Integral Operators

Functional Analysis Examples c-5

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ISBN 978-87-7681-529-5

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1 Hilbert-Schmidt operators

Example 1.1 *Let (e_k) denote an orthonormal basis in a Hilbert space H, and assume that the operator T has the matrix representation (t_{jk}) with respect to the basis (e_k). Show that*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|t_{jk}|^2<\infty$$

implies that T is compact.
Let (f_k) denote another orthonormal basis in H, and let

$$s_{jk}=(Tf_j,f_k)$$

so that (s_{jk}) is the matrix representation of T with respect to the basis (f_k).
Show that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|t_{jk}|^2=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|s_{jk}|^2 .$$

An operator satisfying

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|t_{jk}|^2<\infty$$

is called a general Hilbert-Schmidt operator.

Write $t_{jk}=(Te_j,e_j)$. It follows from Ventus, Hilbert spaces, etc., Example 2.7 that

$$Tx=T\left(\sum_{j=1}^{+\infty}x_je_j\right)=\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}x_jt_{jk}e_k.$$

Define the sequence (T_n) of operators by

$$T_nx=T_n\left(\sum_{j=1}^{+\infty}x_je_j\right)=\sum_{j=1}^{+\infty}\sum_{k=1}^{n}x_jt_{jk}e_k.$$

The range of T_n is finite dimensional, so T_n is compact. Then we conclude from

$$\|(T-T_n)\,x\|^2=\left\|\sum_{j=1}^{+\infty}\sum_{n=1}^{+\infty}x_jt_{jk}e_k\right\|^2=\sum_{k=n+1}^{+\infty}\left|\sum_{j=1}^{+\infty}x_jt_{jk}\right|^2,$$

where

$$\left|\sum_{j=1}^{+\infty}x_jt_{jk}\right|^2\le\left\{\sum_{j=1}^{+\infty}|x_j|^2\right\}\cdot\left\{\sum_{j=1}^{+\infty}|t_{jk}|^2\right\},$$

that

$$\|(T - T_n)x\|^2 \leq \left\{ \sum_{k=n+1}^{+\infty} \sum_{j=1}^{+\infty} |t_{jk}|^2 \right\} \cdot \|x\|^2.$$

It follows that

$$\|T - T_n\|^2 \leq \sum_{k=n+1}^{+\infty} \sum_{j=1}^{+\infty} |t_{jk}|^2 .$$

Putting

$$a_k = \sum_{j=1}^{+\infty} |t_{jk}|^2 \geq 0,$$

it follows from the assumption that

$$\sum_{k=1}^{+\infty} a_k = \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 < +\infty.$$

Hence, to every $\varepsilon > 0$ there is an $n \in \mathbb{N}$, such that

$$\sum_{k=n+1}^{+\infty} a_k < \varepsilon^2,$$

from which

$$\|T-T_n\|^2 \le \sum_{k=n+1}^{+\infty}\sum_{j=1}^{+\infty}|t_{jk}|^2 = \sum_{k=n+1}^{+\infty} a_k < \varepsilon^2,$$

thus $\|T-T_n\| < \varepsilon$, and we have proved that $T_n \to T$. Because all the T_n are compact, we conclude that T is also compact.

Given another orthonormal basis (f_k) of H, and let $s_{jk} = (Tf_j, f_k)$. Then an application of Parseval's equation gives that

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|(Te_k,f_j)|^2 = \sum_{k=1}^{+\infty}\|Te_j\|^2 = \sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty}|(Te_k,e_j)|^2 = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|t_{kj}|^2$$

and

$$\begin{aligned}\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|(Te_k,f_j)|^2 &= \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|(e_k,T^\star f_j)|^2 = \sum_{j=1}^{+\infty}\|T^\star f_j\|^2 = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|(T^\star f_j,f_k)|^2 \\ &= \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|(f_j,Tf_k)|^2 = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|(Tf_j,f_k)|^2 = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|s_{jk}|^2,\end{aligned}$$

hence,

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|t_{jk}|^2 = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|t_{kj}|^2 = \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|s_{jk}|^2.$$

Example 1.2 *For a general Hilbert-Schmidt operator we define the Hilbert-Schmidt norm* $\|\cdot\|_{\mathrm{HS}}$ *by*

$$\|T\|_{\mathrm{HS}} = \left\{\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|t_{jk}|^2\right\}^{\frac12}.$$

Show that this is *a norm, and show that*

$$\|T\| \le \|T\|_{\mathrm{HS}}$$

for a general Hilbert-Schmidt operator T.

Write $t_{jk} = (Te_j, e_k)$, and let

$$\|T\|_{\mathrm{HS}} = \left\{\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}|t_{jk}|^2\right\}^{\frac12}.$$

Then $\|T\|_{\mathrm{HS}} \ge 0$, and if $\|T\|_{\mathrm{HS}} = 0$, then $t_{jk} = (Te_j,e_k) = 0$ for all j, $k \in \mathbb{N}$, thus

$$Te_j = \sum_{k=1}^{+\infty}(Te_j,e_k)\,e_k = \sum_{k=1}^{+\infty}t_{jk}e_k = 0 \qquad \text{for every } j \in \mathbb{N}.$$

It follows that $T = 0$ as required.

We infer from $(\alpha\, T e_j, e_k) = \alpha\,(T e_j, e_k) = \alpha\, t_{jk}$ that

$$\|\alpha\, T\|_{\mathrm{HS}} = \left\{ |\alpha|^2 \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} |t_{jk}|^2 \right\}^{\frac{1}{2}} = |\alpha| \cdot \|T\|_{\mathrm{HS}}.$$

Finally, if $\mathbf{S} = (s_{jk})$ and $\mathbf{T} = (t_{jk})$, then

$$\begin{aligned}
\|S+T\|_{\mathrm{HS}}^2 &= \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} |{+s_{jk}t_{jk}}|^2 \le \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} \left\{ |s_{jk}|^2 + 2\,|s_{jk}| \cdot |t_{jk}| + |t_{jk}|^2 \right\} \\
&= \|S\|_{\mathrm{HS}}^2 + \|T\|^2{}_{\mathrm{HS}} + 2 \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} |s_{jk}| \cdot |t_{jk}| \\
&\le \|S\|_{\mathrm{HS}}^2 + \|T\|^2{}_{\mathrm{HS}} + 2 \left\{ \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} |s_{jk}|^2 \right\}^{\frac{1}{2}} \cdot \left\{ \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} |t_{jk}|^2 \right\}^{\frac{1}{2}} \\
&= \|S\|_{\mathrm{HS}}^2 + \|T\|^2{}_{\mathrm{HS}} + 2\,\|S\|_{\mathrm{HS}} \cdot \|T\|_{\mathrm{HS}} = \left\{ \|S\|\ _{\mathrm{HS}} + \|T\|_{\mathrm{HS}} \right\}^2,
\end{aligned}$$

and we have proved the triangle inequality,

$$\|S+T\|_{\mathrm{HS}} \le \|S\|\ _{\mathrm{HS}} + \|T\|_{\mathrm{HS}}.$$

We have proved that $\|\cdot\|_{\mathrm{HS}}$ is a norm.

Finally,

$$
\begin{aligned}
\|Tx\|^2 &= \left\|\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} x_j t_{jk} e_k\right\|^2 = \sum_{k=1}^{+\infty}\left|\sum_{j=1}^{+\infty} x_j t_{jk}\right|^2 \le \sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty}\sum_{\ell=1}^{+\infty} |x_j|\cdot|t_{jk}|\cdot|x_\ell|\cdot|t_{\ell k}| \\
&= \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}\sum_{\ell=1}^{+\infty} \{|x_j|\cdot|t_{\ell k}|\}\cdot\{|x_\ell|\cdot|t_{jk}|\} \\
&\le \left\{\sum_{j,k,\ell=1}^{+\infty} |x_j|^2\,|t_{\ell j}|^2\right\}^{\frac{1}{2}}\cdot\left\{\sum_{j,k,\ell=1}^{+\infty} |x_\ell|^2\,|t_{jk}|^2\right\}^{\frac{1}{2}} = \|T\|^2{}_{\mathrm{HS}}\cdot\|x\|^2,
\end{aligned}
$$

hence $\|Tx\| \le \|T\|_{\mathrm{HS}}\cdot\|x\|$ for every x, and we find that $\|T\| \le \|T\|_{\mathrm{HS}}$.

Example 1.3 *Define for $f \in L^2(\mathbb{R})$, the operator K by*

$$
Kf(x) = \int_{-\infty}^{\infty} \frac{1}{2}\exp(-|x-t|)\,f(t)\,dt.
$$

Show that $Kf \in L^2(\mathbb{R})$ and that K is linear and bounded, with norm ≤ 1.
Show that the function $\frac{1}{2}\exp(-|x-t|)$ does not belong to $L^2(\mathbb{R}^2)$, so that K is not a Hilbert-Schmidt operator.

First we see that

$$
\begin{aligned}
Kf(x) &= \int_{-\infty}^{+\infty} \frac{1}{2}\exp(-|x-t|)\,f(t)\,dt = \int_{-\infty}^{x} \frac{1}{2}e^{-x}e^{t}f(t)\,dt + \int_{x}^{+\infty} \frac{1}{2}e^{x}e^{-t}f(t)\,dt \\
&= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{t}f(t)\,dt + \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-t}f(t)\,dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
|Kf(x)|^2 &= \left\{\int_{-\infty}^{+\infty} \frac{1}{2}\exp(-|x-t|)\,f(t)\,dt\right\}^2 \\
&\le \int_{-\infty}^{+\infty} \frac{1}{2}\exp(-|x-t|)\,|f(t)|\,dt \cdot \int_{-\infty}^{+\infty} \frac{1}{2}\exp(-|x-u|)\,|f(u)|\,du \\
&= \frac{1}{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp-|x-t|\,\exp(-|x-u|)\cdot|f(t)|\cdot|f(u)|\,dt\,du \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{4}\exp(-|x-t|-|x-u|)\cdot|f(t)|\cdot|f(u)|\,dt\,du.
\end{aligned}
$$

Hence

$$
\int_{-\infty}^{+\infty} |Kf(x)|^2 dx \le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} \frac{1}{4}\exp(-|x-t|-|x-u|)\,dx\right\}|f(t)|\cdot|f(u)|\,dt\,du.
$$

If $t \le u$, then

$$
|x-t|+|x-u| = \begin{cases} t-x+u-x = t+u-2x, & \text{for } x \le t, \\ x-t+u-x = u-t, & \text{for } t \le x \le u, \\ x-t+x-u = 2x-t-u, & \text{for } x \ge u. \end{cases}
$$

This gives the inspiration to the following rearrangement

$$\int_{-\infty}^{+\infty} |Kf(x)|^2 dx \le 2\int_{-\infty}^{+\infty}\left(\int_t^{+\infty}\left\{\int_{-\infty}^{+\infty}\frac{1}{4}\,\exp(-|x-t|-|x-u|)\,dx\right\}|f(u)|du\right)|f(t)|dt,$$

where

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-|x-t|-|x-u|}\,dx &= \int_{-\infty}^{t} e^{2x-t-u}\,dx+\int_t e^{-u+t}\,dx+\int_u^{+\infty} e^{-2x+t+u}\,dx\\ &= \left[\frac{1}{2}\,e^{2x-t-u}\right]_{x=-\infty}^{t}+(u-t)e^{-u+t}+\left[-\frac{1}{2}\,e^{-2x+t+u}\right]_{x=u}^{+\infty}\\ &= \frac{1}{2}\,e^{t-u}+(u-t)e^{t-u}+\frac{1}{2}\,e^{t-u}=(u-t+1)e^{t-u},\end{aligned}$$

and where we have assumed that $t \le u$.

By insertion,

$$\int_{-\infty}^{+\infty} |Kf(x)|^2\,dx \le \frac{1}{2}\int_{-\infty}^{+\infty}\left\{\int_t^{+\infty}(u-t+1)e^{t-u}\,|f(u)\,du\right\}|f(t)|\,dt.$$

Then we change variables $y = u - t$ and $z = t + u$, thus

$$t=\frac{y+z}{2}\qquad \text{og}\qquad u=\frac{y-z}{2},$$

where $y \in [0, +\infty[$ and $z \in \mathbb{R}$. We get

$$\begin{aligned}\int_{-\infty}^{+\infty} |Kf(x)|^2\,dx &\le \frac{1}{4}\int_{-\infty}^{+\infty}\int_0^{+\infty}(y+1)e^{-y}\left|f\left(\frac{y-z}{2}\right)\right|\cdot\left|f\left(\frac{y+z}{2}\right)\right|\,dy\,dz\\ &= \frac{1}{4}\int_0^{+\infty}\left\{\int_{-\infty}^{+\infty}\left|f\left(\frac{y-z}{2}\right)\right|\cdot\left|f\left(\frac{y+z}{2}\right)\right|dz\right\}(y+1)e^{-y}\,dy.\end{aligned}$$

Then for every fixed y it follows by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left|f\left(\frac{y-z}{2}\right)\right|\cdot\left|f\left(\frac{y+z}{2}\right)\right|\,dz\\ &\quad\le\left\{\int_{-\infty}^{+\infty}\left|f\left(\frac{y-z}{2}\right)\right|^2\,dz\right\}^{\frac{1}{2}}\cdot\left\{\int_{-\infty}^{+\infty}\left|f\left(\frac{y+z}{2}\right)\right|^2\,dz\right\}^{\frac{1}{2}}\\ &\quad\left\{2\int_{-\infty}^{+\infty}\left|f\left(\frac{y-z}{2}\right)\right|^2\,d\left(\frac{y-z}{2}\right)\right\}^{\frac{1}{2}}\cdot\left\{2\int_{-\infty}^{+\infty}\left|f\left(\frac{y-z}{2}\right)\right|^2\,d\left(\frac{y+z}{2}\right)\right\}^{\frac{1}{2}}\\ &\quad=2\|f\|_2\cdot\|f\|_2=2\|f\|_2^2,\end{aligned}$$

and we get by insertion the estimate

$$\begin{aligned}\int_{-\infty}^{+\infty} |Kf(x)|^2\,dx &\le \frac{1}{2}\int_0^{+\infty}(y+1)e^{-y}\,dy\cdot\|f\|_2^2\\ &= \frac{1}{2}\left[-e^{-y}(y+1)+\int e^{-y}\,dy\right]_0^{+\infty}\cdot\|f\|_2^2\\ &= \frac{1}{2}\left[-e^{-y}(y+2)\right]_0^{+\infty}\cdot\|f\|_2^2=\|f\|_2^2,\end{aligned}$$

so we have proved that $Kf \in L^2(\mathbb{R})$ and that

$$\|Kf\|_2 \leq \|f\|_2 \qquad \text{for every } f \in L^2(\mathbb{R}),$$

hence $\|K\| \leq 1$.

On the other hand, the kernel $\frac{1}{2}\, e^{-|x-t|}$ does not belong to $L^2(\mathbb{R})$, because we get by a formal computation that

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{4}\, e^{-2|x-t|}\, dx\, dt &= \frac{1}{4}\int_{-\infty}^{+\infty}\left\{2\int_{t}^{+\infty} e^{-2(x-t)}\, dx\right\} dt \\
&= \frac{1}{4}\int_{-\infty}^{+\infty}\left\{\int_{0}^{+\infty} e^{-x}\, dx\right\} dt = \frac{1}{4}\int_{-\infty}^{+\infty} 1\, dt = +\infty.
\end{aligned}$$

Example 1.4 *Let K denote the Hilbert-Schmidt operator with kernel*

$$k(x,y) = \sin(x)\,\cos(t), \qquad 0 \leq x,\, t \leq 2\pi.$$

Show that the only eigenvalue for K is 0.
Find an orthonormal basis for $\ker(K)$.

First notice that

$$Kf(x) = \int_0^{2\pi} k(x,t)\, f(t)\, dt = \sin(x) \cdot \int_0^{2\pi} \cos(t)\cdot f(t)\, dt,$$

hence $Kf(x) = a(f)\cdot \sin(x)$, where

$$a(f) = \int_0^{2\pi} \cos(t)\cdot f(t)\, dt \in \mathbb{C}.$$

If $\lambda \in \sigma_p(K)$, then the corresponding eigenfunction must be $f(x) = \sin(x)$. Then by insertion,

$$(K\sin)(x) = \sin(x)\int_0^{2\pi} \cos(t)\cdot\sin(t)\, dt = 0,$$

proving that $\lambda = 0$ is the only eigenvalue.

Now,

$$\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\cos(x),\ \frac{1}{\sqrt{\pi}}\sin(x),\ \dots,\ \frac{1}{\sqrt{\pi}}\cos(nx),\ \frac{1}{\sqrt{\pi}}\sin(nx),\ \dots,$$

is an ortonormalbasis for $L^2([0,2\pi])$, so $\ker(K)$ is spanned by all these with the exception of $\frac{1}{\sqrt{\pi}}\cos(x)$, in which case

$$\begin{aligned}
K\left(\frac{1}{\sqrt{\pi}}\cos\right)(x) &= \sqrt{\pi}\int_0^{2\pi}\frac{1}{\sqrt{\pi}}\cos(t)\cdot\frac{1}{\sqrt{\pi}}\cos(t)\, dt\cdot\sin(x) \\
&= \sqrt{\pi}\cdot\sin(x) = \pi\cdot\frac{1}{\sqrt{\pi}}\sin(x),
\end{aligned}$$

and we get in particular, $K^2 \equiv 0$.

Note that

$$\begin{aligned} k_2(x,t) &= \int_0^{2\pi} k(x,s)k(s,t)\,ds = \int_0^{2\pi} \sin(x)\cdot\cos(s)\cdot\sin(s)\cdot\cos(t)\,ds \\ &= \sin(x)\cdot\cos(t)\cdot\int_0^{2\pi}\sin(s)\cdot\cos(s)\,ds = 0, \end{aligned}$$

which agrees with $K^2 \equiv 0$.

Example 1.5 *Let K denote the Hilbert-Schmidt operator with continuous kernel k on $L^2(I)$, where I is a closed and bounded interval. Show that all the iterated kernels K_n are continuous on I^2 and show that*

$$\|k_n\|_2 \leq \|k\|_2^n.$$

Show that if $|\lambda| \, \|k\|_2 < 1$, then the series

$$\sum_{n=1}^{\infty} \lambda^n \, k_n$$

is convergent in $L^2(I)$.

Write $I = [a, b]$. It is well-known that

$$k_n(x,t) = \int_a^b f(x,s) \, k_{n-1}(s,t) \, ds.$$

The first claim is proved by induction. Assume that both $k(x,s)$ and $k_{n-1}(s,t)$ are continuous. By subtracting something and then adding it again we get

$$\begin{aligned}
k_n(x,t) - k_n(x_0,t_0) &= \int_a^b \{k(x,s)k_{n-1}(s,t) - k(x_0,s)k_{n-1}(s,t)\} \; ds \\
&\qquad + \int_a^b \{k(x_0,s)k_{n-1}(s,t) - k(x_0,s)k_{n-1}(s,t_0)\} \; ds \\
&= \int_a^b \{k(x,s) - k(x_0,s)\} \, k_{n-1}(s,t) \, ds \\
&\qquad + \int_a^b k(x_0,s) \cdot \{k_{n-1}(s,t) - k_{n-1}(s,t_0)\} \; ds.
\end{aligned}$$

To every $\varepsilon > 0$ there is a $\delta > 0$, such that

$$|k(x,s) - k(x_0,s)| < \varepsilon \qquad \text{for } |x - x_0| < \delta \text{ and all } s \in [a,b],$$

and

$$|k_{n-1}(s,t) - k_{n-1}(s,t_0)| < \varepsilon \qquad \text{for } |t - t_0| < \delta \text{ and all } s \in [a,b].$$

If therefore $|x - x_0| < \delta$ and $|t - t_0| < \delta$, then we get the following estimate,

$$\begin{aligned}
|k_n(x,t) - k_n(x_0,t_0)| &\leq \int_a^b \varepsilon \cdot \|k_{n-1}\|_\infty \, dx + \int_a^b \|k\|_\infty \cdot \varepsilon \, ds \\
&= (b-a) \, \{\|k\|_\infty + \|k_{n-1}\|_\infty\} \, \varepsilon,
\end{aligned}$$

and we conclude that $k_n(x,t)$ is continuous, and the claim follows by induction.

Furthermore,

$$
\begin{aligned}
\|k_n\|_2^2 &= \int_a^b \int_a^b |k_n(x,t)|^2 \, dx \, dt \\
&= \int_a^b \int_a^b \left| \int_a^b k(x,s) k_{n-1}(s,t) \, ds \right| \cdot \left| \int_a^b k(x,r) k_{n-1}(r,t) \, dr \right| \, dx \, dt \\
&\leq \int_a^b \int_a^b \int_a^b \int_a^b |k(x,s)| \cdot |k_{n-1}(s,t)| \cdot |k(x,r)| \cdot |k_{n-1}(r,t)| \, ds \, dr \, dx \, dt \\
&\leq \frac{1}{2} \int_a^b \int_a^b \int_a^b \int_a^b \left\{ |k(x,s)|^2 |k_{n-1}(r,t)|^2 + |k_{n-1}(s,t)|^2 |k(x,r)|^2 \right\} ds \, dr \, dx \, dt \\
&= \frac{1}{2} \left\{ \|k\|_2^2 \|k_{n-1}\|_2^2 + \|k_{n-1}\|_2^2 \|k\|_2^2 \right\} = \|k\|_2^2 \|k_{n-1}\|_2^2,
\end{aligned}
$$

and we have proved that

$$\|k_n\|_2 \leq \|k\|_2 \|k_{n-1}\|_2.$$

Hence we get for $n = 2$ that $\|k_2\|_2 \leq \|k\|_2^2$.

Assume that $\|k_{n-1}\|_2 \leq \|k\|_2^{n-1}$. Then

$$\|k_n\|_2 \leq \|k\|_2 \|k_{n-1}\|_2 \leq \|k\|_2 \cdot \|k\|_2^{n-1} = \|k\|_2^n,$$

and the claim follows by induction.

The remaining claim is now trivial, because

$$\left\|\sum_{n=1}^{+\infty}\lambda^n\,k_n(x,t)\right\|_2 \le \sum_{n=1}^{+\infty}|\lambda|^n\|k_n\|_2 \le \sum_{n=1}^{+\infty}|\lambda|^n\|k\|_2^n = \sum_{n=1}^{+\infty}\{|\lambda|\cdot\|k\|_2\}^n = \frac{1}{1-|\lambda|\cdot\|k\|_2},$$

where we have used that the geometric series is convergent for $|\lambda|\cdot\|k\|_2 < 1$.

Example 1.6 *Let K and L denote the Hilbert-Schmidt operators with continuous kernels k and ℓ on $L^2(I)$, where I is a closed and bounded interval. We define the* trace *of K, $\operatorname{tr}(K)$ by*

$$\operatorname{tr}(K) = \int_I k(x,x)\,dx,$$

and similarly for K.
Show that

$$|\operatorname{tr}(KL)| \le \|K\|_{\mathrm{HS}}\|L\|_{\mathrm{HS}},$$

and

$$|\operatorname{tr}(K^n)| \le \|K\|_{\mathrm{HS}}^n, \qquad n \ge 2.$$

Moreover, if (K_n), (L_n) denote sequences of Hilbert-Schmidt operators like above, where

$$\|K_n - K\|_{\mathrm{HS}} \to 0 \quad \text{and} \quad \|L_n - L\|_{\mathrm{HS}} \to 0,$$

then

$$\operatorname{tr}(K_nL_n) \to \operatorname{tr}(KL).$$

Remark 1.1 We first show that the claim is not true, if we replace the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ by the operator norm.

Let

$$k(x,t) = \ell(x,t) = x+t$$

be the kernel of self adjoint Hilbert-Schmidt operators K and L on $L^2([0,1])$. It follows from Example 1.7 below that $\frac{1}{2} \pm \frac{1}{\sqrt{3}}$ are the two eigenvalues different from zero of both K and L, and the norm of K (and L) is given by the absolute value of the numerically largest eigenvalue,

$$\|K\| = \|L\| = \frac{1}{2} + \frac{1}{\sqrt{3}}.$$

Furthermore,l

$$\begin{aligned}\|k\|_2^2 &= \|\ell\|_2^2 = \int_0^1\int_0^1 (x+t)^2\,dx\,dt = \int_0^1\int_0^1 \left(x^2+2xt+t^2\right)\,dx\,dt = \int_0^1\left[\frac{x^3}{3}+x^2t+xt^2\right]_{x=0}^1 dt \\ &= \int_0^1\left\{\frac{1}{3}+t+t^2\right\}dt = \frac{1}{3}+\frac{1}{2}+\frac{1}{3} = \frac{7}{6}.\end{aligned}$$

Finally,

$$\operatorname{tr}(KL) = \int_0^1 \left\{ \int_0^1 (x+s)(s+x)\,ds \right\} dx = \int_0^1 \left\{ \int_0^1 (x+s)^2\,ds \right\} dx = \|k\|_2^2 = \frac{7}{6}.$$

Thus, in this example,

$$\operatorname{tr}(KT) = \frac{7}{6} = \|k\|_2^k > \|K\|^2 = \|K\| \cdot \|L\| = \left\{ \frac{1}{2} + \frac{1}{\sqrt{3}} \right\}^2 = \frac{1}{4} + \frac{1}{3} + \frac{\sqrt{3}}{3},$$

which either can be shown numerically, or of course must follow from the theory, because we always have that $\|K\| \le \|k\|_2$. Here we cannot have equality, if $\sigma_p(K)$ contains at least two different points $\neq 0$. $\Diamond$

Then we turn to the example itself.

Write $I = [a,b]$, and let

$$Ku(x) = \int_a^b k(x,t)u(t)\,dt \qquad \text{and} \qquad Lu(x) = \int_a^b \ell(x,t)u(t)\,dt$$

for $u \in L^2([a,b])$. Then

$$\begin{aligned} ((KL)u)(x) &= K(Lu)(x) = \int_a^b k(x,t)\,Lu(t)\,dt = \int_a^b k(x,t) \left\{ \int_a^b \ell(t,s)u(s)\,ds \right\} dt \\ &= \int_a^b \left\{ \int_a^b k(x,t)\ell(t,s)\,dt \right\} u(s)\,ds, \end{aligned}$$

and it follows that the composition KL has the kernel

$$m(x,t) = \int_a^b k(x,s)\ell(s,t)\,ds.$$

Then

$$\begin{aligned} |\operatorname{tr}(KL)| &= \left| \int_a^b m(x,x)\,dx \right| = \left| \int_a^b \left\{ \int_a^b k(x,t)\ell(t,x)\,dt \right\} dx \right| \\ &\le \int_a^b \left\{ \int_a^b |k(x,t)|^2 dt \right\}^{\frac{1}{2}} \cdot \left\{ \int_a^b |\ell(t,x)|^2 dt \right\}^{\frac{1}{2}} dx. \end{aligned}$$

Putting

$$k_1(x) = \left\{ \int_a^b |k(x,t)|^2 dt \right\}^{\frac{1}{2}} \qquad \text{og} \qquad \ell_1(x) = \left\{ \int_a^b |\ell(t,x)|^2 dt \right\}^{\frac{1}{2}},$$

we get k_1, $\ell_1 \in L^2([a,b])$, and it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} |\operatorname{tr}(KL)| &\le \int_a^b k_1(x)\ell_1(x)\,dx \le \left\{ k_1(x)^2\,dx \right\}^{\frac{1}{2}} \left\{ \int_a^b \ell_1(x)^2 dx \right\}^{\frac{1}{2}} \\ &= \left\{ \int_a^b \left(\int_a^b |k(x,t)|^2 dt \right) dx \right\}^{\frac{1}{2}} \left\{ \int_a^b \left(\int_a^b |\ell(t,x)|^2 dt \right) dx \right\}^{\frac{1}{2}} \\ &= \|k\|_2 \cdot \|\ell\|_2 = \|K\|_{\mathrm{HS}} \cdot \|L\|_{\mathrm{HS}}, \end{aligned}$$

and the first claim is proved.

We note that since KL has the kernel

$$m(x,t)=\int_a^b k(x,s)\ell(s,t)\,ds,$$

we have

$$\begin{aligned}
\|KL\|_{\mathrm{HS}}^2 &\le \int_a^b\int_a^b |m(x,t)|^2 dx\,dt = \int_a^b\left\{\int_a^b\left|\int_a^b k(x,s)\ell(s,t)\,ds\right|^2 dx\right\}dt\\
&\le \int_a^b\left(\int_a^b\left\{\left(\int_a^b |k(x,s)|^2 ds\right)^{\frac12}\left(\int_a^b |\ell(s,t)|^2 ds\right)^{\frac12}\right\}^2 dx\right)dt\\
&= \int_a^b\left(\int_a^b\left\{\left(\int_a^b |k(x,s)|^2\,ds\right)\cdot\left(\int_a^b |\ell(s,t)|^2 ds\right)\right\}dx\right)dt\\
&= \int_a^b\int_a^b |k(x,s)|^2 ds\,dx\cdot\int_a^b\int_a^b |\ell(s,t)|^2 ds\,dt = \|k\|_2^2\cdot\|\ell\|_2^2 = \|K\|_{\mathrm{HS}}^2\cdot\|L\|_{\mathrm{HS}}^2.
\end{aligned}$$

This proves that we always have

(1) $\|KL\|_{\mathrm{HS}} \le \|K\|_{\mathrm{HS}}\cdot\|L\|_{\mathrm{HS}}\,.$

Recall for $n=1$ that

$$\operatorname{tr}(K)=\int_a^b k(x,x)\,dx.$$

Choosing $k(x,x)=1$ and $k(x,t)$ continuous, such that $\|k\|_2<\varepsilon$, we get

$$\operatorname{tr}(K)=b-a \qquad \text{and} \qquad \|K\|^2_{\mathrm{HS}}<\varepsilon,$$

which shows that the formula is not true for $n=1$.

On the other hand, if $n\ge 2$, then it follows from the first question and (1) that

$$\left|\operatorname{tr}\left(K^n\right)\right| = \left|\operatorname{tr}\left(K\,K^{n-1}\right)\right| \le \|K\|_{\mathrm{HS}}\,\|K^{n-1}\|_{\mathrm{HS}} \le \|K\|_{\mathrm{HS}}\|K\|_{\mathrm{HS}}^{n-1} = \|K\|_{\mathrm{HS}}^n.$$

Finally, we note that for any scalar λ and any Hilbert-Schmidt operators,

$$\operatorname{tr}(K+\lambda\,L)=\int_a^b\{k(x,x)+\lambda\,\ell(x,x)\}\,dx=\operatorname{tr}(K)+\lambda\operatorname{tr}(L),$$

proving that the *trace* is linear on the vector space of all Hilbert-Schmidt operators. Then we get

$$\begin{aligned}
\operatorname{tr}(KL)-\operatorname{tr}\left(K_n\,L_n\right) &= \operatorname{tr}\left(KL-K_nL_n\right) = \operatorname{tr}\left(KL-KL_n+KL_n-K_nL_n\right)\\
&= \operatorname{tr}\left(K\left(L-L_n\right)\right)+\operatorname{tr}\left(\left(K-K_n\right)L_n\right)\\
&= \operatorname{tr}\left(K\left(L-L_n\right)\right)+\operatorname{tr}\left(\left(K-K_n\right)\left(L_n-L\right)\right)+\operatorname{tr}\left(\left(K-K_n\right)L\right),
\end{aligned}$$

and it follows from the assumptions and the first part of the example that

$$\begin{aligned}
&\left|\operatorname{tr}(KL)-\operatorname{tr}\left(K_nL_n\right)\right|\\
&\quad\le \|K\|_{\mathrm{HS}}\|L-L_n\|_{\mathrm{HS}}+\|K-K_n\|_{\mathrm{HS}}\|L-L_n\|_{\mathrm{HS}}+\|K-K_n\|_{\mathrm{HS}}\|L\|_{\mathrm{HS}}\to 0 \qquad \text{for } n\to+\infty.
\end{aligned}$$

Example 1.7 *Let K denote the Hilbert-Schmidt operator on $L^2([0,1])$ with kernel*

$$k(x,t) = x + t.$$

Find all eigenvalues and eigenfunctions for K.
Solve the equation

$$Ku = \mu\, u + f, \qquad f \in L^2([0,1]),$$

when μ is not in the spectrum for K.

It follows from

$$(2)\quad Kf(x) = x \int_0^1 f(t)\,dt + \int_0^1 t \cdot f(t)\,dt,$$

that every eigenfunction corresponding to an eigenvalue $\lambda \neq 0$ must have the form $f(x) = ax + b$. By insertion into (2) we get

$$Kf(x) = x \int_0^1 (at+b)\,dt + \int_0^1 \left(at^2 + bt\right)\,dt = \left\{\frac{a}{2} + b\right\}\, x + \left\{\frac{a}{3} + \frac{k}{2}\right\}.$$

This expression is equal to $\lambda(ax+b)$, if and only if (a,b) and $\left(\frac{a}{2}+b, \frac{a}{3}+\frac{b}{2}\right)$ are proportion, thus if and only if

$$0 = \left| \begin{array}{cc} \frac{a}{2}+b & \frac{a}{3}+\frac{b}{2} \\ a & b \end{array} \right| = \frac{ab}{2} + b^2 - \frac{a^3}{3} - \frac{ab}{2} = b^2 - \frac{a^2}{3},$$

hence if and only if $b = \pm \frac{1}{\sqrt{3}}\, a$. Since

$$\lambda\, a = \frac{a}{2} + b = \left\{ \frac{1}{2} \pm \frac{1}{\sqrt{3}} \right\} a,$$

the corresponding eigenvalues are $\lambda = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$.

For $\lambda_1 = \frac{1}{2} + \frac{1}{\sqrt{3}}$ we get the eigenfunction $f_1(x) = x + \frac{1}{\sqrt{3}}$.

For $\lambda_2 = \frac{1}{2} - \frac{1}{\sqrt{3}}$ we get the eigenfunction $f_2(x) = x - \frac{1}{\sqrt{3}}$.

Finally, K is trivially self adjoint, thus $\lambda = 0$ is an eigenvalue for every function

$$f \in \left\{ \text{span}\left(x + \frac{1}{\sqrt{3}}, x - \frac{1}{\sqrt{3}} \right) \right\}^{\perp} = \{\text{span}(1,x)\}^{\perp},$$

hence for every function $f \in L^2([0,1])$, for which

$$\int_0^1 f(t)\, dt = 0 \qquad \text{og} \qquad \int_0^1 t\, f(t)\, dt = 0.$$

Now, $k(x,t) = \overline{k(t,x)}$, so K is self adjoint. Therefore, if we put

$$\varphi_1(x) = \frac{f_1}{\|f_1\|_2} \qquad \text{and} \qquad \varphi_2 = \frac{f_2}{\|f_2\|_2},$$

then the operator K is described by

(3) $Ku = \lambda_1\ (u,\varphi_1)\ \varphi_1 + \lambda_2\ (u,\varphi_2)\, \varphi_2$.

If $(f,\varphi_1) = (f,\varphi_2) = 0$, then it follows by a simple check that the solution of the equation

$$Ku = \mu\, u + f, \qquad \text{hvor } \mu \notin \left\{ 0, \frac{1}{2} + \frac{1}{\sqrt{3}}, \frac{1}{2} - \frac{1}{\sqrt{3}} \right\},$$

is given by $u = -\frac{1}{\mu}\, f$.

Then assume that $f = a\,\varphi_1 + b\,\varphi_2$. The equation $Ku = \mu\, u + f$ can now be written in the form

$$\lambda_1\ (u,\varphi_1)\, \varphi_1 + \lambda_2\ (u,\varphi_2)\, \varphi_2 = \mu \sum_{n=1}^{\infty} (u,\varphi_n) + a\,\varphi_1 + b\,\varphi_2,$$

which implies that

$$u = c_1\,\varphi_1 + c_2\,\varphi_2,$$

where

$$c_1 = (u,\varphi_1) = \frac{a}{\lambda_1-\mu} = \frac{1}{\lambda_1-\mu}\,(f,\varphi_1)\,,$$

and

$$c_2 = (u,\varphi_2) = \frac{b}{\lambda_2-\mu} = \frac{1}{\lambda_2-\mu}\,(f,\varphi_2)\,.$$

The equation being linear, it follows in general from the rewriting

$$Ku - \mu\,u = f = (f,\varphi_1)\,\varphi_1 + (f,\varphi_2)\,\varphi_2 + \{f - (f,\varphi_1)\,\varphi_1 - (f,\varphi_2)\,\varphi_2\}\,,$$

that

$$\begin{aligned} u &= \frac{1}{\lambda_1-\mu}\,(f,\varphi_1)\,\varphi_1 + \frac{1}{\lambda_2-\mu}\,(f,\varphi_2)\,\varphi_2 - \frac{1}{\mu}\,f + \frac{1}{\mu}\,(f,\varphi_1)\,\varphi_1 + \frac{1}{\mu}\,(f,\varphi_2)\,\varphi_2 \\ &= \frac{\lambda_1}{\mu(\lambda_1-\mu)}\,(f,\varphi_1)\,\varphi_1 + \frac{\lambda_2}{\mu(\lambda_2-\mu)}\,(f,\varphi_2)\,\varphi_2 - \frac{1}{\mu}\,f = A\,\varphi_1 + B\,\varphi_2 - \frac{1}{\mu}\,f, \end{aligned}$$

which *in principle* can be written explicitly by means of the functions $f_i(x)$, $i = 1, 2$. We shall, however, not waste our time on that, because the result will look extremely nasty.

Example 1.8 *Lad K denote the Hilbert-Schmidt operator on $L^2\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$ with kernel*

$$k(x,t) = \cos(x-t).$$

Find all eigenvalues and eigenfunctions for K.
Solve the equation

$$Ku = \mu\,u + f, \qquad f \in L^2\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right),$$

when μ is not in the spectrum for K.

Obviously, K is self adjoint.

It follows in general from

$$\cos(x-t) = \cos(x)\cdot\cos(t) + \sin(x)\cdot\sin(t),$$

that

$$(4)\quad Kf(x) = \cos(x)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\cos(t)\,dt + \sin(x)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\sin(t)\,dt.$$

Then any eigenfunction corresponding to some eigenvalue $\lambda \neq 0$ must be of the structure

$$f(x) = a\cdot\cos(x) + b\cdot\sin(x).$$

By insertion into (4),

$$
\begin{aligned}
Kf(x) &= \cos(x)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{a\cdot\cos^2 t + b\cdot \sin t\cos t\, dt\right\} + \sin(x)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{a\cdot \sin t\cos t + b\cdot\sin^2 t\right\}\, dt \\
&= \left\{\frac{a\pi}{2}+0\right\}\cos(x) + \left\{0+\frac{b\pi}{2}\right\}\sin(x) = \frac{\pi}{2}\left\{a\,\cos(x)+b\,\sin(x)\right\} = \frac{\pi}{2}\,f(x),
\end{aligned}
$$

hence $f(x) = a\cdot\cos(x) + b\cdot\sin(x)$ is for every pair $(a,b)\neq(0,0)$ an eigenfunction corresponding to the eigenvalue $\lambda = \dfrac{\pi}{2}$.

For $\lambda = 0$ we get the eigenspace $\{\cos(x),\sin(x)\}^{\perp}$ i $L^2\left(\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\right)$.

Alternatively, we see that

$$\cos(x-t) = \frac{1}{2}\,e^{ix}e^{-it} + \frac{1}{2}\,e^{-ix}e^{it}.$$

We get from

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|e^{\pm ix}\right|^2\,dx = \pi,$$

the normed functions

$$\varphi_1(x) = \frac{1}{\sqrt{\pi}}\,e^{ix} \qquad \text{and} \qquad \varphi_{-1} = \frac{1}{\sqrt{\pi}}\,e^{-ix},$$

where

$$(\varphi_1,\varphi_{-1}) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi_1(x)\,\overline{\varphi_{-1}(x)}\,dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2ix}\,dx = \frac{1}{2i\pi}\,\left\{e^{i\pi}-e^{-i\pi}\right\} = 0,$$

hence

$$k(x,t) = \cos(x-t) = \frac{\pi}{2}\,\varphi_1(x)\,\overline{\varphi_1(t)} + \frac{\pi}{2}\,\varphi_{-1}(x)\,\overline{\varphi_{-1}(t)}.$$

We obtain directly that $\lambda = \dfrac{\pi}{2}$ is the only eigenvalue $\neq 0$, thus $\|K\| = \dfrac{\pi}{2}$, and the eigenfunctions are φ_1 and φ_{-1}.

Remark 1.2 A basis for $L^2\left(\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\right)$ is e.g.

$$\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\,\cos 2x,\ \frac{1}{\sqrt{\pi}}\,\sin 2x,\ \frac{1}{\sqrt{\pi}}\,\cos 4x,\ \frac{1}{\sqrt{\pi}}\,\sin 4x,\ \dots,$$

from which it follows that $\{\cos(x),\sin(x)\}^{\perp}$ may be difficult to describe. $\Diamond$

It follows from $\overline{k(t,x)} = k(x,t)$ that K is self adjoint, which also was noted previously. We may therefore apply the standard method where we expand after the eigenfunctions.

First choose f, such that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\cos t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\sin t\,dt = 0.$$

Then $Kf = 0$, and we conclude that $u = -\dfrac{1}{\mu}\,f$ is the only solution.

We get in the general case that

$$\begin{aligned} u &= \sum_{n=1}^{+\infty} (u,\varphi_n)\,\varphi_n = \frac{1}{\frac{\pi}{2}-\mu}\,\{(f,\varphi_1)\,\varphi_1 + (f,\varphi_2)\,\varphi_2\} - \frac{1}{\mu}\,f + \frac{1}{\mu}\,(f,\varphi_1)\,\varphi_1 + \frac{1}{\pi}\,(f,\varphi_2)\,\varphi_2 \\ &= \frac{\frac{\pi}{2}}{\mu\left(\frac{\pi}{2}-\mu\right)}\,\{(f,\varphi_1)\,\varphi_1 + (f,\varphi_2)\,\varphi_2\} - \frac{1}{\mu}\,f. \end{aligned}$$

Now,

$$\varphi_i = \frac{f_i}{\|f_i\|_2}, \qquad i = 1,\, 2,$$

where $f_1(x) = \cos x$ and $f_2(x) = \sin x$, and $\|f_1\|_2^2 = \|f_2\|_2^2 = \frac{\pi}{2}$, hence

$$\begin{aligned} u &= \frac{\frac{\pi}{2}}{\mu\left(\frac{\pi}{2}-\mu\right)} \cdot \frac{1}{\frac{\pi}{2}} \{(f,\cos t)\cos(x) + (f,\sin t)\sin(x)\} - \frac{1}{\mu} f \\ &= \frac{1}{\mu\left(\frac{\pi}{2}-\mu\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\cos t\,dt \cdot \cos(x) + \frac{1}{\mu\left(\frac{\pi}{2}-\mu\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)\,\sin t\,dt \cdot \sin(x) - \frac{1}{\mu} f(x). \end{aligned}$$

Notice that this expression can be written as

$$u = \frac{1}{\mu\left(\frac{\pi}{2}-\mu\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x-t)\, f(t)\,dt - \frac{1}{\mu} f(x) = \frac{1}{\mu\left(\frac{\pi}{2}-\mu\right)} Kf - \frac{1}{\mu} f.$$

We have assumed that

$$\mu \notin \sigma(K) = \sigma_p(K) = \left\{0, \frac{\pi}{2}\right\}.$$

Example 1.9 *Let K denote the Hilbert-Schmidt operator on $L^2([-\pi,\pi])$ with kernel*

$$k(x,t) = \{\cos(x) + \cos(t)\}^2.$$

Find all eigenvalues and eigenfunctions for K, and find an orthonormal basis for $\ker(K)$.

By a simple computation,

$$\begin{aligned} k(x,t) &= (\cos x + \cos t)^2 = \cos^2 x + 2\,\cos x\,\cos t + \cos^2 t \\ &= \frac{1}{2}\,\cos 2x + 2\,\cos x\,\cos t + \frac{1}{2}\,\cos 2t + \frac{1}{2} \\ &= \frac{1}{2}\,\cos 2x + 2\,\cos x\,\cos t + \left\{1 + \frac{1}{2}\,\cos 2t\right\} \cdot 1. \end{aligned}$$

Hence

$$\begin{aligned} (5)\; Kf(x) &= \cos 2x \int_{-\pi}^{\pi} \frac{1}{2} f(t)\,dt + \cos x \int_{-\pi}^{\pi} 2\,f(t)\,\cos t\,dt \\ &\quad + \int_{-\pi}^{\pi} f(t)\,dt + \int_{-\pi}^{\pi} \frac{1}{2} f(t)\,\cos 2t\,dt. \end{aligned}$$

Therefore, any eigenfunction corresponding to an eigenvalue $\lambda \neq 0$ must be of the form

$$f(x) = a \cdot \cos 2x + b \cdot \cos x + c,$$

where we shall find the constants a, b and c. We get by insertion into (5) that

$$\begin{aligned} Kf(x) &= \cos 2x \int_{-\pi}^{\pi} \frac{1}{2}\,(a\cdot\cos 2t + b\cdot\cos t + c)\,dt + \cos x\int_{-\pi}^{\pi} 2(a\,\cos 2t + b\,\cos t + c)\cos t\,dt \\ &\qquad + \int_{-\pi}^{\pi} (a\cdot\cos 2t + b\cdot\cos t + c)\,dt \\ &\qquad + \int_{-\pi}^{\pi} \frac{1}{2}\,(a\cdot\cos 2t + b\cdot\cos t + c)\cdot\cos 2t\,dt \\ &= c\pi\cdot\cos 2x + 2b\pi\,\cos x + 2\pi\,c + \frac{a\pi}{2}. \end{aligned}$$

This expression is equal to $\lambda a\cdot\cos 2x + \lambda b\cdot\cos x + \lambda c$, if and only if

$$\lambda\, a = c\pi, \qquad \lambda\, b = 2\pi b, \qquad \lambda\, c = 2\pi\, c + \frac{a\pi}{2}.$$

We immediately get the eigenvalue $\lambda = 2\pi$ with its corresponding eigenfunction $\cos x$.

The other eigenfunctions are found in the following way: The vectors (a, c) and $\left(c\pi, 2c\pi + \dfrac{a\pi}{2}\right)$ must be proportional, so

$$0 = \begin{vmatrix} c & 2c + \frac{a}{2} \\ a & c \end{vmatrix} = c^2 - 2ac - \frac{a^2}{2} = (c-a)^2 - \frac{3}{2}\,a^2,$$

hence

$$c = a \pm \sqrt{\frac{3}{2}}\,a = \left\{1 \pm \sqrt{\frac{3}{2}}\right\} a,$$

corresponding to

$$\lambda = \frac{c\pi}{a} = \left\{1 \pm \sqrt{\frac{3}{2}}\right\} \pi.$$

For $\lambda_1 = \left\{1 + \sqrt{\dfrac{3}{2}}\right\} \pi$ we get the eigenfunction

$$f_1(x) = \cos 2x + 1 + \sqrt{\frac{3}{2}} \qquad \left[= 2\cos^2 x + \sqrt{\frac{3}{2}}\right].$$

For $\lambda_2 = \left\{1 - \sqrt{\dfrac{3}{2}}\right\} \pi$ we get the eigenfunction

$$f_2(x) = \cos 2x + 1 - \sqrt{\frac{3}{2}} \qquad \left[= 2\cos^2 x - \sqrt{\frac{3}{2}}\right].$$

For $\lambda = 2\pi$ we get the eigenfunction $f_3(x) = \cos x$.

There is no reason here to norm these eigenfunctions. We only notice that they span the same subspace of $L^2([-\pi,\pi])$ as 1, $\cos x$, and $\cos 2x$ do.

It follows from $\overline{k(t,x)} = k(x,t)$ that K is self adjoint, so the null-space is simply the orthogonal complement of the subspace mentioned above. Thus we conclude that $\ker(K)$ is spanned by

$$\sin x,\ \sin 2x,\ \cos 3x,\ \sin 3x,\ \cos 4x,\ \sin 4x,\ \dots,$$

i.e. of the usual trigonometric basis with the exception of 1, $\cos x$ and $\cos 2x$.

Example 1.10 *Let K denote a self adjoint Hilbert-Schmidt operator on $L^2(I)$ with kernel k. Show that $\|K\| = \|k\|_2$ if and only if the spectrum for K consists of at most two points.*

It follows from K being self adjoint that $\overline{k(t,x)} = k(x,t)$ and there exist an ortonormal sequence (φ_n) in $L^2(I)$ and a sequence (λ_n) of real numbers with $|\lambda_1| \ge |\lambda_2| \ge \cdots$, where either $\lambda_n = 0$ eventually, or $\lambda_n \to 0$, such that

$$(6)\quad Ku = \sum_{n=1}^{+\infty} \lambda_n\ (u, \varphi_n)\ \varphi_n \qquad \text{for } u \in L^2(I),$$

where every φ_n is an eigenfunction of the corresponding $\lambda_n \in \sigma_p(K)$, and where 0 is either an eigenvalue or belongs to the continuous spectrum $\sigma_c(K)$, and where

$$\sigma(K) = \{0\} \cup \sigma_p(K).$$

We shall prove that $\|K\| = \|k\|_2$, if and only if $\sigma(K)$ contains at most two points.

1) If $\sigma(K)$ only consists of one point, then $\sigma(K) = \{0\}$, and $Ku \equiv 0$, thus $k(x,t) = 0$ almost everywhere, and it follows trivially that $\|K\| = \|k\|_1 = 0$.

2) If $\sigma(K)$ contains two points, then it follows from the introducing argument that we necessarily must have

$$\sigma(M) = \{0, \lambda\},$$

so the operator is described by

$$Ku = \ (u, \varphi)\, \varphi = \lambda \int_a^b \varphi(x)\, \overline{\varphi(t)}\, u(t)\, dt,$$

from which we derive that

$$k(x,t) = \lambda\, \varphi(t) \varphi(x).$$

Clearly, $\|K\| = \lambda$. Because $\|\varphi\|_2 = 1$, we get

$$\|k\|_2^2 = \int_a^b \int_a^b |k(x,t)|^2 dx\, dt = |\lambda|^2 \int_a^b \int_a^b |\varphi(x)|^2 |\varphi(t)|^2 dx\, dt = |\lambda|^2.$$

Hence $\|k\|_2 = |\lambda| = \|K\|$ in this case.

3) If $\sigma(K)$ contains more than two points, then

$$\|K\| = \max |\lambda_n| = |\lambda_1|\,.$$

Furthermore, we get by the computation

$$Ku(x) = \int_I k(x,y)\, u(t)\, dt = \sum_{n=1}^{+\infty} \lambda_n \; (u, \varphi_n) \; \varphi_n(x) = \int_I \sum_{n=1}^{+\infty} \lambda_n \, \varphi_n(x) \, \overline{\varphi_n(t)} \, u(t)\, dt,$$

that

$$\|k\|_2^2 = \sum_{n=1}^{+\infty} \lambda_n^2 > \lambda_1^2 = \|K\|^2,$$

and the claim is proved.

Example 1.11 *Let $\{e_1, e_2, \dots, e_p\}$ denote a finite orthonormal set in $L^2(I)$, and let the Hilbert-Schmidt operator K be given by the kernel*

$$k(x,y) = \sum_{i=1}^{p}\sum_{j=1}^{p} k_{ij}\, e_i(x)\, e_j(t).$$

Find the trace $\mathrm{tr}(K)$.
We say that the operator K has a canonical kernel of finite rank.

This example is trivial,

$$\mathrm{tr}(K) = \int_I k(x,x)\,dx = \int_I \sum_{i=1}^{p}\sum_{j=1}^{p} k_{ij}\, e_i(x)\, e_j(x)\,dx = \sum_{i=1}^{p}\sum_{j=1}^{p} k_{ij}\,\delta_{ij} = \sum_{i=1}^{p} k_{ii}.$$

Note that this corresponds to the trace of matrix (k_{ij}).

Example 1.12 *Denote by K a self adjoint Hilbert-Schmidt operator on $L^2(I)$ of kernel k.*
Prove that K is a general Hilbert-Schmidt operator (cf. the definition in Example 1.1*), and find the Hilbert-Schmidt norm $\|K\|_{\mathrm{HS}}$.*

Put

$$Ku = \sum_{n=1}^{+\infty} \lambda_n\, (u,\varphi_n)\, \varphi_n.$$

It follows from Ventus, Hilbert spaces etc., Example 2.7 that

$$t_{jk} = (K\varphi_j,\varphi_k) = \left(\sum_{n=1}^{+\infty} \lambda_n\, (\varphi_j,\varphi_n)\, \varphi_n, \varphi_k\right) = (\lambda_j\,\varphi_j,\varphi_k) = \lambda_j\,\delta_{jk},$$

thus $t_{jj} = \lambda_j$ and $t_{jk} = 0$ for $j \neq 0$.

Then by Example 1.1, K is a general Hilbert-Schmidt operator, if

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} |t_{jk}|^2 < +\infty,$$

because it was proved that this number is independent of the choice of orthonormal basis. Furthermore, it follows from Example 1.2 that

$$\|K\|_{\mathrm{HS}} = \left\{\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} |t_{jk}|^2\right\}^{\frac{1}{2}}.$$

In the present case we get

$$\|K\|_{\mathrm{HS}} = \left\{\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} |\lambda_j|^2\,\delta_{jk}\right\}^{\frac{1}{2}} = \left\{\sum_{j=1}^{+\infty} |\lambda_j|^2\right\}^{\frac{1}{2}} = \|k\|_2.$$

Example 1.13 *Let*

$$k(x,t)=\{\sin(x)+\sin(t)\}^2-\frac{1}{8}$$

be the kernel for a Hilbert-Schmidt operator K on the complex Hilbert space $L^2([-\pi,\pi])$. Show that K is self adjoint and express the range $K\left(L^2([-\pi,\pi])\right)$ of K with the help of the non-normalized basis

$$1,\ \cos(x),\ \sin(x),\ \cos(2x),\ \sin(2x),\ \dots .$$

Find all non-zero eigenvalues and corresponding eigenfunctions for K, and determine $\sigma(K)$. Solve the equation $Ku=\pi\,u-\dfrac{5\pi}{4}$ in $L^2([-\pi,\pi])$.

1) Clearly, $k(x,t)\in L^2([-\pi,\pi]\times[-\pi,\pi])$, and

$$\overline{k(t,x)}=(\sin t+\sin x)^2-\frac{1}{8}=k(x,t),$$

thus $k(x,t)$ is Hermitian, and K is a self adjoint Hilbert-Schmidt-operator. It follows from

$$\begin{aligned}k(x,t)&=(\sin x+\sin t)^2-\frac{1}{8}=\sin^2x+2\sin x\cdot\sin t+\sin^2t-\frac{1}{8}\\&=-\frac{1}{2}\cos 2x+2\sin x\cdot\sin t-\frac{1}{2}\cos 2t+\frac{7}{8},\end{aligned}$$

that

$$\begin{aligned}(7)\ Kf(x)&=\left\{-\frac{1}{2}\int_{-\pi}^{\pi}f(t)\,dt\right\}\cos 2x+\left\{2\int_{-\pi}^{\pi}f(t)\,\sin t\,dt\right\}\sin x\\&\quad+\left\{-\frac{1}{2}\int_{-\pi}^{\pi}f(t)\,\cos 2t\,dt+\frac{7}{8}\int_{-\pi}^{\pi}f(t)\,dt\right\}\cdot 1,\end{aligned}$$

and we conclude that the range $K\left(L^2([-\pi,\pi])\right)$ is spanned by 1, $\sin x$ and $\cos 2x$. (Choose e.g. suitable linear combinations of these three functions in order to conclude that the dimension is 3).

2) An eigenfunction f corresponding to an eigenvalue $\lambda\neq 0$ must necessarily lie in the range, thus it is of the form

$$f(x)=a\cdot\cos 2x+b\cdot\sin x+c,\qquad a,\ b,\ c\in\mathbb{C}.$$

When we insert this expression into (7) and then apply that 1, $\sin x$ and $\cos 2x$ are mutually orthogonal, we get

$$\begin{aligned}Kf(x)&=\left\{-\frac{1}{2}\,c\cdot 2\pi\right\}\cos 2x+\left\{2b\cdot\frac{2\pi}{2}\right\}\sin x+\left\{-\frac{1}{2}\,a\cdot\frac{2\pi}{2}+\frac{7}{8}\,c\cdot 2\pi\right\}\cdot 1\\&=-c\pi\cdot\cos 2x+2b\pi\cdot\sin x+\left\{\frac{7\pi}{4}\,c-\frac{\pi}{2}\,a\right\}\cdot 1.\end{aligned}$$

We have for comparison,

$$\lambda\, f(x) = \lambda\, a \cdot \cos 2x + \lambda\, b \cdot \sin x + \lambda\, c \cdot 1.$$

The coefficient b occurs only in connection with $\sin x$, hence we conclude that $\sin x$ is an eigenfunction corresponding to the eigenvalue $\lambda = 2\pi$.

Assume that $b = 0$. If $a \cdot \cos 2x + c$ is an eigenfunction, then the vectors

$$\left(-c\pi, \frac{7\pi}{4}\, c - \frac{\pi}{2}\, a\right) = \pi \left(-c, \frac{7}{4}\, c - \frac{1}{2}\, a\right) \quad \text{og} \quad (a, c)$$

must be proportional with the eigenvalue $\lambda = -\dfrac{c}{a}\,\pi$ as the factor of proportion. Thus we get the condition

$$\begin{vmatrix} a & -c \\ c & \frac{7}{4}\, c - \frac{1}{2}\, a \end{vmatrix} = c^2 + \frac{7}{4}\, ac - \frac{1}{2}\, a^2 = 0.$$

By solving this equation with respect to c we get

$$c = -\frac{7}{8}\, a \pm \sqrt{\frac{49}{64}\, a^2 + \frac{1}{2}\, a^2} = -\frac{7}{8}\, a \pm \sqrt{\frac{81}{64}\, a^2} = -\frac{7}{8}\, a \pm \frac{9}{8}\, a.$$

We have now two possibilities:

a) For $c = -\dfrac{7}{8}\, a - \dfrac{9}{8}\, a = -2a$ we get $\lambda = -\dfrac{c}{a}\,\pi = 2\pi$, corresponding to the eigenfunction $\cos 2x - 2$.

b) For $c = -\dfrac{7}{8}\, a + \dfrac{9}{8}\, a = \dfrac{1}{4}\, a$ we get $\lambda = -\dfrac{c}{a}\,\pi = -\dfrac{\pi}{4}$, corresponding to the eigenfunction $\cos 2x + \dfrac{1}{4}$.

Summing up,

$$\begin{aligned} &\lambda_1 = 2\pi, && \varphi_1(x) = \sin x, \\ &\lambda_2 = 2\pi, && \varphi_2(x) = \cos 2x - 2, \\ &\lambda_3 = -\frac{\pi}{4}, && \varphi_3(x) = \cos 2x + \frac{1}{4}. \end{aligned}$$

Notice that $\lambda_1 = \lambda_2$,and that the eigenfunctions are not normed.

It follows e.g. from $(K \cos)(x) = 0$ that $\ker(K) \neq \emptyset$, thus

$$\sigma(K) = \sigma_p = \left\{0, -\frac{\pi}{2}, 2\pi\right\}.$$

3) The equation $Ku = \pi\, u - \dfrac{5\pi}{4}$ can be solved in several ways:

First method. The coefficient π of u on the right hand side of the equation does not belong to the spectrum, $\pi \notin \sigma(K)$, hence the solution is unique. Because

$$-\frac{5\pi}{4} = \frac{5\pi}{9}\,(\cos 2x - 2) - \frac{5\pi}{9}\left(\cos 2x + \frac{1}{4}\right),$$

we see that $-\dfrac{5\pi}{4}$ lies in the subspace spanned by the eigenvectors

$$\varphi_2(x) = \cos 2x - 2 \qquad \text{and} \qquad \varphi_3(x) = \cos 2x + \frac{1}{4}.$$

Thus we *guess* a solution of the structure

$$u(x) = a \cdot (\cos 2x - 2) + b \cdot \left(\cos 2x + \frac{1}{4}\right).$$

We get by insertion of this structure that

$$\begin{aligned} Ku(x) - \pi\, u(x) &= 2\pi a \cdot (\cos 2x - 2) - \frac{\pi}{4}\, b \cdot \left(\cos 2x + \frac{1}{4}\right) \\ &\qquad -\pi a(\cos 2x - 2) - \pi b \left(\cos 2x + \frac{1}{4}\right) \\ &= \pi a(\cos 2x - 2) - \frac{5\pi}{4}\, b \left(\cos 2x + \frac{1}{4}\right) \\ &= \pi \left(a - \frac{5}{4}\, b\right) \cos 2x - \pi \left(2a + \frac{5}{16} - b\right). \end{aligned}$$

This expression is equal to $-\dfrac{5\pi}{4}$, if

$$a = \frac{5}{4}\,b \qquad \text{and} \qquad 2\cdot\frac{5}{4}\,b + \frac{1}{4}\cdot\frac{5}{4}\,b = \frac{5}{4},$$

hence $\dfrac{9}{4}\,b = 1$ and $b = \dfrac{4}{9}$, $a = \dfrac{5}{9}$. Finally, we get by insertion,

$$u(x) = \frac{5}{9}\,(\cos 2x - 2) + \frac{4}{9}\left(\cos 2x + \frac{1}{4}\right) = \cos 2x - 1 = -2\sin^2 x.$$

Method 1a. A variant of the FIRST METHOD is to guess a solution of the form

$$u(x) = a\cdot\cos 2x + c.$$

Then apply the previous computation from (2) to get

$$Ku(x) = -c\pi\cdot\cos 2x + \left\{\frac{7\pi}{4}\,c - \frac{\pi}{2}\,a\right\},$$

and

$$-\pi\,u(x) = -a\pi\cdot\cos 2x - c\pi,$$

hence

$$Ku(x) - \pi\,u(x) = -(a+c)\cos 2x + \frac{3\pi}{4}\,c - \frac{\pi}{2}\,a.$$

This expression is equal to $-\dfrac{5\pi}{4}$, if and only if

$$c = -a \qquad \text{and} \qquad -\frac{5\pi}{4} = \frac{3\pi}{4}\,c - \frac{\pi}{2}\,a = -\frac{5\pi}{4}\,a,$$

thus $a = 1$ and $c = -1$, and the unique solution is given by

$$u(x) = \cos 2x - 1 = -2\sin^2 x.$$

Second method. It is also possible to apply the standard method. A straightforward computation where we explicitly use the previously found eigenfunctions (these should then be normed), would demand a lot of energy, although one at different stages could apply one of the two variants above.

We shall show below how this might be carried out. First put

$$\varphi_1(x) = \sin x, \quad \varphi_2(x) = \cos 2x - 2, \quad \varphi_3(x) = \cos 2x + \frac{1}{4}.$$

Let $\{\varphi_n \mid n \geq 4\}$ denote an orthonormal basis of the null-space $\ker(K)$. Then a solution of the equation

$$Ku = \pi\,u - \frac{5\pi}{4},$$

has the structure

$$u = \sum_{n=1}^{+\infty} a_n \, \varphi_n, \qquad \text{where } \sum_{n=4}^{+\infty} |a_n|^2 < +\infty.$$

Put $f(x) = -\frac{5\pi}{4}$. It follows from

$$(f, \varphi_n) = \left(-\frac{5\pi}{4}, \varphi_n \right) = 0 \qquad \text{for } n \in \mathbb{N} \setminus \{2, 3\},$$

and

$$f(x) = -\frac{5\pi}{4} = c_2(\cos 2x - 2) + c_3 \left(\cos 2x + \frac{1}{4} \right) = (c_2 + c_3) \cos 2x - \left(2c_2 - \frac{1}{4} \, c_3 \right),$$

that $c_3 = -c_2$, and

$$2c_2 - \frac{1}{4} \, c_3 = 2c_2 + \frac{1}{4} \, c_2 = \frac{9}{4} \, c_2 = \frac{5\pi}{4},$$

thus

$$c_2 = \frac{5\pi}{9} \qquad \text{and} \qquad c_3 = -\frac{5\pi}{9}.$$

Then we get by insertion into the equation

$$Ku - \pi \, u = -\frac{5\pi}{4}$$

that

$$\begin{aligned}
Ku - \pi \, u &= \lambda_1 a_1 \varphi_1 + \lambda_2 a_2 \varphi_2 + \lambda_3 a_3 \varphi_3 - \sum_{n=1}^{+\infty} a_n \, \varphi_n \\
&= (2\pi - \pi) a_1 \varphi_1 + (2\pi - \pi) a_2 \varphi_2 - \left(\frac{\pi}{4} + \pi \right) a_3 \varphi_3 - \pi \sum_{n=4}^{+\infty} a_n \varphi_n \\
&= \pi a_1 \varphi_1 + \pi a_2 \varphi_2 - \frac{5\pi}{4} \, a_3 \varphi_3 - \pi \sum_{nm=4}^{+\infty} a_n \varphi_n \\
&= -\frac{5\pi}{4} = c_2 \varphi_2 + c_3 \varphi_3,
\end{aligned}$$

and we derive that

$$a_1 = 0, \quad a_2 = \frac{1}{\pi} \, c_2 = \frac{5}{9}, \quad a_3 = -\frac{4}{5\pi} \, c_3 = \frac{4}{9}, \quad a_n = 0 \text{ for } n \geq 4,$$

hence

$$u(x) = \frac{5}{9} \, (\cos 2x - 2) + \frac{4}{9} \left(\cos 2x + \frac{1}{4} \right) = \cos 2x - 1 = -2 \sin^2 x.$$

Example 1.14 *Let $k(x,t) = x + t + 2xt$ be the kernel for the Hilbert-Schmidt operator K on the Hilbert space $H = L^2([-1,1])$.*
Show that K is self adjoint and determine the range $K(H)$.
Find all non-zero eigenvalues and corresponding eigenfunctions for K, and determine $\sigma(K)$ as well as $\|K\|$.
Express Kf, $f \in H$, with the help of the Legendre polynomials (P_n).
Let $f(x) = \cosh(1)\cosh(x) - \cosh(2x)$. Show that $(f, P_0) = (f, P_1) = 0$ and solve the equation

$$Ku(x) + u(x) = f(x).$$

1) It follows from

$$\overline{k(t,x)} = \overline{t + x + 2tx} = x + t + 2xt = k(x,t),$$

that the kernel is Hermitian, thus K is self adjoint. We conclude from

$$Kf(x) = \int_{-1}^{1} (x + t + 2xt) f(t)\, dt = x \int_{-1}^{1} (1 + 2t) f(t)\, dt + \int_{-1}^{1} t\, f(t)\, dt,$$

that the range is $K\left(L^2([-1,1])\right) = \text{span}\{1, x\}$.

2) The only possible eigenfunctions must be of the form $f(x) = ax + b$. We get by insertion the condition

$$\lambda\, f(x) = \lambda a x + \lambda b = Kf(x) = x \int_{-1}^{1} (1 + 2t)(at + b)\, dt + \int_{-1}^{1} t(qt + b) = \, dt,$$

hence

$$\lambda a = \int_{-1}^{1} (1 + 2t)(at + b)\, dt = \int_{-1}^{1} \left\{2at^2 + (a + 2b)t + b\right\}\, dt = \frac{4}{3}\, a + 2b$$

and

$$\lambda b = \int_{-1}^{1} \left(at^2 + bt\right)\, dt = \frac{2a}{3}.$$

Hence,

$$\lambda^2 a = \frac{4}{3}\, a\lambda + 2\lambda b = \frac{4}{3}\, \lambda a + \frac{4}{3}\, a.$$

If $a = 0$, then $2b = \left(\lambda - \dfrac{4}{3}\right) a = 0$, which leads to nothing, so we may assume that $a \neq 0$, e.g. $a = 1$. Then

$$\lambda^2 - \frac{4}{3}\, \lambda - \frac{4}{3} = 0,$$

i.e.

$$\lambda = \frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{4}{3}} = \frac{2}{3} \pm \sqrt{\frac{16}{9}} = \frac{2}{3} \pm \frac{4}{3} = \left\{ \begin{array}{c} 2, \\ \\ -\dfrac{2}{3}. \end{array} \right.$$

If $\lambda_1 = 2$ and $a = 1$, then $b = \dfrac{1}{\lambda_1} \cdot \dfrac{2a}{3} = \dfrac{1}{3}$, and the corresponding eigenfunction is

$$\varphi_1(x) = x + \frac{1}{3}, \qquad \lambda_1 = 2.$$

If $\lambda_2 = -\dfrac{2}{3}$ and $a = 1$, then $b = \dfrac{1}{\lambda_2} \cdot \dfrac{2a}{3} = -\dfrac{3}{2} \cdot \dfrac{2}{3} = -1$, and the corresponding eigenfunction is

$$\varphi_2(x) = x - 1, \qquad \lambda_2 = -\frac{2}{3}.$$

Since K is self adjoint and of Hilbert-Schmidt-type, $\|K\|$ is the absolute value of the eigenvalue of largest absolute value,

$$\|K\| = 2.$$

Finally,

$$\sigma(K) = \sigma_p(K) = \left\{ -\frac{2}{3}, 0, 2 \right\},$$

and every function, which is orthogonal on both φ_1 and φ_2, i.e. on both 1 and x by a change of basis, must lie in the eigenspace corresponding to $\lambda = 0$.

3) It is well-known that the Legendre polynomials form an orthogonal system on $L^2([-1,1])$. We have in particular,

$$P_0(t) = 1 \qquad \text{and} \qquad P_1(t) = t,$$

and since $\text{span}\{P_0, P_1\} = K\left(L^2([-1,1])\right)$, we infer that

$$KP_n = 0 \qquad \text{for every } n \geq 2.$$

It follows that if $f = \sum_{n=0}^{+\infty} a_n P_n$, then

$$\begin{aligned}
Kf(x) &= K\left(\sum_{n=0}^{+\infty} a_n P_n\right)(x) = K\left(\sum_{n=0}^{1} a_n P_n\right)(x) \\
&= K\left(a_0 + a_1 t\right)(x) = \int_{-1}^{1} \left(a_0 + a_1 t\right)(x + t + 2xt)\, dt \\
&= \int_{-1}^{1} \left\{a_0 x + a_0 t + 2a_0 x \cdot t + a_1 x \cdot t + a_1(1+2x)t^2\right\} dt \\
&= 2a_0 x + \frac{2}{3}\, a_1(1+2x) = \left(2a_0 + \frac{4}{3}\, a_1\right) x + \frac{2}{3}\, a_1 \\
&= \left(2a_0 + \frac{4}{3}\, a_1\right) P_1(x) + \frac{2}{3}\, a_1 P_0(x).
\end{aligned}$$

4) Let $f(x) = \cosh 1 \cdot \cosh x - \cosh 2x$. Then

$$\begin{aligned}
(f, P_0) &= \int_{-1}^{1} \{\cosh 1 \cdot \cosh x - \cosh 2x\} dx = \cosh 1 \cdot [\sinh x]_{-1}^{1} - \left[\frac{1}{2}\, \sinh 2x\right]_{-1}^{1} \\
&= \cosh 1 \cdot 2 \sinh 1 - \frac{1}{2} \cdot 2 \sinh 2 = \sinh 2 - \sinh 2 = 0,
\end{aligned}$$

and

$$(f, P_1) = \int_{-1}^{1} \{\cosh 1 \cdot \cosh x - \cosh 2x\} \cdot x\, dx = 0,$$

because the integrand is an odd function, and because we integrate over a finite symmetric interval.

Finally, we shall solve the equation

$$Ku(x) + u(x) = \cosh 1 \cdot \cosh x - \cosh 2x.$$

If

$$u = \sum_{n=0}^{+\infty} a_n P_n \quad \text{and} \quad \cosh 1 \cdot \cosh x - \cosh 2x = \sum_{n=2}^{+\infty} b_n P_n,$$

then it follows from the above that

$$\begin{aligned}
&\frac{2}{3}\, a_1 P_0 + \left(2a_0 + \frac{4}{3}\, a_1\right) P_1 + a_0 P_0 + a_1 P_1 + \sum_{n=2}^{+\infty} a_n P_n \\
&\qquad = \sum_{n=2}^{+\infty} b_n P_n = \cosh 1 \cdot \cosh x - \cosh 2x,
\end{aligned}$$

and we conclude that $a_n = b_n$ for $n \geq 2$ and that

$$\begin{cases} a_0 + \dfrac{2}{3}\, a_1 = 0, \\ 2a_0 + \dfrac{7}{3}\, a_1 = 0, \end{cases} \qquad \text{hence } a_0 = a_1 = 0,$$

and whence

$$u = \sum_{n=2}^{+\infty} a_n P_n = \sum_{n=2}^{+\infty} b_n P_n = \cosh 1 \cdot \cosh x - \cosh 2x.$$

Example 1.15 *In $L^2([-\pi,\pi])$ we consider the orthonormal basis (e_n), $n \in \mathbb{Z}$, where*

$$e_n(t) = \frac{1}{\sqrt{2\pi}}\, e^{int}.$$

1. *Let $\varphi : \mathbb{R} \to \mathbb{C}$ denote a continuous function with period 2π, and assume that $\varphi(-x) = \overline{\varphi(x)}$ for all $x \in \mathbb{R}$. Show that*

$$Ku(x) = \int_{-\pi}^{\pi} \varphi(x-t)\, u(t)\, dt$$

defines a selfadjoint Hilbert-Schmidt operator on $L^2([-\pi,\pi])$.

2. *Show that all e_n are eigenfunctions for K.*

From now on we assume that φ is the periodic extension from $[-\pi,\pi]$ to $\mathbb{R}$ of the function

$$\varphi(x) = 1 - \frac{|x|}{\pi}.$$

3. *Calculate the spectrum of K.*

4. *Solve the equation*

$$Ku = \frac{2}{\pi}\, u + f \qquad \text{in } L^2([-\pi,\pi]),$$

where $f(x) = \sin^2(x) + \sin(x)$.

5. *Solve the equation*

$$Ku = \frac{4}{\pi}\, u + 1 \qquad \text{in } L^2([-\pi,\pi]).$$

1) The kernel is

$$k(x,t) = \varphi(x-t), \qquad x,\, t \in [-\pi,\pi],$$

where

$$\begin{aligned}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\varphi(x-t)|^2\, dt\, dx &= \int_{-\pi}^{\pi}\left\{\int_{-\pi-t}^{\pi-t} |\varphi(u)|^2\, du\right\} dx = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\varphi(u)|^2\, du\, dx \\ &= 2\pi\, \|\varphi\|_2^2 < +\infty,\end{aligned}$$

proving that K is a Hilbert-Schmidt operator.

Alternatively, φ is continuous on a compact set, hence $|\varphi(x)| \le c$ for $x \in [-\pi,\pi]$. Then apply the periodicity to get the estimate

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\varphi(x-t)|^2\, dt\, dx \le c^2(2\pi)^2 = 4\pi^2 c^2 < +\infty. \quad \Diamond$$

From $\varphi(-x) = \overline{\varphi(x)}$ follows that

$$\overline{k(t,x)} = \overline{\varphi(t-x)} = \varphi(x-t) = k(x,t),$$

which shows that the kernel is Hermitian, thus K is self adjoint.

2) By insertion of $e_n(x)$ follows by a change of variable,

$$\begin{aligned} Ke_n(x) &= \int_{-\pi}^{\pi} \varphi(x-t)\, e_n(t)\, dt = \int_{x-\pi}^{x+\pi} \varphi(u)\, e_n(x-u)\, du \\ &= \int_{x-\pi}^{x+\pi} \varphi(u) \cdot e^{-inu}\, du \cdot \frac{1}{\sqrt{2\pi}}\, e^{inx} = \int_{-\pi}^{\pi} \varphi(u)\, e^{-inu}\, du \cdot e_n(x), \end{aligned}$$

from which follows that every $e_n(x)$, $n \in \mathbb{Z}$, is an eigenfunction for K.

Conversely, if ψ is an eigenfunction, then $\psi = \sum c_n e_n$, hence ψ must lie in the subspace corresponding to the e_n, which have the same eigenvalue. This means that the eigenvalues are

$$\int_{-\pi}^{\pi} \varphi(u)\, e^{-inu}\, du, \qquad n \in \mathbb{Z},$$

and it suffices only to look at the eigenfunctions $e_n(x)$, $n \in \mathbb{Z}$, in the following.

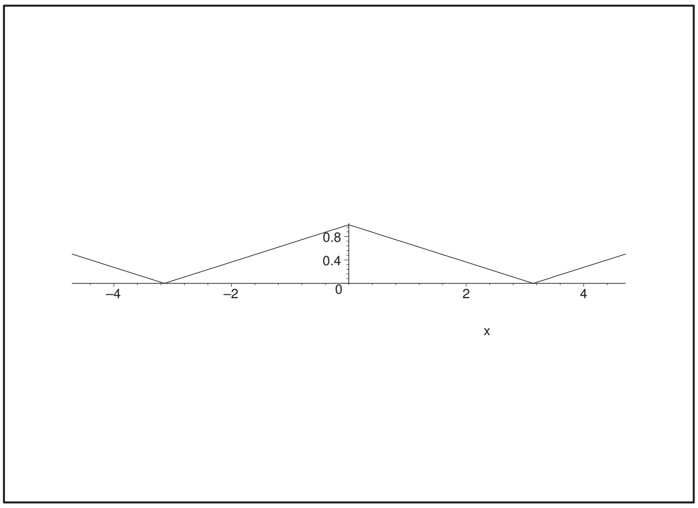


Figure 1: The graph of the function φ.

3) If $\varphi(x) = 1 - \dfrac{|x|}{\pi}$ for $x \in [-\pi, \pi]$, then we have in particular that $\varphi(-x) = \overline{\varphi(x)}$, and that φ is continuous – also after a periodic extension. Therefore, we are again in the situation above. If $n \neq 0$, then the eigenvalues are given by

$$\begin{aligned} \int_{-\pi}^{\pi} \left(1 - \frac{|x|}{\pi}\right) e^{-inx}\, dx &= -\int_{-\pi}^{\pi} \frac{|x|}{\pi}\, e^{-inx}\, dx = -\frac{2}{x} \int_0^{\pi} x\, \cos(nx)\, dx \\ &= 0 + \frac{2}{n\pi} \int_0^{\pi} \sin(nx)\, dx = \frac{2\,\{1-(-1)^n\}}{\pi\, n^2}. \end{aligned}$$

For $n = 0$ we instead get by considering an area on the figure,

$$\int_{-\pi}^{\pi} \left(-\frac{|x|}{\pi}\right) dx = \pi.$$

ALTERNATIVELY,

$$\int_{-\pi}^{\pi}\left(a-\frac{|x|}{\pi}\right)dx=2\pi-\frac{2}{\pi}\int_0^{\pi}x\,dx=2\pi-\frac{2\pi^2}{2\pi}=\pi.$$

Summing up,

$$\lambda_0=\pi,\qquad \begin{cases}\lambda_{2n}=0, & n\in\mathbb{Z}\setminus\{0\},\\[2mm] \lambda_{2n+1}=\dfrac{4}{\pi(2n+1)^2}, & n\in\mathbb{Z},\end{cases}$$

and we conclude that the spectrum is

$$\sigma(K)=\sigma_p(K)=\{0,\pi\}\cup\left\{\frac{4}{\pi(2n+1)^2}\;\middle|\;n\in\mathbb{N}_0\right\}.$$

Notice that the eigenspace corresponding to each eigenvalue of the form $\dfrac{4}{\pi(2n+1)^2}$ is of dimension 2, while the eigenspace corresponding to $\lambda_0=\pi$ is only of dimension 1.

4) Let

$$u=\sum c_ne_n=c_0e_00\sum_{n\neq 0}c_{2n}e_{2n}+\sum_{n\in\mathbb{Z}}c_{2n+1}e_{2n+1}.$$

Then

$$\begin{aligned}
f(x)&=\sin^2x+\sin x=\frac{1-\cos 2x}{2}+\sin 2=\frac{1}{2}+\frac{e^{ix}-e^{-ix}}{2i}-\frac{e^{2ix}+e^{-2ix}}{4}\\
&=\frac{\sqrt{2\pi}}{2}\,e_0(x)+i\,\frac{\sqrt{2\pi}}{2}\,e_{-1}(x)-i\,\frac{\sqrt{2\pi}}{2}\,e_1(x)-\frac{\sqrt{2\pi}}{4}\,e_2(x)-\frac{\sqrt{2\pi}}{4}\,e_{-2}(x)\\
&=Ku-\frac{2}{\pi}\,u\\
&=\left(\pi-\frac{2}{\pi}\right)c_0e_0(x)+\sum_{n\in\mathbb{Z}\setminus\{0\}}\left(-\frac{2}{\pi}\right)c_{2n}e_{2n}(x)+\sum_{n\in\mathbb{Z}}\left\{\frac{4}{(2n+1)^2\pi}-\frac{2}{\pi}\right\}c_{2n+1}e_{2n+1}(x).
\end{aligned}$$

It follows from $\dfrac{2}{\pi}\notin\sigma_p(K)=\sigma(K)$ by identification that

$$c_0=\frac{\sqrt{2\pi}}{2}\cdot\frac{1}{\pi-\dfrac{2}{\pi}}=\sqrt{2\pi}\cdot\frac{\pi}{2\left(\pi^2-2\right)},$$

and

$$c_{-1}=i\,\frac{\sqrt{2\pi}}{2}\cdot\frac{1}{\dfrac{4}{\pi}-\dfrac{2}{\pi}}=i\sqrt{2\pi}\cdot\frac{\pi}{4},\qquad c_1=\overline{c_{-1}}=-i\sqrt{2\pi}\cdot\frac{\pi}{4},$$

and

$$c_{-2}=c_2=-\frac{\sqrt{2\pi}}{4}\cdot\frac{1}{-\dfrac{2}{\pi}}=\sqrt{2\pi}\cdot\frac{\pi}{8},\qquad\text{and } c_n=0\quad\text{otherwise.}$$

This implies that

$$\begin{aligned} u(x) &= \frac{\pi}{2\left(\pi^2-2\right)}\sqrt{2\pi}\,e_0(x)+\frac{\pi}{2}\cdot\frac{\sqrt{2\pi}}{2i}\left\{e_1(x)-e_{-1}(x)\right\}+\frac{\pi}{4}\,\frac{\sqrt{2\pi}}{2}\left\{e_2(x)+e_{-2}(x)\right\} \\ &= \frac{\pi}{2\left(\pi^2-2\right)}+\frac{\pi}{2}\,\sin x+\frac{\pi}{4}\,\cos 2x. \end{aligned}$$

5) In this case, $\frac{4}{\pi}$ is an eigenvalue corresponding to the eigenvectors $e_1(x)$ and $e_{-1}(x)$. Since $1 = \sqrt{2\pi}\,e_0$ is orthogonal to e_1 and e_{-1}, we get

$$u = c_{-1}e_{-1} + c_1e_1 + c_0e_0,$$

where c_{-1} and c_1 are arbitrary constants, and

$$1 = K\left(c_0e_0\right) - \frac{4}{\pi}\,c_0e_0 = \left(\pi - \frac{4}{\pi}\right)c_0e_0 = \left(\pi - \frac{4}{\pi}\right)c_0\cdot\frac{1}{\sqrt{2\pi}},$$

hence

$$c_0 = \frac{\sqrt{2\pi}}{\pi - \dfrac{4}{\pi}} = \frac{\pi\sqrt{2\pi}}{\pi^2-4},$$

and we get the solutions

$$u(x) = \frac{\pi\sqrt{2\pi}}{\pi^2 - 4} + \tilde{c}_1\, e^{ix} + \tilde{c}_{-1} e^{-ix},$$

where $\tilde{c}_1$ and $\tilde{c}_{-1} \in \mathbb{C}$ are arbitrary constants.

Example 1.16 *Let H denote the Hilbert space $L^2([0, 2\pi])$ with the subspace $F = C([0, 2\pi])$, and let K denote the integral operator on H with the kernel*

$$k(x,t) = \begin{cases} \dfrac{i}{2}\exp\left(\dfrac{i}{2}(x-t)\right), & \text{if } 0 \le t < x \le 2\pi, \\[2ex] 0 & \text{if } \quad 0 \le t = x \le 2\pi, \\[2ex] -\dfrac{i}{2}\exp\left(\dfrac{i}{2}(x-t)\right), & \text{if } \quad 0 \le x < t \le 2\pi. \end{cases}$$

1) *Show that K is a self adjoint Hilbert-Schmidt operator.*

2) *Assume that F is equipped with the* sup*-norm. Show that $K : H \to F$ is continuous.*

3) *Now let S denote the restriction of K to F (considered as a subspace of H). Show that S is injective and that S^{-1} is given by*

$$D\left(S^{-1}\right) = \{g \in C^1([0, 2\pi]) \mid g(0) = g(2\pi)\},$$

and

$$S^{-1}g = -i\, g' - \frac{1}{2}\, g \qquad \text{for } g \in D\left(S^{-1}\right).$$

4) *Find all normalized eigenfunctions and associated eigenvalues for S^{-1}. Show that all eigenvalues are simple and that the set of normalized eigenfunctions is an orthonormal system in H.*

5) *Show that the eigenfunctions for S^{-1} are also eigenfunctions for K and find the associated eigenvalues. Justify that all eigenfunctions for K are given this way, and write the kernel for K using the normalized eigenfunctions.*

6) *Let $f \in H$ be given by the Fourier expansion*

$$f = \sum_{n=-\infty}^{\infty} c_n\, e^{inx}.$$

Expand Kf using the Fourier coefficients c_n instead of f.

1) The kernel $k(x,t)$ is bounded and continuous for $t \neq x$ in the compact set $[0, 2\pi]^2$, hence $k \in L^2\left([0, 2\pi]^2\right)$ with

$$\|k\|_2^2 = \int_0^{2\pi} \left\{ \int_0^{2\pi} |k(x,t)|^2\, dt \right\} dx = \frac{1}{4} \cdot (2\pi)^2 = \pi^2,$$

i.e. $\|k\|_2 = \pi$. This shows that K is a Hilbert-Schmidt operator.

We see from

$$\overline{k(t,x)} = \begin{cases} -\dfrac{i}{2}\exp\left(-\dfrac{i}{2}(t-x)\right), & \text{for } 0 \le x < t \le 2\pi, \\[2mm] 0 & \text{for } 0 \le x = t \le 2\pi, \\[2mm] \dfrac{i}{2}\exp\left(-\dfrac{i}{2}\,(t-x)\right), & \text{for } 0 \le t < x \le 2\pi, \end{cases}$$

$$= \begin{cases} \dfrac{i}{2}\exp\left(\dfrac{i}{2}\,(x-t)\right), & \text{for } 0 \le t < x \le 2\pi, \\[2mm] 0 & \text{for } 0 \le t = x \le 2\pi, \\[2mm] -\dfrac{i}{2}\exp\left(\dfrac{i}{2}\,(x-t)\right), & \text{for } 0 \le x < t \le 2\pi, \end{cases}$$

$$= k(x,t),$$

that $k(x,t)$ is Hermitian,, thus K is a self adjoint Hilbert-Schmidt operator.

2) The operator K is described by

$$\begin{aligned} Kf(x) &= \int_0^{2\pi} k(x,t)\,f(t)\,dt = \frac{i}{2}\int_0^x \exp\left(\frac{i}{2}\,(x-t)\right) f(t)\,dt - \frac{i}{2}\int_x^{2\pi} \exp\left(\frac{i}{2}\,(x-t)\right) f(t)\,dt \\ &= \frac{i}{2}\exp\left(i\,\frac{x}{2}\right)\int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt - \frac{i}{2}\exp\left(i\,\frac{x}{2}\right)\int_x^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt \\ &= \frac{i}{2}\exp\left(i\,\frac{x}{2}\right)\left\{\int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt + \int_{2\pi}^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt\right\}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality over $[x, x+\Delta x]$ we get

$$\left|\int_x^{x+\Delta x} \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt\right| \le \|f\|_2 \cdot \sqrt{\Delta x},$$

where obviously the latter factor in the expression for $Kf(x)$ is continuous. The former factor is also continuous, so $K : H \to F$ is a mapping of H into F.

Then we get the estimate

$$\begin{aligned} |Kf(x)| &\le \frac{1}{2}\cdot 1 \cdot \left\{\int_0^x 1\cdot|f(t)|\,dt + \int_x^{2\pi} 1\cdot|f(t)|\,dt\right\} \\ &\le \frac{1}{2}\,\|f\|_2\left\{\sqrt{x} + \sqrt{2\pi - x}\right\} \le \frac{1}{2}\,\|f\|_2 \cdot \left\{\sqrt{\pi} + \sqrt{\pi}\right\} = \sqrt{\pi}\cdot\|f\|_2, \end{aligned}$$

because $\sqrt{x}+\sqrt{2\pi-x}$ has its maximum in the interval $[0,2\pi]$ at $x=\pi$. Then

$$\|Kf\|_\infty \le \sqrt{\pi}\cdot\|f\|_2, \qquad \text{hence} \qquad \|K\|\le\sqrt{\pi},$$

and the linear operator $K : H \to F$ is continuous.

3) Assume that $f \in F$ with $Kf \equiv 0$. Then by (2),

$$\int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt + \int_{2\pi}^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt = 0,$$

for all $x \in [0,2\pi]$. Both integrands are continuous, and the sum of the integrals are C^1 and constant, hence by differentiation,

$$0 = \exp\left(-i\,\frac{x}{2}\right) f(x) + \exp\left(-i\,\frac{x}{2}\right) f(x) = 2\exp\left(-i\,\frac{x}{2}\right) f(x),$$

and we get $f \equiv 0$, so $S = K_{|F}$ is injective.

It was mentioned above that $Kf \in C^1$, if $f \in C$. Furthermore,

$$Kf(0) = \frac{i}{2}\cdot 1\left\{0 - \int_0^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt\right\} = -\frac{i}{2}\int_0^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt,$$

and

$$\begin{aligned} Kf(2\pi) &= \frac{i}{2}\,\exp\left(i\cdot\frac{2\pi}{2}\right)\left\{\int_0^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt + 0\right\} \\ &= -\frac{i}{2}\int_0^{2\pi} \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt = Kf(0), \end{aligned}$$

so we infer that

$$D\left(S^{-1}\right) = KF \subseteqq \{g \in C^1([0,2\pi]) \mid g(0) = g(2\pi)\}.$$

If on the other hand $g \in C^1([0,2\pi])$ satisfies $g(0) = g(2\pi)$, then we shall check if the equation

$$Kf(x) = \frac{i}{2}\,\exp\left(i\,\frac{x}{2}\right)\left\{\int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt + \int_{2\pi}^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt\right\} = g(x)$$

has a solution $f \in F$. This equation is equivalent to

$$(8)\quad \int_0^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt + \int_{2\pi}^x \exp\left(-i\,\frac{t}{2}\right) f(t)\,dt = -2i\,\exp\left(-i\,\frac{x}{2}\right) g(x),$$

so we get by differentiation,

$$(9)\quad 2\exp\left(-i\,\frac{x}{2}\right) f(x) = -2i\,\exp\left(-i\,\frac{x}{2}\right)\left\{-\frac{i}{2}\,g(x) + g'(x)\right\},$$

where (9) is equivalent to that the candidate $f(x)$ must have the structure

$$f(x) = -\frac{1}{2}\,g(x) - i\,g'(x).$$

It is obvious that f given in this way is continuous, when $g \in C^1$. The proof will be concluded, if we can prove that the additional condition $g(0) = g(2\pi)$ combined with (9) implies (8). The trick is that we write

$$2\exp\left(-i\,\frac{x}{2}\right) f(x) = \exp\left(-i\,\frac{x}{2}\right) f(x) + \exp\left(-i\,\frac{x}{2}\right) f(x),$$

where we integrate the former term on the right hand side from 0 to x, and the latter from 2π to x. This construction is guaranteed by the assumption $g(0) = g(2\pi)$.

ALTERNATIVELY one may compute explicitly,

$$Kf(x) = -i\,K(g')(x) - \frac{1}{2}\,K(g)(x),$$

and then convince oneself by some partial integration that the result is $g(x)$. $\Diamond$

4) The equation $S^{-1}g(x) = \lambda\,g(x)$ for $g \in D\left(S^{-1}\right)$ is rewritten as

$$-i\,g'(x) - \frac{1}{2}\,g(x) = \lambda\,g(x), \qquad g(0) = g(2\pi), \quad g \in C^1([0,2\pi]),$$

i.e.

$$g'(x) = i\left\{\lambda + \frac{1}{2}\right\} g(x), \qquad g(0) = g(2\pi).$$

The complete solution without the boundary condition is

$$g(x) = c \cdot \exp\left(i\left(\lambda + \frac{1}{2}\right)x\right).$$

Choosing $c = 1$ and inserting into the boundary condition, we get

$$\exp\left(i\left(\lambda + \frac{1}{2}\right)0\right) = 1 = \exp\left(i\left(\lambda + \frac{1}{2}\right)\cdot 2\pi\right),$$

the solutions of which are $\lambda_n + \dfrac{1}{2} = n \in \mathbb{Z}$.

The eigenvalues are

$$\sigma_p\left(S^{-1}\right) = \left\{\lambda_n = n - \frac{1}{2} \;\middle|\; n \in \mathbb{Z}\right\},$$

with the corresponding normalized eigenfunctions

$$e_n(x) = \frac{1}{\sqrt{2\pi}}\, e^{in\pi}, \qquad n \in \mathbb{Z}.$$

5) It follows from $S^{-1}e_n(x) = \lambda_n e_n(x)$ that

$$\lambda_n K e_n(x) = e_n(x), \qquad \text{thus} \qquad K e_n(x) = \frac{1}{\lambda_n}\, e_n(x),$$

and K has the same eigenfunctions as S^{-1}, and the corresponding eigenvalues are

$$\left\{\frac{1}{\lambda_n} = \frac{1}{n-\frac{1}{2}} = \frac{2}{2n-1} \;\middle|\; n \in \mathbb{Z}\right\} \subseteqq \sigma_p(K).$$

Using that K is a self adjoint Hilbert-Schmidt operator, we get that the spectrum is given by

$$\sigma(K) = \{0\} \cup \left\{\frac{2}{2n-1} \;\middle|\; n \in \mathbb{Z}\right\},$$

where each $\dfrac{2}{2n-1}$ is an eigenvalue. Now, K is injective according to (3), so 0 is not an eigenvalue, thus

$$\sigma_c(K) = \{0\} \quad \text{and} \quad \sigma_p(K) = \left\{\frac{2}{2n-1} \;\middle|\; n \in \mathbb{Z}\right\}.$$

Finally,

$$k(x,t) = \sum_{n=-\infty}^{+\infty} \frac{1}{\lambda_n}\, e_n(x) \cdot \overline{e_n(t)} = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \frac{1}{2n-1}\, e^{in(x-t)}.$$

6) Let $f \in H$ be given by the Fourier expansion

$$f = \sum_{n=-\infty}^{+\infty} c_n e^{inx}.$$

Since e^{inx} is an eigenfunction for K corresponding to the eigenvalue $\dfrac{1}{\lambda_n} = \dfrac{2}{2n-1}$, it follows by a termwise application of K that

$$Kf = \sum_{-\infty}^{+\infty} c_n K\left(e^{in\star}\right) = \sum_{n=-\infty}^{+\infty} \frac{2}{2n-1}\, c_n e^{inx}.$$

2 Other types of integral operators

Example 2.1 *We shall consider* $H = L^2([0,1])$ *as a* real *Hilbert space, and define* $T : H \to H$ *by*

$$Tf(x) = \int_0^x f(t)\,dt.$$

Show that

$$|Tf(x)| \le \sqrt{x}\,\|f\|_2,$$

and use this to show that $\|T\| < 1$.
Show that

$$T^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\,f(t)\,dt.$$

Show that $\log(I+T)$ *is a well-defined operator of Volterra type, and find an explicit expression for the kernel of this operator, using only known functions, that is, find* k *such that*

$$\log(I+T)f(x) = \int_0^x k(x,t)\,f(t)\,dt.$$

1) It follows form the Cauchy-Schwarz inequality that

$$\begin{aligned} |Tf(x)| &= \left|\int_0^x f(t)\,dt\right| = \left|\int_0^1 1_{[0,x]}(t)\,f(t)\,dt\right| \le \left|1_{[0,x]}\right|\big\|_2\,\|f\|_2 \\ &= \left(\int_0^1 \left\{1_{[0,x]}(t)\right\}^2 dt\right)^{\frac{1}{2}} \|f\|_2 = \left\{\int_0^x dt\right\}^{\frac{1}{2}} \|f\|_2 = \sqrt{x}\cdot\|f\|_2. \end{aligned}$$

(There are more variants of this computation).

2) It follows from the estimate above that

$$\|Tf\|_2^2 = \int_0^1 |Tf(x)|^2 dx \le \int_0^1 x\,\|f\|_2^2 dx = \left[\frac{x^2}{2}\right]_0^1 \|f\|_2^2 = \frac{1}{2}\,\|f\|_2^2,$$

and we conclude that

$$\|T\| \le \frac{1}{\sqrt{2}} < 1.$$

3) The formula clearly holds for $n = 1$. Assume that for some $n \in \mathbb{N}$,

$$T^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\,f(t)\,dt, \qquad f \in L^2([0,1]).$$

Interchanging the order of integration in the computation below we get

$$\begin{aligned} T^{n+1}f(x) &= T^n(Tf)(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\,Tf(t)\,dt = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\int_0^t f(s)\,ds\,dt \\ &= \int_0^x \left\{\int_s^x \frac{(x-t)^{n-1}}{(n-1)!}\,dt\right\} f(s)\,ds = \int_0^x \left[-\frac{(x-t)^n}{n!}\right]_{t=s}^{t=x} f(s)\,ds \\ &= \int_0^x \frac{(x-s)^n}{n!}\,f(s)\,ds, \end{aligned}$$

and it follows that the formula also holds, when n is replaced by $n+1$. Then the claim follows by induction.

4) Now,

$$\varphi(\lambda) = \log(1+\lambda) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n}\,\lambda^n, \qquad \text{for } |\lambda| < 1,$$

and $T \in B(L^2([0,1])$ with $\|T\| \le \dfrac{1}{\sqrt{2}} < 1$, so the operator $\log(I+T)$ is indeed defined by

$$\varphi(T) = \log(I+T) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n}\,T^n.$$

Each of the T^n is of Volterra type, and $\varphi(T)$ contains only T^n for $n \ge 1$, hence $\varphi(T)$ is also of Volterra type.

5) When we insert the expression for $T^n f$ from (3), we get by purely formal computations that

$$\log(I+T)f(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n} \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\, f(t)\,dt = \sum_{n=1}^{+\infty} \int_0^x \frac{(t-x)^{n-1}}{n!}\, f(t)\,dt.$$

However, the series $\sum_{n=1}^{+\infty} \dfrac{(t-x)^{n-1}}{n!}$ is *uniformly* convergent for $0 \le t \le x \le 1$. (Notice that we get the sum 1 for $t = x$). Therefore it is indeed legal to interchange summation and integration. The we get for $0 \le t < x$ the sum

$$\sum_{n=1}^{+\infty} \frac{(t-x)^{n-1}}{n!} = \frac{1}{t-x}\left\{\sum_{n=0}^{+\infty} \frac{(t-x)^n}{n!} - 1\right\} = \frac{e^{t-x}-1}{t-x} = e^{-x} \cdot \frac{e^x - e^t}{x-t}.$$

Note that we for $t \to x$ get the limit $e^{-x} \cdot e^x = 1$.

We get by interchanging summation and integration,

$$\log(I+T)f(x) = \int_0^x e^{-x} \cdot \frac{e^x - e^t}{x-t}\, f(t)\,dt,$$

so the kernel of the Volterra operator $\log(I+T)$ is given by

$$k(x,t) = \begin{cases} e^{-x} \cdot \dfrac{e^x - e^t}{x-t} & \text{for } 0 \le t < x \le 1, \\[2mm] 1 & \text{for } 0 \le t = x \le 1, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

Example 2.2 *In this example it is allowed to change the order of integrations without justification. Consider the operator*

$$Af(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}}\, dt, \qquad x \in [0,1],$$

whenever this expression gives sense.

1) Show that $Af \in L^\infty([0,1])$ if $f \in L^p([0,1])$, $p > 2$.

2) Find the operator $B = A^2$, that is find the kernel $k(x,t)$ such that

$$Bf(x) = A^2 f(x) = \int_0^x k(x,t)\, f(t)\, dt$$

for $f \in L^p([0,1])$, $p > 2$.

3) Show that $B : L^p([0,1]) \to L^\infty([0,1])$, $1 \le p \le \infty$ is bounded.

4) Solve the equation

$$(I - A)f(x) = 1$$

formally by a Neumann series, and express f as

$$f(x) = g(x) + Ah(x),$$

where g and h are known functions. (Here it is not possible to express $Ah(x)$ as a known function.) Insert and show that this formal solution is *a solution.*

Remark 2.1 First note that the kernel does *not* belong to $L^2([0,1]^2)$. In fact, it follows from

$$k(x,t) = \begin{cases} \dfrac{1}{\sqrt{x-t}} & \text{for } 0 \le t < x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

that

$$\int_0^1 \int_0^1 |k(x,t)|^2 dt\, dx = \int_0^1 \left\{ \int_0^x \frac{dt}{x-t} \right\} dx = \int_0^1 [-\ln(x-t)]_{t=0}^x\, dx = +\infty,$$

so we cannot apply the theory of the Hilbert-Schmidt operators. Part of the example is to use other methods. $\Diamond$

1) Given $f \in L^p([0,1])$, where $p > 2$, thus $1 < q < 2$, where q is the conjugated number of p, i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then by the *Hölder inequality*

$$\begin{aligned} |Af(x)| &\le \frac{1}{\sqrt{\pi}} \int_0^x \frac{|f(t)|}{\sqrt{x-t}}\, dt \le \frac{1}{\sqrt{\pi}} \left\{ \int_0^x |f(t)|^p\, dt \right\}^{\frac{1}{p}} \left\{ \int_0^x \frac{dt}{(x-t)^{q/2}} \right\}^{\frac{1}{q}} \\ &\le \frac{1}{\sqrt{\pi}} \|f\|_p \left\{ \frac{-1}{1-\frac{q}{2}} \left[(x-t)^{1-\frac{q}{2}} \right]_{t=0}^x \right\}^{\frac{1}{q}} = \frac{1}{\sqrt{\pi}} \|f\|_p \left\{ \frac{1}{1-\frac{q}{2}}\, x^{1-\frac{q}{2}} \right\}^{\frac{1}{q}} \\ &\le \frac{1}{\sqrt{\pi}} \cdot \left\{ 1 - \frac{q}{2} \right\}^{-\frac{1}{q}} \|f\|_p, \end{aligned}$$

where we have used that $1 - \frac{q}{2} > 0$, because $p > 2$. This holds for all $x \in [0,1]$, so

$$\|Af\|_\infty \le \frac{1}{\sqrt{\pi}} \cdot \left\{1 - \frac{q}{2}\right\}^{-\frac{1}{q}} \|f\|_p,$$

and $Af \in L^\infty([0,1])$ for $f \in L^p([0,1])$, when $2 < p < +\infty$.

If instead $p = +\infty$, then we get the following estimate,

$$\begin{aligned} |Af(x)| &\le \frac{1}{\sqrt{\pi}} \int_0^x \frac{|f(t)|}{\sqrt{x-t}}\,dt = \frac{1}{\sqrt{\pi}}\,\|f\|_\infty \int_0^x \frac{dt}{\sqrt{x-t}} \\ &= \frac{1}{\sqrt{\pi}}\,\|f\|_\infty \cdot \left[\frac{-1}{1-\frac{1}{2}}\sqrt{x-t}\right]_0^x = \frac{2}{\sqrt{\pi}}\sqrt{x}\cdot\|f\|_\infty \le \frac{2}{\sqrt{\pi}}\,\|f\|_\infty, \end{aligned}$$

and we get in this case that

$$\|Af\|_\infty \le \frac{2}{\sqrt{\pi}}\,\|f\|_\infty,$$

hence $Af \in L^\infty([0,1])$ for $f \in L^\infty([0,1])$.

2) Assume again that $f \in L^p([0,1])$, where $p > 2$. Then $Af \in L^\infty([0,1])$ according to (1). From $p_1 = \infty > 2$ follows by another application of (1) that $A^2 f \in L^\infty([0,1])$.

Compute

$$Bf(x) = A^2 f(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-t}}\, Af(t)\, dt = \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-t}} \left\{ \frac{1}{\sqrt{\pi}} \int_0^t \frac{f(u)}{\sqrt{t-u}}\, du \right\} dt.$$

From $0 \le u \le t \le x \le 1$ we infer by an interchange of the integrals fås follows by the change of variable $xs = t - u$ that

$$\begin{aligned} Bf(x) &= \frac{1}{\pi} \int_0^x \left\{ \int_u^x \frac{dt}{\sqrt{(x-t)(t-u)}} \right\} f(u)\, du = \frac{1}{\pi} \int_0^x \left\{ \int_0^{x-u} \frac{ds}{\sqrt{\{(x-u)-s)s}} \right\} f(u)\, du \\ &= \frac{1}{\pi} \int_0^x \pi\, f(u)\, du = \int_0^x f(t)\, dt, \end{aligned}$$

where we have used that

$$\int_0^a \frac{ds}{\sqrt{(a-s)s}} = \pi \qquad \text{for } a = x - u > 0.$$

Remark 2.2 We prove for completeness this formula. We get by the monotonous substitution $s = a\, \sin^2\theta$, $\theta \in \left[0, \frac{\pi}{2}\right]$,

$$\begin{aligned} \int_0^a \frac{ds}{\sqrt{(a-s)s}} &= \int_0^{\frac{\pi}{2}} \frac{1 \cdot 2\, \sin\theta\, \cos\theta}{\sqrt{(a - a\, \sin^2\theta) \cdot a\, \sin^2\theta}}\, d\theta = 2a \int_0^{\frac{\pi}{2}} \frac{\sin\theta\, \cos\theta}{\sqrt{a^2(1-\sin^2\theta)\sin^2\theta}}\, d\theta \\ &= \frac{2a}{|a|} \int_0^{\frac{\pi}{2}} \frac{\cos\theta\, \sin\theta}{|\cos\theta\, \sin\theta|}\, d\theta = 2 \int_0^{\frac{\pi}{2}} d\theta = \pi. \qquad \Diamond \end{aligned}$$

The operator is therefore a well-known integral operator, and A corresponds to "integrating one half time from 0". The kernel is explicitly given by

$$k(x,t) = \begin{cases} 1 & \text{for } 0 \le t \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

3) This follows easily from the Hölder inequality,

$$|Bf(x)| \le \int_0^x |f(t)|\, dt \le \int_0^1 |f(t)| \cdot 1\, dt \le 1 \cdot \|f\|_p,$$

hence $\|Bf\|_\infty \le \|f\|_p$, and $\|B\| \le 1$.

4) The Neumann series is given by

$$(I-A)^{-1} = \sum_{n=0}^{+\infty} A^n,$$

so the formal solution is

$$\begin{aligned} f(x) &= \sum_{n=0}^{+\infty} A^n 1(x) = \sum_{n=0}^{+\infty} A^{2n} 1(x) + \sum_{n=0}^{+\infty} A^{2n+1} 1(x) \\ &= \sum_{n=0}^{+\infty} B^n 1(x) + A \sum_{n=0}^{+\infty} B^n 1(x) = g(x) + Ag(x), \end{aligned}$$

hence

$$\begin{aligned} h(x) = g(x) &= \sum_{n=0}^{+\infty} B^n 1(x) = 1 + \sum_{n=1}^{+\infty} B^n 1(x) = 1 + \sum_{n=1}^{+\infty} \int_0^x \frac{t^{n-1}}{(n-1)!} \cdot 1 \, dt \\ &= 1 + \sum_{n=1}^{+\infty} \frac{x^n}{n!} = e^x, \end{aligned}$$

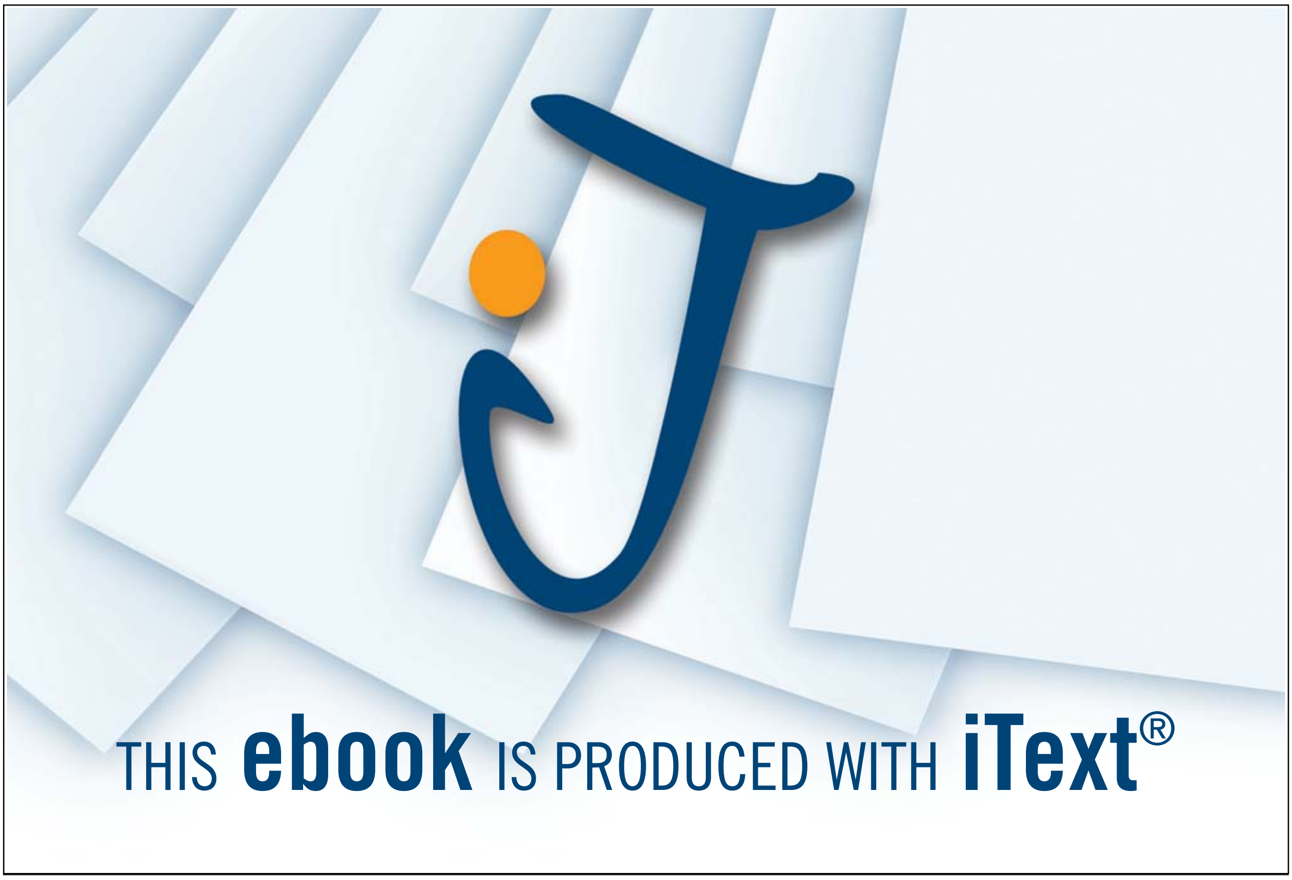
and the formal solution is

$$f(x) = e^x + Ae^x.$$

Then we get by insertion

$$\begin{aligned} (I - A)f(f) &= f(x) - Af(f) = e^x + Ae^x - Ae^x - A^2 e^x \\ &= e^x - Be^x = e^x - \int_0^x e^t \, dt = e^x - \left[e^t\right]_0^x = e^x - (e^x - 1) = 1, \end{aligned}$$

and we have proved that we have found a solution.



Alternatively (and more elegantly),

$$(I-A)(I+A) = (I+A)(I-A) = I - A^2 = I - B.$$

Since B is a Volterra operator, we have that $(I-B)^{-1} = \sum_{n=0}^{+\infty} B^n$ is bounded. Clearly, A and $B = A^2$ commutes, so

$$(I-A)\left\{(I+A)(I-B)^{-1}\right\} = \left\{(I+A)(I-B)^{-1}\right\}(I-A) = I,$$

proving that

$$(I-A)^{-1} = (I+A)(I-B)^{-1}.$$

Hence the equation $(I-A)f = 1$ is equivalent to

$$f(x) = (I-A)^{-1}A(x) = (I+A)\sum_{n=0}^{+\infty} B^n 1(x) = (I+A)e^x = e^x + Ae^x,$$

where we have applied the computation above.

Example 2.3 *Let $H = L^2([0,1])$ and consider the integral operator*

$$Bf(x) = \int_0^x f(t)\,dt, \qquad \textit{for } f \in H.$$

1) Show that

$$k(x,t) = \min\{x,t\}, \qquad 0 \le x,\, t \le 1,$$

is the kernel for the self adjoint Hilbert-Schmidt operator $K = BB^\star$.

2) Let φ be an eigenfunction for K associated with a non-zero eigenvalue λ. Justify that φ can be taken as a C^∞-function.
Next, show that φ must satisfy the equation

$$\lambda\,\varphi''(x) = -\varphi(x),$$

and use this to find all non-zero eigenvalues for K and all the associated eigenfunctions.

3) Assuming the $\|BB^\star\| = \|B^\star|^2$, show that $\|K\| = \|B\|^2$, and find both $\|K\|$ and $\|B\|$.

1) The operator B has the kernel

$$b(x,t) = \begin{cases} 1 & \text{for } 0 \le t \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

so

$$b^\star(x,t) = \overline{b(t,x)} = b(t,x) = \begin{cases} 1 & \text{for } 0 \le x \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the kernel $k(x,t)$ for $K = BB^\star$ is given by

$$\begin{aligned} k(x,t) &= \int_0^1 b(x,s)b^\star(s,t)\,ds = \int_0^1 b(x,s)b(t,s)\,ds \\ &= \int_0^1 b(\min\{x,t\},s)\,ds = \min\{x,t\}, \qquad x,\,t \in [0,1]. \end{aligned}$$

2) Since $k(x,t)$ is continuous, we can choose the eigenfunctions continuous. Hence, if $\varphi(x)$ is an eigenfunction corresponding to an eigenvalue $\lambda \neq 0$, then

(10) $\lambda\lambda\,\varphi(x) = \displaystyle\int_0^1 k(x,t)\,\varphi(t)\,dt = \int_0^x t\,\varphi(t)\,dt + x\int_x^1 \varphi(t)\,dt.$

If φ is continuous, then the right hand side of (10) is differentiable. If φ is of class C^n, then the right hand side of (10) is of class C^{n+1}, hence φ is also of class C^{n+1}. Then the claim follows by induction, hence $\varphi \in C^\infty$.

When we differentiate (10), we get

$$\lambda\,\varphi'(x) = x\,\varphi(x) + \int_x^1 \varphi(t)\,dt - x\,\varphi(x) = \int_x^1 \varphi(t)\,dt,$$

hence by another differentiation,

(11) $\lambda\,\varphi''(x) = -\varphi(x),$

and the claim is proved.

3) Let $\alpha \in \mathbb{C}\setminus\{0\}$ satisfy the condition $\alpha^2 = \dfrac{1}{\lambda}$. Then the equation (11) has the complete solution

(12) $\varphi(x) = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x}.$

When (12) is put into (10), and we apply that $\dfrac{1}{\alpha^2} = \lambda$, then

$$\begin{aligned}
\lambda\,\varphi(x) &= \lambda\left\{C_1 e^{i\alpha x} + C_2 e^{-i\alpha x}\right\} \\
&= \int_0^x t\left\{C_1 e^{i\alpha t} + C_2 e^{-i\alpha t}\right\} dt + x\int_x^1 \left\{C_1 e^{i\alpha t} + C_2 e^{-i\alpha t}\right\} dt \\
&= \left[t\left\{\frac{C_1}{i\alpha}\,e^{i\alpha t} - \frac{C_2}{i\alpha}\,e^{-i\alpha t}\right\}\right]_0^x - \int_0^x \left\{\frac{C_1}{i\alpha}\,e^{i\alpha t} - \frac{C_2}{i\alpha}\,e^{-i\alpha t}\right\} dt \\
&\qquad + x\left[\frac{C_1}{i\alpha}\,e^{i\alpha t} - \frac{C_2}{i\alpha}\,e^{-i\alpha t}\right]_x^1 \\
&= x\left\{\frac{C_1}{i\alpha}\,e^{i\alpha x} - \frac{C_2}{i\alpha}\,e^{-i\alpha x}\right\} - \left[\frac{C_1}{i^2\alpha^2}\,e^{i\alpha t} + \frac{C_2}{i^2\alpha^2}\,e^{-i\alpha t}\right]_0^x \\
&\qquad + x\left\{\frac{C_1}{i\alpha}\,e^{i\alpha} - \frac{C_2}{i\alpha}\,e^{-i\alpha}\right\} - x\left\{\frac{C_1}{i\alpha}\,e^{i\alpha x} - \frac{C_2}{i\alpha}\,e^{-i\alpha x}\right\} \\
&= \frac{1}{\alpha^2}\left\{C_1 e^{i\alpha x} + C_2 e^{-i\alpha x}\right\} - \frac{1}{\alpha^2}\left\{C_1 + C_2\right\} + \frac{x}{i\alpha}\left\{C_1 e^{i\alpha} - C_2 e^{-i\alpha}\right\} \\
&= \lambda\,\varphi(x) - \lambda\left\{C_1 + C_2\right\} + \frac{x}{i\alpha}\left\{C_1 e^{i\alpha} - C_2 e^{-i\alpha}\right\}.
\end{aligned}$$

This equation holds for every x, and $\lambda \neq 0$ and $\alpha \neq 0$, so we conclude that

$$C_1 + C_2 = 0 \qquad \text{and} \qquad C_1 e^{i\alpha} - C_2 e^{-i\alpha} = 0,$$

hence $C_2 = -C_1$, and $C_1\left\{e^{i\alpha} + e^{-i\alpha}\right\} = 2C_1 \cos\alpha = 0$, thus

$$\alpha = \frac{\pi}{2} + n\,\pi, \qquad n \in \mathbb{Z}.$$

It follows from

$$\varphi(x) = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} = C_1 \left\{ e^{i\alpha x} - e^{-i\alpha x} \right\} = 2i\, C_1 \sin \alpha x,$$

that the eigenfunctions for K corresponding to a $\lambda \in \sigma_p(K) \setminus \{0\}$ are some constant times

$$\varphi_n(x) = \sin\left(\left(n - \frac{1}{2} \right) \pi\, x \right), \qquad n \in \mathbb{N},$$

corresponding to the eigenvalue

$$\lambda_n = \frac{1}{\alpha_n^2} = \frac{4}{\pi^2} \cdot \frac{1}{(2n+1)^2}, \qquad n \in \mathbb{N}.$$

4) Now, $\|K\|$ is the absolute value of the numerically largest eigenvalue $|\lambda_1|$, so

$$\|K\| = \|BB\star\| = \lambda_1 = \frac{4}{\pi^2} \cdot \frac{1}{(2-1)^2} = \left(\frac{2}{\pi} \right)^2 .$$

On the other hand, $BB^\star$ is self adjoint, hence

$$\begin{aligned} |BB\star\| &= \sup\{ |(BB^\star f, f)| \mid f \in L^2([0,1]),\ \|f\|_2 = 1 \} \\ &= \sup\{ (B^\star f, B^\star f) \mid f \in L^2([0,1]), \|f\|_2 = 1 \} \\ &= \sup\{ \|B^\star f\|^2 \mid f \in L^2([0,1]), \|f\|_2 = 1 \} = \|B^\star\|^2 . \end{aligned}$$

Finally, $B \in B(H)$, hence also $B^\star \in B(H)$ with $\|B^\star\| = \|B\|$, and whence

$$\|K\| = \|BB^\star\| = \|B^\star|^2 = \|B\|^2 = \left(\frac{2}{\pi}\right)^2.$$

Then

$$\|B\| = \frac{2}{\pi},$$

where

$$Bf(x) = \int_0^x f(t)\,dt, \qquad f \in L^2([0,1]).$$

Example 2.4 *Let $H = L^2([0,1])$ and consider the operator K with domain $D(K) = C([0,1])$ given by*

$$Kf(x) = x\int_0^x f(t)\,dt + \int_x^1 t\,f(t)\,dt, \qquad f \in D(K).$$

1) Show that $K : D(K) \to C^2([0,1])$, and that

$$(Kf)'(0) = 0 \qquad \text{and} \qquad (Kf)'(1) = (Kf)(1).$$

2) Show that K is injective and that K^{-1} has the domain

$$D\left(K^{-1}\right) = \{u \in C^2([0,1]) \mid u'(0) = 0,\, u(1) = u'(1)\},$$

and the action $K^{-1}u = u''$.

3) Show that K is an integral operator with continuous and symmetric kernel and find this kernel.

4) Let φ and ψ denote eigenfunctions for K associated to the same eigenvalue λ. Define the function f by

$$f(x) = \psi(0)\,\varphi(x) - \varphi(0)\,\psi(x),$$

and use the existence and uniqueness theorem for ordinary differential equations to argue that $f = 0$.
Next show that all eigenspaces for K are of dimension one.

5) Let $\sigma_p(K) = (\lambda_n)$ denote the sequence of eigenvalues for K. Find

$$\sum_{n=1}^{\infty} \lambda_n^2.$$

6) Let λ be a positive eigenvalue and let $\mu = \dfrac{1}{\sqrt{\lambda}}$. Express the associated eigenfunction with μ a transcendent equation for μ.
Use a graph argument to show that K has at most one positive eigenvalue.

1) If $f \in C([0,1])$, then we get immediately that Kf is of class $C^1([0,1])$ and

$$(Kf)'(x) = \int_0^x f(t)\,dt + x\,f(x) - x\,f(x) = \int_0^x f(t)\,dt.$$

This shows that we even have $(Kf)' \in C^1([0,1])$, hence $Kf \in C^2([0,1])$, and

(13) $(Kf)''(x) = f(x)$.

Furthermore,

$$(Kf)'(0) = \int_0^0 f(t)\,dt = 0,$$

and

$$(Kf)(1) = 1 \cdot \int_0^1 f(t)\,dt + \int_1^1 t\,f(t)\,dt = \int_0^1 f(t)\,dt = (Kf)'(1).$$

2) Now, K is linear, hence K is injective, If $Kf(x) \equiv 0$ implies that $f = 0$. This follows from (13) in (1), because

$$f(x) = (Kf)''(x) = 0.$$

Assume that $u \in C^2([0,1])$ satisfies $u'(0) = 0$ and $u(1) = u'(1)$. We shall prove that there is an $f \in C([0,1])$, for which $u = Kf$. According to (13) the only possibility is $f = u''$, which we now check. Using that $u'' \in C([0,1])$, we get

$$\begin{aligned} Ku''(x) &= x\int_0^x u''(t)\,dt + \int_x^1 t\,u''(t)\,dt = x\,\{u'(x) - u'(0)\} + [t\,u'(t)]_x^1 - \int_x^1 1 \cdot u'(t)\,dt \\ &= x\,u'(x) + u'(1) - x\,u'(x) - [u(t)]_x^1 = u'(1) - u(1) + u(x) = u(x), \end{aligned}$$

and the claim is proved.

3) We get from the expression for Kf that

$$Kf(x) = \int_0^1 k(x,t)\,f(t)\,dt = \int_0^x f(t)\,dt + \int_x^1 t\,f(t)\,dt = \int_0^1 \max\{x,t\}\,f(t)\,dt,$$

thus

$$k(x,t) = \max\{x,t\} \qquad \text{for } x,\ t \in [0,1],$$

and $k(x,t)$ is clearly continuous in $[0,1]^2$, hence of class $L^2\left([0,1]^2\right)$.

We note that $k(x,t) = \overline{k(t,x)}$, hence the kernel is Hermitian and K is a self adjoint Hilbert-Schmidt operator.

4) This is trivial. We know that K is injective, so $0 \notin \sigma_p(K)$, and if $\lambda \in \sigma_p(K)$, $\lambda \neq 0$, and $K\varphi = \lambda\,\varphi$, it follows by an application of K^{-1} that

$$\varphi = \lambda\,K^{-1}\varphi, \qquad \text{i.e.} \qquad K^{-1}\varphi = \frac{1}{\lambda}\,\varphi.$$

5) Assume that φ and ψ are eigenvectors for K with the same eigenvalue λ. Then

$$f(x) = \psi(0)\,\varphi(x) - \varphi(0)\,\psi(x)$$

is also an eigenfunction corresponding to λ, hence f is according to (4) an eigenvector corresponding to the operator $K^{-1} = \dfrac{d^2}{dx^2}$ with the eigenvalue $\dfrac{1}{\lambda}$, so

$$f''(x) = \frac{1}{\lambda}\,f(x).$$

Now, $(K\varphi)'(0) = 0 = \lambda\varphi'(0)$, and analogously for ψ, so we conclude from (1) that

$$f(0) = \psi(0)\,\varphi(0) - \varphi(0)\,\psi(0) = 0$$

and

$$f'(0) = \psi(0)\,\varphi-(0) - \varphi(0)\,\psi'(0) = 0.$$

It follows from the existence and uniqueness theorem for linear second order differential equations that

$$(14)\quad \frac{d^2 f}{dx^2} - \frac{1}{\lambda}\,f(x) = 0, \qquad f(0) = 0, \quad f'(0) = 0,$$

does only have the solution $f(x) \equiv 0$, hence

$$(15)\quad \psi(0)\,\varphi(x) = \varphi(0)\,\psi(x).$$

Then assume that $\varphi(0) = 0$ for every eigenfunction. Then also $\varphi'(0) = 0$, cf. the above, so φ is a solution of (14), and $\varphi \equiv 0$. This means that φ is not an eigenfunction, contradicting the assumption. Therefore, we conclude that $\varphi(0) \neq 0$ for every eigenfunction. Then it follows from (15) that all eigenfunctions of the same eigenvalue are mutually proportional, hence every eigenspace for K has dimension 1.

6) When we use that K is self adjoint and of Hilbert-Schmidt type, cf. (3), we get that all eigenvalues are real, and

$$\sum_{n=1}^{+\infty} \lambda_n^2 = \|k\|_2^2,$$

where we have used (5) that every eigenspace has dimension 1. Then

$$\begin{aligned}
\sum_{n=1}^{+\infty} \lambda_n^2 &= \|k\|_2^2 = \int_0^1 \int_0^1 \max\{x,t\}^2\,dt\,dx = \int_0^1 \left\{ \int_0^x x^2\,dt + \int_x^1 t^2\,dt \right\} dt \\
&= \int_0^1 \left\{ x^3 + \left[\frac{t^3}{3}\right]_x^1 \right\} dx = \int_0^1 \left\{ x^3 + \frac{1}{3} - \frac{x^3}{3} \right\} dx = \frac{1}{3}\int_0^1 \left(2x^3+1\right)\,dx \\
&= \frac{1}{3}\left[\frac{x^4}{2} + x\right]_0^1 = \frac{1}{3}\left\{\frac{1}{2}+1\right\} = \frac{1}{2}.
\end{aligned}$$

7) It follows from (4) that if $\lambda > 0$ and $\mu = \dfrac{1}{\sqrt{\lambda}}$, then

$$\varphi''(x) = \frac{1}{\lambda}\,\varphi(x) = \mu^2\,\varphi(x),$$

the complete solution of which is

$$\varphi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

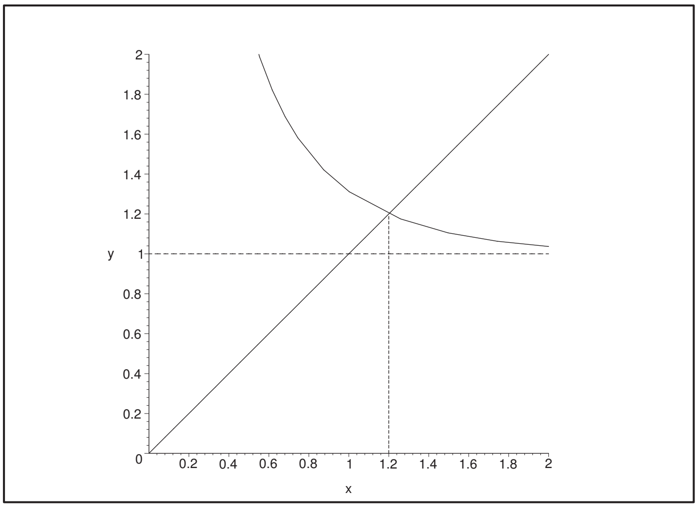


Figure 2: The graphs of $x = \mu$ and $x = \coth\mu$ intersect at $\mu \approx 1.199\,678\,640$.

We shall find the values of C_1, C_2 and μ, for which $\varphi \in D\left(K^{-1}\right)$. We compute

$$\varphi'(x) = \mu\left\{C_1 e^{\mu x} - C_2 e^{-\mu x}\right\},$$

and get the conditions (because $\mu > 0$)

$$\varphi'(0) = \mu\left\{C_1 - C_2\right\} = 0, \qquad \text{i.e. } C_1 = C_2 = C,$$

and

$$\varphi(1) = C\left\{e^{\mu} + e^{-\mu}\right\} = C\mu\left\{e^{\mu} - e^{-\mu}\right\} = \varphi'(1),$$

so μ is a solution of the equation

$$\cosh\mu = \mu\sinh\mu,$$

which we write as

$$\coth\mu = \mu.$$

Considering the graphs we see that this equation has only one solution $\mu > 0$.

Remark 2.3 Using the iteration

$$\mu_{n+1} = \frac{1}{\tan\mu_n}$$

we get on a pocket calculator that

$$\mu \approx 1.199\,678\,640.$$

Note that

$$\lambda_1^2 = \frac{1}{\mu^4} \approx 0.482\,770\,022 < 0,.5,$$

so

$$\sum_{n=2}^{+\infty} \lambda_n^2 = 0.017\,229\,978 \ll \lambda_1^2.$$

The norm of K is approximately

$$\|K\| = \lambda_1 \approx 0.694\,82.$$

We have for any other eigenvalue $\lambda \in \mathbb{R}$ that $\lambda < 0$, so $\mu = \dfrac{1}{\sqrt{\lambda}}$ is purely imaginary. $\Diamond$

Example 2.5 *Let $K \in B(H)$, where $H = L^2([0,1])$, be given by*

$$Kf(x) = \int_{1-x}^{1} f(t)\,dt.$$

1) Show that K is actually bounded.

2) Show that the kernel $k(x,t)$ for K is Hermitian, and calculate

$$\|k\|^2 = \int_0^1 \int_0^1 |k(x,t)|^2\,dt\,dx.$$

3) Show that the kernel $k_2(x,t)$ for K^2 is $\min\{x,t\}$.

4) Show that an eigenfunction for K is an eigenfunction for K^2.
Now, let f denote an eigenfunction for K associated with the eigenvalue λ. Calculate $\left(K^2 f\right)''$, justify that it belongs to H and show that f is a solution to the equation

$$\lambda^2 f'' + f = 0.$$

5) Find all eigenvalues and associated eigenfunctions for K.

6) Determine $\|K\|$.

1) Apply the Cauchy-Schwarz inequality in $L^2([1-x,1])$ for $f \in H$. This gives

$$\|Kf\|_2^2 = \int_0^1 \left| \int_{1-x}^{1} 1 \cdot f(t)\,dt \right|^2 dx \le \int_0^1 \left\{\sqrt{x} \cdot \|f\|_2\right\}^2 dx = \|f\|_2^2 \int_0^1 x\,dx = \frac{1}{2}\,\|f\|_2^2,$$

and we conclude that $\|K\| \le \dfrac{1}{\sqrt{2}}$, thus K is bounded.

2) It follows from

$$Kf(x) = \int_0^1 k(x,t)\,f(t)\,dt = \int_{1-x}^{2} f(t)\,dt = \int_0^1 1_{[1-x,1]}(t)\,f(t)\,dt,$$

that

$$k(x,t) = 1_{[1-x,1]}(t) = \begin{cases} 1 & \text{for } 1-x \le t \le 1, \quad x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $k(x,t) = 1$, if and only if $x + t \ge 1$, x, $t \in [0,1]$, and 0 otherwise, i.e. if and only if

$$(x,t) \in B = \{(x,t) \in [0,1]^2 \mid x + t \ge 1\},$$

so we get (cf. the figure)

$$k(x,t) = 1_B(x,t) = \overline{1_B(t,x)} = \overline{k(t,x)},$$

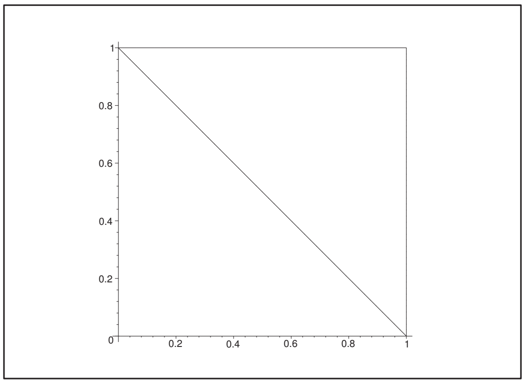


Figure 3: The domain B, where $k(x,t) = 1$, is the upper triangle.

which shows that the kernel is Hermitian.

Then we get

$$\|k\|_2^2 = \int_0^1 \int_0^1 |k(x,t)|^2 \, dt \, dx = \int_0^1 \int_0^1 k(x,t) \, dt \, dx = \text{ area}(B) = \frac{1}{2},$$

possibly in the variant

$$\|k\|_2^2 = \int_0^1 \int_0^1 k(x,t) \, dt \, dx = \int_0^1 (K1)(x) \, dx = \int_0^1 \left\{ \int_{1-x}^1 dt \right\} dx = \int_0^1 x \, dx = \frac{1}{2}.$$

3) The kernel for K^2 is given by

$$k_2(x,t) = \int_0^1 k(x,s)k(s,t) \, ds,$$

where the integrand is $\neq 0$, if and only if

$$1 - x \leq s \leq 1 \qquad \text{and} \qquad 1 - s \leq t \leq 1.$$

This provides us with the bounds

$$1 - x \leq s \leq 1 \qquad \text{and} \qquad 1 - t \leq s \leq 1,$$

hence $s \leq 1$ and

$$s \geq \max\{1-x, 1-t\} = 1 - \min\{x,t\}.$$

Then by insertion

$$\begin{aligned} k_2(x,t) &= \int_0^1 k(x,s)k(s,t) \, ds = \int_{1-\min\{x,t\}}^1 k(x,s)k(s,t) \, ds \\ &= \int_{1-\min\{x,t\}}^1 ds = \min\{x,t\}, \end{aligned}$$

i.e.

$$k_2(x,t) = \min\{x,t\}, \qquad (x,t) \in [0,1]^2.$$

4) If $Kf = \lambda\, f$, then of course

$$K^2 f = \lambda\, K f = \lambda^2 f,$$

so if f is an eigenfunction for K corresponding to the eigenvalue λ, then f is an eigenfunction for K^2 corresponding to the eigenvalue λ^2.

We get, the kernel for K^2 being k_2,

$$K^2 f(x) = \int_0^1 \min\{x,t\}\, f(t)\, dt = \int_0^x t\, f(t)\, dt + x \int_x^1 f(t)\, dt.$$

Obviously, $K^2 f$ is differentiable in the weak sense, and we get

$$\left(K^2 f\right)'(x) = x\, f(x) + \int_x^1 f(t)\, dt - x\, f(x) = \int_x^1 f(t)\, dt.$$

This shows that $\left(K^2 f\right)'$ also is weakly differentiable, so

$$\left(K^2 f\right)''(x) = -f(x).$$

If f is an eigenvalue for K corresponding to the eigenvalue λ, i.e. $Kf = \lambda f$, then it follows from the above that

$$\left(K^2 f\right)(x) = \lambda^2 f(x)$$

and f is differentiable. It follows by induction that f is infinitely often differentiable, so we get from the above that

$$\lambda^2 f''(x) = \left(K^2 f\right)''(x) = -f(x),$$

hence by a rearrangement,

$$(16)\quad \lambda^2 f''(x) + f(x) = 0.$$

Therefore, if f is an eigenfunction for K with eigenvalue λ, then f must also fulfil (16). In particular, $\lambda \neq 0$, if f is an eigenfunction. It is well-known that the solutions of (16) are

$$f(x) = c_1 \exp\left(\frac{i}{\lambda}\,x\right) + c_2 \exp\left(-\frac{i}{\lambda}\,x\right).$$

From $K^2 f(0) = 0 = \lambda^2 f(0)$ follows that $f(0) = 0$, so we conclude that $c_1 + c_2 = 0$. Putting $c_1 = \dfrac{c}{2i}$, we get $c_2 = -\dfrac{c}{2i}$, and the only possibility of an eigenfunction is

$$f(x) = \frac{c}{2i}\left\{\exp\left(\frac{i}{\lambda}\,x\right) - \exp\left(-\frac{i}{\lambda}\,x\right)\right\} = c \cdot \sin\left(\frac{x}{\lambda}\right).$$

5) It remains to find the possible eigenvalues λ.

Put $c = 1$ and $\alpha = \dfrac{1}{\lambda}$. It follows from $Kf(x) = \lambda f(x)$ that

$$f(x) = \sin\left(\frac{x}{\lambda}\right) = \sin(\alpha\, x) = \frac{1}{\lambda}\, Kf(x) = \alpha \cdot K\sin(\alpha\cdot)(x),$$

hence by insertion into the definition of K,

$$\begin{aligned}\sin(\alpha\, x) &= \alpha \int_{1-x}^{1} \sin(\alpha\, t)\, dt = \left[-\cos(\alpha t)\right]_{1-x}^{1} = \cos(\alpha(1-x)) - \cos\alpha \\ &= \cos\alpha \cdot \cos\alpha x + \sin\alpha \cdot \sin\alpha x - \cos\alpha,\end{aligned}$$

so

$$(1 - \sin\alpha)\sin\alpha x = \cos\alpha \cdot (\cos\alpha x - 1).$$

This equation is fulfilled for all x, if either $\alpha = 0$, which is not possible because $\alpha = \dfrac{1}{\lambda}$, or if $\cos\alpha = 0$ and $\sin\alpha = 1$, hence

$$\alpha_p = \frac{\pi}{2} + 2p\pi, \qquad p \in \mathbb{Z},$$

and we get

$$\lambda_p = \frac{1}{\alpha_p} = \frac{1}{\frac{\pi}{2} + 2p\pi} = \frac{1}{\pi} \cdot \frac{1}{4p+1}, \qquad p \in \mathbb{Z}.$$

Then we derive the point spectrum and the continuous spectrum,

$$\sigma_p(K) = \left\{ \frac{2}{\pi} \cdot \frac{1}{4p+1} \;\middle|\; p \in \mathbb{Z} \right\} \quad \text{and} \quad \sigma_c(K) = \{0\}.$$

The eigenfunction corresponding to

$$\lambda_p = \frac{2}{\pi} \cdot \frac{1}{4p+1}, \qquad p \in \mathbb{Z},$$

is

$$f_p(x) = \sin\left(\left(\frac{\pi}{2} + 2p\pi \right) x \right), \qquad x \in [0,1]; \quad p \in \mathbb{Z}.$$

6) The numerically largest eigenvalue is $\lambda_0 = \dfrac{2}{\pi} > 0$, hence

$$\|K\| = \max\{|\lambda_p| \mid p \in \mathbb{Z}\} = \frac{2}{\pi}.$$

Check. As a *check* we use that we should have

$$\frac{1}{2} = \|k\|_2^2 = \sum_{p\in\mathbb{Z}} |\lambda_p|^2 .$$

We get

$$\sum_{p\in\mathbb{Z}} |\lambda_p|^2 = \frac{4}{\pi^2} \sum_{p=-\infty}^{+\infty} \frac{1}{(4p+1)^2} = \frac{4}{\pi^2} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} = \frac{4}{\pi^2} \cdot \frac{\pi^2}{8} = \frac{1}{2} = \|k\|_2^2,$$

because it follows from

$$\begin{aligned} \frac{\pi^2}{6} &= \sum_{n=1}^{+\infty} \frac{1}{n^2} = \left\{ 1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots \right\} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} = \sum_{n=0}^{+\infty} \frac{1}{4^n} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} \\ &= \frac{4}{3} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2}, \end{aligned}$$

that

$$\sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}.$$

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