Examples of Eigenvalue Problems

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Calculus 4c-2

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ISBN 978-87-7681-381-9

Contents

Introduction

Here we present a collection of examples of eigenvalue problems. The reader is also referred to *Calculus 4b* as well as to *Calculus 3c-2*.

It should no longer be necessary rigourously to use the ADIC-model, described in *Calculus 1c* and *Calculus 2c*, because we now assume that the reader can do this himself.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
20th May 2008

1 Initial and boundary value problems

Example 1.1 *Solve the following eigenvalue problem*

$$y'' + \lambda y = 0, \qquad x \in [0, L], \quad y(0) = y'(0) = 0.$$

This is a pure **initial value problem**

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0,$$

hence the solution is unique. Obviously, the zero solution is the only solutions.

Example 1.2 *Prove that the boundary value problem*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0, \quad x \in [0, \pi], \quad y(0) = 1, \quad y(\pi) = -e^{-\pi},$$

has infinitely many solutions and find these. Sketch the graphs of some of these solution.

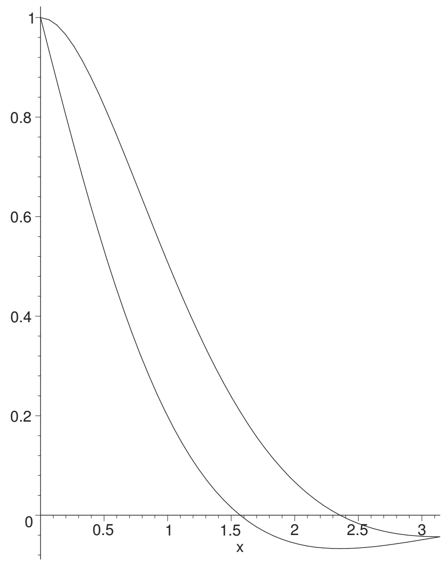
The characteristic polynomial

$$R^2 + 2R + 2 = (R+1)^2 + 1$$

has the roots $R = -1 \pm i$.

The complete solution is given by

$$y(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x, \quad x \in [0, \pi], \quad c_1, c_2 \in \mathbb{R}.$$



It follows from the boundary values that

$$y(0) = c_1 = 1 \quad \text{og} \quad y(\pi) = -c_1 e^{-\pi} = -e^{\pi}.$$

We get in both cases that $c_1 = 1$, and we have no requirement on $c_2 \in \mathbb{R}$.

The complete solution of the boundary value problem is

$$y(x) = e^{-x}\cos x + ce^{-x}\sin x, \quad x \in [0,\pi], \quad c \in \mathbb{R} \text{ arbitrær}.$$

Example 1.3 *For a loaded column at equilibrium, one can as a mathematical model for a (small) bending $y(x)$ in a convenient coordinate system use the following linear boundary value problem,*

$$EI\frac{d^2y}{dx^2} + Py = -Pe, \quad x \in [0, L], \quad y(0) = 0, \quad y'(L) = 0.$$

Here, E, I, L, P and e are given positive constants. For convenience we write $P/(EI) = k^2$.

1) Find the solution of the boundary value problem.

2) Prove that $y(L) \to \infty$ for $P \to EI\dfrac{\pi^2}{4L^2}$, no matter how small the fixed constant e is.

3) Sketch $y(L)$ as a function of $kl = \sqrt{\dfrac{P}{EI}} \cdot L$, $0 \le kL < \dfrac{\pi}{2}$.

1) By a division with $EI > 0$ the equation is transferred into the inhomogeneous equation

$$\frac{d^2y}{dx^2} + k^2y = -k^2e, \qquad k^2 = \frac{P}{EI} > 0.$$

(a) First find the complete solution. The characteristic equation

$$R^2 + k^2 = 0, \quad \text{i.e. } R = \pm ik, \quad [\text{NB } k > 0]$$

provides us with the following solution of the corresponding *homogeneous equation*

$$c_1 \cos kx + c_2 \sin kx, \qquad c_1, c_2 \text{ are arbitrary.}$$

We *guess* a particular solution as the constant $y = -e$. Since the equation is linear, the complete solution is

$$y = -e + c_1 \cos(kx) + c_2 \sin(kx), \quad x \in [0, L], \quad c_1, c_2 \text{ arbitrary.}$$

NB. Unfortunately e is a constant which has nothing to do with the usual mathematical constant $2,718\dots$.

(b) Insert into the boundary conditions.

We get

$$y(0) = 0 = -e + c_1, \qquad \text{dvs. } c_1 = e,$$

and

$$y'(L) = 0 = -c_1k\sin(kL) + c_2k\cos(kL),$$

hence [because $k > 0$]

$$c_2 \cos(kL) = e \cdot (kL).$$

If $kL = \dfrac{\pi}{2} + p\pi$, then the left hand side is 0, and the right hand side is $\pm e$. *Therefore we do not have any solution* for $kL = \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{N}_0$.

If $kL \neq \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{N}_0$, then $\cos(kL) \neq 0$, so

$$c_2 = e \cdot \tan(kL).$$

By insertion of $c_1 = e$ and $c_2 = e \cdot \tan(kL)$ we get the solution

$$\begin{aligned} y &= -e + e\cos(kx) + e\tan(kL)\sin(kx) \\ &= e\left\{\frac{1}{\cos(kL)}\left(\cos(kL)\cdot\cos(kx) + \sin(kL)\cdot\sin(kx)\right) - 1\right\} \\ &= e\left\{\frac{\cos(k(L-x))}{\cos(kL)} - 1\right\}, \qquad x \in [0, L]. \end{aligned}$$

2) If $P \to EI\dfrac{\pi^2}{4L^2}$ from below, then $k^2 = \dfrac{P}{EI} \to \left(\dfrac{\pi}{2}\right)^2 \cdot \dfrac{1}{L^2}$ from below, so

$$kL \to \frac{\pi}{2} - .$$

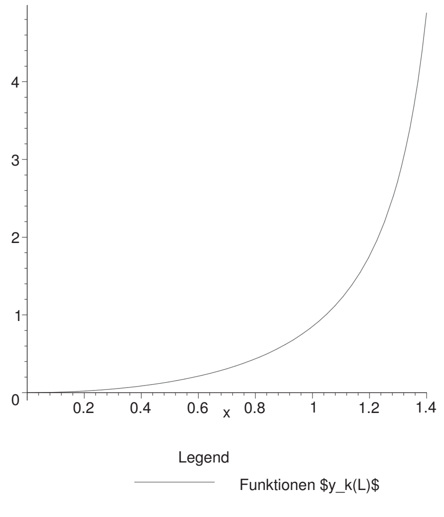
By insertion of $x = L$ we get

$$y(L) = e\left\{\frac{1}{\cos(kL)} - 1\right\} \to \infty \qquad \text{for } P \to EI\frac{\pi^2}{4L^2} - .$$

3) The function

$$y_k(L) = e\left\{\frac{1}{\cos(kL)} - 1\right\} = e\{\sec(kL) - 1\}$$

(secant $= 1/$cosine) is easily sketched on a figure.



Example 1.4 *Consider the boundary value problem*

$$y'' + 3y = 0, \qquad x \in [0,\pi], \qquad y(0) = y(\pi) = 0.$$

Set up the linear system of equations

$$\mathbf{Bc} = \mathbf{z},$$

and check it.

The characteristic polynomial $R^2 + 3$ has the two simple roots $R = \pm i\sqrt{3}$, so the complete solution is

$$y = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x), \quad x \in [0,\pi], \quad c_1, c_2, \text{ arbitrary}.$$

It follows from the boundary conditions,

$$\begin{cases} c_1 & +0 \cdot c_2 & = y(0) = 0, \\ \\ c_1 \cdot \cos(\sqrt{3}\pi) & +c_2 \cdot \sin(\sqrt{3}\pi) & = y(\pi) = 0. \end{cases}$$

The matrix equation is

$$\mathbf{Bc} = \begin{pmatrix} 1 & 0 \\ \cos(\sqrt{3}\pi) & \sin(\sqrt{3}\pi) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since

$$\det \mathbf{B} = \sin(\sqrt{3}\pi) \neq 0,$$

the solution $c_1 = c_2 = 0$ is unique end the zero solution is the only solution.

Example 1.5 *Consider the boundary value problem*

$$y'' + 4y = 0, \qquad x \in [0,\pi], \quad y(0) = y(\pi) = 0.$$

Set up the linear system of equations

$$\mathbf{Bc} = \mathbf{z},$$

and check it.

Since the characteristic polynomial $R^2 + 4$ has the two simple roots $R = \pm 2i$, the complete solution is

$$y = c_1 \cos(2x) + c_2 \sin(2x), \qquad x \in [0,\pi], \quad c_1, c_2 \text{ arbitary}.$$

It follows from the boundary conditions that

$$\begin{cases} c_1 + 0 \cdot c_2 = y(0) = 0, \\ c_1 + 0 \cdot c_2 = y(\pi) = 0. \end{cases}$$

The matrix equation becomes

$$\mathbf{Bc} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\operatorname{rang}(\mathbf{B}|\mathbf{z}) = \operatorname{rang}(\mathbf{B}) = 1 < n = 2.$$

It follows immediately that the boundary value problem has infinitely many solutions,

$$y = c \cdot \sin(2x), \qquad x \in [0,\pi], \quad x \in \mathbb{R} \text{ arbitrary.}$$

Example 1.6 *Prove that the boundary value problem*

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0, \qquad x\in[0,1], \quad y(0)=y'(1)=0,$$

has a nontrivial solution and find all its complete solution.

The characteristic polynomial

$$R^2+2R+1=(R+1)^2$$

has the double root $R=-1$, so the complete solution is

$$y=c_1e^{-x}+c_2xe^{-x}, \qquad x\in[0,1],$$

where c_1 and c_2 are arbitrary constants.

From the boundary value $y(0)=0$ follows that

$$y(0)=c_1=0,$$

so the candidates must have the form

$$y(x)=c_2xe^{-x}.$$

Since

$$y'(x)=c_2(1-x)e^{-x},$$

it follows from the boundary value $y'(1)=0$ that

$$y'(1)=c_2\cdot 0=0,$$

which is fulfilled for every $c_2\in\mathbb{R}$.

The complete solution of the boundary value problem is

$$y=c\cdot xe^{-x}, \qquad x\in[0,1], \quad c \text{ an arbitrary constant.}$$

Example 1.7 *Given the boundary value problem*

$$y'' + \lambda^2 y = 0, \qquad x \in [0,1], \quad \lambda \in \mathbb{R}_+,$$

with the boundary conditions

$$y(0) = 1, \qquad y'(0) = -1, \qquad y(1) + y'(1) = 0,$$

where λ is considered as a parameter.

Find all the possible values of the parameter $\lambda \in \mathbb{R}_+$, and the corresponding functions $y(x)$.

This example is a boundary value problem, very much like an eigenvalue problem without being one. The differences are

1) we have three conditions for an equation of second order,

2) the boundary conditions are not zero.

Clearly, the complete solution is

$$y = c_1 \cos \lambda x + c_2 \sin \lambda x$$

where

$$y' = -\lambda c_1 \sin \lambda x + \lambda c_2 \cos \lambda x.$$

It follows from the boundary conditions that

$$y(0) = c_1 = 1, \qquad y'(0) = c_2 \lambda = -1,$$

$$y(1) + y'(1) = c_1\{\cos \lambda - \lambda \sin \lambda\} + c_2\{\sin \lambda + \lambda \cos \lambda\} = 0.$$

Hence $c_1 = 1$, $c_2 = -\dfrac{1}{\lambda}$, which we put into the latter equation,

$$0 = \cos \lambda - \lambda \sin \lambda - \frac{1}{\lambda} \sin \lambda - \cos \lambda = -\frac{\lambda^2+1}{\lambda} \sin \lambda.$$

The latter equation is fulfilled if $\lambda_n = n\pi$, $n \in \mathbb{N}$. If we e.g. put

$$y_n(x) = \cos(n\pi x) - \frac{1}{n\pi} \sin(n\pi x), \qquad n \in \mathbb{N},$$

then all eigenfunctions corresponding to $\lambda_n = n\pi$, $n \in \mathbb{N}$, are given by

$$y(x) = c \cdot y_n(x) = c\left\{\cos(n\pi x) - \frac{1}{n\pi} \sin(n\pi x)\right\}, \quad c \text{ arbitrary.}$$

2 Eigenvalue problems

Example 2.1 (Cf. Example 1.1). *Solve the following eigenvalue problem*

$$y'' + \lambda y = 0, \qquad x \in [0, L], \quad y(0) = 0, \quad y'(L) = 0.$$

The characteristic polynomial $R^2 + \lambda$ has the roots:

(a) If $\lambda = -k^2$, then $R = \pm k$, $k > 0$.

(b) If $\lambda = 0$, then $R = 0$ is a double root.

(c) If $\lambda = k^2$, then $R = \pm ik$, $k > 0$.

We treat each of the three cases separately.

(a) If $\lambda = -k^2$, $k > 0$, then the complete solution is

$$y = c_1 \sinh(kx) + c_2 \cosh(kx).$$

It follows from the boundary condition $y(0) = 0$ that $c_2 = 0$, hence

$$y = c_1 \sinh(kx) \quad \text{where} \quad y'(x) = c_1 k \cosh(kx).$$

Applying the boundary condition $y'(L) = 0$ we get $c_1 k = 0$, so $c_1 = 0$. The zero solution is the only solution, and no $\lambda = -k^2 < 0$ is an eigenvalue.

(b) If $\lambda = 0$, then the complete solution is

$$y = c_1 x + c_2 \qquad \text{where} \quad y'(x) = c_1.$$

It follows from the boundary conditions that

$$y(0) = c_2 = 0 \qquad \text{og} \qquad y'(L) = c_1 = 0,$$

and again we only get the zero solution, so $\lambda = 0$ is not an eigenvalue.

(c) If $\lambda = k^2$, $k > 0$, then the complete solution is

$$y(x) = c_1 \sin(kx) + c_2 \cos(kx).$$

Using the boundary condition $y(0) = c_2 = 0$ we see that the candidates should be searched among

$$y(x) = c_1 \sin(kx) \qquad \text{where} \quad y'(x) = c_1 \cdot k \cos(kx).$$

It follows from the latter boundary condition that

$$y'(L) = 0 = c_1 k \cdot \cos(kL).$$

We find proper solutions, when $\cos(kL) = 0$, i.e. when

$$k_n L = \frac{\pi}{2} + n\pi, \qquad n \in \mathbb{N}_0,$$

so the eigenvalues are

$$\lambda_n = k_n^2 = \frac{1}{L^2}\left(\frac{\pi}{2} + n\pi\right)^2 = \frac{\pi^2 (2n+1)^2}{4L^2}, \quad n \in \mathbb{N}_0,$$

and a generating eigenfunction is

$$y_n(x) = \sin(k_n x) = \sin\left((2n+1)\frac{\pi}{2}x\right).$$

Example 2.2 *Solve the following eigenvalue problem*

$$y'' + \lambda y = 0, \qquad x \in [0,1], \quad y(0) = y'(1) - \lambda y'(0) = 0.$$

The characteristic polynomial is $R^2 + \lambda$.

1) If $\lambda = -k^2 < 0$, $k > 0$, then the characteristic polynomial has the two real roots $R = \pm k$, and the complete solution is

$$y(x) = c_1 \sinh(kx) + c_2 \cosh(kx).$$

It follows immediately from the boundary condition $y(0) = 0$ that $c_2 = 0$, so the set of candidates is limited to

$$y(x) = c_1 \sinh(kx) \quad \text{where} \quad y'(x) = c_1 k \cdot \cosh(kx).$$

By insertion into the boundary condition

$$y'(1) - \lambda y'(0) = y'(1) + k^2 y'(0) = 0$$

we get

$$0 = c_1 k\{\cosh(k) + k^2\},$$

hence $c_1 = 0$, and we only get the zero solution, hence no $\lambda = -k^2 < 0$ is an eigenvalue.

2) If $\lambda = 0$, then the complete solution is

$$y(x) = c_1 x + c_2.$$

It follows from the boundary conditions that

$$y(0) = c_2 = 0 \quad \text{and} \quad y'(1) - 0 \cdot y'(0) = c_1 = 0,$$

and we obtain again only the zero solution, so $\lambda = 0$ is not an eigenvalue.

3) If $\lambda = k^2 > 0$, $k > 0$, the the characteristic equation $R^2 + k^2 = 0$ has the two complex solutions $R = \pm ik$. The complete solution is

$$y(x) = c_1 \sin(kx) + c_2 \cos(kx).$$

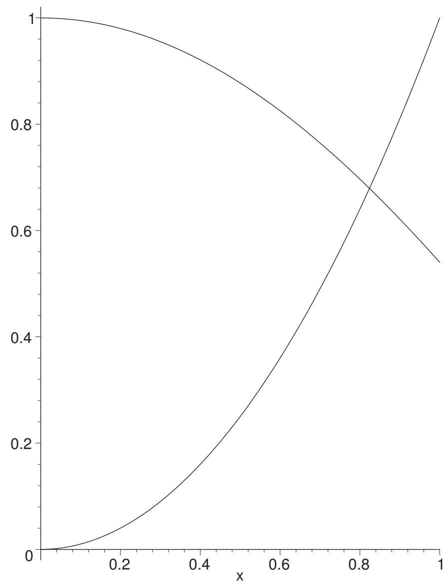
The boundary condition $y(0) = 0$ implies that $c_2 = 0$, so the set of candidates shall be found among

$$y(x) = c_1 \sin(kx) \quad \text{where} \quad y'(x) = c_1 k \cos(kx).$$

By insertion into the second boundary condition we get

$$0 = y'(1) - \lambda y'(0) = y'(1) - k^2 y'(0) = c_1 k\{\cos(k) - k^2\}.$$

We get proper solutions, when $\cos(k) = k^2$. By considering a graph we see that there is precisely one solution $k > 0$, namely $k \approx 0,824$.



More explicitly we apply the Newton-Raphson iteration formula on the equation

$$F(k) = k^2 - \cos k \quad \text{where} \quad F'(k) = 2k + \sin k.$$

The iteration formula is

$$k_{n+1} = k_n - \frac{F(k_n)}{F'(k_n)} = k_n - \frac{k_n^2 - \cos k_n}{2k_n + \sin k_n}.$$

By putting $k_1 = 1$ we get

$$k_2 = 0,838218, \quad k_3 = 0,824242, \quad k_4 = k_5 = 0,824132,$$

corresponding to the eigenvalue

$$\lambda = k^2 \approx 0,679194,$$

and a generating eigenfunction is

$$y_0(x) = \sin(kx) \approx \sin(0,824x).$$

Example 2.3 *Solve the following eigenvalue problem*

$$y'' + \lambda y' = 0, \quad x \in [0, L], \quad y(0) = y(L) = 0.$$

The characteristic polynomial

$$R^2 + \lambda R = R(R + \lambda)$$

has the roots $R = 0$ and $R = -\lambda$.

1) If $\lambda = 0$, then $R = 0$ is a double root, and the complete solution is

$$y(x) = c_1 x + c_2.$$

It follows from $y(0) = 0 = c_2$ that the candidates are limited to $y = c_1 x$. However, since $y(L) = c_1 L = 0$ implies $c_1 = 0$, we only get the zero solution, and $\lambda = 0$ is not an eigenvalue.

2) If $\lambda \neq 0$, then the complete solution is

$$y(x) = c_1 \exp(-\lambda x) + c_2.$$

It follows from the boundary conditions that

$$\begin{aligned} y(0) = c_1 + c_2 = 0, &\quad \text{thus} \quad c_2 = -c_1, \\ y(L) = c_1 \exp(-\lambda L) + c_2 = 0, &\quad \text{thus} \quad c_1\{\exp(-\lambda L) - 1\} = 0. \end{aligned}$$

Since $\exp(-\lambda L) \neq 1$, we have $c_1 = 0$, which implies that $c_2 = 0$. Again, we only obtain the zero solution, hence no $\lambda \neq 0$ is an eigenvalue.

Summing up we see that the eigenvalue problem does not have any eigenvalue.

Example 2.4 *Solve the following eigenvalue problem*

$$y^{(4)} + \lambda y^{(2)} = 0, \quad x \in [0, 1], \quad y''(0) = y'''(0) = y''(1) = y'''(1) = 0.$$

The characteristic polynomial is $R^4 + \lambda R^2 = R^2(R^2 + \lambda)$.

1) If $\lambda = -k^2 < 0$, $k > 0$, then $R = 0$ is a double root, and we have furthermore two simple, real roots $R = \pm k$. The complete solution is

$$y(x) = c_1 + c_2 x + c_3 \cosh(kx) + c_4 \sinh(kx).$$

Since the terms $c_1 + c_2 x$ disappear after at least two differentiations, every $\lambda \in \mathbb{R}$ is an eigenvalue, and $c_1 + c_2 x$ is the corresponding eigenfunction.

We shall then check if there are other eigenfunctions. We first calculate

$$y''(x) = c_3 k^2 \cosh(kx) + c_4 k^2 \sinh(kx),$$

$$y'''(x) = c_3 k^3 \sinh(kx) + c_4 k^3 \cosh(kx).$$

It follows from the first condition $y''(0) = 0$ that $c_3 = 0$. Then it follows from the second condition $y'''(0) = 0$ that $c_4 = 0$.

If $\lambda = -k^2 < 0$, $k > 0$, then λ is an eigenvalue with the eigenfunctions

$$y(x) = c_1 + c_2 x, \qquad c_1, c_2 \text{ arbitrary.}$$

2) If $\lambda = 0$, then $R = 0$ is a multiple root of multiplicity four. The complete solution is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^4$$

where

$$y''(x) = 2c_3 + 6c_4 x, \quad \text{and} \quad y'''(x) = 6c_4.$$

We conclude as above that $\lambda = 0$ is an eigenvalue with the corresponding eigenfunctions $c_1 + c_2 x$. There are no other eigenfunctions, because

$$y''(0) = 2c_3 = 0 \quad \text{and} \quad y'''(0) = 6c_4 = 0$$

imply that $c_3 = c_4 = 0$.

3) If $\lambda = k^2$, then $R = 0$ is a double root, and $R = \pm ik$ are simple, complex conjugated roots. The complete solution is

$$y(x) = c_1 + c_2 x + c_3 \sin(kx) + c_4 \cos(kx).$$

We conclude as above that $c_1 + c_2 x$ are eigenfunctions for every such $\lambda = k^2$.

We shall now check if there exist other eigenfunctions. We first calculate

$$y''(x) = -c_3 k^2 \sin(kx) - c_4 k^2 \cos(kx).$$

It follows from $y''(0) = 0$ that $c_4 = 0$, so only $y''(x) = -c_3 k^2 \sin(kx)$ is relevant where

$$y'''(0) = -c_3 k^3 \cos(kx).$$

It follows from $y'''(0) = -c_3 k^3 = 0$ that $c_3 = 0$, hence the only eigenfunctions are $c_1 + c_2 x$.

Summing up we see that every $\lambda \in \mathbb{R}$ is an eigenvalue with

$$y(x) = c_1 + c_2 x, \qquad c_1, c_2 \text{ arbitrary},$$

as the corresponding eigenfunctions.

Example 2.5 *Solve the following eigenvalue problem*

$$y^{(4)} + \lambda y^{(2)} = 0, \quad x \in [0, L], \quad y'(0) = y'''(0) = y'(L) = y'''(L) = 0.$$

The characteristic polynomial is $R^2(R^2 + \lambda)$.

1) If $\lambda = -k^2 < 0$, $k > 0$, then $R = 0$ is a double root, and $R = \pm k$ are simple, real roots. The complete solution is

$$y(x) = c_1 + c_2 x + c_3 \cosh(kx) + c_4 \sinh(kx).$$

Clearly, the constants $y(x) = c_1$ are always eigenfunctions, hence $\lambda = -k^2$, $k > 0$ is always an eigenvalue.

We shall now check if there exist other eigenfunctions. We first calculate

$$y'(x) = c_2 + c_3 k \sinh(kx) + c_4 k \cosh(kx)$$

and

$$y'''(x) = c_3 k^3 \sinh(kx) + c_4 k^3 \cosh(kx).$$

It follows from the former two boundary conditions that

$$0 = y'(0) = c_2 + c_4 k \quad \text{and} \quad 0 = y'''(0) = c_4 k^3,$$

hence $c_4 = 0$ and thus $c_2 = 0$. This reduces the set of possible candidates to

$$y(x) = c_1 + c_3 \cosh(kx)$$

where

$$y'(x) = c_3 k \sinh(kx) \quad \text{and} \quad y'''(x) = c_3 k^3 \sinh(kx).$$

It follows from the next boundary condition that $y'(L) = c_3 k \sinh(kL) = 0$, hence $c_3 = 0$.

Every $\lambda < 0$ is an eigenvalue with $y_0(x) = 1$ as the corresponding generating eigenfunction.

2) If $\lambda = 0$, then $R = 0$ is a root of multiplicity four. The complete solution is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

where

$$y'(x) = c_2 + 2c_3 x + 3c_4 x^2 \quad \text{and} \quad y'''(x) = 6c_4.$$

It is immediately seen that $y_0(0) = 1$ is a generating eigenfunction, so $\lambda = 0$ is an eigenvalue.

We shall now check if there are other eigenfunctions. We get by insertion into the first two boundary conditions that

$$y'(0) = c_2 = 0 \quad \text{and} \quad y'''(0) = 6c_4 = 0, \quad \text{hence } c_2 = c_4 = 0.$$

Finally, $y'(L) = 2c_3 L = 0$, so $c_3 = 0$.

Summing up we see that there do not exist any other eigenfunctions than the constants.

3) If $\lambda = k^2 > 0$, $k > 0$, then $R = 0$ is a double root, and $R = \pm ik$ are simple, complex conjugated roots. The complete solution is

$$y(x) = c_1 + c_2 x + c_3 \sin(kx) + c_4 \cos(kx).$$

It follows again that the constants are eigenfunctions. Then we check if there are other eigenfunctions. We first calculate

$$y'(x) = c_2 + c_3 k \cos(ks) - c_4 k \sin(kx)$$

and

$$y'''(x) = -c_3 k^3 \cos(kx) + c_4 k^3 \sin(kx).$$

We get from the first two boundary conditions that

$$y'(0) = c_2 + c_3 k = 0 \quad \text{and} \quad y'''(0) = -c_3 k^3 = 0,$$

hence $c_3 = 0$, and thus $c_2 = 0$.

It remains to consider

$$y(x) = c_1 + c_4 \cos(kx)$$

where

$$y'(x) = -c_4 k \sin(kx) \quad \text{and} \quad y'''(x) = c_4 k^3 \sin(kx).$$

The latter two boundary conditions, $y'(L) = y'''(L) = 0$, will both give us the condition

$$\sin(kL) = 0, \qquad \text{thus } k_n L = n\pi, \quad n \in \mathbb{N}.$$

For the particular eigenvalues

$$\lambda_n = k_n^2 = \left(\frac{n\pi}{L}\right)^2, \qquad n \in \mathbb{N},$$

we also get the eigenfunctions

$$y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad n \in \mathbb{N}.$$

Summing up we see that every $\lambda \in \mathbb{R}$ is an eigenvalue with the corresponding generation eigenfunction $y_0(x) = 1$.

Furthermore, when $\lambda_n = (n\pi/L)^2$, $n \in \mathbb{N}$, we get the generating eigenfunctions

$$y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad n \in \mathbb{N}.$$

Example 2.6 *Solve the following eigenvalue problem*

$$y^{(4)} + \lambda y^{(2)} = 0, \quad x \in [0,1], \quad y(0) = y'(0) = y''(0) = y(1) = 0.$$

The characteristic polynomial is $R^4 + \lambda R^2 = R^2(R^2 + \lambda)$.

1) If $\lambda = -k^2 < 0$, $k > 0$, then $R = 0$ is a double root, and $R = \pm k$ are two real simple roots. The complete solution is

$$y(x) = c_1 + c_2 x + c_3 \sinh(kx) + c_4 \cosh(kx)$$

where

$$y'(x) = c_2 + c_3 k \cosh(kx) + c_4 k \sinh(kx),$$

$$y''(x) = c_3k^2 \sinh(kx) + c_4k^2 \cosh(kx).$$

By inspection we see that we should start with the boundary condition

$$y''(0) = 0 = c_4k^2, \qquad \text{thus } c_4 = 0.$$

Then we get

$$y(0) = 0 = c_1 + c_4 = c_1, \qquad \text{i.e. } c_1 = 0.$$

We have furthermore

$$y'(0) = 0 = c_2 + c_3k, \qquad \text{i.e. } c_2 = -kc_3,$$

so the candidates must necessarily have the structure

$$y(x) = c_3\{-kx + \sinh(kx)\}.$$

Then the latter boundary condition gives

$$y(1) = 0 = c_3\{\sinh(k) - k\}, \qquad k > 0.$$

The function $\varphi(t) = \sinh(t) - t$ is strictly increasing for $t > 0$ (because $\varphi'(t) = \cosh t - 1 > 0$), and $\varphi(0) = 0$, so $\sinh(k) - k > 0$, and $c_3 = 0$. Hence we only get the zero solution, and we conclude that no $\lambda < 0$ can be an eigenvalue.

2) If $\lambda = 0$, then the root $R = 0$ has multiplicity four. The complete solution becomes

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

where

$$y'(x) = c_2 + 2c_3 x + 3c_4 x^2 \quad \text{og} \quad y''(x) = 2c_3 + 6c_4 x.$$

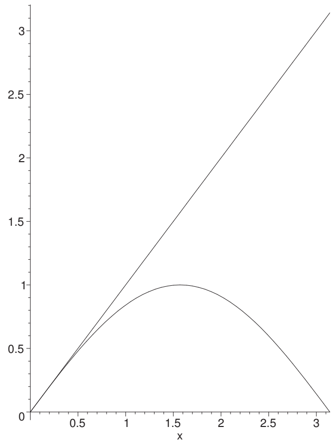
It follows from $y(0) = 0$ that $c_1 = 0$.
It follows from $y'(0) = 0$ that $c_2 = 0$.
It follows from $y''(0) = 0$ that $c_3 = 0$.
Since $c_1 = c_2 = c_3 = 0$, we also get $y(1) = c_4 = 0$, and the zero solution is the only solution. Therefore we conclude that $\lambda = 0$ is not an eigenvalue.

3) If $\lambda = k^2 > 0$, $k > 0$, then the root $R = 0$ has multiplicity two, and we have furthermore the two simple and complex conjugated roots $R = \pm ik$, $k > 0$. The complete solution is

$$y(x) = c_1 + c_2 x + c_3 \sin(kx) + c_4 \cos(kx)$$

where

$$y'(x) = c_2 + c_3 k \cos(kx) - c_4 k \sin(kx),$$
$$y''(x) = -c_3 k^2 \sin(kx) - c_4 k^2 \cos(kx).$$



We have concerning the boundary conditions:
It follows from $y''(0) = -c_4 k^2 = 0$ that $c_4 = 0$.
It follows from $y(0) = 0 = c_1 + c_4 = c_1$ that $c_1 = 0$.
It follows from $y'(0) = 0 = c_2 + c_3 k$ that $c_2 = -c_3 k$.
The possible candidates then necessarily have the structure

$$y(x) = c_3\{-kx + \sin(kx)\}$$

We conclude from $y(1) = \{-k + \sin k\} c_3 = 0$ by considering a graph that $-k + \sin k < 0$, så $c_3 = 0$. Again we only obtain the zero solution.

Summing up it follows that no $\lambda \in \mathbb{R}$ is an eigenvalue.

Example 2.7 *Solve the following eigenvalue problem*

$$y'' + \lambda y = 0, \quad x \in [0,1], \quad y(0) - y(1) = 0, \quad y'(0) + y'(1) = 0.$$

The characteristic polynomial is $R^2 + \lambda$.

1) If $\lambda = -k^2 > 0$, $k > 0$, then the complete solution is

$$y(x) = c_1 \cosh(kx) + c_2 \sinh(kx)$$

where

$$y'(x) = c_1 k \sinh(kx) + c_2 \cosh(kx).$$

It follows from the boundary conditions that

$$y(0) - y(1) = c_1\{1 - \cosh(k)\} - c_2 \sinh(k) = 0,$$

$$y'(0) + y'(1) = c_1 k \sinh(k) + c_2 k\{1 + \cosh(k)\} = 0,$$

hence on matrix form

$$\begin{pmatrix} 1 - \cosh(k) & -\sinh(k) \\ k\sinh(k) & k\{1 + \cosh(k)\} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows from

$$\det \mathbf{B} = \begin{vmatrix} 1 - \cosh(k) & -\sinh(k) \\ k\sinh(k) & k\{1 + \cosh(k)\} \end{vmatrix} = k\{1 - \cosh^2(k) + \sinh^2(k)\} = 0,$$

that there exist proper solutions $(c_1, c_2) \neq (0,0)$, e.g.

$$c_1 = \sinh(k) \qquad \text{and} \qquad c_2 = 1 - \cosh(k).$$

Every $\lambda = -k^2 < 0$, $k > 0$, is an eigenvalue and the corresponding generating eigenfunction is

$$y_k(x) = \sinh(k)\cosh(kx) + (1 - \cosh(k))\sinh(kx) = \sinh(k\{1 - x\}) + \sinh(kx).$$

2) If $\lambda = 0$, then the root $R = 0$ has multiplicity 2 and the complete solution is

$$y(x) = c_1 x + c_2 \qquad \text{where} \qquad y'(x) = c_1.$$

It follows from the boundary values that

$$y(0) - y(1) = -c_1 = 0 \quad \text{and} \quad y'(0) + y'(1) = 2c_1 = 0,$$

hence $c_1 = 0$, and c_2 can be chosen arbitrarily.

We conclude that $\lambda = 0$ is an eigenvalue and that we can choose the generating eigenfunction $y_0(x) = 1$.

3) If $\lambda = k^2 > 0$, $k > 0$, then the complete solution is

$$y(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

where

$$y'(x) = -c_1 k \sin(kx) + c_2 k \cos(kx).$$

It follows from the boundary conditions that

$$y(0) - y(1) = c_1(1 - \cos k) - c_2 \sin k = 0,$$

$$y'(0) + y'(1) = -c_1 k \sin k + c_2 k(1 + \cos k) = 0,$$

hence written in the form of a matrix,

$$\begin{pmatrix} 1 - \cos k & -\sin k \\ -k \sin k & k(1 + \cos k) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows from

$$\det \mathbf{B} = \begin{vmatrix} 1 - \cos k & -\sin k \\ -k \sin k & k(1 + \cos k) \end{vmatrix} = k(1 - \cos^2 k - \sin^2 k) = 0,$$

that we have proper solutions $(c_1, c_2) \neq (0, 0)$, e.g.

$$c_1 = \sin k \quad \text{and} \quad c_2 = 1 - \cos k, \quad \text{for } k \neq 2n\pi, \quad n \in \mathbb{N}.$$

Every $\lambda = k^2 > 0$, $k > 0$ is an eigenvalue and a corresponding eigenfunction can be chosen as

$$y_k(x) = \sin k \cdot \cos(kx) + (1 - \cos k)\sin(kx) = \sin(k\{1 - x\}) + \sin(kx), \qquad \text{for } k \neq n\pi,$$

and

$$y_{n,0}(x) = \cos(2n\pi x) \qquad \text{for } k = 2n\pi,$$

and

$$y_{n,1}(x) = \sin(2n + 1)\pi x \qquad \text{for } k = (2n + 1)\pi.$$

Example 2.8 *Consider the problem of the column from Example 1.3 and put $e = 0$. Solve this eigenvalue problem and sketch $y(L)$ as a function of $kL = \sqrt{P/(EI)} \cdot L$.*

When we put $e = 0$ in Example 1.3, then

$$EI \frac{d^2y}{dx^2} + Py = 0, \quad x \in [0,1], \quad y(0) = 0, \quad y'(L) = 0.$$

Write $P/(EI) = k^2$. Then we get by a division by EI,

$$\frac{d^2y}{dx^2} + k^2 y = 0, \quad x \in [0,L], \quad y(0) = 0, \quad y'(L) = 0.$$

The complete solution is

$$y(x) = c_1 \cos kx + c_2 \sin kx$$

where

$$y'(x) = -c_1 k \sin kx + c_2 k \cos kx.$$

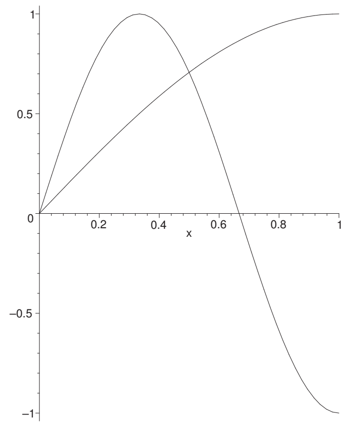
It follows from the boundary conditions that

$$y(0) = 0 = c_1,$$

$$y'(L) = 0 = -c_1 k \sin kL + c_2 k \cos kL = c_2 k \cos kL.$$

Clearly, $c_1 = 0$, so we only obtain proper solutions $y(x) = c_2 \sin kx$, if

$$\cos kL = 0, \qquad \text{thus} \qquad k_n L = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{N}_0.$$



the eigenvalues are

$$\lambda_n = k_n^2 = \frac{\pi^2(2n+1)^2}{4L^2}, \qquad n \in \mathbb{N}_0,$$

and a corresponding eigenfunction may be chosen as

$$y_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right), \qquad n \in \mathbb{N}_0.$$

There are infinitely many eigenfunctions $c \cdot y_n(x)$, $c \in \mathbb{R} \setminus \{0\}$, of which $y_0(x)$ and $y_1(x)$ are sketched on the figure.

Example 2.9 *Consider the eigenvalue problem*

$$y'' + \lambda y = 0, \quad x \in [0,1], \quad y'(0) = 0, \quad y(1) + y'(1) = 0.$$

1) Prove that we have separated (Sturm) boundary conditions.

2) Prove that $\lambda < 0$ and $\lambda = 0$ cannot be eigenvalues.

3) Find an equation which an eigenvalue λ must fulfil. The put $\lambda = \alpha^2$, and sketch the roots (α_n) of the equation above and find the corresponding eigenfunctions (y_n).

4) Explain that all of the conclusions of the eigenvalue theorem (Sturm's oscillation theorem) are fulfilled.

1) If we write the equations of the boundary values as

$$\begin{cases} 0 \cdot y(0) + 1 \cdot y'(0) = 0 \\ \\ 1 \cdot y(1) + 1 \cdot y'(1) = 0, \end{cases},$$

we see that we have separated (Sturm) boundary conditions. Since $r(x) = 1$, we even have a regular Sturm-Liouville problem.

2) If $\lambda = -\alpha^2$, $\alpha < 0$, then the complete solution is

$$y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$

where

$$y'(x) = c_1 \alpha \sinh(\alpha x) + c_2 \alpha \cosh(\alpha x).$$

It follows from the former boundary condition that

$$y'(0) = c_2 \alpha = 0, \qquad \text{dvs. } c_2 = 0.$$

Hence we only need to consider the candidates

$$y(x) = c_1 \cosh(\alpha x) \quad \text{med} \quad y'(x) = c_1 \alpha \sinh(\alpha x).$$

Then by the latter boundary condition,

$$y(1) + y'(1) = c_1\{\cosh(\alpha) + \alpha \sinh(\alpha)\} = 0.$$

Since $\cosh(\alpha) + \alpha \sinh(\alpha) > 0$ for $\alpha > 0$, we must have $c_1 = 0$, so we only obtain the zero solution, thus no $\lambda < 0$ can be an eigenvalue.

If $\lambda = 0$, then the equation is reduced to $y'' = 0$, so the complete solution is

$$y(x) = c_1 x + c_2, \qquad \text{with} \qquad y'(x) = c_1.$$

It follows from $y'(0) = 0$ that $c_1 = 0$, and $y(x) = c_2$ must be a constant. Then

$$y(1) + y'(1) = c_2 + 0 = c_2 = 0,$$

and we also here only get the zero solution. Thus $\lambda = 0$ cannot be an eigenvalue either.

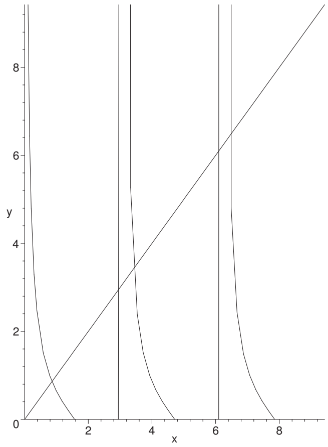
3) Finally, if $\lambda = \alpha^2$, $\alpha > 0$, then the complete solution is

$$y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

where

$$y'(x) = -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x),$$

It follows from $y'(0) = c_2\alpha = 0$ that $c_2 = 0$, so the candidates must necessarily fulfil



$$y(x) = c_1 \cos(\alpha x) \qquad \text{where} \qquad y'(x) = -c_1 \alpha \sin(\alpha x).$$

Then by the second boundary condition,

$$y(1) + y'(1) = c_1\{\cos\alpha - \alpha \sin\alpha\} = 0.$$

We get proper solutions $c_1 \neq 0$ when

$$\cos\alpha = \alpha \sin\alpha, \qquad \alpha > 0.$$

Since $\cos\alpha \neq 0$ for every solution, this is also written

$$\cot\alpha = \alpha, \qquad \alpha > 0,$$

which is easily solved graphically.

It follows that there exists precisely one root α_n in every interval $]n\pi, (n+1)\pi[$, $n \in \mathbb{N}$, and that

$$\alpha_n \approx n\pi \qquad \text{for large } n \in \mathbb{N},$$

or more precisely,

$$\alpha_n = n\pi + \varepsilon(n), \qquad n \in \mathbb{N},$$

where $\varepsilon(0) \in \left]0, \frac{\pi}{2}\right[$, and $\varepsilon(n) \to 0$ decreasingly.

A corresponding generating eigenfunction is e.g.

$$y_n(x) = \cos(\alpha_n x), \qquad n \in \mathbb{N}_0.$$

4) Finally, we shall check the conclusions of Sturm's oscillation theorem.

a) Since $\lambda_n = \alpha_n^2$, $n \in \mathbb{N}_0$, we clearly have

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots ,$$

and $\lambda_n \to \infty$ for $n \to \infty$.

b) To every eigenvalue λ_n there corresponds (modulo an arbitrary constant factor) precisely one eigenfunction,

$$y_n(x) = \cos(\alpha_n x) = \cos(\sqrt{\lambda_n} x).$$

c) Since $n\pi < \alpha_n < \left(n + \frac{1}{2}\right)\pi$, $n \in \mathbb{N}$, we see that $\psi_n(x) = \alpha_n x$ satisfies

$$[0, n\pi] \subset \psi([0,1]) \subset \left[0, \left(n + \frac{1}{2}\right)\pi\right[, \qquad n \in \mathbb{N}.$$

Since $\cos t$ has precisely n zeros in $[0, n\pi]$ and $\left[0, \left(n + \frac{1}{2}\right)\pi\right[$, the function $y_n(x) = \cos(\alpha_n x)$ must have precisely n zeros in $[0, 1]$, and it is obvious that $y_n(x)$ changes its sign whenever we cross a zero.

Remark 2.1 We can now find the values of the first α_n. However, this cannot be done by a direct application of either Newton-Raphson's iteration formula or Banach's fixpoint theorem, because they cannot be used on

$$\cos\alpha - \alpha\sin\alpha = 0 \qquad \text{eller} \qquad \cot\alpha = \alpha.$$

Instead one may use the alternative form

$$\alpha_n = n\pi + \text{Arccot }\alpha_n, \qquad n \in \mathbb{N}_0,$$

and then the methods mentioned can be applied.

Example 2.10 *Consider the eigenvalue problem*

$$y'' + \lambda y = 0, \quad x \in [0,1], \quad y(0) = 0, \quad y(1) = y'(1).$$

1) Prove that we have no negative eigenvalues.

2) Prove that $\lambda = 0$ is an eigenvalue and find a corresponding eigenfunction.

3) Prove that the remaining eigenfunctions are given by $y_n(x) = \sin\alpha_n x$, where α_n is the n-th positive root of the equation $\tan z = z$. Sketch the roots.

1) Put $\lambda = -k^2 < 0$, where $k > 0$.

- **The complete solution**.
  The characteristic equation

  $$R^2 + \lambda = R^2 - k^2 = (R-k)(R+k) = 0$$

  has the solutions $R = \pm k$, and the differential equation is homogeneous, so the complete solution is

  $$y(x) = c_1\cosh(kx) + c_2\sinh(kx)$$

  where

  $$y'(x) = c_1 k\sinh(kx) + c_2 k\cosh(kx).$$

  Remark 2.2 Masochists would probably here choose the variant

  $$y = \tilde{c}_1 e^{kx} + \tilde{c}_2 e^{-kx} \quad \text{med } y' = \tilde{c}_1 k e^{kx} - \tilde{c}_2 k e^{-kx}.$$

  This variant will of course give the same result after much bigger calculations.

- **Insert into the boundary conditions.**
  It follows from the first boundary condition that

  $$y(0) = 0 = c_1 \qquad [\text{possibly} \quad 0 = \tilde{c}_1 + \tilde{c}_2].$$

  The candidates must then necessarily satisfy

  $$y(x) = c_2\sinh(kx), \quad \text{where } y'(x) = c_2 k\cosh(kx).$$

Then we get from the second boundary condition,

$$y(1) = c_2 \sinh(k) = y'(1) = c_2 k \cdot \cosh(k),$$

hence by a rearrangement,

$$c_2\{\sinh(k) - k\cosh(k)\} = 0.$$

If there are proper solutions (i.e. $c_2 \neq 0$), then

(1) $\sinh(k) - k\cosh(k) = 0.$

The function

$$\varphi(t) = \sinh(t) - t\cosh(t)$$

has the derivative

$$\varphi'(t) = -t\sinh t < 0 \qquad \text{for } t > 0,$$

so $\varphi(t)$ is decreasing! Now, $\varphi(0) = 0$, so

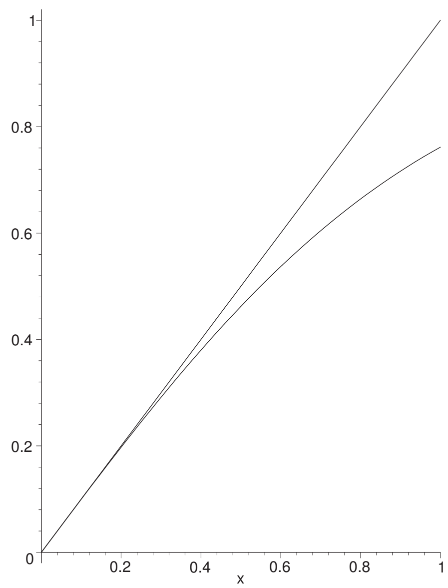
$$\sinh(k) - k\cosh(k) < 0 \quad \text{for alle } k > 0,$$

and we only get the solution $c_2 = 0$. Thus, no $\lambda < 0$ can be an eigenvalue.

Alternatively we see that (1) is equivalent to

$$\tanh(k) = k,$$

where a graphical analysis shows that $k = 0$ is the only solution.



2) Let $\lambda = 0$, so the equation is reduced to $\dfrac{d^2y}{dx^2} = 0$.

- **The complete solution** follows by two integrations,

$$y = c_1 x + c_2 \qquad \text{where} \qquad y' = c_1.$$

- **Insertion into the boundary conditions**:

$$y(0) = 0 = c_2, \qquad \text{thus } y = c_1 x \text{ where } y' = c_1.$$

The latter boundary condition is now trivial,

$$y(1) = c_1 = y'(1).$$

- **The complete set of eigenfunctions** is

$$y(x) = c_1 x, \qquad x \in [0,1], \quad c_1 \text{ arbitrary}.$$

3) Let $\lambda = k^2 > 0$, $k > 0$.

- **The complete solution** of

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

is

$$y(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

where

$$y'(x) = -c_1 k \sin(kx) + c_2 k \cos(kx).$$

- **Insertion into the boundary conditions**.
It follows from the first boundary condition that

$$y(0) = 0 = c_1,$$

so we must necessarily have

$$y(x) = c_2 \sin(kx) \quad \text{where} \quad y'(x) = c_2 k \cos(kx).$$

It follows from the second boundary condition $y(1) = y'(1)$ that

$$c_2 \sin(k) = c_2 k \cos(k).$$

We only obtain proper solutions, $c_2 \neq 0$, if

$$F(k) = \sin(k) - k\cos(k) = 0, \quad \text{thus } \tan(k) = k.$$

By a graphical consideration we see that there is no solution in $\left]0, \frac{\pi}{2}\right[$, and that there is precisely one solution $\alpha_n \in \left]n\pi, n\pi + \frac{\pi}{2}\right[$, $n \in \mathbb{N}$, where it follows from the geometry that

$$\left(n\pi + \frac{\pi}{2}\right) - \alpha_n \to 0 \qquad \text{for } n \to \infty.$$

- Since $c_1 = 0$, we find the **eigenvalues** $\lambda_n = \alpha_n^2$ with the **generating eigenfunctions**

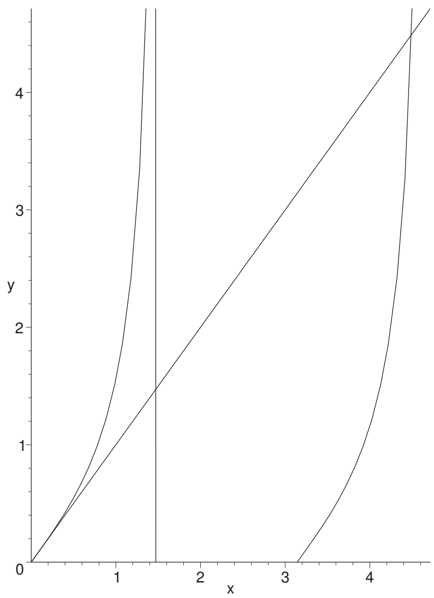
$$y_n(x) = \sin(\alpha_n x), \quad x \in [0,1] \text{ og } n \in \mathbb{N}.$$

Remark 2.3 The zeros of $F(z) = \sin z - z\cos z$ can be found very fast by a Newton-Raphson iteration. In fact, since

$$F'(z) = z \sin z,$$

we get the iteration scheme

$$z_{n+1} = z_n - \frac{F(z_n)}{F'(z_n)} = z_n + \frac{z_n \cos z_n - \sin z_n}{z_n \sin z_n} = z_n + \cot z_n - \frac{1}{z_n}.$$



Using the initial values $z_0^{(0)} = \dfrac{\pi}{2} + n\pi$ we get for the first zeros,

n	1	2	3	4
$z_0^{(n)}$	$\dfrac{3\pi}{2} = 4{,}71239$	$\dfrac{5\pi}{2} = 7{,}85398$	$\dfrac{7\pi}{2} = 10{,}99557$	$\dfrac{9\pi}{2} = 14{,}13717$
$z_1^{(n)}$	4,50018	7,72666	10,90463	14,06643
$z_2^{(n)}$	4,49342	7,72525	10,90412	14,06619
$z_3^{(n)}$	4,49341	7,72525	10,90412	14,06619
α_n	4,49341	7,72525	10,90412	14,06619

Example 2.11 *Consider the eigenvalue problem*

$$y'' + 2y' + \lambda y = 0, \quad x \in [0,1], \quad y(0) = y(1) = 0.$$

1) Prove that $\lambda = 1$ is not an eigenvalue.

2) Prove that there does not exist any eigenvalue $\lambda < 1$.

3) Prove that the n-th positive eigenvalue is $\lambda_n = n^2\pi^2 + 1$, and find a corresponding eigenfunction.

1) Let $\lambda = 1$. Then the characteristic equation is

$$R^2 + 2R + 1 = (R+1)^2 = 0.$$

- Since $R = -1$ is a double root, the **complete solution** is

$$y(x) = c_1 x e^{-x} + c_2 e^{-x}.$$

- **Insertion into the boundary conditions**:
  It follows from the first boundary condition that

  $$y(0) = 0 = c_2,$$

  hence we shall only look for candidates of the structure $y = c_1 x e^{-x}$.
  It follows from the second boundary condition that

  $$y(1) = 0 = c_1 \cdot 1 \cdot e^{-1}, \qquad \text{thus } c_1 = 0.$$

  Since $(c_1, c_2) = (0, 0)$ is the only solution, we conclude that $\lambda = 1$ is not an eigenvalue.

2) Assume that $\lambda = 1 - k^2 < 1$, $k > 0$.

- **The complete solution**:
  The characteristic equation

  $$R^2 + 2R + 1 - k^2 = (R+1)^2 - k^2 = 0$$

  has the two simple roots $R = -1 \pm k$.
  The complete solution is

  $$y(x) = c_1 e^{-x} \cosh(kx) + c_2 e^{-x} \sinh(kx).$$

- **Insertion into the boundary conditions**:
  It follows from $y(0) = 0$ that

  $$y(0) = c_1 = 0.$$

  Then we shall only look for candidates of the form $y(x) = c_2 e^{-x} \sinh(kx)$. Then it follows from $y(1) = 0$ that

  $$y(1) = 0 = c_2 e^{-1} \sinh(k).$$

  Now, $e^{-1} \sinh(k) > 0$ for $k > 0$, so $c_2 = 0$ is the only solution. Hence, no $\lambda < 1$ is an eigenvalue.

3) Assume that $\lambda = 1 + k^2 > 1$, $k > 0$.

- **The complete solution**.
  The characteristic equation

  $$R^2 + 2R + 1 + k^2 = (R+1)^2 + k^2 = 0$$

  has the two simple roots $R = -1 \pm ik$, so the complete solution is

  $$y(x) = c_1 e^{-x} \cos(kx) + c_2 e^{-x} \sin(kx).$$

- **Insertion into the boundary conditions**.
  We get from $y(0) = 0$ that $c_1 = 0$, so we need only in the following to consider functions of the form

  $$(2) \quad y(x) = c_2 e^{-x} \sin(kx).$$

  We get from $y(1) = 0$ that

  $$c_2 \cdot \frac{1}{e} \cdot \sin k = 0.$$

  We get proper solutions $c_2 \neq 0$, when $k_n = n\pi$, $n \in \mathbb{N}$.

- **Eigenvalues and eigenfunctions**.
  The eigenvalues are $\lambda_n = 1 + k_n^2 = n^2\pi^2 + 1$, $n \in \mathbb{N}$.
  A corresponding eigenfunction is by (2) given by

$$y_n(x) = e^{-x}\sin(n\pi x), \qquad x \in [0,1],$$

and all eigenfunctions corresponding to λ_n are given by $c \cdot y_n(x)$, where $c \neq 0$ is an arbitrary constant.

Example 2.12 *Consider the eigenvalue problem*

$$y'' + \lambda y = 0, \quad x \in [0,1], \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

1) Prove that we do not have negative eigenvalues.

2) Prove that $\lambda = 0$ is not an eigenvalue.

3) Find all positive eigenvalues and the corresponding eigenfunctions.

1) Let $\lambda = -k^2$, $k > 0$.

- The characteristic equation $R^2 - k^2 = 0$ has the two simple roots $R = \pm k$, so the **complete solution** is

  $$y(x) = c_1 \cosh(kx) + c_2 \sinh(kx)$$

  where

  $$y'(x) = c_1 k \cdot \sinh(kx) + c_2 k \cdot \cosh(kx).$$

- **Insertion into the boundary conditions**:
  It follows immediately from $y(0) = 0$ that $c_1 = 0$, so the candidates must have the structure

  $$y(x) = c_2 \sinh(kx) \quad \text{med} \quad y'(x) = c_2 k \cdot \cosh(kx).$$

  By insertion into the second boundary condition we get

  $$0 = y(1) + y'(1) = c_2\{\sinh(k) + k \cdot \cosh(k)\}.$$

  From $\sinh(k) + k \cdot \cosh(k) > 0$ for every $k > 0$ follows that $c_2 = 0$. Since $(c_1, c_2) = (0, 0)$, we conclude that we only have the zero solution, hence no $\lambda < 0$ is an eigenvalue.

2) If $\lambda = 0$, the differential equation is reduced to $y'' = 0$.

- **The complete solution is** (by two integrations)

  $$y(x) = c_1 x + c_2 \qquad \text{where} \qquad y'(x) = c_1.$$

- **Insertion into the boundary conditions**:
  It follows from $y(0) = 0 = c_2$ that $y(x) = c_1 x$.
  It follows from $0 = y(1) + y'(1) = c_1 + c_1 = 2c_1$ that $c_1 = 0$, hence we only get the zero solution, and $\lambda = 0$ is not an eigenvalue.

3) Let $\lambda = k^2$, $k > 0$.

- The characteristic equation $R^2 + k^2 = 0$ has the two simple, complex conjugated roots $R = \pm ik$, hence the **complete solution** is

  $$y(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

  where

  $$y'(x) = -c_1 k \cdot \sin(kx) + c_2 k \cdot \cos(kx).$$

- **Insertion into the boundary conditions**:
  It follows immediately from $y(0) = 0$ that $c_1 = 0$, so the candidates must have the structure

  $$y(x) = c_2 \sin(kx) \quad \text{where} \quad y'(x) = c_2 k \cdot \cos(kx).$$

  Then it follows from the second boundary condition that

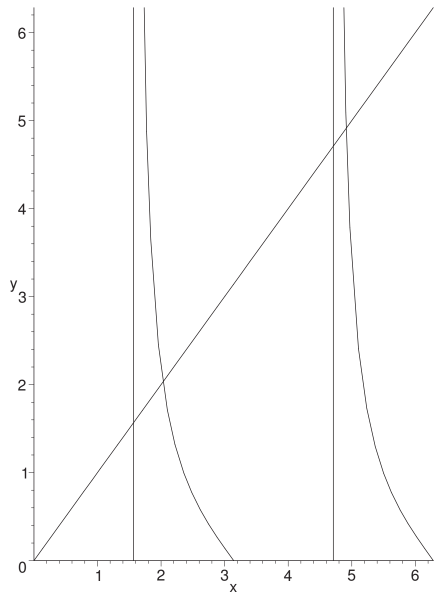
  $$0 = y(1) + y'(1) = c_2\{\sin(k) + k \cdot \cos(k)\}.$$

  We obtain proper solutions, $c_2 \neq 0$, when

  $$\sin k + k \cdot \cos k = 0, \quad \text{thus} \quad k = -\tan k.$$

  By considering a graph we see that there is precisely one solution

  $$\alpha_n \in \left]n\pi - \frac{\pi}{2}, n\pi\right[\quad \text{for every } n \in \mathbb{N}.$$



- **Eigenvalues and eigenfunctions**.
  The eigenvalues are $\lambda_n = \alpha_n^2$, $n \in \mathbb{N}$, and the corresponding generating eigenfunctions are $y_n = \sin(\alpha_n x)$. All eigenfunctions are of course given by $c \cdot y_n(x)$, where $c \neq 0$ is an arbitrary constant.

Remark 2.4 It follows from the figure that

$$\alpha_n - \left(n\pi - \frac{\pi}{2}\right) \to 0 \qquad \text{for } n \to \infty.$$

Newton-Raphson's iteration formula becomes a little complicated, if we choose

$$F(z) = \sin z + z \cdot \cos z,$$

though this choice does not harm the convergence,

$$z_{n+1} = z_n - \frac{F(z_n)}{F'(z_n)} = z_n + \frac{\sin z_n + z_n \cdot \cos z_n}{z_n \cdot \sin z_n - 2\cos z_n}.$$

One may here choose the initial values

$$z_0^{(p)} = p\pi - \frac{\pi}{2}, \qquad p \in \mathbb{N}.$$

Example 2.13 *Consider the eigenvalue problem*

$$y'' + \lambda y = 0, \quad x \in [0, L], \quad y(0) = 0, \quad cy(L) - y'(L) = 0, \quad c \in \mathbb{R}.$$

1) Prove that $\lambda = 0$ is an eigenvalue, if and only if $cL = 1$, and find in that particular case a corresponding generating eigenfunction.

2) Prove that there exists just one negative eigenvalue, if and only if $cL > 1$. Find in the case of $cL = 6$ an approximate value of the negative eigenvalue and a corresponding generating eigenfunction.

3) Find in case of $cL = -1$ an approximate value of the smallest positive eigenvalue and a corresponding generating eigenfunction.

1) Let $\lambda = 0$. The complete solution is

$$y(x) = c_1 x + c_2 \qquad \text{med} \qquad y'(x) = c_1.$$

It follows from the boundary conditions that

$$\begin{cases} y(0) = 0 = c_2, \\ cy(L) - y'(L) = c_1 cL - c_2 c - c_1 = 0, \end{cases} \quad \text{thus} \quad \begin{cases} c_2 = 0, \\ c_1(cL - 1) = 0. \end{cases}$$

It follows that $\lambda = 0$ is an eigenvalue, if and only if $cL = 1$. If so, then $y = x$ is a generating eigenfunction corresponding to $\lambda = 0$.

2) Then assume that $\lambda = -\alpha^2$, $\alpha > 0$, is an eigenvalue. The complete solution of the differential equation is

$$y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

It follows from the boundary condition $y(0) = c_1 = 0$ that if $\lambda = -\alpha^2$ is an eigenvalue, then [where we put $c_2 = 1$]

$$y_\alpha(x) = \sinh(\alpha x), \qquad x \in [0, L],$$

is a corresponding generating eigenfunction.

This eigenfunction must also fulfil the second boundary condition,

$$(3) \quad c \cdot y_\alpha(L) - y_\alpha'(L) = c \cdot \sinh(\alpha L) - \alpha \cdot \cosh(\alpha L) = 0.$$

The equation (3) is a little tricky. In the first case, $\alpha > 0$ was given, and we should find a connection between c and L, which assures that (3) is satisfied. It is, however, difficult to give a direct solution of the equation, because we end up with a discussion of another equation of the form

$$e^{2\alpha L} = \frac{c+\alpha}{c-\alpha}.$$

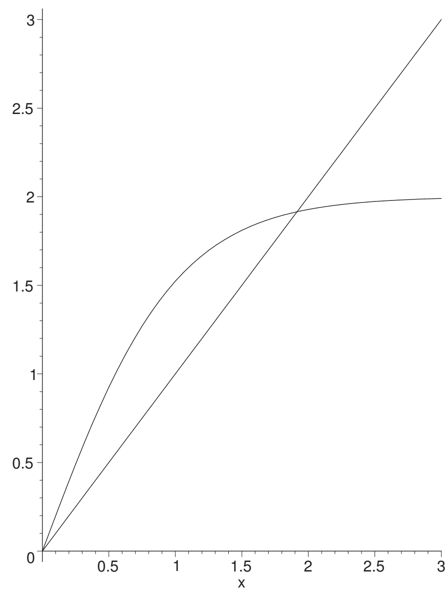
An **alternative** procedure is the following: Let $t = \alpha > 0$ be the *variable*, and let c and L be *constants*. Then define an auxiliary function by

$$\varphi_{c,L}(t) = c \cdot \sinh(tL) - t \cdot \cosh(tL), \qquad t \geq 0.$$

We see that $\varphi_{c,L}(0) = 0$ and

$$\begin{aligned}\varphi'_{c,L}(t) &= c \cdot L \cosh(tL) - \cosh(tL) - tL \cdot \sinh(tL) = (cL-1)\cosh(tL) - tL \cdot \sinh(tL) \\ &= (cL-1)\cosh u - u \cdot \sinh u, \quad u = t \cdot L \geq 0.\end{aligned}$$

If therefore $cL \leq 0$, then $\varphi'_{c,L}(t) < 0$ for $t > 0$, and $\varphi_{c,L}(t)$ is decreasing, so (3) is never fulfilled.



A *necessary* condition for (3) is therefore that $cL > 1$. We shall now assume this. Then (3) is also written

$$L \cdot \varphi_{c,L}(\alpha) = c \cdot L \cdot \sinh(\alpha L) - \alpha L \cdot \cosh(\alpha L) = 0,$$

hence with $u = t \cdot L = \alpha \cdot L > 0$,

$$cL \cdot \sinh u - u \cosh u = 0.$$

Now, $\cosh u \geq 1$, so we rewrite the equation above to

(4) $u = cL \tanh u, \qquad u > 0.$

The curve $z = cL \tanh u$ has $z = cL$ as an horizontal asymptote. Its derivative is $cL > 1$ for $u = 0$, and it decreases towards 0 for u increasing. Hence this curve has precisely one intersection with the curve $z = u$, which again means that the curve given by (4) has precisely one solution$u = \alpha L$.

We have now proved that if $cL > 1$, then there is just one negative eigenvalue $\lambda = -\alpha^2$, where $\alpha = u/L$, and where u is the unique positive solution of (4).

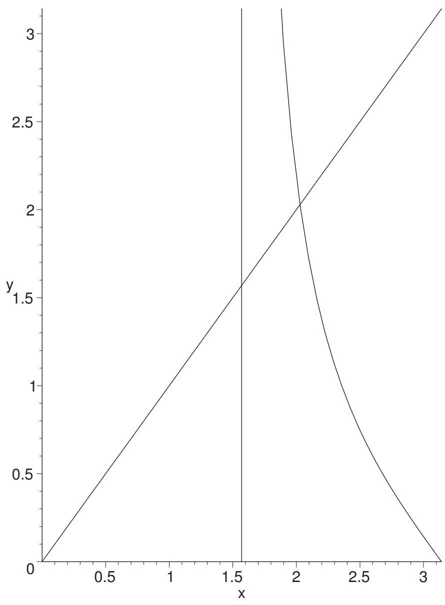
A corresponding generating eigenfunction is $y_\alpha(x) = \sinh(\alpha x)$.

Now put $cL = 6$, so (4) is written

$$u = 6 \tanh u, \qquad u > 0.$$

Since $\tanh u \to 1$ for $u \to \infty$, we get $u \approx 6$. Then by Newton-Raphson iteration, or just by *regula falsi* on a pocket calculator (i.e. successive interpolation between $u = 5,9$ and $u = 6,0$ etc.) we get

$$u = 5,999926, \qquad \text{thus} \qquad u \approx 6.$$



Then $\alpha \approx 6/L$ and $\lambda = -\alpha^2 = -36/L^2$, and a corresponding generating eigenfunction is approximatively

$$y(x) = \sinh\left(\frac{6x}{L}\right).$$

3) Let $\lambda = \alpha^2$, $\alpha > 0$. The complete solution is

$$y(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x).$$

It follows from $y(0) = c_2 = 0$ that we may only consider

$$y(x) = c_1 \sin(\alpha x) \quad \text{where} \quad y'(x) = c_1 \alpha \cos(\alpha x).$$

Then by the second boundary condition,

$$cy(L) - y'(L) = c_1\{c \sin(\alpha L) - \alpha \cos(\alpha L)\} = 0.$$

We only obtain proper solutions (where $c_1 \neq 0$) if

$$cL \sin(\alpha L) - \alpha L \cos(\alpha L) = 0.$$

If we put $cL = -1$ and $t = \alpha L$, it follows that we shall find the smallest positive solution of

$$\sin t + t \cos t = 0, \quad \text{thus} \quad t = -\tan t.$$

We get by a graphical consideration that $t \in \left]\frac{\pi}{2}, \pi\right[$.

Remark 2.5 When we apply the Newton-Raphson iteration method we put

$$F(t) = \sin t + t\cos t \quad \text{where} \quad F'(t) = 2\cos t - t\sin t.$$

Then

$$g(t) = t + \frac{\sin t + t\cos t}{t\sin t - 2\cos t},$$

and the iteration formula becomes

$$\alpha_{n+1} = \alpha_n + \frac{\sin\alpha_n + \alpha_n\cos\alpha_n}{\alpha_n\sin\alpha_n - 2\cos\alpha_n}.$$

Choosing the initial value $\alpha_1 = 2$ we get $\alpha_2 = 2,029048$ and $\alpha_3 = 2,028758 = \alpha_4$, hence

$$\alpha = 2,028758 \cdot \frac{1}{L} \quad \text{where} \quad \lambda = \alpha^2 = 4,115858 \cdot \frac{1}{L^2}.$$

A generating eigenfunction is

$$y_1(x) = \sin\left(2,028758 \cdot \frac{x}{L}\right).$$

Example 2.14 *Consider an axle which is simply supported at its endpoints $x = 0$ and $x = L$. The axle is rotating with the constant angular speed ω. For some values of ω, called the* critical angular speeds*, the axle may rotate in a bent form. The model equation for small bendings of the rotating axle is*

$$(5)\quad EI\frac{d^4u}{dx^4} - \omega^2\varrho u = 0, \qquad x \in [0, L],$$

where E is the elasticity module of the axle, I is the moment of inertia, and ϱ is the mass per length. Given the boundary conditions

$$u(0) = u''(0) = u(L) = u''(L) = 0,$$

we shall find the critical angular speeds and their corresponding bendings $u(x)$. We therefore consider (5) together with the boundary conditions above as an eigenvalue problem where the eigenvalue is defined as $\lambda = \omega^2$, and where we shall find the positive eigenvalues and their corresponding eigenfunctions $u(x)$. (It may be convenient to introduce $k^4 = \omega^2\varrho/(EI)$.)

When we divide by $EI > 0$ the text above is transformed into the following shorter and equivalent eigenvalue problem,

$$\begin{cases} \dfrac{d^4u}{dx^4} - k^4u = 0, \\ \\ u(0) = 0, \quad u''(0) = 0, \quad , u(L) = 0, \quad u''(L) = 0. \end{cases}$$

1) **The complete solution**.

The characteristic equation

$$0 = R^4 - k^4 = (R^2 + k^2)(R^2 - k^2)$$

has the four simple roots $R = \pm ik$ and $R = \pm k$, so the complete solution of the differential equation is

$$u(x) = c_1 \cos(kx) + c_2 \sin(kx) + c_3 \cosh(kx) + c_4 \sinh(kx).$$

Since we later on also shall consider the boundary conditions, we here also compute for convenience,

$$u''(x) = k^2\{-c_1 \cos(kx) - c_2 \sin(kx) + c_3 \cosh(kx) + c_4 \sinh(kx)\}.$$

2) **Insertion into the boundary conditions.**

It follows from the first two boundary conditions that

$$\begin{cases} u(0) = c_1 + c_3 = 0, \\ u''(0) = k^2\{-c_1 + c_3\} = 0, \end{cases}$$

hence $c_1 = c_3 = 0$. Then the candidates must have the structure

$$u(x) = c_2 \sin(kx) + c_4 \sinh(kx)$$

where

$$\frac{u''(x)}{k^2} = -c_2 \sin(kx) + c_4 \sinh(kx).$$

Then it follows from the latter two boundary conditions that

$$u(L) = c_2 \sin(kL) + c_4 \sinh(kL) = 0,$$

$$\frac{u''(L)}{k^2} = -c_2 \sin(kL) + c_4 \sinh(kL) = 0.$$

If we shall have proper solutions, then we must have

$$0 = \begin{vmatrix} \sin(kL) & \sinh(kL) \\ -\sin(kL) & \sinh(kL) \end{vmatrix} = 2\sinh(kL)\cdot\sin(kL).$$

Since $\sinh(kL) > 0$, the only possibility is $\sin(kL) = 0$, thus

$$k_n L = n\pi, \qquad n \in \mathbb{N}.$$

We get e.g. by insertion

$$u(L) = c_2 \cdot 0 + c_4 \cdot \sinh(n\pi) = 0,$$

so $c_4 = 0$ and c_2 is arbitrary.

3) **Eigenvalues and eigenfunctions**.
We have seen in 2) that the eigenvalues are

$$\lambda_n = \omega_n^2 = \frac{k_n^4 EI}{\varrho} = \frac{n^4\pi^4 EI}{\varrho}, \qquad n \in \mathbb{N}.$$

The corresponding generating eigenfunctions are then

$$u_n(x) = \sin(k_n x) = \sin\left(n\pi \cdot \frac{x}{L}\right), \qquad x \in [0, L].$$

The complete set of corresponding eigenfunctions is then given by $c \cdot u_n(x)$, where $c \neq 0$ is an arbitrary constant.

Example 2.15 *Consider the eigenvalue problem*

$$x^2 y'' + xy' + \lambda y = 0, \quad x \in [1, e], \quad y(1) = 0, \quad y(e) = 0.$$

The differential equation is a so-called Euler differential equation. Prove that the eigenvalues are $\lambda_n = n^2\pi^2$, $n \in \mathbb{N}$, *and find the corresponding eigenfunctions.*

The Euler differential equations are characterized by each term of the equation has the structure

$$x^j \frac{d^j y}{dx^j}.$$

We have here two possible methods of solution:

1) **The method of guesses**, i.e. we guess the structure $y = x^\alpha$. Then we typically obtain a polynomial in α, which we put equal to 0.

   - If the order – as in the present case – is 2, and we have two different real roots α_1 and α_2, then we can immediately write down the complete solution, namely that it is generated by the two linearly independent solutions x^{α_1} and x^{α_2}.
   - If α is a (real) *double root*, two linearly independent solutions are x^α and $x^\alpha \ln|x|$.
   - If the roots are *complex conjugated*, $\alpha \pm i\beta$, we have two linearly independent solutions given by

$$x^\alpha \cos(\beta \ln|x|) \text{ and } x^\alpha \sin(\beta \ln|x|).$$

2) The **standard method**. We apply the substitution $u = \ln x$, $x > 0$, thus $x = e^u$. Then by the chain rule,

$$x\frac{dy}{dx} = \frac{dy}{dx} \quad \text{and} \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}.$$

By this substitution it follows by insertion that an Euler differential equation is *always* transferred into a differential equation of constant coefficients, thus

$$x^2\frac{d^2y}{dx^2} + a_1 x\frac{dy}{dx} + a_2 y = 0$$

is transferred into

$$\frac{d^2y}{du^2} + (a_1 - 1)\frac{dy}{du} + a_2 y = 0, \qquad u = \ln x, \quad x > 0.$$

First method. By insertion of $y = x^\alpha$ we get

(6) $x^\alpha\{\alpha(\alpha-1) + \alpha + \lambda\} = x^\alpha(\alpha^2 + \lambda) = 0,$

so we obtain a solution, if $\alpha^2 + \lambda = 0$. (This corresponds to the usual characteristic equation).

1) If $\lambda = -k^2$, $k > 0$, then $\alpha = \pm k$, hence x^k and x^{-k} are two linearly independent solutions. Then it follows by the existence and uniqueness theorem for linear differential equations of second order that the complete solution is

$$y(x) = c_1 x^k + c_2 x^{-k}.$$

It follows from the initial conditions that

$$\left\{\begin{array}{l} y(1) = 0 = c_1 + c_2, \\ \\ y(e) = 0 = e^k c_1 + e^{-k} c_2 = e^{-k}\{e^{2k}c_1 + c_2\}, \end{array}\right. \qquad \text{dvs. } c_1 = c_2 = 0.$$

Hence, no $\lambda > 0$ can be an eigenvalue.

2) If $\lambda = 0$, we rewrite the equation is rewritten in the following way,

$$0 = x^2 y'' + xy' = x\left\{x\frac{d(y')}{dx} + 1 \cdot y'\right\} = x\frac{d}{dx}(xy') = 0.$$

Then by integration,

$$x\frac{dy}{dx} = c_1, \qquad \text{hence } y(x) = c_1 \ln x + c_2.$$

Then by the boundary conditions,

$$y(1) = 0 = c_2 \quad \text{and} \quad y(e) = 0 = c_1 + c_2,$$

hence $c_1 = c_2 = 0$, and $\lambda = 0$ is not an eigenvalue.

3) If $\lambda = k^2$, $k > 0$, then $\alpha = \pm ik$. The corresponding solutions are

$$x^{\pm ik} = \exp(\pm ik \ln x), \qquad x \in [1, e],$$

hence the complete solution is

$$y(x) = c_1 \sin(k \ln x) + c_2 \cos(k \ln x).$$

Since $y(1) = 0 = c_2$, the candidates must have the structure $y(x) = c_1 \sin(k \ln x)$. Then it follows from $y(e) = 0$ that

$$c_1 \sin(k \ln e) = c_1 \sin(k) = 0.$$

If there exists eigenvalues, then we must have $\sin k = 0$, hence $k_n = n\pi$, $n \in \mathbb{N}$. The eigenvalues are $\lambda_n = k_n^2 = n^2\pi^2$, $n \in \mathbb{N}$, and the corresponding generating eigenfunctions are

$$y_n(x) = \sin(k_n \ln x) = \sin(n\pi \ln x), \quad n \in \mathbb{N}.$$

Second method. When we apply the substitution $u = \ln x$, the problem is transferred into

$$\frac{d^2y}{du^2} + \lambda y = 0, \quad u \in [0, 1], \quad y_{|u=0} = y_{|u=1} = 0.$$

1) If $\lambda = -k^2$, $k > 0$, then the complete solution is

$$y(u) = c_1 \cosh(ku) + c_2 \sinh(ku).$$

It follows from the boundary conditions that

$$y_{|u=0} = 0 = c_1 \text{ og } y_{|u=1} = c_1 \cosh(k) + c_2 \sinh(k) = c_2 \sinh(k) = 0,$$

hence $c_1 = c_2 = 0$, and no $\lambda < 0$ is an eigenvalue.

2) If $\lambda = 0$, then $y(u) = c_1 u + c_2$. Then we get by the boundary conditions that

$$y_{|u=0} = c_2 = 0 \quad \text{and} \quad y_{|u=1} = c_1 + c_2 = 0,$$

hence $c_1 = c_2 = 0$, and $\lambda = 0$ is not an eigenvalue.

3) If $\lambda = k^2$, $k > 0$, then the complete solution is

$$y(u) = c_1 \cos(ku) + c_2 \sin(ku).$$

It follows from the boundary conditions that

$$y_{|u=0} = c_1 = 0 \text{ og } y_{|u=1} = c_1 \cos k + c_2 \sin k = c_2 \sin k = 0,$$

and it follows that the eigenvalues correspond to $\sin k = 0$, thus $k_n = n\pi$, $n \in \mathbb{N}$. We conclude that the eigenvalues are $\lambda_n = k_n^2 = n^2\pi^2$, $n \in \mathbb{N}$, and the corresponding generating eigenfunctions are

$$y_n(x) = \sin(k_n u) = \sin(n\pi \ln x), \quad x \in [1, e], \quad n \in \mathbb{N}.$$

Example 2.16 *Consider the eigenvalue problem*

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + \lambda y = 0, \quad x \in [1, e], \quad y(1) = 0, \quad y(e) = 0.$$

The differential equation is an Euler differential equation. Prove that every eigenvalue is bigger than 4, that $\lambda_n = n^2\pi^2 + 4$, and that the corresponding eigenfunctions are $y_n(x) = x^2 \sin(n\pi \ln x)$.
Hint: Apply the substitution $u = \ln x \in [0, 1]$ and derive the eigenvalue problem where u is the variable. Put $z(u) = y(x)$.

The different methods of solution of an Euler differential equation have already been described in the beginning of Example 2.15.

When we apply the monotonous substitution $u = \ln x \in [0, 1]$ we get by the chain rule,

$$\frac{dy}{dx} = \frac{du}{dx}\,\frac{dz}{du} = \frac{1}{x}\,\frac{dz}{du}, \qquad x = e^u$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{1}{x}\,\frac{dz}{du}\right\} = -\frac{1}{x^2}\,\frac{dz}{du} + \frac{1}{x}\cdot\frac{1}{x}\,\frac{d^2z}{du^2} = \frac{1}{x^2}\left\{\frac{d^2z}{du^2} - \frac{dz}{du}\right\}.$$

Then by insertion into the differential equation,

$$0 = x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} - \lambda y = x^2 \cdot \frac{1}{x^2} \left\{ \frac{d^2z}{du^2} - \frac{dz}{du} \right\} - 3x \cdot \frac{1}{x} \frac{dz}{du} + \lambda z(u) = \frac{d^2z}{du^2} - \frac{dz}{du} - 3\frac{dz}{du} + \lambda z,$$

and the transformed equation becomes

$$\frac{d^2z}{du^2} - 4\frac{dz}{du} + \lambda z = 0$$

with the boundary conditions

$$z(0) = y(1) = 0 \qquad \text{and} \qquad z(1) = y(e) = 0.$$

The characteristic polynomial is

$$R^2 - 4R + \lambda = (R-2)^2 + \lambda - 4.$$

1) If $\lambda = 4-k^2 < 4$, $k > 0$, then the characteristic polynomial has the two real simple roots $R = 2 \pm k$, so the complete solution is

$$z = c_1\, e^{(2+k)u} + c_2\, e^{(2-k)u} = e^{2u} \left\{ c_1\, e^{ku} + c_2\, e^{-ku} \right\}.$$

It follows from the boundary conditions that

$$z(0) = c_1 + c_2 = 0 \text{ og } z(1) = e^2 \left\{ e^k \cdot c_1 + e^{-k} \cdot c_2 \right\} = 0,$$

thus

$$1 \cdot c_1 + 1 \cdot c_2 = 0 \quad \text{og} \quad e^k \cdot c_1 + e^{-k} \cdot c_2 = 0.$$

Now, $e^k \neq e^{-k}$, so it follows immediately that $c_1 = c_2 = 0$, corresponding to the zero solution, and no $\lambda = 4 - k^2 < 4$ is an eigenvalue.

2) If $\lambda = 4$, then the characteristic polynomial has the double root $R = 2$. The complete solution is then

$$z(u) = c_1\, e^{2u} + c_2\, u\, e^{2u} = e^{2u}(c_1 + c_2 u).$$

It follows from the boundary conditions that

$$z(0) = c_1 = 0 \quad \text{and} \quad z(1) = e^2(c_1 + c_2) = 0,$$

hence $c_1 = c_2 = 0$, corresponding to the zero solution, and $\lambda = 4$ cannot be an eigenvalue.

3) If $\lambda = 4 + k^2 > 4$, $k > 0$, then the characteristic polynomial has the complex conjugated roots $R = 2 \pm ik$. The complete solution is

$$z(u) = c_1\, e^{2u} \sin(ku) + c_2\, e^{2u} \cos(ku).$$

It follows from the first boundary condition that

$$z(0) = 0 = c_2,$$

so each candidate must have the structure

$$z(u) = c_1\, e^{2u} \sin(ku).$$

Then it follows from the second boundary condition that

$$z(1) = 0 = c_1\, e^2 \sin(k).$$

Here we only obtain proper solutions, if $k_n = n\pi$, $n \in \mathbb{N}$. If so, then the eigenvalue is

$$\lambda_n = 4 + k_n^2 = 4 + n^2\pi^2, \qquad n \in \mathbb{N}.$$

A generating eigenfunction is

$$z_n(u) = e^{2u} \sin(k_n u) = e^{2u} \sin(n\pi u).$$

This is transformed back to y_n by $u = \ln x$, thus

$$y_n(x) = z_n(u) = z_n(\ln x) = x^2 \sin(n\pi \ln x), \qquad x \in [1, e].$$

Example 2.17 *Consider the eigenvalue problem*

$$y'' + \lambda y = 0, \quad x \in [-\pi, \pi], \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(-\pi).$$

1) Prove that $\lambda = 0$ is an eigenvalue and find a corresponding eigenfunction.

2) Prove that there are no negative eigenvalues.

3) Find all the positive eigenvalues and prove that each of them has two corresponding linearly independent eigenfunctions. Explain why this is not a counterexample to Sturm's oscillation theorem.

1) If $\lambda = 0$, then the complete solution is

$$y(x) = c_1 x + c_2 \quad \text{med} \quad y'(x) = c_1.$$

It follows from the boundary conditions that

$$-c_1\pi + c_2 = c_1\pi + c_2 \qquad \text{and} \qquad c_1 = c_1,$$

hence $c_1 = 0$, while c_2 is arbitrary. It follows that $\lambda = 0$ is an eigenvalue with a corresponding generating eigenfunction $y_0(x) = 1$.

2) If $\lambda = -k^2$, $k > 0$, then the complete solution is

$$y(x) = c_1 \cosh(kx) + c_2 \sinh(kx)$$

where

$$y'(x) = kc_1 \sinh(kx) + kc_2 \cosh(kx).$$

It follows from the boundary conditions that

$$c_1 \cosh(k\pi) - c_2 \sinh(kx) = c_1 \cosh(k\pi) + c_2 \sinh(k\pi),$$

hence $c_2 = 0$ after a reduction, and

$$k\{-c_1 \sinh(k\pi) + c_2 \cosh(k\pi)\} = k\{c_1 \sinh(k\pi) + c_2 \cosh(k\pi)\},$$

from which $c_1 = 0$. Now $c_1 = c_2 = 0$ corresponds to the zero solution, so no $\lambda < 0$ can be an eigenvalue.

3) If $\lambda = k^2$, $k > 0$, the complete solution is

$$y(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

where

$$y'(x) = -kc_1 \sin(kx) + kc_2 \cos(kx).$$

It follows from the boundary conditions that

$$y(-\pi) = c_1 \cos(k\pi) - c_2 \sin(k\pi) = y(\pi) = c_1 \cos(k\pi) + c_2 \sin(k\pi),$$

$$y'(-\pi) = kc_2 \cos(k\pi) + kc_1 \sin(k\pi) = y'(\pi) = kc_2 \cos(k\pi) - kc_1 \sin(k\pi),$$

hence

$$2c_2 \sin(k\pi) = 0 \quad \text{and} \quad 2c_1 k \sin(k\pi) = 0.$$

These equations are satisfied for all (c_1, c_2), if $\sin(k\pi) = 0$, thus if $k \in \mathbb{N}$.

We conclude that $\lambda_n = n^2$, $n \in \mathbb{N}$, is an eigenvalue with the corresponding two linearly independent eigenfunctions

$$y_{n,1}(x) = \cos nx \qquad \text{and} \qquad y_{n,2}(x) = \sin nx.$$

Since the boundary conditions are not separated, the assumptions of Sturm's oscillation theorem are not fulfilled, thus it cannot be applied. For that reason the example is not a counterexample to this theorem.

Example 2.18 *The bending $u(x)$ of a column can be modelled as an eigenvalue problem in the following way by convenient choices of the geometry, the spring constant and the material constant,*

$$\frac{d^4u}{dx^4} + a^2 \frac{d^2u}{dx^2} = 0, \qquad u \in [0,1],$$

$$u(1) = 0, \quad u'(1) = 0, \quad u''(0) = 0, \quad a^2 u'(0) + u(0) + u^{(3)}(0) = 0.$$

1) *Consider a as an eigenvalue. Prove that the positive eigenvalues are the roots of the equation*

$$\tan a = a(1 - a^2).$$

2) *Find the smallest positive eigenvalue (approximatively) graphically as well as by means of an iteration with 2 decimals.*

3) *Find a corresponding eigenfunction $u(s)$ for the smallest positive eigenvalue.*

We assume that $a > 0$. This implies that the characteristic polynomial

$$R^4 + a^2 R^2 = R^2(R^2 + a^2)$$

has the simple imaginary roots $\pm ia$ supplied with the og double root $R = 0$. The complete solution is

$$u(x) = c_1 \sin(ax) + c_2 \cos(ax) + c_3 x + c_4$$

where

$$u'(x) = ac_1\cos(ax) - ac_2\sin(ax) + c_3$$

and

$$u''(x) = -a^2c_1\sin(ax) - a^2c_2\cos(ax),$$

and

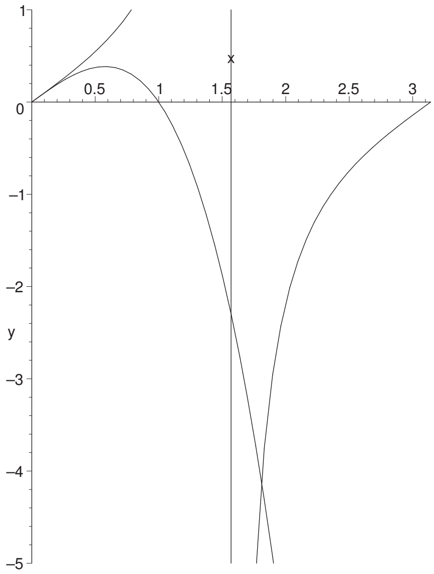
$$u^{(3)}(x) = -a^3c_1\cos(ax) + a^3c_2\sin(ax).$$

1) We get by insertion into the boundary conditions,

$$\begin{aligned} u(1) &= 0 = c_1\sin a + c_2\cos a + c_3 + c_4, \\ u'(1) &= 0 = ac_1\cos a - ac_2\sin a + c_3, \\ u''(0) &= 0 = -a^2c_2, \end{aligned}$$

and

$$a^2u'(0) + u(0) + u^{(3)}(0) = 0 = a^3c_1 + a^2c_3 + c_2 + c_4 - a^3c_1.$$



It follows immediately that $c_2 = 0$, so the system is reduced to

$$\begin{cases} \sin a \cdot c_1 & + & c_3 & + & c_4 & = & 0, \\ a\cos \cdot c_1 & + & c_3 & & & = & 0, \\ & & a^2 c_3 & + & c_4 & = & 0. \end{cases}$$

Then it follows from the last equation that $c_4 = -a^2 c_3$, so the first two are reduced to

$$(7)\quad \begin{cases} \sin a \cdot c_1 & + & (1-a^2)c_3 & = & 0, \\ a\cos a \cdot c_1 & + & 1 \cdot c_3 & = & 0. \end{cases}$$

The determinant condition for proper solutions is

$$\begin{vmatrix} \sin a & 1-a^2 \\ a\cos a & 1 \end{vmatrix} = \sin a - a(1-a^2)\cos a = 0.$$

If $\cos a = 0$, then $\sin a \neq 0$, and there is no solution. We can therefore assume that $\cos a \neq 0$. Then the determinant condition can be written

$$(8)\quad \tan a = a(1-a^2).$$

2) A graphical consideration shows that the smallest positive solution of (8) lies very close to $\pi/2$ in the interval $]\pi/2, \pi[$. It is, however, difficult to create a figure which shows that we actually have $a \approx 1,8$.

We shall use the Newton-Raphson iteration method to find the smallest positive zero $a \in]\pi/2, \pi[$ of

$$F(a) = \tan a - a(1-a^2) = \tan a + a^3 - a.$$

Since

$$F'(a) = 1 + \tan^2 a + 3a^2 - 1 = \tan^2 a + 3a^2,$$

the iteration is given by

$$a_{n+1} = a_n - \frac{F(a_n)}{F'(a_n)} = a_n - \frac{\tan a_n + a_n(a_n^2 - 1)}{\tan^2 a_n + 3a_n^2}, \quad n \in \mathbb{N}.$$

Notice that the denominator $F'(a) = \tan^2 + 3a^2 \geq 3(\pi/2)^2 \gg 0$, so we may expect a very fast convergence.

If we e.g. choose $a_1 = 2$ (a more energetic choice would of course be $\tilde{a}_1 = 1,8$), then we get

$$a_2 = 1,772572, \quad a_3 = 1,805332, \quad a_4 = 1,809239$$

$$a_5 = 1,809278, \quad a_6 = 1,809279.$$

Then with 2 decimals $a \approx 1,81$.

We shall continue in 3) to work with the better value

$$a \approx a_6 = 1,809279.$$

3) When we calculate the eigenfunction we choose $c_1 = 1$. As mentioned above we use the improved value $a \approx 1,809279$ in order to minimize the rounding errors. The final results will only be given with 2 decimals.

We have from above that $c_2 = 0$, and we have furthermore chosen $c_3 = 1$. We shall therefore only calculate

$$c_4 = -a^2 c_3 \quad \text{and from (7),} \quad c_1 = -\frac{c_3}{a \cos a},$$

thus

$$c_4 = -a^2 = -3,273491 \approx -3,27$$

and

$$c_1 = -\frac{1}{a \cos a} = 2,339710 \approx 2,34.$$

With 2 decimals an eigenfunction corresponding to the smallest positive eigenvalue $a \approx 1,81$ is approximately given by

$$u(x) = c_1 \sin(ax) + c_2 \cos(ax) + c_3 x + c_4 \approx 2,34 \sin(1,81x) + x - 3,27.$$

Remark 2.6 From a practical point of view the result cannot be correct, because we get $u(0) = -3,27$. If the spring constant is the same for the two springs, then we should get 0 by the symmetry. An analysis of the boundary conditions shows that there is "something wrong** with

$$a^2 u'(0) + u(0) + u^{(3)} = 0.$$

In fact, the physical dimensions do not agree. For instance, $u(0)$ has dimension ℓ, and $u^{(3)}$ has dimension $\ell/\ell^3 = 1/\ell^2$. One should therefore *always* check the physical dimensions of a model, before one starts on solving it. Inside pure mathematics, however, this is an excellent example.

Example 2.19 *In the investigation of the stability of rotating thin columns one may as a linearized model of the bending $u(x)$ perpendicular to the plane of rotation use the following eigenvalue problem,*

$$u^{(3)} + \Omega^2(r-1)u' = 0, \quad 0 \le x \le 1, \quad u(0) = u'(0) = u''(1) = 0.$$

Here, r is a positive constant, and $\lambda = \Omega^2$ denotes the eigenvalue. We shall only be concerned with the positive eigenvalues.
Find in the three cases $r > 1$, $r = 1$ and $r < 1$ the possible solutions of the eigenvalue problem (i.e. both såvel eigenvalues and eigenfunctions).

We consider here an eigenvalue problem for a differential equation of third order,

$$\frac{d^3u}{dx^3} + \Omega^2(r-1)\frac{du}{dx} = 0, \qquad 0 \le x \le 1.$$

The characteristic polynomial is

$$R^3 + \Omega^2(r-1)R = R\{R^2 + \Omega^2(r-1)\}.$$

1) If $r > 1$, the characteristic polynomial has the roots

$$R = 0 \qquad \text{and} \qquad R = \pm i\Omega\sqrt{r-1}.$$

The complete solution is

$$u = c_1 \sin(\Omega\sqrt{r-1}\cdot x) + c_2 \cos(\Omega\sqrt{r-1}\cdot x) + c_3$$

where

$$u' = c_1\Omega\sqrt{r-1}\cos(\Omega\sqrt{r-1}\cdot x) - c_2\Omega\sqrt{r-1}\sin(\Omega\sqrt{r-1}\cdot x)$$

and

$$u'' = -c_1\Omega^2(r-1)\sin(\Omega\sqrt{r-1}\cdot x) - c_2\Omega^2(r-1)\cos(\Omega\sqrt{r-1}\cdot x).$$

It follows from the boundary conditions that

$$u(0) = 0 = c_2 + c_3, \qquad \text{thus} \qquad c_3 = -c_2,$$

and

$$u'(0) = 0 = c_1\Omega\sqrt{r-1}, \qquad \text{i.e.} \qquad c_1 = 0.$$

Since $c_1 = 0$, it follows from the latter boundary condition that

$$u''(1) = 0 = -0 - c_2\Omega^2(r-1)\cos(\Omega\sqrt{r-1}).$$

Now $c_1 = 0$ and $c_3 = -c_2$, so we only obtain proper solutions when

$$\cos(\Omega\sqrt{r-1}) = 0, \quad \text{thus } \Omega\sqrt{r-1} = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{N}_0.$$

This corresponds to the eigenvalues

$$\lambda_n = \Omega_n^2 = \frac{1}{r-1}\,\frac{\pi^2}{4}\,(2n+1)^2, \qquad n \in \mathbb{N}_0,$$

with the corresponding generating eigenfunction (i.e. $c_2 = 1$)

$$u_n(x) = \cos\left(\frac{\pi}{2}(2n+1)x\right) - 1, \qquad n \in \mathbb{N}_0.$$

2) If $r = 1$, then the characteristic polynomial is reduced to R^3 in which $R = 0$ is a root of multiplicity three. The complete solution is

$$u(x) = c_1 x^2 + c_2 x + c_3,$$

where

$$u'(x) = 2c_1 x + c_2 \qquad \text{og} \qquad u''(x) = 2c_1.$$

It follows from the boundary conditions that

$$u(0) = 0 = c_3, \quad u'(0) = 0 = c_2, \quad u''(1) = 0 = 2c_1,$$

so there does not exist any proper solution, hence not eigenvalue or eigenfunction.

Notice that since $r - 1 = 0$ we see that Ω has totally disappeared from the problem.

3) If $0 < r < 1$, then the characteristic polynomial has the three real roots

$$R = 0 \qquad \text{and} \qquad R = \pm\Omega\sqrt{1-r}.$$

The complete solution is

$$u = c_1 \sinh(\Omega\sqrt{1-r}x) + c_2 \cosh(\Omega\sqrt{1-r}x) + c_3$$

where

$$u' = \Omega\sqrt{1-r}\{c_1 \cosh(\Omega\sqrt{1-r}x) + c_2 \sinh(\Omega\sqrt{1-r}x)\},$$

and

$$u'' = \Omega^2(1-r)\{c_1 \sinh(\Omega\sqrt{1-r}x) + c_2 \cosh(\Omega\sqrt{1-r}x)\}.$$

It follows from the boundary conditions that

$$u(0) = c_2 + c_3 = 0, \qquad \text{thus } c_3 = -c_2,$$

$$u'(0) = c_1\Omega\sqrt{1-r} = 0, \quad \text{i.e. } c_1 = 0.$$

since $c_1 = 0$, it follows from the latter boundary condition that

$$u''(1) = 0 + c_2\Omega^2(1-r)\cdot 1 = c_2\Omega^2(1-r) = 0.$$

Since $1-r > 0$ and $\Omega > 0$, we must have $c_2 = 0$ and thus $c_3 = 0$, and we only get the zero solution, so we have no eigenvalue when $0 < r < 1$.

Example 2.20 *Given the differential equation*

$$y'' + 2\lambda y' + 2\lambda^2 y = 0, \qquad 0 \le x \le \pi,$$

with $y(0) - y'(0) = 0$ and $y(\pi) - y'(\pi) = 0$, and where the parameter $\lambda \in \mathbb{R}$ is considered as a possible eigenvalue.

1) Prove that $\lambda = 0$ is not an eigenvalue.

2) Find all the eigenvalues and the corresponding eigenfunctions.

1) If $\lambda = 0$, then the equation is reduced to $y'' = 0$, the complete solution of which is

$$y = c_1 + c_2 x \qquad \text{where } y' = c_2.$$

Then by insertion into the boundary conditions,

$$y(0) - y'(0) = c_1 - c_2 = 0,$$

hence $c_1 = c_2$, and

$$y(\pi) - y'(\pi) = c_1 + c_2\pi - c_2 = 0,$$

so $c_1 = -(\pi - 1)c_2$.

Since $c_1 = c_2 = 0$ is the only solution, we conclude that $\lambda = 0$ is not an eigenvalue.

2) If $\lambda \neq 0$, then the characteristic polynomial

$$R^2 + 2\lambda R + 2\lambda^2 = (R+\lambda)^2 + \lambda^2,$$

has the simple roots $R = -\lambda \pm i\lambda$. The complete solution is

$$y = c_1 e^{-\lambda x}\cos(\lambda x) + c_2 e^{-\lambda x}\sin(\lambda x)$$

where

$$y' = \lambda(c_2 - c_1)e^{-\lambda x}\cos(\lambda x) - \lambda(c_1 + c_2)e^{-\lambda x}\sin(\lambda x).$$

It follows from the boundary conditions that

$$0 = y(0) - y'(0) = c_1 - \lambda(c_2 - c_1) = (1+\lambda)c_1 - \lambda c_2.$$

Now $\lambda \neq 0$ by (1), so $c_2 = \dfrac{1+\lambda}{\lambda}\, c_1$, which by insertion gives

$$\begin{aligned}
0 &= y(\pi) - y'(\pi) \\
&= c_1 e^{-\lambda\pi}\cos(\lambda\pi) + c_1 \cdot \frac{1+\lambda}{\lambda}\, e^{-\lambda\pi}\sin(\lambda\pi) \\
&\qquad - \left(\lambda \cdot \frac{1+\lambda}{\lambda} - \lambda\right) c_1 e^{-\lambda\pi}\cos(\lambda\pi) + \left(\lambda + \lambda \cdot \frac{1+\lambda}{\lambda}\right) c_1 e^{-\lambda\pi}\sin(\lambda\pi) \\
&= c_1 e^{-\lambda\pi}\left\{(1-1)\cos(\lambda\pi) + \left(\lambda + \frac{(\lambda+1)^2}{\lambda}\right)\sin(\lambda\pi)\right\} \\
&= c_1 e^{-\lambda\pi} \cdot \frac{\lambda^2 + (\lambda+1)^2}{\lambda}\,\sin(\lambda\pi).
\end{aligned}$$

Since $\lambda^2 + (\lambda+1)^2 > 0$ for all $\lambda \in \mathbb{R}\setminus\{0\}$, we only obtain proper solutions, $c_1 \neq 0$, if $\lambda = n \in \mathbb{Z}\setminus\{0\}$. We get for $\lambda_n = n \in \mathbb{Z}\setminus\{0\}$ and $c_1 = n$ that $c_2 = n+1$, so an eigenfunction corresponding to n is

$$y_n(x) = n e^{-nx}\cos nx + (n+1)e^{-nx}\sin nx, \qquad n \in \mathbb{Z}\setminus\{0\}.$$

All the eigenfunctions corresponding to $\lambda_n = n \in \mathbb{Z}\setminus\{0\}$ are then given by $c \cdot y_n(x)$, where c is an arbitrary constant.

Remark 2.7 We get for $n = -1$,

$$y_{-1}(x) = -e^x \cos x$$

without any sine term.

Example 2.21 *Given the eigenvalue problem*

$$\frac{d^4y}{dx^4} + \lambda^2 \frac{d^2y}{dx^2} = 0, \quad x \in [0,1], \quad y(0) = y''(0) = y(1) = y'(1) = 0.$$

1) Check if $\lambda = 0$ is an eigenvalue.

2) Prove that every eigenvalue $\lambda \neq 0$ must fulfil the equation $\tan\lambda = \lambda$.

1) For $\lambda = 0$ the equation is reduced to $\dfrac{d^4y}{dx^4} = 0$, the complete solution of which is

$$y = c_0 + c_1x + c_2x^2 + c_3x^3$$

where

$$\frac{dy}{dx} = c_1 + 2c_2x + 3c_3x^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 2c_2 + 6c_3x.$$

It follows from the boundary conditions that

$$y(0) = c_0 = 0, \qquad y(1) = c_0 + c_1 + c_2 + c_3 = 0,$$

$$y''(0) = 2c_2 = 0, \qquad y'(1) = c_1 + 2c_2 + 3c_3 = 0,$$

which is reduced to $c_0 = c_2 = 0$ and

$$c_1 + c_3 = 0, \qquad c_1 + 3c_3 = 0,$$

hence also $c_1 = c_3 = 0$.

Since the zero solution is the only solution, we conclude that $\lambda = 0$ is not an eigenvalue.

2) If $\lambda \neq 0$, then the characteristic polynomial

$$R^4 + \lambda^2R^2 = R^2(R^2 + \lambda^2)$$

has the double root $R = 0$ and the two simple and complex conjugated roots $R = \pm i\lambda$. The complete solution is

$$y = c_0 + c_1x + c_2\cos\lambda x + c_3\sin\lambda x$$

where

$$\frac{dy}{dx} = c_1 - c_2\lambda\sin\lambda x + c_3\lambda\cos\lambda x$$

and

$$\frac{d^2y}{dx^2} = -c_2\lambda^2\cos\lambda x - c_3\lambda^2\sin\lambda x.$$

It follows from the boundary conditions that

$$y(0) = c_0 + c_2 = 0, \qquad y''(0) = -c_2\lambda^2 = 0,$$

$$y(1) = c_0 + c_1 + c_2 \cos \lambda + c_3 \sin \lambda = 0,$$

$$y'(1) = c_1 - c_2 \lambda \sin \lambda + c_3 \lambda \cos \lambda = 0.$$

Since the two values $\pm\lambda$ correspond to the same square λ^2, we may of course assume that $\lambda > 0$. Then by the first two equations, $c_2 = 0$ and $c_0 = 0$, and the two remaining equations are reduced to

$$\left\{ \begin{array}{l} c_1 + c_3 \sin \lambda = 0, \\ c_1 + c_3 \lambda \cos \lambda = 0, \end{array} \right. \qquad \text{i.e.} \quad \begin{pmatrix} 1 & \sin \lambda \\ 1 & \lambda \cos \lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We only get proper solutions, if the matrix is singular, i.e. if

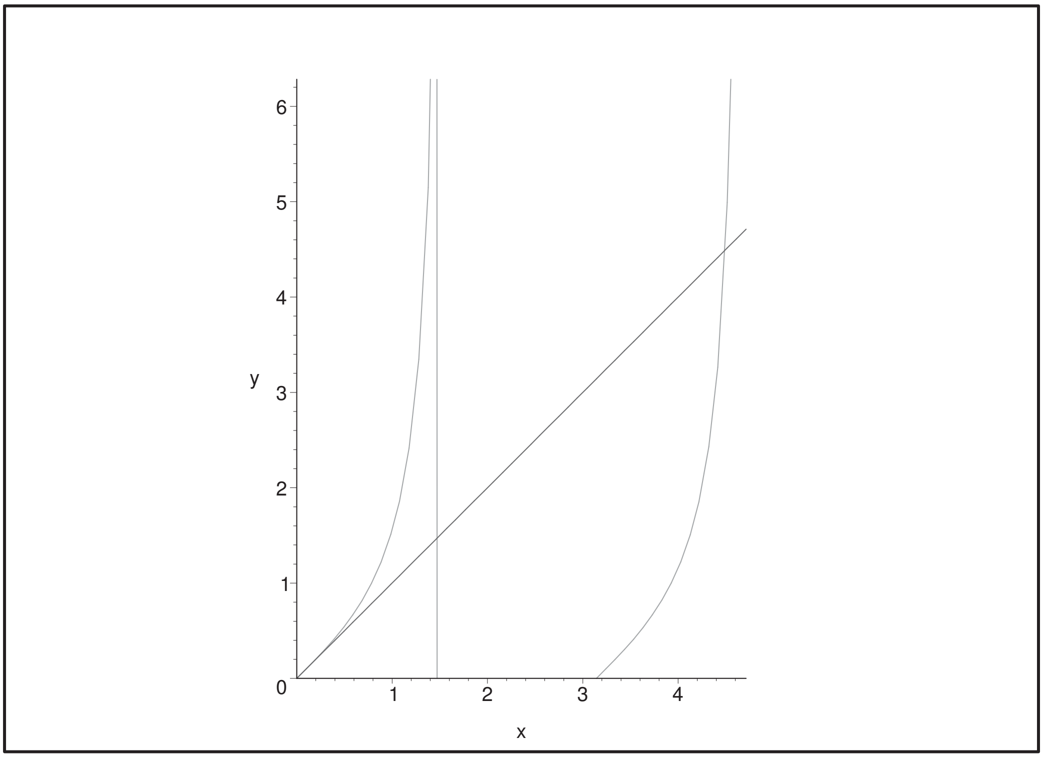
$$\begin{vmatrix} 1 & \sin \lambda \\ 1 & \lambda \cos \lambda \end{vmatrix} = \lambda \cos \lambda - \sin \lambda = 0,$$

hence

$$\lambda \cos \lambda - \sin \lambda = 0.$$

Since $\sin \lambda \neq 0$, when $\cos \lambda = 0$, we must have $\cos \lambda \neq 0$ for every solution. Then the equation is rewritten as

$$\tan \lambda = \lambda.$$



It follows from the figure that we have precisely one solution $\lambda_n > 0$ in each interval $\left]n\pi, n\pi + \frac{\pi}{2}\right[$, $n \in \mathbb{N}$, and that $\lambda_n \sim \left(n + \frac{1}{2}\right)\pi$ for $n \to \infty$.

A corresponding eigenfunction is e.g.

$$\varphi_n(x) = \sin(\lambda_n x) - (\sin \lambda_n) \cdot x,$$

where we have chosen $c_1 = -\sin \lambda_n$ and $c_3 = 1$.

Example 2.22 *Given the eigenvalue problem*

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad 0 \le x \le \frac{1}{2},$$

$$y(0) + y'(0) = 0, \qquad y\left(\frac{1}{2}\right) = 0.$$

1) Prove that we have no negative eigenvalues.

2) Find an equation from with one in principle can find the smallest eigenvalue (the calculation is not required).

1) If $\lambda < 0$, $\lambda = -k^2$, then the complete solution is

$$y = c_1 \cosh(kx) + c_2 \sinh(kx)$$

where

$$y' = k\{c_2 \cosh(kx) + c_1 \sinh(kx)\}.$$

We get by insertion into the boundary conditions that

$$y(0) + y'(0) = c_1 + kc_2 = 0,$$

$$y\left(\frac{1}{2}\right) = \cosh\left(\frac{k}{2}\right) c_1 + \sinh\left(\frac{k}{2}\right) c_2 = 0.$$

This linear system of equations in (c_1, c_2) has the determinant

$$\begin{vmatrix} 1 & k \\ \cosh\frac{k}{2} & \sinh\frac{k}{2} \end{vmatrix} = \sinh\frac{k}{2} - k\cosh\frac{k}{2} = \cosh\frac{k}{2}\left\{\tanh\frac{k}{2} - k\right\} < \cosh\frac{k}{2}\left\{\frac{k}{2} - k\right\} < 0 \quad \text{for } k > 0.$$

Since this determinant is $\neq 0$, the system has only the zero solution, so no $\lambda < 0$ can be an eigenvalue.

2) If $\lambda = 0$, then the complete solution is

$$y = c_1 + c_2 x \qquad \text{where } y'(x) = c_2.$$

It follows from the boundary conditions that

$$y(0) + y'(0) = c_1 + c_2 = 0 \quad \text{og} \quad y\left(\frac{1}{2}\right) = c_1 + \frac{1}{2}c_2 = 0.$$

The only solution is $c_1 = c_2 = 0$, so $\lambda = 0$ cannot be an eigenvalue either.

If $\lambda = k^2 > 0$, $k > 0$, then the complete solution is

$$y = c_1 \cos(kx) + c_2 \sin(kx)$$

where

$$y' = kc_2 \cos(kx) - kc_1 \sin(kx).$$

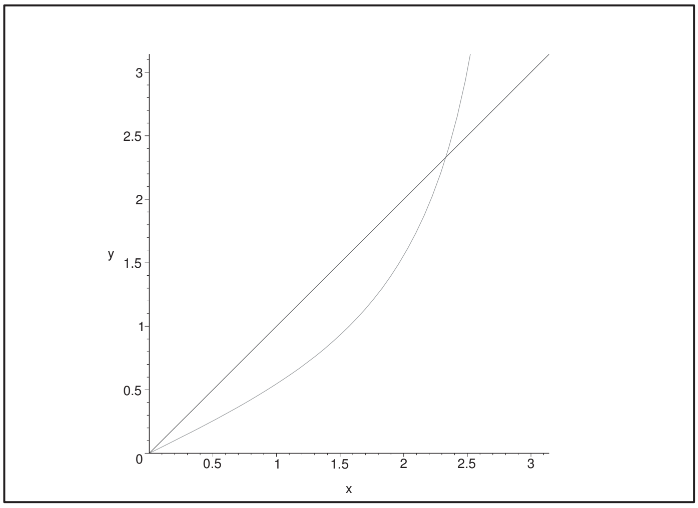
We get by insertion into the boundary conditions that

$$y(0) + y'(0) = c_1 + kc_2 = 0,$$

$$y\left(\frac{1}{2}\right) = \cos\left(\frac{k}{2}\right) c_1 + \sin\left(\frac{k}{2}\right) c_2 = 0.$$

This system has proper solutions $(c_1, c_2) \neq (0,0)$, if and only if the corresponding determinant is zero, thus

$$0 = \begin{vmatrix} 1 & k \\ \cos\frac{k}{2} & \sin\frac{k}{2} \end{vmatrix} = \sin\left(\frac{k}{2}\right) - k \cdot \cos\left(\frac{k}{2}\right), \qquad k > 0.$$



Since $\cos \dfrac{k}{2} \neq 0$ for every solution, this condition is equivalent to the equation

$$\tan \frac{k}{2} = k.$$

Since $\tan \dfrac{k}{2} \approx \dfrac{k}{2} < k$ in the neighbourhood of 0, and $\tan \dfrac{k}{2} \to \infty$ for $k \to \pi-$, this equation must by the continuity have a solution $k \in]0, \pi[$. It follows from the figure that it has precisely one solution.

Remark 2.8 It can be proved by a Newton-Raphson iteration that the first, i.e. the smallest positive eigenvalue is

$$\lambda_1 = k_1^2 \approx 5,434.$$

Example 2.23 *Consider the eigenvalue problem*

$$\frac{d^2y}{dx^2} + \lambda\frac{dy}{dx} - (\lambda+1)y = 0, \qquad x \in [0,1], \qquad y(0)-y'(0) = y(1)-y'(1) = 0.$$

1) Prove that $\lambda = -2$ is an eigenvalue and find all its corresponding eigenfunctions.

2) Prove that every $\lambda \in \mathbb{R}$ is an eigenvalue for the eigenvalue problem under consideration, and that $y = e^x$, $x \in [0,1]$, is a corresponding eigenfunction.

1) We get by insertion of $\lambda = -2$ that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0, \qquad x \in [0,1],$$

with the characteristic polynomial $R^2 - 2R + 1 = (R-1)^2$. Since the root $R = 1$ has multiplicity 2, the complete solution is

$$y = c_1 e^x + c_2 x e^x \quad \text{where} \quad y' = (c_1 + c_2)e^x + c_2 x e^x.$$

First calculate

$$y(0) = c_1 \qquad \text{and} \qquad y'(0) = c_1 + c_2,$$

and

$$y(1) = c_1 + c_2 \qquad \text{and} \qquad y'(1) = c_1 + 2c_2.$$

It follows from the boundary conditions that

$$c_1 = c_1 + c_2 \qquad \text{and} \qquad c_1 + c_2 = c_1 + 2c_2$$

for $c_2 = 0$ and c_1 arbitrary constants. Hence, $\lambda = -2$ is an eigenvalue, and every eigenfunction has the form ce^x.

2) It is immediately seen that the function $y = e^x$ fulfils both the equation and the boundary conditions, no matter the choice of $\lambda \in \mathbb{R}$.

Example 2.24 *Consider the eigenvalue problem*

$$\frac{d^2y}{dx^2} + \lambda \frac{dy}{dx} = 0, \quad x \in [0,1], \quad y(0) = 0, \quad y'(1) = 0.$$

Prove that $\lambda = 0$ is not an eigenvalue. Then prove that the eigenvalue problem does not have an eigenvalue.

1) When $\lambda = 0$, the equation is reduced to $\dfrac{d^2y}{dx^2} = 0$, the complete solution of which is

$$y = c_1 x + c_2.$$

By insertion into the boundary conditions we get

$$y(0) = c_2 = 0 \qquad \text{and} \qquad y'(1) = c_1 = 0.$$

This shows that $\lambda = 0$ is not an eigenvalue.

2) If $\lambda \neq 0$, then the characteristic polynomial

$$R^2 + \lambda R = R(R + \lambda),$$

has the roots $R = 0$ and $R = -\lambda$. The complete solution is

$$y = c_1 + c_2 e^{-\lambda x} \quad \text{where} \quad y' = -\lambda c_2 e^{-\lambda x}.$$

By insertion into the boundary values we get

$$y(0) = c_1 + c_2 = 0 \quad \text{og} \quad y'(1) = -\lambda c_2 e^{-\lambda} = 0.$$

Since $\lambda \neq 0$, we get $c_2 = 0$, and hence $c_1 = -c_2 = 0$.

It follows in particular that no $\lambda \neq 0$ can be an eigenvalue.

Summing up, the eigenvalue problem does not have any eigenvalue.

3 Nontypical eigenvalue problems

We collect in this chapter some eigenvalue problems which for some reason are nontypical. In some of the cases there is required a lot more of the reader than one could expect. In other cases I have found some eigenvalue problems in the literature, which I feel very strange. They have only been included here, because some of the readers may come across them.

Example 3.1 *Consider the eigenvalue problem*

$$u''' + \Omega^2(r-1)u' + \Omega^2 u(1) = 0, \quad x \in [0,1], \quad u(0) = u'(0) = u''(0) = 0,$$

where r is a positive constant, and Ω is the eigenvalue.

1) Prove for every fixed $r > 1$ that the positive eigenvalues fulfil the equation

$$(9)\ \ \tan \Omega\sqrt{r-1} = r\Omega\sqrt{r-1}.$$

(Hint: Use the three boundary conditions and furthermore the identity $u(1) = u(1)$).

2) Find for $r = 2$ the smallest positive eigenvalue with three decimals.

3) We again assume that $r > 1$. Prove that the smallest positive eigenvalue Ω_0 satisfies $\Omega_0 \to \sqrt{3}$ for $r \to 1$.

(Hint: Apply (9), put $x = \sqrt{r-1}$ and use Taylor's formula for $\tan \Omega x$).

4) Find in the case of $r = 1$ all the positive eigenvalues and their corresponding eigenfunctions.

1) Since $u(1)$ occurs, the equation is not a usual differential equation. If we consider for a while $u(1)$ just as some constant c ("independent of $u(t)$"), it makes sense to guess a particular solution of the form $u_0(x) = ax + b$.

Since $r > 1$, we get by insertion

$$0 + \Omega^2(r-1)a + \Omega^2(a+b) = \Omega^2(ra+b) = 0.$$

Now, $\Omega > 0$, so $b = -ra$, and $u_0(x) = a(x-r)$ where $u(1) = a(1-r)$, thus

$$a = \frac{u(1)}{1-r}.$$

For given $u(1)$, a particular solution is

$$u_(x) = \frac{u(1)}{1-r}\,(x-r).$$

The "homogeneous" equation where we neglect the term $\Omega^2 u(1)$, has the characteristic polynomial

$$R^3 + \Omega^2(r-1)R = R\{R^2 + \Omega^2(r-1)\}.$$

Since $r > 1$, the complete solution is

$$u(x) = c_1 \sin(\Omega\sqrt{r-1}x) + c_2 \cos(\Omega\sqrt{r-1}x) + c_3 + \frac{u(1)}{1-r}(x-r)$$

where

$$u(1)=c_1\sin(\Omega\sqrt{r-1})+c_2\cos(\Omega\sqrt{r-1})+c_3+u(1),$$

and we derive the additional condition

$$(10)\quad c_1\sin(\Omega\sqrt{r-1})+c_2\cos(\Omega\sqrt{r-1})+c_3=0.$$

Now,

$$u'(x)=\Omega\sqrt{r-1}\{c_1\cos(\Omega\sqrt{r-1}x)-c_2\sin(\Omega\sqrt{r-1})\}+\frac{u(1)}{1-r},$$

$$u''(x)=-\Omega^2(r-1)\{c_1\sin(\Omega\sqrt{r-1}x)+c_2\cos(\Omega\sqrt{r-1}x)\}.$$

It follows from the boundary conditions that

$$u(0)=c_2+c_3+\frac{r}{r-1}\,u(1)=0,$$

$$u'(0)=c_1\Omega\sqrt{r-1}-\frac{1}{r-1}\,u(1)=0,$$

$$u''(1)=-\Omega^2(r-1)\{c_1\sin(\Omega\sqrt{r-1})+c_2\cos(\Omega\sqrt{r-1})\}=0.$$

When we compare the latter equation and (10) we get $c_3=0$, and the system is reduced to the three equations

$$c_2+\frac{r}{r-1}\,u(1)=0,$$

$$c_1\Omega\sqrt{r-1}-\frac{1}{r-1}\,u(1)=0,$$

$$c_1\sin(\Omega\sqrt{r-1})+c_2\cos(\Omega\sqrt{r-1})=0.$$

It follows from the first two equations that

$$c_2=-\frac{r}{r-1}\,u(1)\quad\text{and}\quad c_2=-r\Omega\sqrt{r-1}\,c_1.$$

Then by insertion into the last equation,

$$c_1\{\sin(\Omega\sqrt{r-1})-r\Omega\sqrt{r-1}\cos(\Omega\sqrt{r-1})\}=0.$$

A *necessary* condition for Ω being an eigenvalue is therefore

$$(11)\quad \sin(\Omega\sqrt{r-1})=r\Omega\sqrt{r-1}\cos(\Omega\sqrt{r-1}).$$

Clearly, a solution of this equation also satisfies $\cos(\Omega\sqrt{r-1})\neq 0$, so we get as required (9),

$$\tan(\Omega\sqrt{r-1})=r\Omega\sqrt{r-1}.$$

The condition (11) is also *sufficient*. Assume that it holds, and let

$$u(x)=c_1\sin(\Omega\sqrt{r-1}x)+c_2\cos(\Omega\sqrt{r-1}x)+c_3+\frac{u(1)}{1-r}\,(x-r).$$

If $u(x)$ is an eigenfunction, then we have already proved above that $c_3 = 0$ and

$$c_2 = -\frac{r}{r-1}\,u(1), \qquad c_1\Omega\sqrt{r-1} = \frac{1}{r-1}\,u(1),$$

and by (11),

$$\begin{aligned} c_1 \sin(\Omega\sqrt{r-1}) + c_2\cos(\Omega\sqrt{r-1}) &= c_1 r\Omega\sqrt{r-1}\cos(\Omega\sqrt{r-1}) + c_2\cos(\Omega\sqrt{r-1}) \\ &= \cos(\Omega\sqrt{r-1}\{c_1 r\Omega\sqrt{r-1} + c_2\} = 0. \end{aligned}$$

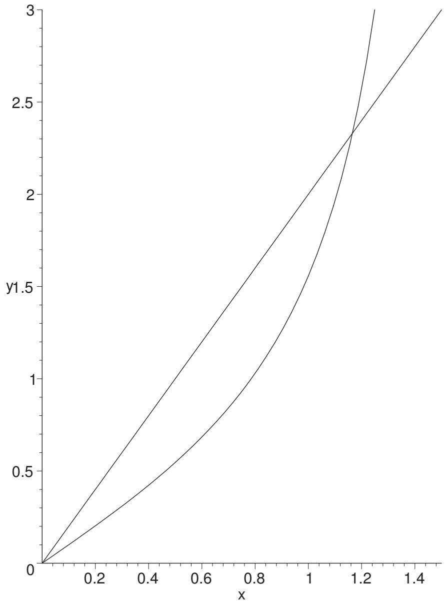
Since $\cos(\Omega\sqrt{r-1}) \neq 0$, this is reduced to

$$c_1\Omega\sqrt{r-1} = \frac{1}{r-1}\,u(1), \qquad c_2 = -\frac{r}{r-1}\,u(1),$$

$$0 = c_1 r\Omega\sqrt{r-1} + c_2 = r\cdot\frac{1}{r-1}\,u(1) - \frac{r}{r-1}\,u(1) = 0,$$

which is fulfilled for whatever the choice of $u(1)$. If we choose e.g. $u(1) = 1$, then we obtain the corresponding generating eigenfunction

$$u_\Omega(x) = \frac{\sin(\Omega\sqrt{r-1}x)}{\Omega(r-1)\sqrt{r-1}} - \frac{r}{r-1}\cos(\Omega\sqrt{r-1}x) + \frac{1}{1-r}(x-r).$$



2) Now let $r = 2$. Then (9) is written in the form

$$\tan \Omega = 2\Omega.$$

By a graphical consideration we see that there is precisely one solution in the interval $]0, \pi/2[$, and since

$$\tan \frac{\pi}{4} = 1 < 2 \cdot \frac{\pi}{4} = \frac{\pi}{2},$$

the solution must even lie in the interval $]\pi/4, \pi/2[$.

We shall now find the zero in $]\pi/4, \pi/2[$ of the function

$$F(\Omega) = 2\Omega \cos \Omega - \sin \Omega.$$

Here we apply Newton-Raphson iteration . From

$$F'(\Omega) = 2 \cos \Omega - 2\Omega \sin \Omega - \cos \Omega = -2\Omega \sin \Omega + \cos \Omega,$$

follows that the iteration formula becomes

$$\Omega_{n+1} = \Omega_n - \frac{F(\Omega_n)}{F'(\Omega_n)} = \Omega_n + \frac{2\Omega_n \cos \Omega_n - \sin \Omega_n}{2\Omega_n \sin \Omega_n - \cos \Omega_n}.$$

Thus with the initial value $\Omega_1 = 1$,

$$\Omega_2 = 1,209282,\ \Omega_3 = 1,167398,\ \Omega_4 = 1,165565,\ \Omega_5 = 1,165561,$$

hence $\Omega \approx 1,1656$.

3) Let us return to the equation (9),

$$\tan(\Omega\sqrt{r-1}) = r\Omega\sqrt{r-1}.$$

If we put $x = \sqrt{r-1}$, then $x \to 0+$ for $r \to 1+$.

Let $\varphi(u) = \tan u$, $\varphi(0) = 0$. Then

$$\begin{aligned}
\varphi'(u) &= 1 + \tan^2 u, && \varphi'(0) = 1 \\
\varphi''(u) &= 2\tan u(1 + \tan^2 u), && \varphi''(0) = 0, \\
\varphi^{(3)}(0) &= 2(1 + \tan^2 u)^2 + \tan u \cdot \{\cdots\}, && \varphi^{(3)}(0) = 2,
\end{aligned}$$

so by a Taylor expansion,

$$\varphi(u) = \tan u = u + \frac{1}{3}\, u^3 + u^3 \varepsilon(u).$$

Then put $u = \Omega_0\sqrt{r-1} > 0$. It follows from (9) that

$$r\Omega_0\sqrt{r-1} = \tan(\Omega_0\sqrt{r-1}) = \Omega_0\sqrt{r-1} + \frac{1}{3}\,\Omega_0^3\sqrt{r-1}(r-1) + (\sqrt{r-1})^3\varepsilon(\sqrt{r-1}).$$

When this equation is divided by $\Omega_0\sqrt{r-1} > 0$, then

$$r = 1 + \frac{1}{3}\,\Omega_0(r-1) + (r-1)\varepsilon(\sqrt{r-1}),$$

hence by a rearrangement,

$$\Omega_0^2 \cdot (r-1) = 3(r-1) + (r-1)\varepsilon(\sqrt{r-1}).$$

This equation is then divided by $r - 1 > 0$. This gives

$$\Omega_0(r)^2 = 3 + \varepsilon(\sqrt{r-1}),$$

hence by taking the limit,

$$\Omega_0 = \lim_{r\to 1+} \Omega_0(r) = \sqrt{3}.$$

4) If we put $r = 1$, the eigenvalue problem is reduced to

$$u^{(3)} + \Omega^2\, u(1) = 0, \quad x \in [0,1], \quad u(0) = u'(0) = u''(1) = 0.$$

If we again just consider $u(1)$ as a constant, the corresponding *homogeneous* equation becomes $u^{(3)} = 0$, the complete solution of which is

$$u(x) = c_2x^2 + c_3x + c_4.$$

This should inspire us to *guess* on the structure of the solution

$$u(x) = c_1x^3 + c_2x^2 + c_3x + c_4$$

of the original equation. We see that

$$u(1) = c_1 + c_2 + c_3 + c_4.$$

If we put this function $u(x)$ into the differential equation (thus testing it), we get

$$u^{(3)} + \Omega^2 u(1) = 6c_1 + \Omega^2 (c_1 + c_2 + c_3 + c_4) = 0.$$

Furthermore,

$$u'(x) = 3c_1 x^2 + 2c_2 x + c_3, \qquad u''(x) = 6c_1 x + 2c_2.$$

Then by the boundary conditions,

$$u(0) = 0 = c_4, \quad u'(0) = 0 = c_3, \quad u''(1) = 6c_1 + 2c_2 = 0.$$

Now, $c_3 = c_4 = 0$, so

$$u(x) = c_1 x^3 + c_2 x^2,$$

where c_1, c_2 and $u(1)$ satisfy

$$c_1 + c_2 - u(1) = 0, \qquad (\Omega^2 + 6)c_1 + \Omega^2 c_2 = 0, \qquad 6c_1 + 2c_2 = 0.$$

We find the eigenvalues which this system is singular. We see that $u(1) = c_1 + c_2$ only occurs in the first equation. Hence, the condition becomes

$$0 = \begin{vmatrix} \Omega^2 + 6 & \Omega^2 \\ 6 & 2 \end{vmatrix} = \begin{vmatrix} 6 & \Omega^2 \\ 4 & 2 \end{vmatrix} = 4 \begin{vmatrix} 3 & \Omega^2 \\ 1 & 1 \end{vmatrix} = 4(3 - \Omega^2),$$

thus $\Omega^2 = 3$. From $\Omega > 0$ follows that $\Omega = \sqrt{3}$, which was already indicated in 3).

Now let $\Omega = \sqrt{3}$. We shall now express c_1 and c_2 by $u(1)$. The equations

$$\begin{cases} c_1 + c_2 = u(1) \\ \\ 3c_1 + c_2 = 0, \end{cases} \qquad \text{imply} \qquad \begin{cases} c_1 = -\dfrac{1}{2}\, u(1), \\ \\ c_2 = \dfrac{3}{2}\, u(1). \end{cases}$$

For $\Omega = \sqrt{3}$ the only eigenfunctions are

$$u(x) = -\frac{1}{2}\, u(1) x^3 + \frac{3}{2}\, u(1) x^2 = \frac{u(1)}{2}\, x^2 (3 - x), \quad x \in [0, 1].$$

It is left to the reader to test this solution, i.e. prove that the obtained function $u(x)$ is an eigenfunction for $r = 1$ corresponding to $\Omega = \sqrt{3}$.

The eigenvalue problem of this example is of a type, which is usually *not* included in the textbooks.

Example 3.2 In some cases one may also be forced to use the power series method in eigenvalue problems. We shall here illustrate this in a (very big and complicated) example.

Consider the eigenvalue problem

$$\frac{d^4y}{dx^4}+(\lambda-x)\frac{d^2y}{dx^2}-\frac{dy}{dx}=0, \qquad x\in[0,\lambda].$$

$$y(0)=y'(0)=y''(\lambda)=y'''(\lambda)=0.$$

This is the model equation of the bending of a vertical thin column of length λ, clamped in one end and under the influence of the weight of the column. One wants to find the smallest positive eigenvalue λ.

1) First **inspect** the equation. Since

$$\frac{d}{dx}\left\{(\lambda-x)\frac{dy}{dx}\right\}=(\lambda-x)\frac{d^2y}{dx^2}-\frac{dy}{dx},$$

the differential equation is also written

$$\frac{d^4y}{dx^4}+\frac{d}{dx}\left\{(\lambda-x)\frac{dy}{dx}\right\}=0.$$

This can immediately be integrated,

$$\frac{d^3y}{dx^3}+(\lambda-x)\frac{dy}{dx}=c, \qquad c \text{ arbitrær}.$$

2) **The determination of c by using the boundary value** $y'''(\lambda) = 0$ follows from the equation

$$c = y'''(\lambda) + (\lambda - \lambda)y'(\lambda) = 0.$$

The problem is then reduced to the simpler homogeneous equation

$$\frac{d^3y}{dx^3} + (\lambda - x)\frac{dy}{dx} = 0,$$

which is a camouflaged differential equation of second order in $\dfrac{dy}{dx}$. We therefore put $z = \dfrac{dy}{dx}$, so

$$\frac{d^2z}{dx^2} + (\lambda - x)z = 0,$$

where the boundary values for z are

$$z(0) = y'(0) = 0 \quad \text{and} \quad z'(\lambda) = y''(\lambda) = 0.$$

Remark 3.1 We have already applied the boundary value $y'''(\lambda) = z''(\lambda) = 0$, and we see that it now also follows from the equation. Furthermore, $y(0) = 0$ is not at all relevant for $z = y'$.

3) **Change of variable**. The factor $\lambda - x$ is annoying, so we change the variable to $t = \lambda - x$. If we put

$$u(t) = z(x), \qquad \text{thus} \qquad u(\lambda - x) = z(x),$$

then the equation is transferred into

$$\frac{d^2u}{dt^2} + tu(t) = 0 \quad \text{where} \quad u(\lambda) = 0 \text{ and } u'(0) = 0.$$

4) We shall neglect the boundary condition $u(\lambda) = 0$ for a while, when we find a power series solution of this equation. We shall later come back to the condition $u(\lambda) = 0$. It follows from $u'(0) = 0$ that $a_1 = 0$. By inserting the formal power series

$$u(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad \frac{d^2u}{dt^2} = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$$

into the differential equation we get

$$\begin{aligned} 0 &= \frac{d^2u}{dt^2} + tu(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + \sum_{n=1}^{\infty} a_{n-1}t^n \\ &= 2a_2 + \sum_{n=1}^{\infty} \{(n+2)(n+1)a_{n+2} + a_{n-1}\}t^n. \end{aligned}$$

Then we get by the identity theorem that $a_2 = 0$ (we have already proved that $a_1 = 0$), and for $n \in \mathbb{N}$ (the summation domain)

$$(n+2)(n+1)a_{n+2} + a_{n-1} = 0 \qquad \text{for } n \in \mathbb{N}.$$

This is by $n \mapsto n+1$ transformed to

$$(n+3)(n+2)a_{n+3} + a_n = 0 \qquad \text{for } n \in \mathbb{N}_0.$$

There is a leap of 3 in the indices, hence we conclude by induction from $a_1 = 0$ and $a_2 = 0$ that

$$a_{3n+1} = 0 \quad \text{and} \quad a_{3n+2} = 0 \qquad \text{for } n \in \mathbb{N}_0.$$

We can now write the power series solution in the form

$$u(t) = \sum_{n=0}^{\infty} a_{3n} t^{3n} = \sum_{n=0}^{\infty} b_n t^{3n},$$

where the recursion formula for $a_{3n} = b_n$ is obtained by $n \mapsto 3n$, thus

$$(3n+3)(3n+2)a_{3n+3} + a_{3n} = 0, \qquad n \in \mathbb{N}_0,$$

so

$$b_{n+1} = -\frac{1}{(3n+3)(3n+2)}\, b_n, \qquad b_n = a_{3n}, \quad n \in \mathbb{N}_0.$$

The radius of convergence for $b_0 \neq 0$ (and hence for $b_n \neq 0$) and $t \neq 0$ is found by the **criterion of quotients**

$$\left|\frac{a_{n+1}(t)}{a_n(t)}\right| = \frac{|b_{n+1}||t|^{3(n+1)}}{|b_n||t|^{3n}} = \frac{|t|^3}{(3n+3)(3n+2)} \to 0 \text{ for } n \to \infty.$$

It follows that the series is convergent for every $t \in \mathbb{R}$, and that $\varrho = \infty$.

Since we are actually considering a **boundary value problem**, the coefficients $a_0 = b_0 \neq 0$ are "free". We choose $a_0 = b_0 = 1$. Then by induction,

$$b_n = a_{3n} = (-1)^n \cdot \frac{1}{(3n)!} \prod_{j=0}^{n-1} (3j+1), \qquad n \in \mathbb{N}.$$

5) We have now proved that

$$(12) \quad \frac{dy}{dx} = z(x) = u(\lambda - x) = \sum_{n=0}^{\infty} a_{3n}(\lambda - x)^{3n}, \qquad x \in \mathbb{R},$$

where we have found a_{3n} in (4). The function cannot be expressed by elementary functions. It can, however, be termwise integrated. Since $y(0) = 0$, we get by termwise integration and a rearrangement that

$$\begin{aligned} y(x) = \;\; &= \sum_{n=0}^{\infty} a_{3n} \int_0^x (\lambda - t)^{3n} dt = \sum_{n=0}^{\infty} a_{3n} \left[-\frac{1}{3n+1}(\lambda - t)^{3n+1} \right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{a_{3n}}{3n+1}\, \lambda^{3n+1} - \sum_{n=0}^{\infty} \frac{a_{3n}}{3n+1}(\lambda - x)^{3n+1}, \end{aligned}$$

which is the structure of the eigenfunctions, if only we can find the eigenvalues.

6) We still miss to find the smallest (positive) $\lambda = \lambda_{\text{crit}}$, for which we have a proper solution, i.e. where $a_0 \neq 0$. Here we use the boundary condition $y'(0) = 0$, thus by (12),

$$y'(0) = \sum_{n=0}^{\infty} a_{3n}\lambda^{3n} = 0 \qquad \text{where } a_0 = 1.$$

This transcendent equation is solved approximatively in the following way:

We write for convenience $\eta = \lambda^3$, and then we find successively the smallest root of each of the polynomials

$$P_n(\eta) = \sum_{k=0}^{n} a_{3k}\eta^k, \qquad n \in \mathbb{N}.$$

Since the a_{3k} have alternating signs, the possible real roots can only be positive. The first polynomials may only have complex roots, but if two succeeding polynomials $P_n(\eta)$ and $P_{n+1}(\eta)$ have here (smallest) real roots η_n and η_{n+1}, then every following polynomial $P_{n+m}(\eta)$ will also have a (smallest) real root η_{n+m}. Since a_{3n} is alternating, it is easy to prove that η_{n+m}, $m > 1$, always lies between η_n and η_{n+1}, so we get a convergent sequence. The following numerical computations show that the convergence is very fast.

7) **Numerical computations**. No text needed.

$$n = 1: \quad P_1(\eta) = 1 - \frac{1}{3 \cdot 2}\eta, \qquad \eta_1 = 6 \quad \text{and} \quad \lambda_1 = \sqrt[3]{6} = 1,81712.$$

$$n = 2: \quad P_2(\eta) = 1 - \frac{\eta}{6}\left(1 - \frac{\eta}{6 \cdot 5}\right), \qquad \eta_2 = 8,29180 \quad \text{and} \quad \lambda_2 = \sqrt[3]{\eta_2} = 2,02403.$$

$$n = 3: \quad P_3(\eta) = 1 - \frac{\eta}{6}\left(1 - \frac{\eta}{30}\left(1 - \frac{\eta}{9 \cdot 8}\right)\right), \qquad \eta_3 = 7,814712 \quad \text{and} \quad \lambda_3 = \sqrt[3]{\eta_3} = 1,98444.$$

$$n = 4: \quad P_4(\eta) = 1 - \frac{\eta}{6}\left(1 - \frac{\eta}{30}\left(1 - \frac{\eta}{72}\left(1 - \frac{\eta}{12 \cdot 11}\right)\right)\right),$$

$$\eta_4 = 7,838213 \quad \text{and} \quad \lambda_4 = \sqrt[3]{\eta_4} = 1,98643.$$

$$n=5: \ P_5(\eta)=1-\frac{\eta}{6}\left(1-\frac{\eta}{30}\left(1-\frac{\eta}{72}\left(1-\frac{\eta}{132}\left(1-\frac{\eta}{15 \cdot 14}\right)\right)\right)\right),$$

$$\eta_5 = 7,837325 \quad \text{and} \quad \lambda_5 = \sqrt[3]{\eta_5} = 1,98635.$$

and for $n = 6$,

$$P_6(\eta)=1-\frac{\eta}{6}\left(1-\frac{\eta}{30}\left(1-\frac{\eta}{72}\left(1-\frac{\eta}{132}\left(1-\frac{\eta}{210}\left(1-\frac{\eta}{18 \cdot 17}\right)\right)\right)\right)\right),$$

$$\eta_6 = 7,837348 \quad \text{and} \quad \lambda_6 = \sqrt[3]{\eta_6} = 1,98635.$$

It follows that $\lambda_5 = \lambda_6 = 1,98635$ is an estimate of λ_{crit} with 5 decimals. This result is obtained after six iterations.

Example 3.3 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We can solve this example in many ways. Here we shall give three variants.

1) The **eigenvalue method**. The eigenvalues are the roots of the characteristic polynomial

$$\begin{vmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{vmatrix} = (\lambda-1)(\lambda+3)-5 = \lambda^2+2\lambda-8 = (\lambda+1)^2-9,$$

thus

$$\lambda = -1 \pm 3 = \begin{cases} 2, \\ -4. \end{cases}$$

a) If $\lambda = 2$, then we get the matrix

$$\begin{pmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 1 & -5 \end{pmatrix},$$

and we conclude that an eigenvector can be chosen as e.g. $(5, 1)$.

b) If $\lambda = -4$, we get the matrix

$$\begin{pmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix},$$

and we can choose the eigenvector $(1, -1)$.

Summing up the complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5e^{2t} & e^{-4t} \\ e^{2t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

2) The **fumbling method**. We write the system of equations,

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 5x_2, \\ \dfrac{dx_2}{dt} = x_1 - 3x_2. \end{cases}$$

It follows from the latter equation that

(13) $x_1 = \dfrac{dx_2}{dt} + 3x_2,$

so by insertion into the former,

$$\frac{dx_1}{dt} = \frac{d^2x_2}{dt^2} + 3\frac{dx_2}{dt} = x_1 + 5x_2 = \frac{dx_2}{dt} + 8x_2.$$

Then by a rearrangement,

$$\frac{d^2x_2}{dt^2} + 2\frac{dx_2}{dt} - 8x_2 = 0.$$

The characteristic equation $R^2 + 2R - 8 = 0$ has the roots $R = 2$ and $R = -4$, so

$$x_2 = c_2 e^{2t} + c_2 e^{-4t}.$$

If this is put into (13), then

$$x_1 = \frac{dx_2}{dt} + 3x_2 = (2c_1 e^{2t} - 4c_2 e^{-4t}) + (3c_1 e^{2t} + 3c_2 e^{-4t}) = 5c_1 e^{2t} - c_2 e^{-4t}.$$

Summing up we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5c_1 e^{2t} - c_2 e^{-4t} \\ c_1 e^{2t} + c_2 e^{-4t} \end{pmatrix} = \begin{pmatrix} 5e^{2t} & -e^{-4t} \\ e^{2t} & e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

3) The **exponential matrix**. The characteristic polynomial is

$$(\lambda + 1)^2 - 9.$$

Then we get by **Caley-Hamilton's theorem**,

$$(\mathbf{A} + \mathbf{I})^2 - 9\mathbf{I} = \mathbf{0}, \quad \text{dvs. } \mathbf{B}^2 = 9\mathbf{I}, \text{ where } \mathbf{B} = \mathbf{A} + \mathbf{I}.$$

Since $\mathbf{I}$ trivially commutes with $\mathbf{A}$, we get

$$\begin{aligned}
\exp(\mathbf{A}t) &= \exp((\mathbf{B}-\mathbf{I})t) = e^{-t}\exp(\mathbf{B}t) \\
&= e^{-t}\left\{\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,\mathbf{B}^{2n}t^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,\mathbf{B}^{2n+1}t^{2n+1}\right\} \\
&= e^{-t}\left\{\sum_{n=0}^{\infty}\frac{(3t)^{2n}}{(2n)!}\,\mathbf{I} + \frac{1}{3}\sum_{n=0}^{\infty}\frac{(3t)^{2n+1}}{(2n+1)!}\,\mathbf{B}\right\} \\
&= e^{-t}\left\{\cosh(3t)\mathbf{I} + \frac{1}{3}\sinh(3t)\mathbf{B}\right\} \\
&= e^{-t}\left\{\cosh(3t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\sinh(3t)\begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}\right\} \\
&= \frac{1}{3}\,e^{-t}\begin{pmatrix} 3\cosh 3t + 2\sinh 3t & 5\sinh 3t \\ \sinh 3t & 3\cosh 3t - 2\sinh 3t \end{pmatrix} \\
&= \frac{1}{6}\,e^{-t}\begin{pmatrix} 3e^{3t}+3e^{-3t}+2e^{3t}-2e^{-3t} & 5e^{3t}-5e^{-3t} \\ e^{3t}-e^{-3t} & 3e^{3t}+3e^{-3t}-2e^{3t}+2e^{-3t} \end{pmatrix} \\
&= \frac{1}{6}\,e^{-t}\begin{pmatrix} 5e^{3t}+e^{-3t} & 5e^{3t}-5e^{-3t} \\ e^{3t}-e^{-3t} & e^{3t}+5e^{-3t} \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 5e^{2t}+e^{-4t} & 5e^{2t}-5e^{-4t} \\ e^{2t}-e^{-4t} & e^{2t}+5e^{-4t} \end{pmatrix}.
\end{aligned}$$

Hence the complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 5e^{2t}+e^{-4t} \\ e^{2t}-e^{-4t} \end{pmatrix} + c_2\begin{pmatrix} 5e^{2t}-5e^{-4t} \\ e^{2t}+5e^{-4t} \end{pmatrix}.$$

Example 3.4 *Prove that $\lambda = 3$ is an eigenvalue for the eigenvalue problem*

$$\frac{d^2y}{dx^2} - 2\lambda\frac{dy}{dx} + (\pi^2 - 9 + 6\lambda)y = 0, \quad x \in [0,1], \quad y(0) = 0, \quad y(1) = 0,$$

and find a corresponding eigenfunction.

If we immediately put $\lambda = 3$, then

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + (9 + \pi^2)y = 0, \quad x \in [0,1], \quad y(0) = 0, \quad y(1) = 0.$$

The characteristic polynomial

$$R^2 - 6R + 9 + \pi^2 = (R-3)^2 + \pi^2$$

has the roots $3 \pm i\pi$, and the complete solution is

$$y = c_1 e^{3x}\cos(\pi x) + c_2 e^{3x}\sin(\pi x).$$

It follows from the the boundary conditions that

$$y(0) = c_1 = 0 \qquad \text{and} \qquad y(1) = -e^3 c_1 = 0,$$

so $c_1 = 0$, and c_2 can be chosen arbitrarily. We get a corresponding eigenfunction by $c_1 = 0$ and $c_2 = 1$,

$$y(x) = e^{3x}\sin(\pi x), \qquad x \in [0,1].$$

Remark 3.2 The example is *tricky*, because if one does not immediately put $\lambda = 3$, then we get the characteristic polynomial

$$R^2 - 2\lambda R + \pi^2 - 9 + 6\lambda,$$

the (real or complex) roots are

$$R = \lambda \pm \sqrt{\lambda^2 - \pi^2 + 9 - 6\lambda} = \lambda \pm \sqrt{(\lambda-3)^2 - \pi^2}.$$

The complete solution is for $\lambda \neq 3 \pm \pi$ in a *complex form*

$$\begin{aligned} y &= c_1 \exp((\lambda + \sqrt{(\lambda-3)^2-\pi^2})x) + c_2 \exp((\lambda - \sqrt{(\lambda-3)^2-\pi^2})x) \\ &= e^{\lambda x}\left\{ c_1 \exp(\sqrt{(\lambda-3)^2-\pi^2}\,x) + c_2 \exp(-\sqrt{(\lambda-3)^3-\pi^2}\,x)\right\}. \end{aligned}$$

If we put $x = 0$, then $y(0) = 0$ gives that $c_2 = -c_1$. If $c_1 = 1$, then it follows from $y(1) = 0$ that

$$\exp(\sqrt{(\lambda-3)^2-\pi^2}) - \exp(-\sqrt{(\lambda-3)^2-\pi^2}) = 0,$$

thus

$$\exp(2\sqrt{(\lambda-3)^2-\pi^2}) = 1 = e^{2ip\pi}, \qquad p \in \mathbb{Z}.$$

Hence

$$2\sqrt{(\lambda-3)^2-\pi^2} = 2ip\pi, \qquad \text{thus } \sqrt{(\lambda-3)^2-\pi^2} = ip\pi, \quad p \in \mathbb{Z}.$$

In particular we must have $(\lambda-3)^2 \le \pi^2$, thus $-\pi+3 \le \lambda \le \pi+3$, and

$$(\lambda-3)^3 = (1-p^2)\pi^2 \ge 0.$$

The only possibility is $p = 0$, where $\lambda = 3 \pm \pi$, and $p = \pm 1$, where $\lambda = 3$.

We have already checked $\lambda = 3$.

If $\lambda = 3 \pm \pi$, then λ is a double root in characteristic polynomial, and the complete solution is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}, \qquad \lambda = 3 \pm \pi.$$

It follows from $y(0) = 0$ that $c_1 = 0$ and $y(1) = 0$ we get $c_2 = 0$, and none of these possible values is an eigenvalue.

We have with this additional remark shown that $\lambda = 3$ is the *only* eigenvalue of the problem.

Example 3.5 *Consider the eigenvalue problem*

$$\frac{d^2y}{dx^2} + \lambda\frac{dy}{dx} - (\lambda+1)y = 0, \quad x \in [0,1], \quad y(1) = 0, \quad y'(0) = 0.$$

Prove that $\lambda = -2$ is an eigenvalue and find a corresponding eigenfunction.

First variant. If we immediately put $\lambda = -2$, then we obtain

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

The characteristic equation $R^2 - 2R + 1 = (R-1)^2 = 0$ has the double root $R = 1$, so the complete solution is

$$y = ae^x + bxe^x \qquad \text{where } y' = (a+b)e^x + bxe^x.$$

It follows from the boundary values that

$$\begin{cases} y(1) = (a+b)e = 0, \\ y'(0) = 0, \end{cases}$$

which are satisfied if $a + b = 0$, e.g. if $a = 1$ and $b = -1$. This proves that $\lambda = -2$ is an eigenvalue and that a corresponding eigenfunction is

$$y = e^x - xe^x = (1-x)e^x, \qquad x \in [0,1].$$

A check shows that the conditions indeed are satisfied.

Second variant. If one does not start by putting $\lambda = -2$, then we must go through the following considerations: The general characteristic equation

$$R^2 + \lambda R - (\lambda+1) = (R+\lambda+1)(R-1) = 0$$

has the roots $R = 1$ and $R = -\lambda - 1$. If $\lambda = -2$ we get the double root $R = 1$. If $\lambda \neq -2$ the roots are simple.

If $\lambda \neq -2$, the complete solution is

$$y = ae^x + be^{-(\lambda+1)x} \quad \text{where } y' = ae^x - (\lambda+1)be^{-(\lambda+1)x}.$$

It follows from the boundary values that

$$\begin{cases} y(1) = ae + be^{-\lambda-1} = 0, \\ y'(0) = a - (\lambda+1)b = 0, \end{cases}$$

which we write in form of a matrix

$$\begin{pmatrix} e & e^{-\lambda-1} \\ 1 & -(\lambda+1) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If $\lambda \neq -2$, the eigenvalues are those values for which the matrix is singular, hence

$$\varphi(\lambda) = \begin{vmatrix} e & e^{-\lambda-1} \\ 1 & -(\lambda+1) \end{vmatrix} = -e(\lambda+1) - e^{-(\lambda+1)} = 0.$$

It follows from

$$\varphi'(\lambda) = -e + e^{-(\lambda+1)} = e\left\{e^{-(\lambda+2)} - 1\right\},$$

that $\varphi'(\lambda) = 0$ for $\lambda = -2$, corresponding to a (global) maximum

$$\varphi(-2) = -e(-2+1) - e^{-(-2+1)} = e - e = 0,$$

hence $\varphi(\lambda)$ is only zero at the exceptional value $\lambda = -2$, and no $\lambda \neq -2$ is an eigenvalue.

If $\lambda = -2$, we just repeat the **first variant**, and we see that $\lambda = -2$ is the only eigenvalue and a generating eigenfunction is

$$y(x) = (1-x)e^x, \qquad x \in [0,1].$$

Example 3.6 *Consider the eigenvalue problem*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + (1+\lambda)y = 0, \quad x \in [0,1], \quad y(0) - y'(0) = y(1) = 0.$$

Prove that $\lambda = \dfrac{\pi^2}{4}$ *is an eigenvalue and find a corresponding eigenfunction.*

If we immediately put $\lambda = \dfrac{\pi^2}{4}$, then we get the characteristic equation

$$R^2 - 2R + 1 + \frac{\pi^2}{4} = 0,$$

the solutions of which are $R = 1 \pm i\,\dfrac{\pi}{2}$. Then we get the complete solution

$$y = c_1 e^x \cos\frac{\pi x}{2} + c_2 e^x \sin\frac{\pi x}{2}$$

where

$$y' = \left(c_1 + \frac{\pi}{2}\,c_2\right) e^x \cos\frac{\pi x}{2} + \left(c_2 - \frac{\pi}{2}\,c_1\right) e^x \sin\frac{\pi x}{2}.$$

Then by the boundary conditions,

$$y(1) = c_2 \cdot e \cdot 1 = 0, \qquad \text{thus } c_2 = 0,$$

and

$$y(0) = y'(0) = c_1 - \left(c_1 + \frac{\pi}{2}\,c_2\right) = -\frac{\pi}{2}\,c_2 = 0,$$

so $c_2 = 0$, and c_1 is arbitrary.

It follows that $\lambda = \dfrac{\pi^2}{4}$ is an eigenvalue and that a corresponding eigenfunction is obtained for $c_2 = 0$ and e.g. $c_1 = 1$, hence

$$y = e^x \cos\frac{\pi x}{2}.$$

Remark 3.3 It is possible to prove that all eigenvalues are given by

$$\lambda_n = \pi^2\left(n + \frac{1}{2}\right)^2, \qquad n \in \mathbb{N}_0$$

with the corresponding generating eigenfunctions

$$\varphi_n(x) = \cos(\sqrt{\lambda_n}x) = \cos\left(\pi\left(n + \frac{1}{2}\right)x\right), \qquad n \in \mathbb{N}_0.$$

Example 3.7 *Consider the eigenvalue problem*

$$\begin{cases} \dfrac{d^2y}{dx^2} + (\lambda+3)\dfrac{dy}{dx} + 3\lambda y = 0, & x \in [0,1], \\ 5y(0) + y'(0) = y(1) + y'(1) = 0. \end{cases}$$

Prove that $\lambda = 3$ *is an eigenvalue and find a corresponding eigenfunction.*

If we immediately put $\lambda = 3$, then we get the simpler equation,

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0, \qquad x \in [0,1],$$

the characteristic equation of which $R^2 + 6R + 9 = (R+3)^2 = 0$ has the double root $R = -3$. The complete solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}$$

where

$$\frac{dy}{dx} = (-3c_1 + c_2)e^{-3x} - 3c_2 x e^{-3x}.$$

By insertion into the boundary conditions we get

$$5y(0) + y'(0) = 5c_1 - 3c_1 + c_2 = 2c_1 + c_2 = 0,$$

and

$$y(1) + y'(1) = e^{-3}\{c_1 + c_2 - 3c_1 + c_2 - 3c_2\} = -e^{-3}(2c_1 + c_2) = 0.$$

These two conditions are simultaneously fulfilled if and only if $c_2 = -2c_1$, where c_1 can be chosen arbitrarily. We conclude that $\lambda = 3$ is an eigenvalue and that a corresponding eigenfunction is obtained be e.g. choosing $c_1 = 1$,

$$y(x) = e^{-3x} - 2xe^{-3x} = (1-2x)e^{-3x}.$$

Example 3.8 *Consider the eigenvalue problem*

$$\frac{d^2y}{dx^2} - (\lambda+2)\frac{dy}{dx} + 5y = 0, \qquad x \in \left[0, \frac{\pi}{2}\right],$$

$$y(0) - y'(0) = 3y\left(\frac{\pi}{2}\right) - y'\left(\frac{\pi}{2}\right) = 0.$$

1) Prove that $\lambda = 2$ is an eigenvalue and find all its corresponding eigenfunctions.

2) Is $y = e^{2x}\cos\left(x + \frac{\pi}{4}\right)$, $x \in \left[0, \frac{\pi}{2}\right]$, an eigenfunction corresponding to the eigenvalue $\lambda = 2$?

1) If we put $\lambda = 2$, then we get the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$$

with constant coefficients. The characteristic equation

$$R^2 - 4R + 5 = (R-2)^2 + 1 = 0$$

has the two simple complex conjugated roots $R = 2 \pm i$. The complete solution is

$$y = c_1 e^{2x}\cos x + c_2 e^{2x}\sin x.$$

Then by a differentiation,

$$\begin{aligned} y'(x) &= 2c_1e^{2x}\cos x - c_1e^{2x}\sin x + 2c_2e^{2x}\sin x + c_2e^{2x}\cos x \\ &= (2c_1 + c_2)e^{2x}\cos x + (2c_2 - c_1)e^{2x}\sin x. \end{aligned}$$

When we put these into the boundary conditions, we get

$$y(0) - y'(0) = c_1 - (2c_1 + c_2) = -c_1 - c_2 = -(c_1 + c_2) = 0,$$

$$3y\left(\frac{\pi}{2}\right) - y'\left(\frac{\pi}{2}\right) = 3c_2e^{\pi} - (2c_2 - c_1)e^{\pi} = (c_1 + c_2)e^{\pi} = 0.$$

We obtain the eigenfunctions when $c_2 = -c_1$, hence the eigenfunctions are c_1 times the generating eigenfunction

$$y_0(x) = e^{2x}(\cos x - \sin x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right) = \sqrt{2}\,e^{2x}\cos\left(x + \frac{\pi}{4}\right).$$

2) If we choose $c_1 = \dfrac{1}{\sqrt{2}}$, we get the eigenfunction

$$\frac{1}{\sqrt{2}}\, y_0(x) = e^{2x} \cos \left(x + \frac{\pi}{4} \right),$$

and we see that the answer is "yes".