, this expression can be further reduced to

$$\frac{d}{dx} \left\{ x(1-x)^2 e^{-x} \frac{d}{dx} \left(\frac{e^x}{1-x} \cdot y \right) \right\} = 0,$$

hence by an integration,

$$x(1-x)^2 e^{-x}\frac{d}{dx}\left(\frac{e^x}{1-x}\,y\right) = c_2, \qquad \text{dvs.} \quad \frac{d}{dx}\left(\frac{e^x}{1-x}\,y\right) = c_2\cdot\frac{e^x}{x(1-x)^2}.$$

By another integration, followed by a multiplication by $(1-x)e^{-x}$, we get the complete solution

$$y = c_1(1-x)e^{-x} + c_2(1-x)e^{-x}\int \frac{e^x}{x(1-x)^2}\,dx.$$

We have a power series expansion of the first term from $x = 0$. This does not exist for the second term because of the factor x in the denominator of the integrand. On the other hand, the singularity at $x = 1$ is removed, because the singularity of the integrand of second order (in the denominator) by integration becomes a singularity of first order, and this is cancelled by the factor $1 - x$, which has a zero of first order at $x = 1$. $\Diamond$

Example 2.8 *Given the differential equation*

$$(29)\quad x(x^2+1)\frac{d^2y}{dx^2}+(4x^2+2)\frac{dy}{dx}+2xy=0, \qquad x\in\mathbb{R}.$$

1) Prove that if the power series of radius of convergence $\varrho>0$,

$$(30)\quad \sum_{n=0}^{\infty} a_n x^n, \qquad x\in\left]-\varrho,\varrho\right[,$$

is a solution of (29), then

$$2a_1+(2a_0+6a_2)x+(6a_1+12a_2)x^2+\sum_{n=2}^{\infty}(n+2)[(n+3)a_{n+2}+(n+1)a_n]x^{n+1}=0$$

for $x\in\left]-\varrho,\varrho\right[$.

2) Prove that (29) has a solution $y=\varphi(x)$ of the form (30), satisfying

$$\varphi(0)=1, \qquad \varphi'(0)=0.$$

3) Find the sum function of the power series solution of (2).

1) By insertion of the usual formal series

$$y=\sum_{n=0}^{\infty}a_nx^n, \quad y'=\sum_{n=1}^{\infty}na_nx^{n-1}, \quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

and by addition of some zero terms, we get

$$\begin{aligned}
0 &= (x^3+x)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+(4x^2+2)\sum_{n=1}^{\infty}na_nx^{n-1}+2x\sum_{n=0}^{\infty}a_nx^n\\
&= \sum_{\substack{n=2\\(n=0)}}^{\infty}n(n-1)a_nx^{n+1}+\sum_{\substack{n=2\\(n=1)}}^{\infty}n(n-1)a_nx^{n-1}\\
&\qquad +\sum_{\substack{n=1\\(n=0)}}^{\infty}4na_nx^{n+1}+\sum_{n=1}^{\infty}2na_nx^{n-1}+\sum_{n=0}^{\infty}2a_nx^{n+1}\\
&= \sum_{n=0}^{\infty}\{n(n-1)+4n+2\}a_nx^{n+1}+\sum_{n=1}^{\infty}\{n(n-1)+2n\}a_nx^{n-1}\\
&= \sum_{n=0}^{\infty}(n^2+3n+2)a_nx^{n+1}+\sum_{n=1}^{\infty}(n^2+n)a_nx^{n-1}\\
&= \sum_{n=0}^{\infty}(n+1)(n+2)a_nx^{n+1}+\sum_{n=1}^{\infty}n(n+1)a_nx^{n-1}\\
&= \sum_{n=0}^{\infty}(n+1)(n+2)a_nx^{n+1}+\sum_{n=-1}^{\infty}(n+2)(n+3)a_{n+2}x^{n+1}\\
&= 2a_1+\sum_{n=0}^{\infty}\{(n+3)a_{n+3}+(n+1)a_n\}x^{n+1}.
\end{aligned}$$

Remark 2.6 If we here remove the terms for $n = 0$ and $n = 1$, we get precisely the desired form. The following investigation becomes, however, more easy, if we keep the equivalent form, derived here, which is obtained by adding zero terms.

2) It follows from the **identity theorem** that $a_1 = 0$. Since $n + 2 \neq 0$ for $n \geq 0$, we derive the recursion formula

$$(n+3)a_{n+2} + (n+1)a_n = 0, \qquad \text{for } n \geq 0.$$

The recursion formula has a leap of 2 in the indices. Since already $a_1 = 0$, we conclude by induction that $a_{2n+1} = 0$ for all odd indices.

For even indices the recursion formula is written

$$(2n+3)a_{2(n+1)} + (2n+1)a_{2n} = 0, \qquad \text{for } n \geq 0,$$

hence by recursion,

$$a_{2(n+1)} = -\frac{2n+1}{2n+3}\, a_{2n} = (-1)^{n+1} \cdot \frac{2n+1}{2n+3} \cdot \frac{2n-1}{2n+1} \cdots \frac{1}{3}\, a_0 = \frac{(-1)^{n+1}}{2n+3}\, a_0.$$

Then by a change of index,

$$a_{2n} = \frac{(-1)^n}{2n+1}\, a_0 = \frac{(-1)^n}{2n+1}, \qquad n \in \mathbb{N}, \text{ og for } n = 0.$$

The formal power series solution is given by

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n}.$$

The radius of convergence is $\varrho = 1$, so the series is convergent for $]-1, 1[$.

3) If $x \neq 0$, then

$$y = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1} = \frac{\text{Arctan } x}{x},$$

and we see that the sum function is

$$y = \begin{cases} \dfrac{1}{x}\,\text{Arctan } x & \text{for } 0 < |x| < 1, \\[2mm] \quad 1 & \text{for } x = 0. \end{cases}$$

Remark 2.7 Again the example can be treated **alternatively** with some manipulation,

$$\begin{aligned}
0 &= x(x^2+1)\frac{d^2y}{dx^2}+(4x^2+2)\frac{dy}{dx}+2xy \\
&= (x^2+1)\left\{x\frac{d^2y}{dx^2}+1\cdot\frac{dy}{dx}\right\}+(3x^2+1)\frac{dy}{dx}+2xy \\
&= (x^2+1)\frac{d}{dx}\left\{x\frac{dy}{dx}+1\cdot y\right\}-(x^2+1)\frac{dy}{dx}+(3x^2+1)\frac{dy}{dx}+2xy \\
&= (x^2+1)\frac{d}{dx}\left\{\frac{d}{dx}(xy)\right\}+2x^2\frac{dy}{dx}+2xy \\
&= (x^2+1)\frac{d}{dx}\left\{\frac{d}{dx}(xy)\right\}+2x\left\{x\frac{dy}{dx}+1\cdot y\right\} \\
&= (x^2+1)\frac{d}{dx}\left\{\frac{d}{dx}(xy)\right\}+\frac{d}{dx}(x^2+1)\cdot\frac{d}{dx}(xy) \\
&= \frac{d}{dx}\left\{(x^2+1)\frac{d}{dx}(xy)\right\},
\end{aligned}$$

so (29) can be written

$$(31)\quad \frac{d}{dx}\left\{(x^2+1)\frac{d}{dx}(xy)\right\}=0.$$

If we integrate (31), then

$$(x^2+1)\frac{d}{dx}(xy)=c_2, \qquad \text{i.e.} \quad \frac{d}{dx}(xy)=\frac{c_2}{x^2+1}.$$

Then by another integration,

$$xy=c_1+c_2\,\text{Arctan}\,x,$$

hence

$$y=\frac{c_1}{x}+c_2\,\frac{\text{Arctan}\,x}{x} \qquad \text{for } x\neq 0.$$

Since $\dfrac{\text{Arctan}\,x}{x}\to 1$ for $x\to 0$, and $\dfrac{\text{Arctan}\,x}{x}$ is even and differentiable with $\varphi'(0)=0$, and since $\dfrac{1}{x}$ cannot be extended to 0, the answer of (2) becomes

$$y=\varphi(x)=\frac{\text{Arctan}\,x}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\,x^{2n}, \qquad x\in\,]-1,1[,$$

where $\dfrac{\text{Arctan}\,x}{x}$ is interpreted as 1 for $x=0$.

Example 2.9 *Given the differential equation*

$$(32)\quad (x^2+x)\frac{d^2y}{dx^2}+(3x+2)\frac{dy}{dx}+y=0, \qquad x\in\mathbb{R}.$$

1) Prove that if the power series of radius of convergence $\varrho>0$,

$$(33)\quad \sum_{n=0}^{\infty} a_n x^n, \qquad x\in\,]-\varrho,\varrho[,$$

is a solution of (32), then

$$a_0+2a_1+(4a_1+6a_2)x+\sum_{n=2}^{\infty}(n+1)[(n+2)a_{n+1}+(n+1)a_n]x^n=0,$$

for $x\in\,]-\varrho,\varrho[$.

2) Prove that (32) has a solution $y=\varphi(x)$ of the form (33), satisfying the condition $\varphi(0)=1$.

3) Find the sum function for the power series of (2).

Remark 2.8 We first demonstrate the "untraditional" solution, which shows that it pays off just to think about the problem before one starts on some standard procedure.

We get by some small rearrangements of the equation

$$\begin{aligned}0 &= (x^2+x)\frac{d^2y}{dx^2}+(3x+2)\frac{dy}{dx}+y=\left\{(x^2+x)\frac{d^2y}{dx^2}+(2x+1)\frac{dy}{dx}\right\}+\left\{(x+1)\frac{dy}{dx}+1\cdot y\right\}\\ &= \frac{d}{dx}\left\{(x^2+x)\frac{dy}{dx}\right\}+\frac{d}{dx}\{(x+1)y\}=\frac{d}{dx}\left\{(x^2+x)\frac{dy}{dx}+(x+1)y\right\}\\ &= \frac{d}{dx}\left\{(x+1)\left[x\frac{dy}{dx}+1\cdot y\right]\right\}=\frac{d}{dx}\left\{(x+1)\frac{d}{dx}(xy)\right\}.\end{aligned}$$

Hence by an integration,

$$(x+1)\frac{d}{dx}(xy)=c_1, \quad \text{i.e.} \quad \frac{d}{dx}(xy)=\frac{c_1}{1+x} \quad \text{for } x\neq -1.$$

Then by another integration,

$$x\cdot y=c_1\ln|1+x|+c_2.$$

If we also assume that $x\neq 0$, then we get the complete solution

$$y=c_1\,\frac{\ln|1+x|}{x}+c_2\cdot\frac{1}{x} \qquad \text{for } x\neq 0,\,-1.$$

The former function can actually be extended to $x=0$, because the power series expansion for $0<|x|<1$ is

$$\frac{\ln(1+x)}{x}=\frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,x^n=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,x^{n-1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\,x^n.$$

When $x\to 0$ we obtain the value 1.

Then we apply the **standard method**.

1) When we insert the formal power series

$$y=\sum_{n=0}^{\infty}a_nx^n,\quad y'=\sum_{n=1}^{\infty}na_nx^{n-1},\quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

and add some zero terms, we get

$$\begin{aligned}
0 &= (x^2+x)\frac{d^2y}{dx^2}+(3x+2)\frac{dy}{dx}+y\\
&= (x^2+x)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+(3x+2)\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n\\
&= \sum_{\substack{n=2\\(n=0)}}^{\infty}n(n-1)a_nx^n+\sum_{\substack{n=2\\(n=1)}}^{\infty}n(n-1)a_nx^{n-1}+\sum_{\substack{n=1\\(n=0)}}^{\infty}3na_nx^n+\sum_{n=1}^{\infty}2na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n\\
&= \sum_{n=0}^{\infty}\{n(n-1)+3n+1\}a_nx^n+\sum_{n=1}^{\infty}\{n(n-1)+2n\}a_nx^{n-1}\\
&= \sum_{n=0}^{\infty}(n+1)^2a_nx^n+\sum_{n=1}^{\infty}n(n+1)a_nx^{n-1}\\
&= \sum_{n=0}^{\infty}(n+1)^2a_nx^n+\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+1}x^n\\
&= \sum_{n=0}^{\infty}(n+1)\{(n+1)a_n+(n+2)a_{n+1}\}x^n.
\end{aligned}$$

When we remove the first two terms, we get the desired form. We shall, however, here keep the form above, because it will be more convenient in the following.

2) The series above is a power series expansion of 0. This is by the **identity theorem** unique, hence we get by identification of the coefficients the following recursion formula

$$(n+1)\{(n+1)a_n+(n+2)a_{n+1}\}=0\qquad\text{for } n\in\mathbb{N}_0.$$

Now, $n+1\neq 0$ for $n\in\mathbb{N}_0$, so this is immediately reduced to

$$(n+2)a_{n+1}=-(n+1)a_n,\qquad n\in\mathbb{N}_0.$$

This difference equation can be solved in three different ways.

a) **The divine inspiration**. If we introduce

$$b_n=(-1)^n(n+1)a_n,$$

then

$$b_{n+1}=(-1)^{n+1}(n+2)a_{n+1}=(-1)^n(n+1)a_n=b_n,\qquad n\in\mathbb{N},$$

and thus

$$(-1)^n(n+1)a_n=b_n=b_{n-1}=\cdots=b_0=1\cdot a_0=\varphi(0)=1.$$

We then get

$$a_n = \frac{(-1)^n}{n+1} \qquad \text{for } n \in \mathbb{N}_0.$$

b) **Recursion**. It follows from

$$(n+1)a_n = -na_{n-1} \qquad \text{that} \qquad a_n = -\frac{n}{n+1}\,a_{n-1}, \qquad n \in \mathbb{N}.$$

By repeating this formula with n replaced by $n-1$ etc., we get

$$a_n = \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n}\right)\cdots\left(-\frac{1}{2}\right)a_0 = \frac{(-1)^n}{n+1}\,a_0 = \frac{(-1)^n}{n+1}.$$

c) **Induction**. Consider again the recursion formula in the form

$$a_n = -\frac{n}{n+1}\, a_{n-1}, \qquad n \in \mathbb{N}, \quad \text{med } a_0 = 1.$$

If $n = 1$, then $a_1 = -\frac{1}{2}$.
If $n = 2$, then $a_2 = -\frac{2}{3}\left(-\frac{1}{2}\right) = \frac{1}{3}$.
If $n = 3$, then $a_3 = -\frac{3}{4} \cdot \frac{1}{3} = -\frac{1}{4}$.
This gives us the hint of the structure

$$a_n = \frac{(-1)^n}{n+1}, \qquad n \in \mathbb{N}_0.$$

Assume that this formula holds for some $n \in \mathbb{N}_0$. This is at least true for $n = 0, 1, 2, 3$. Then we get by this assumption that

$$a_{n+1} = -\frac{n+1}{n+2}\, a_n = -\frac{n+1}{n+2} \cdot \frac{(-1)^n}{n+1} = \frac{(-1)^{n+1}}{n+2},$$

which is the assumption of induction with n replaced by $n+1$. Hence

$$a_n = \frac{(-1)^n}{n+1}, \qquad \text{for } n \in \mathbb{N}_0.$$

We see that no matter the choice of method of the solution of the recursion formula, the *formal* power series solution of the problem becomes

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\, x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, x^{n-1}.$$

The **radius of convergence** can be found by the **criterion of roots**,

$$\sqrt[n]{\left|\frac{(-1)^n}{n+1}\, x^n\right|} = \frac{|x|}{\sqrt[n]{n+1}} \to |x| < 1 \qquad \text{for } n \to \infty,$$

hence $\varrho = 1$.

3) The formal power series can also be written

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, x^{n-1},$$

so we are very close to a logarithmic series. If $|x| < 1$, then

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, x^n = x \cdot \varphi(x),$$

hence $\varrho = 1$, and

$$\varphi(x) = \begin{cases} \dfrac{\ln(1+x)}{x} & \text{for } 0 < |x| < 1, \\[2mm] 1 & \text{for } x = 0. \end{cases}$$

Example 2.10 *Given the differential equation*

$$(34)\quad x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(x^2+6)y=0, \qquad x\in\mathbb{R}.$$

1) Prove that if the power series of radius of convergence $\varrho>0$,

$$(35)\quad \sum_{n=0}^{\infty}a_nx^n, \qquad x\in\left]-\varrho,\varrho\right[,$$

is a solution of (34), then

$$6a_0+2a_1x+a_0x^2+a_1x^3+\sum_{n=4}^{\infty}\{(n-2)(n-3)a_n+a_{n-2}\}x^n=0,$$

for $x\in\left]-\varrho,\varrho\right[$.

2) Prove that (34) has a solution $y=\varphi(x)$ of the form (35), satisfying the initial conditions

$$\varphi(0)=0,\quad \varphi'(0)=0,\quad \varphi''(0)=2,\quad \varphi'''(0)=0.$$

3) Find the sum function of the power series of (2).

1) When we insert the formal series

$$y=\sum_{n=0}^{\infty}a_nx^n,\quad \frac{dy}{dx}=\sum_{n=1}^{\infty}na_nx^{n-1},\quad \frac{d^2y}{dx^2}=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

and add some zero terms, we get

$$\begin{aligned}0 &= \sum_{\substack{n=2\\(n=0)}}^{\infty}n(n-1)a_nx^n-\sum_{\substack{n=1\\(n=0)}}^{\infty}4na_nx^n+\sum_{n=0}^{\infty}6a_nx^n+\sum_{n=0}^{\infty}a_nx^{n+2}\\ &= \sum_{n=0}^{\infty}\{n^2-5n+6\}a_nx^n+\sum_{n=2}^{\infty}a_{n-2}x^n\\ &= 6a_0+2a_1x+\sum_{n=2}^{\infty}\{(n-2)(n-3)a_n+a_{n-2}\}x^n.\end{aligned}$$

Then by removing two terms we get

$$0=6a_0+2a_1x+a_0x^2+a_1x^3+\sum_{n=4}^{\infty}\{(n-2)(n-3)a_n+a_{n-2}\}x^n,$$

and we can in the following argue on both expressions. We shall here choose the given version from the example.

Remark 2.9 Because possible singular points are given by $x^2=0$, i.e. $x=0$, the possible radii of convergence are $\varrho\in\{0,\infty\}$.

2) It follows from the *initial conditions*,

$$\varphi(0) = a_0 = 0, \qquad \varphi'(0) = 0 = a_1,$$

$$\varphi''(0) = 2!a_2 = 2, \quad \varphi'''(0) = 3!a_3 = 0,$$

thus

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 1, \quad a_3 = 0.$$

Then by the *identity theorem*,

$$6a_0 = 0, \quad 2a_1 = 0, \quad a_0 = 0, \quad a_1 = 0,$$

and

$$(n-2)(n-3)a_n + a_{n-2} = 0, \qquad \text{for } n \geq 4,$$

which fortunately is in agreement with $a_0 = 0$ and $a_1 = 0$ found previously.

If $n \geq 4$, then $(n-2)(n-3) > 0$, and the *recursion formula* is rewritten as

$$a_n = -\frac{1}{(n-2)(n-3)}\, a_{n-2}, \qquad \text{for } n \geq 4.$$

There is here a leap of 2 in the indices, so we shall divide into the two cases of odd and even indices.

From $a_3 = 0$ follows by induction that $a_{2p+1} = 0$ for all odd indices ≥ 3, thus for all odd indices, because also $a_1 = 0$.

For even indices we write $n = 2p \geq 4$, which is satisfied for $p \geq 2$,

$$a_{2p} = -\frac{1}{(2p-2)(2p-3)}\, a_{2(p-1)}, \qquad p \geq 2.$$

In order to find some common pattern we try the first values,

$$a_2 = 1, \quad a_4 = -\frac{1}{2 \cdot 1}, \quad a_6 = +\frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!}.$$

These values inspires us to the hypothesis of induction

$$a_{2p} = \frac{(-1)^{p+1}}{(2p-2)!}, \qquad \text{i.e. } a_{2(p-1)} = \frac{(-1)^p}{(2p-4)!}.$$

a) By the first values it is seen that the hypothesis holds for $p = 1, 2, 3$.
b) *Induction.* Assume that

$$a_{2(p-1)} = \frac{(-1)^p}{(2p-4)!} \qquad \text{for some } p \geq 2.$$

Then by the recursion formula

$$a_{2p} = -\frac{1}{(2p-2)(2p-3)}\, a_{2(p-1)} = -\frac{1}{(2p-2)(2p-3)} \cdot \frac{(-1)^p}{(2p-4)!} = \frac{(-1)^{p+1}}{(2p-2)!},$$

which is precisely the hypothesis of induction for p.

Then we get the formal power series solution,

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{p=1}^{\infty} a_{2p} x^{2p} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-2)!}\, x^{2n}.$$

Radius of convergence.
The series is of course convergent for $x = 0$.
If $x \neq 0$, then $b_n = \left| \frac{(-1)^{n+1}}{(2n-2)!}\, x^{2n} \right| = \frac{x^{2n}}{(2n-2)!} > 0$.
We get by the **criterion of quotients**,

$$\frac{b_{n+1}}{b_n} = \frac{x^{2n+2}}{(2n)!} \cdot \frac{(2n-2)!}{x^{2n}} = \frac{x^2}{2n(2n-1)} \to 0 < 1 \quad \text{for } n \to \infty$$

for every fixed $x \in \mathbb{R}$, hence $\varrho = \infty$, and the interval of convergence is $\mathbb{R}$,

$$y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-2)!}\, x^{2n} \qquad \text{for } x \in \mathbb{R}.$$

3) The sum function is

$$y=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-2)!}\,x^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{(2n)!}\,x^{2n+2}=x^2\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n)!}\,x^{2n}=x^2\cos x.$$

Since we already know that the cosine series is convergent in $\mathbb{R}$, we might from the identification of the sum function immediately obtain the interval of convergence.

Remark 2.10 An **alternative** solution method is the following: The not so obvious trick is for $x \neq 0$ to divide by x^4. Then we get by some manipulation that the equation can be rewritten in the following way,

$$\begin{aligned}0 &= \frac{1}{x^2}\frac{d^2y}{dx^2}-\frac{2}{x^2}\frac{dy}{dx}-\frac{2}{x^3}\frac{dy}{dx}+\frac{1}{x^2}\,y+\frac{6}{x^4}\,y=\frac{d}{dx}\left\{\frac{1}{x^2}\frac{dy}{dx}\right\}-\frac{d}{dx}\left\{\frac{2}{x^3}\,y\right\}+\frac{y}{x^2}\\ &= \frac{d}{dx}\left\{\frac{1}{x^2}\frac{dy}{dx}+\frac{d}{dx}\left(\frac{1}{x^2}\right)\cdot y\right\}+\frac{y}{x^2}=\frac{d^2}{dx^2}\left\{\frac{y}{x^2}\right\}+\frac{y}{x^2}.\end{aligned}$$

If we put $z = \dfrac{y}{x^2}$, the equation is reduced to

$$\frac{d^2z}{dx^2} + z = 0,$$

the complete solution of which is

$$z = \frac{y}{x^2} = c_1 \cos x + c_2 \sin x,$$

hence

$$y = c_1 x^2 \cos x + c_2 x^2 \sin x, \qquad \text{for } x \neq 0.$$

Example 2.11 *Given the differential equation*

$$(36)\quad x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = 0, \qquad x \in \mathbb{R}.$$

1) Prove that if the power series of radius of convergence $\varrho > 0$,

$$(37)\quad \sum_{n=0}^{\infty} a_n x^n, \qquad x \in]-\varrho, \varrho[,$$

is a solution of (36), then

$$2a_0 + a_0 x^2 + \sum_{n=3}^{\infty} [(n-1)(n-2)a_n + a_{n-2}]x^n = 0, \quad x \in]-\varrho, \varrho[.$$

2) Prove that (36) has a $y = \varphi(x)$ of the form (37), satisfying the initial conditions

$$\varphi(0) = 0, \qquad \varphi'(0) = 0, \qquad \varphi''(0) = 2.$$

3) Find the sum function of the power series of (2) expressed by elementary functions.

1) When we insert the formal power series

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

into (36), read from the right hand side to the left hand side, we get by adding some zero terms,

$$\begin{aligned}
0 &= x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + (x^2+2) \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{\substack{n=2\\(n=0)}}^{\infty} n(n-1)a_n x^2 - \sum_{\substack{n=1\\(n=0)}}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} \\
&= \sum_{n=0}^{\infty} \{n^2 - n - 2n + 2\} a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n \\
&= \sum_{n=0}^{\infty} (n-1)(n-2) a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n \\
&= 2a_0 + a_0 x^2 + \sum_{n=3}^{\infty} \{(n-1)(n-2)a_n + a_{n-2}\} x^n,
\end{aligned}$$

and we have proved that (37) necessarily must fulfil

$$(38)\ \ 2a_0 + a_0 x^2 + \sum_{n=3}^{\infty} \{(n-1)(n-2)a_n + a_{n-2}\} x^n = 0.$$

2) It follows from the **identity theorem** that $a_0 = 0$ and that the recursion formula becomes

$$(n-1)(n-2)a_n + a_{n-2} = 0 \qquad \text{for } n \geq 3$$

It is given that

$$\varphi(0) = a_0 = 0, \quad \text{(in agreement with the identity theorem)},$$

$$\varphi'(0) = 1 \cdot a_1 = 0, \qquad \text{i.e. } a_1 = 0,$$

$$\varphi''(0) = 2!a_2 = 2, \qquad \text{i.e. } a_2 = 1.$$

The recursion formula has a leap of 2 in the indices, so it follows from $a_1 = 0$ by induction that $a_{2m+1} = 0$, $m \in \mathbb{N}_0$.

If $n = 2m$ is even, the recursion formula is written

$$(39)\ \ (2m-1)(2m-2)a_{2m} = -a_{2(m-1)} \qquad \text{for } m \geq 2.$$

(Note that $m \neq 1$).

When we multiply by $(-1)^{m-1}(2m-3)! \neq 0$, we get

$$b_m := (-1)^{m-1}(2m-1)!a_{2m} = (-1)^{m-2}(2m-3)!a_{2(m-1)} = b_{m-1},$$

thus

$$b_m = (-1)^{m-1}(2m-1)!a_{2m} = \cdots = b_1 = (-1)^0 \cdot 1!a_2 = 1,$$

and hence

$$a_{2n} = \frac{(-1)^{n-1}}{(2n-1)!}, \qquad n \in \mathbb{N}.$$

Alternatively we get by recursion of (39),

$$\begin{aligned} a_{2m} &= -\frac{1}{(2m-1)(2m-2)}\, a_{2(m-1)} = (-1)^2 \frac{1}{(2m-1)(2m-2)(2m-3)(2m-4)}\, a_{2(m-2)} \\ &= \cdots = \frac{(-1)^{m-1}}{(2m-1)(2m-2)\cdots 3 \cdot 2}\, a_2 = \frac{(-1)^{m-1}}{(2m-1)!}, \qquad m \in \mathbb{N}. \end{aligned}$$

Summing up we obtain the *formal* power series solution

$$y = \sum_{n=1}^{\infty} a_{2n} x^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}\, x^{2n}.$$

If $x \neq 0$, then $a_n(x) = \left|\dfrac{(-1)^{n-1}}{(2n-1)!}\,x^{2n}\right| = \dfrac{x^{2n}}{(2n-1)!} > 0$. We get by the **criterion of quotients** for $x \neq 0$ that

$$\frac{a_{n+1}(x)}{a_n(x)} = \frac{x^{2n+2}}{(2n+1)!}\cdot\frac{(2n-1)!}{x^{2n}} = \frac{x^2}{(2n+1)2n} \to 0 \text{ for } n \to \infty.$$

Hence we conclude by the **criterion of quotiens** that the series is convergent for every $x \in \mathbb{R}$, so the interval of convergence is $\mathbb{R}$.

3) Using a known power series expansion we get the sum function

$$y = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}\,x^{2n} = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,x^{2n+1} = x\sin x, \quad x \in \mathbb{R}.$$

Remark 2.11 One can also solve this equation by a small trick. Since we have a singular point at $x = 0$, and since $x = 0$ clearly gives $y(0) = 0$ for every solution, it seems natural to put $y = xz$ and then derive some differential equation in z. From

$$\frac{dy}{dx} = x\frac{dz}{dx} + z \quad \text{og} \quad \frac{d^2y}{dx^2} = x\frac{d^2z}{dx^2} + 2\frac{dz}{dx},$$

follows by insertion that

$$\begin{aligned} 0 &= x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = \left\{x^3\frac{d^2z}{dx^2} + 2x^2\frac{dz}{dx}\right\} - \left\{2x^2\frac{dz}{dx} + 2xz\right\} + \left\{x^3z + 2xz\right\} \\ &= x^3\frac{d^2z}{dx^2} + x^3z = x^3\left\{\frac{d^2z}{dx^2} + z\right\}. \end{aligned}$$

Thus, we get the equation when $x \neq 0$,

$$\frac{d^2z}{dx^2} + z = 0,$$

the complete solution of which is

$$z = c_1\sin x + c_2\cos x.$$

Therefore, if $x \neq 0$ then the complete solution of the original equation is

$$y = c_1 x\sin x + c_2 x\cos x, \qquad c_1,\ c_2 \text{ arbitrære}.$$

Remark 2.12 If we put $y_1(x) = x\sin x$, then both $y_1(0) = 0$ and $y_1'(0) = 0$. This fact will cause some extension problems at $x = 0$, where we cannot conclude anything from the existence and uniqueness theorem. We shall not go further into this difficult question.

Example 2.12 *1) Prove that if* $y(x) = \sum_{n=0}^{\infty} a_n x^n$ *is a power series solution of*

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 0,$$

then we have the recursion formula

$$n(n-2)a_n = 4a_{n-4} \qquad \textit{for } n \geq 4.$$

2) Find the power series solution, which satisfies the conditions $y(0) = 1$ *and* $y''(0) = 0$*, and find the interval of convergence of the series.*

3) Does there exist a power series solution $y(x)$*, for which* $y'(0) = 1$*?*

1) When we insert the formal series

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

into the differential equation, we get

$$\begin{aligned}
0 &= x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y \\
&= x\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}na_nx^{n-1}-4x^3\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}-a_1-\sum_{n=2}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}4a_nx^{n+3} \\
&= -a_1+\sum_{\substack{n=2\\(n=3)}}^{\infty}n(n-2)a_nx^{n-1}-\sum_{n=4}^{\infty}4a_{n-4}x^{n-1} \\
&= -a_1+3a_3x^2+\sum_{n=4}^{\infty}\{n(n-2)a_n-4a_{n-4}\}x^{n-1}.
\end{aligned}$$

Then by the identity theorem.

(40) $a_1 = 0 \qquad \text{and} \qquad a_3 = 0,$

and the recursion formula (with a leap of 4 in the indices)

(41) $n(n-2)a_n = 4a_{n-4} \qquad \text{for } n \geq 4.$

2) We get by induction from (40) and (41) that $a_{4n+1} = 0$ and $a_{4n+3} = 0$, hence

$$a_{2n+1} = 0 \qquad \text{for } n \in \mathbb{N}_0.$$

If $y(0) = a_0 = 1$ and $y''(0) = 2!a_2 = 0$, thus $a_2 = 0$, it follows again by induction that

$$a_{4n+2} = 0 \qquad \text{for } n \in \mathbb{N}_0.$$

The remaining case is $a_n = a_{4m}$, i.e. $n = 4m$, $m \in \mathbb{N}_0$, where $a_0 = 1$. We write in this case (41) in the form

$$4m(4m-2)a_{4m} = 4a_{4m-4}, \qquad m \geq 1,$$

which is rewritten as

(42) $2m(2m-1)a_{4m} = a_{4(m-1)}, \qquad m \geq 1.$

We can solve this recursion formula in several ways, of which we shall only demonstrate a couple.

a) If (42) is multiplied by $(2m-2)! = (2(m-1))!$, then

$$(2m)!a_{4m} = \{2(m-1)\}!a_{4(m-1)} = \cdots = \{2\cdot 0\}!a_0 = 1,$$

hence

$$a_{4m} = \frac{1}{(2m)!}, \qquad m \in \mathbb{N}_0.$$

b) We get by recursion from (42) that

$$\begin{aligned} a_{4m} &= a_{2\cdot 2m} = \frac{1}{2m(2m-1)} \cdot a_{2\cdot(2m-1)} = \frac{1}{2m(2m-1)(2m-2)(2m-3)}\, a_{2\cdot(2,-4)} \\ &= \cdots = \frac{1}{2m(2m-1)\cdots 2\cdot 1}\, a_{2\cdot 0} = \frac{1}{(2m)!}, \qquad m \in \mathbb{N}_0. \end{aligned}$$

As a conclusion we have obtained the formal power series solution

$$y = \sum_{m=0}^{\infty} a_{4m} x^{4m} = \sum_{m=0}^{\infty} \frac{1}{(2n)!}(x^2)^{2m} = \cosh(x^2).$$

We recognize the series as the series of $\cosh(x^2)$ of radius of convergence $\varrho = \infty$, so the interval of convergence is $\mathbb{R}$.

3) If some power series solution existed with $y'(0) = 1$, then $a_1 = 1$, which violates (40). Hence, we cannot have any power series solution of the equation, for which $y'(0) = 1$.

Alternatively, put $t = 0$ into the differential equation,

$$0 = 0 \cdot y''(0) - y'(0) - 4 \cdot 0^3 y(0) = -y'(0), \quad \text{dvs. } y'(0) = 0.$$

It is immediately seen that one cannot have any solution for which $y'(0) = 1$ whatsoever.

Remark 2.13 By using a general solution formula, a linearly independent solution is given by

$$\begin{aligned} y_2(x) &= \cosh(x^2) \int \frac{1}{\cosh^2(x^2)} \exp\left(\int \frac{1}{x}\, dx \right) dx = \cosh(x^2) \int \frac{x}{\cosh^2(x^2)}\, dx \\ &= \frac{1}{2}\cosh(x^2) \int_{u=x^2} \frac{du}{\cosh^2(u)} = \frac{1}{2}\cosh(x^2)\tanh(x^2) = \frac{1}{2}\sinh(x^2). \end{aligned}$$

The complete solution is

$$y = c_1 \cosh(x^2) + c_2 \sinh(x^2), \quad x \in \mathbb{R}, \quad c_1, c_2 \text{ arbitrære}.$$

Example 2.13 *Consider the differential equation*

$$\frac{d^2y}{dx^2} + x^2y = 0.$$

Given (and shall not be proved) that there exists a power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$.

1) *Find a recursion formula for the coefficients* a_n *of the power series solution (one shall not solve the equation of recursion).*

2) *Find* a_0, a_1, a_2, a_3 *and* a_4 *for the power series solution for which we have the initial conditions* $y(0) = 1$ *and* $y'(0) = 0$.

1) By inserting the formal series

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad \text{og} \qquad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

we get

$$\begin{aligned} 0 &= \frac{d^2y}{dx^2} + x^2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=4}^{\infty} a_{n-4} x^{n-2} \\ &= 2 \cdot 1 \cdot a_2 + 3 \cdot 2 a_3 x + \sum_{n=4}^{\infty} \{n(n-1)a_n + a_{n-4}\} x^{n-2}. \end{aligned}$$

Then by the identity theorem, $a_2 = 0$ and $a_3 = 0$, and the recursion formula

$$n(n-1)a_n + a_{n-4} = 0 \qquad \text{for } n \geq 4,$$

thus

$$(n+4)(n+3)a_{n+4} + a_n = 0 \qquad\qquad \text{for } n \geq 0.$$

2) Clearly, $a_0 = 1$ and $a_1 = 0$. It follows from (1) that $a_2 = 0$ and $a_3 = 0$. Finally, we get from the recursion formula,

$$(0+4)(0+3)a_4 + a_0 = 0 \qquad \text{for } n = 0,$$

so $a_4 = -\dfrac{1}{12}$.
Summing up we have

$$a_0 = 1, \quad a_1 = a_2 = a_3 = 0 \quad \text{and} \quad a_4 = -\frac{1}{12}.$$

3 An eigenvalue problem solved by the power series method

Example 3.1 One can sometimes also use the power series method in more complicated problems. We shall here give one example of an eigenvalue problem, which can be solved by the power series method. If one wants to know more about *eigenvalue problems*, the reader is referred to e.g. *Calculus 4b*.

Consider the eigenvalue problem

$$\frac{d^4y}{dx^4} + (\lambda - x)\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0, \qquad x \in [0,\lambda].$$

$$y(0) = y'(0) = y''(\lambda) = y'''(\lambda) = 0.$$

This is the model equation of the deviation from the vertical of a vertically thin column of length λ under the influence of the weight of the column itself. We shall find the smallest eigenvalue λ.

1) Start by an **inspection** of the equation. Since

$$\frac{d}{dx}\left\{(\lambda - x)\frac{dy}{dx}\right\} = (\lambda - x)\frac{d^2y}{dx^2} - \frac{dy}{dx},$$

we can also write the differential equation in the form

$$\frac{d^4y}{dx^4} + \frac{d}{dx}\left\{(\lambda - x)\frac{dy}{dx}\right\} = 0.$$

This can immediately be integrated

$$\frac{d^3y}{dx^3} + (\lambda - x)\frac{dy}{dx} = c, \qquad c \text{ arbitrær.}$$

2) **Identification of c by the boundary value** $y'''(\lambda) = 0$ gives

$$c = y'''(\lambda) + (\lambda - \lambda)y'(\lambda) = 0.$$

Hence, the problem is reduced to the simpler homogeneous equation

$$\frac{d^3y}{dx^3} + (\lambda - x)\frac{dy}{dx} = 0,$$

which is identified as a second order differential equation in $\frac{dy}{dx}$. If we therefore put $z = \frac{dy}{dx}$, then

$$\frac{d^2z}{dx^2} + (\lambda - x)z = 0,$$

where the boundary values for z are given by

$$z(0) = y'(0) = 0 \quad \text{and} \quad z'(\lambda) = y''(\lambda) = 0.$$

Remark 3.1 We have already used the boundary value $y'''(\lambda) = z''(\lambda) = 0$ above, and we see that it also follows from the equation. Furthermore, $y(0) = 0$ is not at all relevant for $z = y'$.

3) **Change of variable**. The factor $\lambda - x$ is annoying, so we chance the variable to $t = \lambda - x$. If we put

$$u(t) = z(x), \qquad \text{i.e.} \qquad u(\lambda - x) = z(x),$$

the equation is transferred into

$$\frac{d^2u}{dt^2} + tu(t) = 0 \quad \text{med} \quad u(\lambda) = 0 \text{ og } u'(0) = 0.$$

4) We shall now for some time neglect the boundary condition $u(\lambda) = 0$, when we find a power series solution of this equation. We shall of course later apply the condition $u(\lambda) = 0$. Since $u'(0) = 0$, we have $a_1 = 0$. By insertion of the formal series

$$u(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad \frac{d^2u}{dt^2} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

into the differential equation we get

$$\begin{aligned}0 &= \frac{d^2u}{dt^2}+tu(t)=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}a_nt^{n-1}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=1}^{\infty}a_{n-1}t^n\\ &= 2a_2+\sum_{n=1}^{\infty}\{(n+2)(n+1)a_{n+2}+a_{n-1}\}t^n.\end{aligned}$$

It follows from the identity theorem that $a_2 = 0$ (we knew already that $a_1 = 0$), and for $n \in \mathbb{N}$ (the summation domain)

$$(n+2)(n+1)a_{n+2}+a_{n-1}=0 \qquad \text{for } n \in \mathbb{N}.$$

By $n \mapsto n+1$ this is transformed into

$$(n+3)(n+2)a_{n+3}+a_n=0 \qquad \text{for } n \in \mathbb{N}_0.$$

Here we have a leap of 3 in the indices, hence, because $a_1 = 0$ and $a_2 = 0$, we conclude by induction that

$$a_{3n+1}=0 \quad \text{and} \quad a_{3n+2}=0 \qquad \text{for } n \in \mathbb{N}_0.$$

We have now reduced the power series solution to

$$u(t)=\sum_{n=0}^{\infty}a_{3n}t^{3n}=\sum_{n=0}^{\infty}b_nt^{3n},$$

where the recursion formula for $a_{3n} = b_n$ is obtained by the change $n \mapsto 3n$, thus

$$(3n+3)(3n+2)a_{3n+3}+a_{3n}=0, \qquad n \in \mathbb{N}_0,$$

and hence

$$b_{n+1}=-\frac{1}{(3n+3)(3n+2)}\,b_n, \qquad b_n=a_{3n}, \quad n \in \mathbb{N}_0.$$

If $b_0 \neq 0$ and thus $b_n \neq 0$, we find the radius of convergence by an application of the **criterion of quotients**

$$\left|\frac{a_{n+1}(t)}{a_n(t)}\right|=\frac{|b_{n+1}||t|^{3(n+1)}}{|b_n||t|^{3n}}=\frac{|t|^3}{(3n+3)(3n+2)}\to 0 \text{ for } n\to\infty.$$

Hence, the series is convergent for every $t \in \mathbb{R}$, and $\varrho = \infty$.

Now we should in reality consider a **boundary value problem**, so $a_0 = b_0 \neq 0$ are "free". We choose arbitrarily $a_0 = b_0 = 1$. Then by induction,

$$b_n=a_{3n}=(-1)^n\cdot\frac{1}{(3n)!}\prod_{j=0}^{n-1}(3j+1), \qquad n \in \mathbb{N}.$$

5) We have proved that

$$(43)\quad \frac{dy}{dx} = z(x) = u(\lambda - x) = \sum_{n=0}^{\infty} a_{3n}(\lambda - x)^{3n}, \qquad x \in \mathbb{R},$$

where we have found a_{3n} in (4). This function cannot be expressed by elementary functions. We can still perform termwise integration. Since $y(0) = 0$, we get by termwise integration and a rearrangement that

$$\begin{aligned} y(x) = \; &= \sum_{n=0}^{\infty} a_{3n} \int_0^x (\lambda - t)^{3n} dt = \sum_{n=0}^{\infty} a_{3n} \left[-\frac{1}{3n+1}(\lambda - t)^{3n+1} \right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{a_{3n}}{3n+1} \lambda^{3n+1} - \sum_{n=0}^{\infty} \frac{a_{3n}}{3n+1}(\lambda - x)^{3n+1}, \end{aligned}$$

giving us the structure of the eigenfunctions, if only we can find the eigenvalues.

6) It remains to find the smallest (positive) $\lambda = \lambda_{\text{crit}}$, for which we have a proper solution, i.e. for which $a_0 \neq 0$. Here we use the boundary condition $y'(0) = 0$, hence by (43),

$$y'(0) = \sum_{n=0}^{\infty} a_{3n}\lambda^{3n} = 0 \qquad \text{where } a_0 = 1.$$

This transcendent equation is solved approximately in the following way:

We put for convenience $\eta = \lambda^3$, and find successively the smallest root of each of the polynomials

$$P_n(\eta) = \sum_{k=0}^{n} a_{3k}\eta^k, \qquad n \in \mathbb{N}.$$

Since a_{3k} has alternating sign, the possible real roots can only be positive. In the first polynomials we may only get complex roots. However, if two successive polynomials $P_n(\eta)$ and $P_{n+1}(\eta)$ have their (smallest) real roots η_n and η_{n+1}, then every successive polynomial $P_{n+m}(\eta)$ will also have a (smallest) real root η_{n+m}. Since a_{3n} is alternating it is easily proved that η_{n+m}, $m > 1$, always lies between η_n and η_{n+1}, so we obtain a convergent sequence of number. The following numerical calculations show that the convergence is fairly fast.

7) **Numerical calculations**. No text needed.

$$n=1: \quad P_1(\eta) = 1 - \frac{1}{3\cdot 2}\eta, \qquad \eta_1 = 6 \quad \text{and} \quad \lambda_1 = \sqrt[3]{6} = 1,81712.$$

$$n=2: \quad P_2(\eta) = 1 - \frac{\eta}{6}\left(1 - \frac{\eta}{6\cdot 5}\right), \qquad \eta_2 = 8,29180 \quad \text{and} \quad \lambda_2 = \sqrt[3]{\eta_2} = 2,02403.$$

$$n=3: \quad P_3(\eta) = 1 - \frac{\eta}{6}\left(1 - \frac{\eta}{30}\left(1 - \frac{\eta}{9\cdot 8}\right)\right), \qquad \eta_3 = 7,814712 \quad \text{and} \quad \lambda_3 = \sqrt[3]{\eta_3} = 1,98444.$$

$$n=4: \quad P_4(\eta) = 1 - \frac{\eta}{6}\left(1 - \frac{\eta}{30}\left(1 - \frac{\eta}{72}\left(1 - \frac{\eta}{12\cdot 11}\right)\right)\right),$$
$$\eta_4 = 7,838213 \quad \text{and} \quad \lambda_4 = \sqrt[3]{\eta_4} = 1,98643.$$

$$n=5: \quad P_5(\eta) = 1 - \frac{\eta}{6}\left(1 - \frac{\eta}{30}\left(1 - \frac{\eta}{72}\left(1 - \frac{\eta}{132}\left(1 - \frac{\eta}{15\cdot 14}\right)\right)\right)\right),$$
$$\eta_5 = 7,837325 \quad \text{and} \quad \lambda_5 = \sqrt[3]{\eta_5} = 1,98635.$$

$$\text{n=6:} \quad P_6(\eta) = 1 - \frac{\eta}{6}\left(1 - \frac{\eta}{30}\left(1 - \frac{\eta}{72}\left(1 - \frac{\eta}{132}\left(1 - \frac{\eta}{210}\left(1 - \frac{\eta}{18\cdot 17}\right)\right)\right)\right)\right),$$
$$\eta_6 = 7,837348 \quad \text{and} \quad \lambda_6 = \sqrt[3]{\eta_6} = 1,98635.$$

It follows that $\lambda_5 = \lambda_6 = 1,98635$ is a correct estimate of λ_{crit} with 5 decimals. This result is obtained after only 6 iterations.