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Power series

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Introduction

This is the fourth book containing examples from the *Theory of Complex Functions*. In this volume we shall only consider complex power series and their relationship to the general theory, and finally the technique of solving linear differential equations with polynomial coefficients by means of a power series.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
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1 Some simple theoretical results concerning power series

Every analytic function $f(z)$ defined in an open domain Ω can *locally* be described by a convergent power series. Thus, if $z_0 \in \Omega$ is an interior point, and r_{z_0} denotes the distance from z_0 to the boundary of Ω, then we have the alternative description

$$f(z) = \sum_{n=0}^{+\infty} a_n \left(z - z_0\right), \qquad \text{for } |z - z_0| < r_{z_0},$$

where the coefficients a_n, $n \in \mathbb{N}_0$ are uniquely determined corresponding to $f(z)$ and the point of expansion z_0.

The two descriptions complement each other. They have both their advantages and their drawbacks.

First consider complex series without any connection to analytic functions. For given $z_0 \in \mathbb{C}$ and a given complex sequence $\{a_n\}$ such a series is formally given by

$$\sum_{n=0}^{+\infty} a_n \left(z - z_0\right)^n .$$

We define the *number of convergence* λ by

$$0 \leq \lambda := \limsup_{n \to +\infty} \sqrt[n]{|a_n|} \leq +\infty.$$

Then

Theorem 1.1 *The power series*

$$\sum_{n=0}^{+\infty} a_n \left(z-z_0\right)^n$$

is absolutely convergent for every $z \in \mathbb{C}$ *fulfilling*

$$\lambda \left|z-z_0\right| < 1, \qquad \textit{thus for } \left|z-z_0\right| < \frac{1}{\lambda} := R,$$

where R *denotes the radius of convergence, and it is divergent for every* $z \in \mathbb{C}$*, for which* $\lambda \left|z-z_0\right| > 1$*.*

One shall always be more careful, when one considers the points on the circle $\lambda \left|z-z_0\right| = 1$, because almost everything may occur here. concerning convergence/divergence. There exist examples of series being absolute convergent, being divergent everywhere, or conditionally convergent in some points and divergent in all others, and finally, there even exist examples in which the series is conditionally convergent everywhere on the circle of radius R. Notice, however, that it the series is absolutely convergent in just one point on the circle of convergence, then it is absolutely convergent everywhere. Hence, the advice is to avoid this set, unless one is explicitly asked to investigate it.

There are three main types of power series:

1) If $\lambda = +\infty$, then the radius of convergence is $R = 0$. In this case the series is only convergent for $z = z_0$, and since a point never is an open domain, it does not make sense in this case to talk about an analytic function. Hence, this case is not at all interesting in this connection, and we shall avoid it.

2) If $0 < \lambda < +\infty$, then the radius of convergence is finite, $R = 1/\lambda$. The prototype of such series is the *geometric series*,

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad\qquad |z| < 1,$$

with the point of expansion $z_0 = 0$. We note that the distance between $z_0 = 0$ and $z = 1$, where the denominator is zero, is precisely the radius of convergence 1.

In a sense all power series of finite positive radius of convergence is a variant of the geometric series.

3) If $\lambda = 0$, then the radius of convergence is $R = +\infty$, and the series is convergent in $\mathbb{C}$. The prototype for such series is the *exponential series*,

$$\exp z = e^z = \sum_{n=0}^{+\infty} \frac{1}{n!}\, z^n, \qquad\qquad z \in \mathbb{C}.$$

If one can stretch one's imagination one may say that every series of infinite radius of convergence in some sense is very much like the exponential series.

Concerning rules of computation for series one must always be very careful to have the same point of expansion z_0 for all the series involved. This is typically chosen as $z_0 = 0$, so one hardly discovers that one may get a problem here. Furthermore, they shall all be convergent in the same neighbourhood of z_0.

Theorem 1.2 *Choose for convenience $z_0 = 0$. Assume that*

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{+\infty} b_n z^n,$$

are two series, both convergent for $|z| < r$. Then $f(z) \pm g(z)$ and $f(z) \cdot g(z)$ also have convergent power series expansions, which (at least) are convergent in the same disc $|z| < r$. Furthermore, they are given by

$$(f \pm g)(z) := f(z) \pm g(z) = \sum_{n=0}^{+\infty} \{a_n \pm b_n\} z^n, \qquad \text{and} \qquad (fg)(z) := f(z)g(z) = \sum_{n=0}^{+\infty} c_n z^n, \qquad \text{resp.}$$

where we by the Cauchy multiplication *define*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \qquad n \in \mathbb{N}_0.$$

We only *know* that we have convergence in the original domain $|z| < 1$. However, if we roughly speaking, remove a singularity for e.g. $f + g$ or for $f\dot{g}$, then we may get a larger radius of convergence. It is left to the reader to go through the examples

$$f(z) = -g(z) = \frac{1}{1-z} \qquad \text{and} \qquad f(z) + g(z),$$

and

$$f(z) = \frac{1}{1-z} \text{ and } g(z) = 1 - z \qquad \text{og} \qquad f(z) \cdot g(z),$$

where the radii of convergence become bigger than for $f(z)$ or $g(z)$. The readers who have just started on the topic of Complex Functions are advised to *avoid* the Cauchy multiplication. Without some experience one usually makes lots of errors, and the method will only be necessary in very rare cases.

One of the main results concerning power series is

Theorem 1.3 *Given a power series of radius of convergence $r > 0$ and point of expansion z_0. Then the sum function*

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad \text{for } |z - z_0| < r,$$

is an analytic function. Its derivative is obtained by termwise differentiation,

$$f'(z) = \sum_{n=0}^{+\infty} n\, a_n (z - z_0)^{n-1} \qquad \text{for } |z - z_0| < r.$$

It follows by iterating the latter expression that the series, an hence also the analytic function itself, is infinitely often differentiable in its open domain of convergence, and that one obtains all its derivatives by termwise differentiation, i.e. by differentiating under the sum.

By differentiating n times and then putting $z = z_0$, it follows that

$$f^{(n)}(z_0) = n!\, a_n, \qquad \text{dvs.} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0),$$

thus

Theorem 1.4 *Let $f(z)$ be the sum of a power series of point of expansion z_0 and radius of convergence $r > 0$. Then $f(z)$ is equal to its Taylor series expanded from z_0,*

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(z_0) \cdot (z - z_0)^n \qquad \textit{for } |z - z_0| < r.$$

This theorem implies the important

Theorem 1.5 THE IDENTITY THEOREM. *If two power series*

$$\sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad \textit{and} \qquad \sum_{n=0}^{+\infty} A_n (z - z_0)^n,$$

of the same point of expansion z_0, are convergent and equal for $|z - z_0| < r$, where $r > 0$, then $a_n = A_n$ for every $n \in \mathbb{N}_0$, and the two series have the same radius of convergence.

We have so far introduced two parallel theories, partly the analytic functions as continuous differentiable functions in the complex variable z, and partly the analytic functions as the sums of a convergent series. We shall now unite these two theories.

Theorem 1.6 *Assume that Ω is an open domain, and let $f : \Omega \to \mathbb{C}$ be analytic. Given any $z_0 \in \Omega$, the Taylor series of f with z_0 as point of expansion, is convergent in (at least) the largest open disc of centrum z_0 contained in Ω. Furthermore, in this disc,*

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}(z_0) \cdot (z - z_0)^n.$$

In other words: If we start with a convergent series, then it is equal to the Taylor series from the chosen point of expansion of the analytic sum. Conversely, if we start with an analytic function $f(z)$, then the corresponding Taylor series of point of expansion $z_0 \in \Omega$ is precisely the series with the sum $f(z)$.

We assume again that $f(z)$ is analytic in an open domain Ω, and we let $z_0 \in \Omega$. We call z_0 a zero of order n for $f(z)$, if

$$f^{(j)}(z_0) = 0 \qquad \text{for } j = 0, 1, \dots, n-1 \quad \text{and} \quad f^{(n)}(z_0) \neq 0.$$

This definition is supported by the fact that if z_0 is a zero of order n, then the Taylor series can be written

$$f(z) = \sum_{j=n}^{+\infty} a_j (z - z_0)^j = (z - z_0)^n \sum_{j=0}^{+\infty} a_{n+j} (z - z_0)^j.$$

Theorem 1.7 *Assume that $f : \Omega \to \mathbb{C}$ is analytic and not the zero function. Then, to every $z \in \Omega$ there exists an $n = n(z_0) \in \mathbb{N}_0$, such that $f^{(n)}(z_0) \neq 0$.*

Every zero z_0 for an analytic function, which is not identically zero, is isolated.

Contrary to the case in the real analysis it is not possible to have curves in the complex plane, on which the complex function $f(z)$ is zero, unless it is identically zero. Note, however, that it is still possible for its real or imaginary parts to be zero on some curves. This is important for the applications, because this can be used in practice.

A consequence of the theorem above is

Theorem 1.8 THE IDENTITY THEOREM. *Assume that both $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are analytic in the open domain Ω. If the set $\{z \in \Omega \mid f(z) = g(z)\}$ has an accumulation point in Ω, then $f(z) = g(z)$ everywhere in Ω.*

We shall finally mention a strange, and at the same time important property of the non-constant analytic functions $f : \Omega \to \mathbb{C}$. If Ω is open, then the absolute value $|f(z)|$ cannot attain its maximum in an interior point of Ω:

Theorem 1.9 THE MAXIMUM PRINCIPLE. *Assume that $f : \Omega \to \mathbb{C}$ is analytic in an open domain Ω. If $|f(z)|$ has a local maximum at an inner point $z_0 \in \Omega$, then $f(z)$ is constant in Ω.*

Of course, we also have a minimum principle, but this is more complicated:

Theorem 1.10 THE MINIMUM PRINCIPLE. *Assume that the analytic function $f : \Omega \to \mathbb{C}$ is not a constant in the open domain Ω. If $|f(z)|$ has a local minimum at an interior point $z_0 \in \Omega$, then $f(z_0) = 0$.*

The maximum principle does not hold for unbounded domains. There exists, however, a useful version

Theorem 1.11 PHRAGMÉN-LINDELÖF'S THEOREM. *Assume that $f(z)$ is analytic in the right half plane $\operatorname{Re} z > 0$, and assume further that $f(z)$ can be extended continuously and bounded to the boundary, i.e. $|f(i\,y)| \le M$ on the imaginary axis. Furthermore, assume that we can find constants $a < 1$ and $K > 0$, such that we have the estimate*

$$|f(z)| < K \cdot \exp\left(r^a\right), \qquad \textit{for } \operatorname{Re} z \ge 0, \ \textit{hvor } z = r\,e^{i\theta}.$$

Then, everywhere in the right half plane,

$$|f(z)| \le M, \qquad \textit{for } \operatorname{Re} z \ge 0.$$

There exist actually some practical applications of Phragmén-Lindelöf's theorem in the technical literature.

Finally, we mention the following theorem, which again shows that it is a very exclusive property of a function being analytic.

Theorem 1.12 SCHWARZ'S LEMMA. *Assume that $f : B(a, R) \to \mathbb{C}$ is analytic and $f(a) = 0$ and $|f(z)| \le M$ for every $z \in B(a, R)$, i.e. in the open disc of centrum a and radius R. Then we have the estimate,*

$$(1) \quad |f(z)| \le \frac{M}{R}\,|z - a| \qquad \textit{for ethvert } z \in B(a, R).$$

If we have equality at just one point of $z \in B(a, R) \setminus \{a\}$ in (1), then we have equality everywhere in (1), and there exists a constant θ, such that

$$f(z) = e^{i\theta} \cdot \frac{M}{R}\,(z - a), \qquad \textit{for every } z \in B(a, R).$$

2 Simple Fourier series in the Theory of Complex Functions

It follows from the definition that $\exp(in\theta)$ has the period $2\pi/n$. Since also

$$\cos n\theta = \frac{1}{2}\left\{e^{in\theta} + e^{-in\theta}\right\} \qquad \text{og} \qquad \sin n\theta = \frac{1}{2i}\left\{e^{in\theta} - e^{-in\theta}\right\},$$

it follows that every piecewise continuous function $\varphi(\ theta)$, $\theta \in \mathbb{R}$, of period 2π also has a *complex* Fourier series expansion,

$$(2)\quad \varphi(\theta) \sim \sum_{n=-\infty}^{+\infty} c_n\, e^{in\theta},$$

where it can be proved that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)\, e^{-in\theta}\, d\theta, \qquad \text{for every } n \in \mathbb{Z}.$$

We note here with regards to the introduction of the Laurent series in *Complex Functions c-5* that it is quite natural that the summation of (2) is extended to all of $\mathbb{Z}$, i.e. also to the negative integers.

Here we shall only demonstrate the connection with the analytic functions. Assume that $f(z)$ is an analytic function in a neighbourhood of 0 with the power series expansion (which exists)

$$f(z)=\sum_{n=0}^{+\infty} a_n\, z^n, \qquad \text{for } |z|<\varrho.$$

If we here use polar coordinates, $z=r\,e^{i\theta}$, then we get for every fixed $r\in\,]0,\varrho[$ a Fourier series of the function $\varphi(\theta)$ given by

$$\varphi(\theta):=f\left(r\,e^{i\theta}\right)=\sum_{n=0}^{+\infty}\left\{a_n\, r^n\right\} e^{in\theta},$$

thus of the structure (2) for

$$c_n=a_n\, r^n \quad \text{for } n\in\mathbb{N}_0 \qquad \text{and} \qquad c_n=0 \quad \text{for } n\in\mathbb{Z}_-.$$

When we apply this technique on the analytic function e^z, we get

$$e^z=\sum_{n=0}^{+\infty}\frac{r^n}{n!}\cos n\theta+i\sum_{n=0}^{+\infty}\frac{r^n}{n!}\sin n\theta,$$

and since also $e^z=e^x(\cos y+i\,\sin y)$, it follows by another insertion and then a separation of the real and the imaginary parts that

$$e^{r\,\cos\theta}\cos(r\,\sin\theta)=\sum_{n=0}^{+\infty}\frac{r^n}{n!}\cos n\theta, \qquad \text{and} \qquad e^{r\,\cos\theta}\sin(r\,\sin\theta)=\sum_{n=0}^{+\infty}\frac{r^n}{n!}\sin n\theta.$$

When $f(z)=1/(1-z)$,$|z|<1$, is treated in the same way, we obtain after some computation the following important formulæ

$$\frac{1-r\,\cos\theta}{1+r^2-2r\,\cos\theta}=\sum_{n=0}^{+\infty} r^n\cos n\theta, \qquad \text{og} \qquad \frac{r\,\sin\theta}{1+r^2-2r\,\cos\theta}=\sum_{n=0}^{+\infty} r^n\sin n\theta.$$

Finally, it is easy to derive

Theorem 2.1 Parseval's formula. *Assume that*

$$f(z)=\sum_{n=0}^{+\infty} a_n\, z^n, \qquad \textit{and} \qquad g(z)=\sum_{n=0}^{+\infty} b_n\, z^n,$$

are analytic functions for $|z|<\varrho$. *By using polar coordinates,* $z=r\,e^{in\theta}$, *it follows for every fixed* $r\in\,]0,\varrho[$ *that*

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(r\,e^{i\theta}\right)\overline{g\left(r\,e^{i\theta}\right)}\,d\theta=\sum_{n=0}^{+\infty} a_n\,\overline{b_n}\, r^{2n}.$$

In particular, if we here choose $g(z)=f(z)$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(r\,e^{i\theta}\right)\right|^2 d\theta=\sum_{n=0}^{+\infty}|a_n|^2\, r^{2n}.$$

3 Power series

Example 3.1 *Give (without proof) examples of the various possible forms of convergence on the boundary of the domain of convergence.*

We choose the point of expansion $z_0 = 0$ and the radius of convergence $\varrho = 1$, thus the series shall all be convergent for $|z| < 1$, while we are focussing on their behaviour on the circle $|z| = 1$.

1) The series
$$\sum_{n=1}^{+\infty} \frac{1}{n^2}\, z^n, \qquad |z| < 1,$$
is *absolutely convergent* for $|z| = 1$.

2) The series
$$\sum_{n=1}^{+\infty} z^n, \qquad |z| < 1,$$
is *divergent* for every z on the boundary, $|z| = 1$.

3) The series
$$\sum_{n=1}^{+\infty} \frac{1}{n}\, z^n, \qquad |z| < 1,$$
is *divergent* for $z = 1$ and *conditionally convergent* for every $z \neq 1$ on the boundary $|z| = 1$.

4) Let $[\sqrt{n}]$ denote the *integer part* of $\sqrt{n}$, i.e. the largest integer $N \in \mathbb{Z}$, fulfilling $N \leq \sqrt{n}$. It is possible to prove, though far from easy, that the series
$$\sum_{n=1}^{+\infty} \frac{1}{n} \cdot (-1)^{[\sqrt{n}]}\, z^n, \qquad |z| < 1,$$
is *conditionally convergent* for every z on the unit circle $|z| = 1$.

Example 3.2 *Find the radius of convergence for each of the series*
$$(a)\ \sum_{n=0}^{+\infty} 2^n z^n, \qquad (b)\ \sum_{n=0}^{+\infty} n^2 z^n, \qquad (c)\ \sum_{n=1}^{+\infty} \frac{2^n z^{2n}}{n^2+n}.$$

(a) It follows from $c_n = 2^n > 0$ that
$$\lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2},$$
or ALTERNATIVELY,
$$\lim_{n\to+\infty} \frac{1}{\sqrt[n]{|c_n|}} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{2^n}} = \frac{1}{2}.$$
The radius of convergence is $\frac{1}{2}$.

Remark 3.1 The series is convergent for $|z| < \frac{1}{2}$ with the sum

$$\sum_{n=0}^{+\infty} 2^n z^n = \sum_{n=0}^{+\infty} (2z)^n = \frac{1}{1-2z}.$$

Since $2^n z^n$ does not converge towards 0 for $n \to +\infty$, when $|z| \geq \frac{1}{2}$, the series is divergent for $|z| \geq \frac{1}{2}$. $\Diamond$

(b) It follows from $c_n = n^2 > 0$ that

$$\lim_{n\to+\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to+\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to+\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1,$$

or ALTERNATIVELY,

$$\lim_{n\to+\infty} \frac{1}{\sqrt[n]{|c_n|}} = \lim_{n\to+\infty} \left(\frac{1}{\sqrt[n]{n}}\right)^2 = 1.$$

The radius of convergence is 1, and the series is convergent for $|z| < 1$. Since $n^2 z^n \to \infty$ for $n \to +\infty$, when $|z| \geq 1$, the series is divergent for $|z| \geq 1$.

Remark 3.2 If $|z| < 1$, then it follows by termwise differentiation that

$$\begin{aligned} g(z) &= \frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \\ g'(z) &= \frac{1}{(1-z)^2} = \sum_{n=0}^{+\infty} (n+1)z^n, \\ g''(z) &= \frac{2}{(1-z)^3} = \sum_{n=0}^{+\infty} (n+2)(n+1)z^n. \end{aligned}$$

Since

$$n^2 = (n+2)(n+1) - 3(n+1) + 1,$$

it follows that the sum for $|z| < 1$ is given by

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} n^2 z^n = \sum_{n=0}^{+\infty} (n+2)(n+1)z^n - 3\sum_{n=0}^{+\infty} (n+1)z^n + \sum_{n=0}^{+\infty} z^n \\ &= g''(z) - 3g'(z) + g(z) = \frac{2}{(1-z)^3} - \frac{3}{(1-z)^2} + \frac{1}{1-z} = \frac{z(z+1)}{(1-z)^3}. \qquad \Diamond \end{aligned}$$

(c) First note that $c_{2n+1} = 0$. We shall use a small and simple trick. If we change the variable to $t = z^2$, we get the series

$$\sum_{n=1}^{+\infty} \frac{2^n t^n}{n^2+n} = \sum_{n=1}^{+\infty} a_n t^n, \qquad \text{where } a_n = \frac{2^n}{n(n+1)} > 0.$$

We shall first find the *t-radius of convergence*,

$$\lim_{n\to+\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to+\infty} \left\{ \frac{\dfrac{2^n}{n(n+1)}}{\dfrac{2^{n+1}}{(n+1)(n+2)}} \right\} = \lim_{n\to+\infty} \left\{ \frac{1}{2} \cdot \frac{n+2}{n+1} \right\} = \frac{1}{2},$$

or ALTERNATIVELY,

$$\lim_{n\to+\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{\dfrac{2^n}{n(n+1)}}} = \lim_{n\to+\infty} \frac{\sqrt[n]{n} \cdot \sqrt[n]{n+1}}{2} = \frac{1}{2}.$$

This shows that the *t-radius of convergence* is $\frac{1}{2}$, and since $t = z^2$, the original series is convergent for $|z|^2 = |t| < \frac{1}{2}$, hence for

$$|z| < \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

and the *z-radius of convergence* becomes $\frac{\sqrt{2}}{2}$.

Remark 3.3 By using lim sup, it is possible directly to find the radius of convergence of the series:

$$\frac{1}{\limsup_{n\to+\infty}\sqrt[n]{|c_n|}}=\lim_{n\to+\infty}\frac{1}{\sqrt[2n]{\dfrac{2^n}{n(n+1)}}}=\lim_{n\to+\infty}\frac{\sqrt[2n]{n}\cdot\sqrt[2n]{n+1}}{\sqrt{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}. \qquad \Diamond$$

Remark 3.4 Put instead $w = 2t = 2z^2$. Then we get the series

$$\sum_{n=1}^{+\infty}\frac{w^n}{n(n+1)},$$

of w-*radius of convergence* 1. Assume that $0 \le |w| < 1$ (it would be sufficient with $0 < |w| < 1$). Then

$$g(w)=w\sum_{n=1}^{+\infty}\frac{w^n}{(n+1)n}=\sum_{n=1}^{+\infty}\frac{w^{n+1}}{(n+1)n}, \qquad 0<|w|<1,$$

and we get by two successive differentiations that

$$g'(w)=\sum_{n=1}^{+\infty}\frac{w^n}{n} \qquad \text{and} \qquad g''(w)=\sum_{n=0}^{+\infty}w^n=\frac{1}{1-w},$$

hence

$$g'(w)=-\operatorname{Log}(1-w)+c_1, \qquad c_1=g'(0)=0, \qquad |w|<1,$$

and then by another integration

$$g(w)=(1-w)\operatorname{Log}(1-w)+w+c_2, \qquad |w|<1,$$

where

$$c_2=g(0)=0.$$

Therefore, if $0<|z|<\dfrac{1}{\sqrt{2}}$, then the sum is given by

$$\begin{aligned} f(z) &= \sum_{n=1}^{+\infty}\frac{2^n z^n}{n^2+n}=\frac{1}{w}\,g(w)=1+\frac{(1-w)\operatorname{Log}(1-w)}{w}\\ &= 1-\ \operatorname{Log}\left(1-2z^2\right)+\frac{\operatorname{Log}\left(1-2z^2\right)}{2z^2}, \end{aligned}$$

and of course $f(0) = 0$, which can also be obtained by a series expansion and taking the limit in the general expression.

Example 3.3 *Assume that $p \in \mathbb{N}$ and $q \in \mathbb{C}$, $|q| < 1$. Find the radius of convergence for each of the series*

$$(a) \quad \sum_{n=0}^{+\infty} n^p z^n, \qquad (b) \quad \sum_{n=0}^{+\infty} q^{n^2} z^n.$$

(a) It follows from the *criterion of roots* that

$$r = \lim_{n \to +\infty} \frac{1}{\sqrt[n]{|c_n|}} = \lim_{n \to +\infty} \frac{1}{\sqrt[n]{n^p}} = \lim_{n \to +\infty} \frac{1}{\left(\sqrt[n]{n}\right)^p} = 1.$$

Alternatively it follows by the *criterion of quotients*, keeping $p \in \mathbb{N}$ fixed,

$$r = \lim_{n \to +\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to +\infty} \frac{n^p}{(n+1)^p} = lim_{n \to +\infty} \left\{ \frac{1}{1 + \dfrac{1}{n}} \right\}^p = 1.$$

Hence the series is convergent for all $|z| < 1$.
If $|z| = 1$, then

$$|c_n z^n| = n^p \to +\infty \qquad \text{for } n \to +\infty,$$

which shows that the necessary condition for convergence is not fulfilled, and the series is divergent for $|z| \geq 1$.

Remark 3.5 It is possible to find the sum for every given $p \in \mathbb{N}$, though a general expression is difficult to derive. $\Diamond$

(b) If $q = 0$, we define $0^0 := 1$, and we get the trivial series

$$\sum_{n=0}^{+\infty} q^{n^2} z^n \equiv 1,$$

which of course is convergent for every $z \in \mathbb{C}$.

If $0 < |q| < 1$, then it follows by the criterion of roots that

$$r = \lim_{n \to +\infty} \frac{1}{\sqrt[n]{|c_n|}} = \lim_{n \to +\infty} \frac{1}{\sqrt[n]{|q|^{n^2}}} = \lim_{n \to +\infty} \left(\frac{1}{|q|} \right)^n = +\infty,$$

and the series is convergent for every $z \in \mathbb{C}$.

Remark 3.6 It follows immediately that if $|q| = 1$, then the radius of convergence is 1. It is only possible to find the sum for special values of q, $|q| = 1$.
If $|q| > 1$, then the radius of convergence is 0, and the analytic sum function does not exist. $\Diamond$

Example 3.4 *Find the radius of convergence for each of the series*

$$(a)\quad \sum_{n=0}^{+\infty} \frac{3^n z^n}{4^n+5^n}, \qquad (b)\quad \sqrt{n}\cdot(z-i)^n, \qquad (c)\quad \sum_{n=1}^{+\infty} z^{2^n}.$$

Remark 3.7 None of these series has a sum which can be expressed by elementary functions. They define some new functions in there domains of convergence. $\Diamond$

(a) It follows by the *criterion of roots* that

$$\begin{aligned} r &= \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{3^n}{4^n+5^n}\cdot\frac{4^{n+1}+5^{n+1}}{3^{n+1}} = \frac{1}{3}\lim_{n\to+\infty}\frac{5^{n+1}+4^{n+1}}{5^n+4^n} \\ &= \frac{5}{3}\lim_{n\to+\infty}\frac{1+\left(\dfrac{4}{5}\right)^{n+1}}{1+\left(\dfrac{4}{5}\right)^n} = \frac{5}{3}, \end{aligned}$$

because $\left(\dfrac{4}{5}\right)^n \to 0$ for $n\to+\infty$.

ALTERNATIVELY we can use the *criterion of roots*

$$r = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{|c_n|}} = \lim_{n\to+\infty} \sqrt[n]{\frac{4^n+5^n}{3^n}} = \lim_{n\to+\infty} \frac{5}{3}\sqrt[n]{1+\left(\frac{4}{5}\right)^n} = \frac{5}{3}.$$

(b) It follows by the *criterion of quotients* that

$$r = \lim \left|\frac{c_n}{c_{n+1}}\right| = \lim \frac{\sqrt{n}}{\sqrt{n+1}} = \lim \sqrt{\frac{n}{n+1}} = 1.$$

ALTERNATIVELY we get by the *criterion of roots* ,

$$r = \lim \frac{1}{\sqrt[n]{|c_n|}} = \lim \frac{1}{\sqrt[n]{\sqrt{n}}} = \lim \frac{1}{\sqrt[2n]{n}} = 1.$$

Remark 3.8 The essential point is of course that the series has the structure of a power series. It is of no importance for the radius of convergence that the expansion is taken with respect to another point than 0. $\Diamond$

(c) This is a so-called *lacunar series*, which means a series in which infinitely many of the coefficients are 0, and infinitely many of them are $\neq 0$. Here,

$$c_p = 1 \quad \text{for } p = 2^n \qquad \text{and } c_p = 0 \text{ otherwise.}$$

It is not possible to apply the *criterion of quotients* in its usual form, because we must never divide by 0.

Instead we use the *criterion of roots* in its general form,

$$r = \frac{1}{\limsup \sqrt[n]{|c_n|}} = 1,$$

and it follows that the radius of convergence is 1.

ALTERNATIVELY it follows that if $|z| \geq 1$, then $|z|^{2^n}$ does not converge towards 0 for $n \to +\infty$, so the *necessary condition of convergence* is not fulfilled. This shows that the series is divergent for $|z| \geq 1$.

Then assume that $|z| < 1$. We have the trivial estimate

$$|z|^{2^n} \leq |z|^n \qquad \text{for every } n \in \mathbb{N}.$$

Then

$$\left|\sum_{n=1}^{+\infty} z^{2^n}\right| \leq \sum_{n=1}^{+\infty} |z|^{2^n} \leq \sum_{n=1}^{+\infty} |z|^n = \frac{|z|}{1-|z|} < +\infty.$$

Hence the series is convergent in the domain of convergence

$$\{z \in \mathbb{C} \mid |z| < 1\},$$

corresponding to the radius of convergence 1.

Example 3.5 *Find the radius of convergence for each of the series*

$$(a)\ \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{2n-1}\,(z-1)^n, \qquad (b)\ \sum_{n=1}^{+\infty} \frac{(z+i)^n}{(3n)^{\sqrt{n}}}, \qquad (c)\ \sum_{n=1}^{+\infty} \left(1+\frac{1}{n}\right)^{n^2} i^n(z-1)^n.$$

Remark 3.9 It is of no importance for the determination of the radius of convergence that the expansion in all three cases is taken form another point than 0. The sum function of the latter two series cannot be expressed by elementary functions, and the sum function of the first series cannot be expressed as a known function at this stage of the development of the theory. $\Diamond$

(a) It follows from

$$c_n = \frac{(-1)^{n+1}}{2n-1},$$

by the *criterion of quotients* that

$$\left|\frac{c_n}{c_{n+1}}\right| = \frac{2n+1}{2n-1} \to 1 = r \qquad \text{for } n \to +\infty - .$$

ALTERNATIVELY we may apply the *criterion of roots*,

$$\frac{1}{\sqrt[n]{|c_n|}} = \sqrt[n]{2n-1} \to 1 \qquad \text{for } n \to +\infty,$$

hence the radius of convergence is 1.

(b) Since

$$c_n = \frac{1}{(3n)^{\sqrt{n}}} \quad (>0),$$

the *criterion of quotients* does not look too promising. Instead we get by the *criterion of roots*,

$$\frac{1}{\sqrt[n]{|c_n|}} = (3n)^{1/\sqrt{n}} = \exp\left(\frac{1}{\sqrt{n}}\,\ln(3n)\right) \to e^0 = 1 \qquad \text{for } n \to +\infty,$$

where we have used the order of magnitudes. It follows that the radius of convergence is 1.

(c) Since

$$c_n = i^n \left(1+\frac{1}{n}\right)^{n^2},$$

with n^2 in the exponent, it would not be a good idea to use the *criterion of quotients*. We shall instead try the *criterion of roots*, thus we first compute

$$\begin{aligned}\sqrt[n]{|c_n|} &= \left(1+\frac{1}{n}\right)^n = \exp\left(n\cdot\ln\left(1+\frac{1}{n}\right)\right) = \exp\left(n\left\{\frac{1}{n}+o\left(\frac{1}{n}\right)\right\}\right) = \exp\left(1+\frac{o\left(\frac{1}{n}\right)}{\frac{1}{n}}\right) \\ &\to \exp(1) = e = \frac{1}{r} \qquad \text{for } n \to +\infty, \qquad \text{thus } r = \frac{1}{e}.\end{aligned}$$

Example 3.6 *Find the radius of convergence for each of the series*

$$(a)\quad \sum_{n=1}^{+\infty} n^n z^n, \qquad (b)\quad \sum_{n=1}^{+\infty} \frac{n}{2^n}\, z^n, \qquad (c)\quad \sum_{n=0}^{+\infty} \left\{3+(-1)^n\right\}^n z^n.$$

(a) Since $c_n = n^n$, we get by the *criterion of roots* that

$$\frac{1}{\sqrt[n]{n}} = \frac{1}{n} \to 0 \qquad \text{for } n \to +\infty,$$

and the radius of convergence is 0.

ALTERNATIVELY we may use the *criterion of quotients* instead,

$$\lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to+\infty} \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n = 0 = r.$$

Since the radius of convergence is 0, the series does not have a sum function.

(b) Since

$$c_n = \frac{n}{2^n} > 0,$$

it follows by the *criterion of roots* that

$$\frac{1}{\sqrt[n]{c_n}} = \frac{2}{\sqrt[n]{n}} = 2 = r,$$

and the radius of convergence is 2.

ALTERNATIVELY we may apply the *criterion of quotients*,

$$\lim_{n\to+\infty} \frac{c_n}{c_{n+1}} = \lim_{n\to+\infty} \frac{n}{2^n} \cdot \frac{2^{n+1}}{n+1} = 2 \lim_{n\to+\infty} \frac{n}{n+1} = 2.$$

Remark 3.10 In this case it is possible to find an explicit expression of the sum function. If $|w| < 1$, then

$$\frac{1}{1-w} = \sum_{n=0}^{+\infty} w^n \qquad \text{and} \qquad \frac{1}{(1-w)^2} = \frac{d}{dw}\left(\frac{1}{1-w}\right) = \sum_{n=1}^{+\infty} n\, w^{n-1}.$$

Hence by a multiplication by w,

$$\frac{w}{(1-w)^2} = \sum_{n=1}^{+\infty} n\, w^n, \qquad |w| < 1.$$

If we here put $w = \dfrac{z}{2}$ for $|z| < 2$, then

$$\sum_{n=1}^{+\infty} \frac{n}{2^n}\, z^n = \frac{\frac{z}{2}}{\left(1-\frac{z}{2}\right)^2} = \frac{2z}{(2-z)^2}. \qquad \Diamond$$

(c) It follows from the structure $c_n = \{3+(-1)^n\}^n$ (>0) that the *criterion of quotients* is not the right one to apply.

Instead we use the extended *criterion of roots*. First note that

$$\sqrt[n]{c_n} = 3+(-1)^n = \begin{cases} 2 & \text{for } n \text{ odd}, \\ 4 & \text{for } n \text{ even}. \end{cases}$$

This implies that

$$\limsup \sqrt[n]{|c_n|} = \max\{2\,,\,4\} = 4,$$

so the radius of convergence becomes

$$r = \frac{1}{\limsup \sqrt[n]{|c_n|}} = \frac{1}{4}.$$

Remark 3.11 It is in fact also here possible to find the sum. If $|z| < \frac{1}{4}$, then we get by splitting the series into two series (with even and odd indices, respectively), because these series are also convergent for $|z| < \frac{1}{4}$,

$$\begin{aligned}\sum_{n=0}^{+\infty} \{3+(-1)^n\}^n z^n &= \sum_{n=0}^{+\infty} 4^{2n} z^{2n} + \sum_{n=0}^{+\infty} 2^{2n+1} z^{2n+1} = \sum_{n=0}^{+\infty} \left(16z^2\right)^n + 2z \sum_{n=0}^{+\infty} \left(4z^2\right)^n \\ &= \frac{1}{1-16z^2} + \frac{2z}{1-4z^2}. \qquad \Diamond\end{aligned}$$

Example 3.7 *Find the radius of convergence for each of the series*

$$(a)\ \sum_{n=1}^{+\infty} z^{n!}, \qquad (b)\ \sum_{n=1}^{+\infty} 2^n z^{n!}, \qquad \sum_{n=0}^{+\infty} (n+a^n)\, z^n, \quad a \in \mathbb{R}_+.$$

Remark 3.12 The former two series are lacunar series, and it is not possible to express their sum functions, which exist in both cases, by using elementary functions. $\Diamond$

(a) The series is trivially convergent for $|z| < 1$ and divergent for $|z| \geq 1$, hence the radius of convergence must be 1. $\Diamond$

(b) The series is lacunar (infinitely many coefficients are zero in an irregular pattern). This means that the *criterion of quotients* cannot be applied. Instead we use the extended *criterion of roots.* We get from $c_{n!} = 2^n$ and $c_m = 0$ otherwise that

$$\sqrt[n!]{|c_{n!}|} = \left(2^n\right)^{1/(n!)} = 2^{1/((n-1)!)} = \sqrt[(n-1)!]{2} \to 1 \qquad \text{for } n \to +\infty,$$

and $\sqrt[m]{|c_m|} = 0$, if $m \neq n!$, $n \in \mathbb{N}$. Hence $\limsup \sqrt[n]{|c_n|} = 1$, and the radius of convergence becomes $r = \frac{1}{1} = 1$.

(c) Since $c_n = n + a^n > 0$, if follows from the *criterion of roots* that

$$\sqrt[n]{|c_n|} = \sqrt[n]{a^n+n} = \sqrt[n]{a^n\left(1+\frac{n}{a^n}\right)}.$$

If $a \in]0,1]$, then it follows from the first equality sign that the radius of convergence is $r = 1$.

If $a > 1$, it follows from the latter rearrangement that the radius of convergence is $r = \frac{1}{a}$. Summing up we can write

$$r = \min\left\{1\,,\,\frac{1}{a}\right\}.$$

Remark 3.13 Here we find the sum function in the following way: If $|z| < r$, then

$$\sum_{n=0}^{+\infty} (n+a^n)\, z^n = \sum_{n=0}^{+\infty} n z^n + \sum_{n=0}^{+\infty} a^n z^n = z \sum_{n=1}^{+\infty} n\, z^{n-1} + \sum_{n=0}^{+\infty} (a\,z)^n = \frac{z}{(1-z)^2} + \frac{1}{1-a\,z}. \qquad \Diamond$$

Example 3.8 *Find the radius of convergence for each of the series*

$$(a)\quad \sum_{n=1}^{+\infty} \frac{n!}{n^n}\, z^n, \qquad (b)\quad \sum_{n=1}^{+\infty} \frac{\left(\sqrt{3}+i\right)^n}{\left(\sqrt{5}\right)^n}\, z^n, \qquad (c)\quad \sum_{n=1}^{+\infty} \frac{1}{(n+i)\sqrt{n}}\, z^n.$$

Remark 3.14 It is only possible in **(b)** to express the sum function by elementary functions, because we here have a quotient series of quotient and first term equal to

$$\frac{\sqrt{3}+i}{\sqrt{5}}\, z.$$

We shall not write down the sum function, but leave it to the reader as an exercise. $\Diamond$

(a) The structure $c_n = \dfrac{n!}{n^n}$, in which the faculty function occurs, indicates that one should avoid the *criterion of roots*. Instead we apply the *criterion of quotients* to get the radius of convergence

$$r = \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \lim_{n\to+\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e.$$

(b) We have already unveiled this example as a quotient series, so we shall only show the two variants. It follows from

$$c_n = \left(\frac{\sqrt{3}+i}{\sqrt{5}}\right)^n,$$

by the *criterion of roots* that

$$r = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{|c_n|}} = \lim_{n\to+\infty} \left|\frac{\sqrt{5}}{\sqrt{3}+i}\right| = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}.$$

Alternatively we get by the *criterion of quotients*,

$$r = \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \left|\frac{\sqrt{3}+i}{\sqrt{5}}\right|^n \cdot \left|\frac{\sqrt{5}}{\sqrt{3}+i}\right|^{n+1} = \lim_{n\to+\infty} \left|\frac{\sqrt{5}}{\sqrt{3}+i}\right| = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}.$$

(c) Since

$$c_n = \frac{1}{(n+i)\sqrt{n}},$$

it follows by the *criterion of quotients*,

$$\begin{aligned} r &= \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{|n+1+i|\sqrt{n+1}}{|n+i|\sqrt{n}} = \lim_{n\to+\infty} \sqrt{\frac{(n+1)^2+1}{n^2+1} \cdot \frac{n+1}{n}} \\ &= \lim_{n\to+\infty} \sqrt{\frac{\left(1+\dfrac{1}{n}\right)^2 + \dfrac{1}{n^2}}{1+\dfrac{1}{n^2}} \cdot \left(1+\frac{1}{n}\right)} = 1. \end{aligned}$$

ALTERNATIVELY we apply the *criterion of roots*. Then

$$r=\lim_{n\to+\infty}\frac{1}{\sqrt[n]{|c_n|}}=\lim_{n\to+\infty}\sqrt[2n]{n^2+1}\cdot\sqrt[2n]{n}=1.$$

Example 3.9 *Find the radius of convergence for each of the series*

$$(a)\quad\sum_{n=1}^{+\infty}\left(\frac{in}{n+1}\right)^{n^2}z^n,\qquad (b)\quad\sum_{n=1}^{+\infty}\frac{\ln n}{n!}(z-i)^n,\qquad (c)\quad\sum_{n=2}^{+\infty}\frac{n\,i^n}{\ln n}z^n.$$

Remark 3.15 In none of the cases can the sum function be expressed by elementary functions. $\Diamond$

(a) It follows from

$$c_n=\left(\frac{i\,n}{n+1}\right)^{n^2}=i^{n^2}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}},$$

by the *criterion of roots* that

$$r=\lim_{n\to+\infty}\frac{1}{\sqrt[n]{|c_n|}}=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e,$$

where we note that an application of the *criterion of quotients* does not look promising.

(b) Since the faculty function occurs, the *criterion of roots* is not convenient for us here. We get from

$$c_n=\frac{\ln n}{n!}>0\qquad\text{for } n\geq 2,$$

by the *criterion of quotients* that

$$\begin{aligned}r&=\lim_{n\to+\infty}\frac{c_n}{c_{n+1}}=\lim_{n\to+\infty}\frac{\ln n}{n!}\cdot\frac{(n+1)!}{\ln(n+1)}=\lim_{n\to+\infty}(n+1)\cdot\frac{\ln n}{\ln(n+1)}\\&=\lim_{n\to+\infty}(n+1)\cdot\frac{\ln n}{\ln n+\ln\left(1+\frac{1}{n}\right)}=+\infty.\end{aligned}$$

Since

$$c_n=\frac{n\,i^n}{\ln n},$$

it follows by the *criterion of roots* that

$$r=\lim_{n\to+\infty}\frac{1}{\sqrt[n]{|c_n|}}=\lim_{n\to+\infty}\sqrt[n]{\frac{\ln n}{n}}=\lim_{n\to+\infty}\frac{\sqrt[n]{\ln n}}{\sqrt[n]{n}}=1,$$

because

$$1\leq\sqrt[n]{\ln n}\leq\sqrt[n]{n}\qquad\text{for } n\geq 3,$$

and because $\sqrt[n]{n} \to 1$ for $n \to +\infty$.

Alternatively we get by the *criterion of quotients*,

$$r = \lim_{n \to +\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to +\infty} \frac{n}{\ln n} \cdot \frac{\ln(n+1)}{n+1} = \lim_{n \to +\infty} \frac{n}{n+1} \cdot \frac{\ln n + \ln\left(1 + \dfrac{1}{n}\right)}{\ln n} = 1.$$

Example 3.10 *Find the radius of convergence for the series*

$$\sum_{n=1}^{+\infty} \frac{1}{3^n} \left\{ (-1)^n + \sin\left(\frac{n\pi}{2}\right) \right\}^n z^n.$$

We have

$$c_n = \frac{1}{3^n} \left\{ (-1)^n + \sin\left(\frac{n\,\pi}{2}\right) \right\}^n,$$

hence we get the trivial estimate

$$|c_n| \le \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n,$$

where the equality sign holds for e.g. $n = 3 + 4p$, $p \in \mathbb{N}_0$. We conclude that

$$\limsup \sqrt[n]{|c_n|} = \frac{2}{3},$$

and the radius of convergence is

$$r = \frac{1}{\limsup \sqrt[n]{|c_n|}} = \frac{3}{2}.$$

Remark 3.16 It is possible in this case to express the sum function by means of elementary functions. If $|z| < \frac{3}{2}$, then

$$\begin{aligned}
&\sum_{n=1}^{+\infty} \frac{1}{3^n} \left\{(-1)^n + \sin\left(\frac{n\,\pi}{2}\right)\right\}^n z^n \\
&= \sum_{n=0}^{+\infty} \frac{1}{3^{4n+1}} \left\{(-1)^{4n+1} + \sin\left(\frac{(4n+1)\pi}{2}\right)\right\}^{4n+1} z^{4n+1} \\
&\qquad + \sum_{n=0}^{+\infty} \frac{1}{3^{4n+2}} \left\{(-1)^{4n+2} + \sin\left(\frac{(4n+2)\pi}{2}\right)\right\}^{4n+2} z^{4n+2} \\
&\qquad + \sum_{n=0}^{+\infty} \frac{1}{3^{4n+3}} \left\{(-1)^{4n+3} + \sin\left(\frac{(4n+3)\pi}{2}\right)\right\}^{4n+3} z^{4n+3} \\
&\qquad + \sum_{n=0}^{+\infty} \frac{1}{3^{4n+4}} \left\{(-1)^{4n+4} + \sin\left(\frac{(4n+4)\pi}{2}\right)\right\}^{4n+4} z^{4n+4} \\
&= \frac{1}{3}\sum_{n=0}^{+\infty} \frac{1}{3^{4n}} \{-1+1\}^{4n+1} z^{4n+1} + \frac{1}{3^2}\sum_{n=0}^{+\infty} \frac{1}{3^{4n}} \{1+0\}\, z^{4n+2} \\
&\qquad + \frac{1}{3^3}\sum_{n=0}^{+\infty} \frac{1}{3^{4n}} \{-1-1\}^{4n+3} z^{4n+3} + \frac{1}{3^4}\sum_{n=0}^{+\infty} \frac{1}{3^{4n}} \{1+0\}\, z^{4n+4},
\end{aligned}$$

which we reduce to

$$\begin{aligned}
&\sum_{n=1}^{+\infty} \frac{1}{3^n} \left\{(-1)^n + \sin\left(\frac{n\,\pi}{2}\right)\right\}^n z^n = \frac{z^2}{3^2}\sum_{n=0}^{+\infty}\left(\frac{z^4}{3^4}\right)^n - \frac{z^3}{3^3}\cdot 2^3 \sum_{n=0}^{+\infty}\left(\frac{2^4 z^4}{3^4}\right)^n + \frac{z^4}{3^4}\sum_{n=0}^{+\infty}\left(\frac{z^4}{3^4}\right)^n \\
&= \frac{z^2}{3^2}\cdot\frac{1}{1-\frac{z^4}{3^4}} - \frac{2^3 z^3}{3^3}\cdot\frac{1}{1-\frac{2^4 z^4}{3^4}} + \frac{z^4}{3^4}\cdot\frac{1}{1-\frac{z^4}{3^4}} \\
&= \frac{9z^2}{81-z^4} - \frac{24z^3}{81-16z^4} + \frac{z^4}{81-z^4} = \frac{z^4+pz^2}{81-z^4} + \frac{24z^3}{16z^4-81} = \frac{z^2}{9-z^2} + \frac{24z^3}{16z^4-81}. \qquad \Diamond
\end{aligned}$$

Example 3.11 *Find the radius of convergence for each of the series*

$$(a)\quad \sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}+i\right)\, z^n, \qquad (b)\quad \sum_{n=1}^{+\infty} \cos\left(1+\frac{i}{n}\right)\, z^n.$$

Remark 3.17 None of the sum functions can be expressed by elementary functions. $\Diamond$

(a) It follows from

$$\cos\left(\frac{1}{n}+i\right) = \cos\frac{1}{n}\cdot\cosh 1 - i\,\sin\frac{1}{n}\cdot\sinh 1,$$

that

$$\left|\cos\left(\frac{1}{n}+i\right)\right|^2 = \cos^2\frac{1}{n}\cdot\cosh^2 1 + \sin^2\frac{1}{n}\cdot\sinh^2 1 = \cos^2\frac{1}{n} + \sinh^2 1,$$

hence by some calculations

$$\sqrt[n]{|c_n|} = \sqrt[2n]{\cos^2\frac{1}{n}+\sinh^2 1} \to 1 \qquad \text{for } n \to +\infty.$$

We conclude that the radius of convergence is

$$r = \frac{1}{1} = 1.$$

(b) It follows from

$$\cos\left(1+\frac{i}{n}\right) = \cos 1\cdot\cosh\frac{1}{n} - i\,\sin 1\cdot\sinh\frac{1}{n},$$

that

$$\left|\cos\left(1+\frac{i}{n}\right)\right|^2 = \cos^2 1\cdot\cosh^2\frac{1}{n} + \sin^2 1\cdot\sinh^2\frac{1}{n} = \cos^2 1 + \sinh^2\frac{1}{n},$$

hence

$$\sqrt[n]{|c_n|} = \sqrt[2n]{\cos^2 1+\sinh^2\frac{1}{n}} \to 1 \qquad \text{for } n \to +\infty,$$

and we conclude that the radius of convergence is

$$r = \frac{1}{1} = 1.$$

Example 3.12 *Find the radius of convergence for each of the series*

$$(a)\quad \sum_{n=0}^{+\infty} \operatorname{Log}(n+i)\, z^n, \qquad (b)\quad \sum_{n=0}^{+\infty} \cos(1+in)\, z^n.$$

Remark 3.18 None of the sum functions can be expressed by elementary functions. $\Diamond$

(a) We get from

$$|\operatorname{Log}(n+i)|^2 = \left|\ln\sqrt{n^2+1} + i\,\operatorname{Arctan}\frac{1}{n}\right|^2 = \frac{1}{4}\left\{\ln\left(n^2+1\right)\right\}^2 + \left\{\operatorname{Arctan}\frac{1}{n}\right\}^2,$$

the estimates

$$\frac{1}{2}\ln\left(n^2+1\right) < |\operatorname{Log}(n+i)| < C_0 \cdot \ln\left(n^2+1\right)$$

for some constant C_0. Now,

$$\sqrt[n]{C\cdot\ln\left(n^2+1\right)} \to 1 \qquad \text{for } n \to +\infty$$

for every positive constant $C > 0$, so we conclude that the radius of convergence is

$$r = \frac{1}{\lim_{n\to+\infty} |\operatorname{Log}(n+i)|} = \frac{1}{1} = 1.$$

(b) It follows from

$$\cos(1+in) = \cos 1 \cdot \cosh n - i\, \sin 1 \cdot \sinh n,$$

that

$$|\cos(1+in)|^2 = \cos^2 1 \cdot \cosh^2 n + \sin^2 1 \cdot \sinh^2 n = \cos^2 1 + \sinh^2 n,$$

so we get the estimates

$$\sinh n < |\cos(1+in)| < 2\, \sinh n \qquad \text{for } n \geq n_0.$$

The series has the same radius of convergence as the auxiliary series

$$\sum_{n=1}^{+\infty} \sinh n \cdot z^n.$$

Since

$$\sqrt[n]{\sinh n} = \sqrt[n]{\frac{1}{2}\left(e^n - e^{-n}\right)} = e\,\sqrt[n]{\frac{1-e^{-2n}}{2}} \to e \qquad \text{for } n \to +\infty,$$

we conclude that the radius of convergence is

$$r = \frac{1}{\lim_{n\to+\infty} \sqrt[n]{\sinh n}} = \frac{1}{e}.$$

Example 3.13 *We define Riemann's ζ-function by*

$$\zeta(z) = \sum_{n=1}^{+\infty} n^{-z} := \sum_{n=1}^{+\infty} e^{-z \ln n}.$$

Prove that it is analytic in the domain

$$\Omega = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 1\}.$$

Find $f'(z)$ in Ω.

Assume that $\operatorname{Re}(z) \geq k > 1$. Then we have the computation and the estimate,

$$\left|n^{-z}\right| = \left|e^{-z \ln n}\right| = \left|e^{-(x+iy)\ln n}\right| = n^{-x} \leq n^{-k},$$

and we conclude that

$$\sum_{n=1}^{+\infty} n^{-k}$$

is a convergent majoring series in the domain given by

$$\mathrm{Re}(z) \geq k > 1.$$

Then the "smaller series"

$$\zeta(z) = \sum_{n=1}^{+\infty} e^{-z \ln n}$$

is *uniformly convergent* in the same domain. Since every term $e^{-z \ln n}$ is analytic and the series is uniformly convergent, we conclude that $\zeta(z)$ is analytic in the same domain. This holds for every $k > 1$, hence $\zeta(z)$ is analytic for $\mathrm{Re}(z) > 1$, thus in Ω.

If $\mathrm{Re}(z) > 1$, we get by termwise differentiation that the derivative is given by

$$\zeta'(z) = \sum_{n=1}^{+\infty}(-\ln n)e^{-z \ln n} = -\sum_{n=2}^{+\infty} \ln n \cdot n^{-z}.$$

Example 3.14 *Prove that* $\sum -n = 1^{+\infty} e^{-n} \sin(nz)$ *is analytic in the domain*

$$\{z \in \mathbb{C} \mid -1 < \mathrm{Im}(z) < 1\}.$$

Each term $e^{-n}\sin(nz)$ is analytic, so we shall only prove that the series is uniformly convergent in

$$\{z \in \mathbb{C} \mid |\mathrm{Im}(z) \leq k\} \qquad \text{for every } k \in]0,1[.$$

Hence we assume that $|\mathrm{Im}(z)| \leq k$. Then

$$e^{-n}\sin(n\,z) = \frac{1}{2i}\,e^{-n}\left(e^{inz} - e^{-inz}\right) = \frac{1}{2i}\,e^{-n}\left\{e^{-ny+inx} - e^{ny-inx}\right\}.$$

Assuming $|\mathrm{Im}(z)| = |y| \leq k < 1$, we get the estimate

$$\left|e^{-n}\sin(n\,z)\right| \leq \frac{1}{2}\,e^{-n}\left\{e^{nk} + e^{nk}\right\} = e^{-n(1-k)} = a^n,$$

where $a = e^{-(1-k)} \in]0,1[$. Since $\sum_{n=1}^{+\infty} a^n$ is a convergent majoring series, the claim is proved.

Remark 3.19 The sum function cannot be expressed by an elementary function. $\Diamond$

Example 3.15 *Let $a \in \mathbb{R}$ be any real constant, and define n^a, $n \in \mathbb{N}$, by*

$$n^a := \exp(a \ln n).$$

Find the radius of convergence for the series

$$\sum_{n=1}^{+\infty} \frac{z^{2n}}{4^n n^a}.$$

We introduce the variable

$$w = \frac{z^2}{4} = \left(\frac{z}{2}\right)^2.$$

Then

$$\sum_{n=1}^{+\infty} \frac{z^{2n}}{4^n n^a} = \sum_{n=1}^{+\infty} \frac{w^n}{n^a}.$$

Putting $c_n = \dfrac{1}{n^a}$ it follows that the w*-radius of convergence* is

$$\varrho_w = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{|c_n|}} = \lim_{n\to+\infty} n^{a/n} = \lim_{n\to+\infty} \exp\left(\frac{a}{n} \ln n\right) = \exp(0) = 1.$$

It follows that the series is convergent for

$$|w| = \left|\frac{z}{2}\right| < 1,$$

i.e. for $|z| < 2$, and divergent for

$$|w| = \left|\frac{z}{2}\right|^2 > 1,$$

i.e. for $|z| > 2$. Then it follows from the definition that the z*-radius of convergence* is

$$\varrho_z = 2.$$

Example 3.16 *Given a series $\sum_{n=0}^{+\infty} c_n z^n$ of radius of convergence $\varrho \in \mathbb{R}_+$. Find the radius of convergence for each of the series*

$$(a)\quad \sum_{n=0}^{+infty} c_n z^{2n}, \qquad (b)\quad \sum_{n=0}^{+\infty} n^n c_n z^n, \qquad (c)\quad \sum_{n=0}^{+\infty} (2^n - 1)\, c_n z^n.$$

It follows from the assumption that

$$\limsup \sqrt[n]{|c_n|} = \frac{1}{\varrho},$$

where $0 < \varrho < +\infty$.

(a) Since $a_{2n} = c_n$ and $a_{2n+1} = 0$, we get

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|} = \limsup \sqrt[2n]{|a_{2n}|} = \sqrt{\limsup \sqrt[n]{|c_n|}} = \frac{1}{\sqrt{\varrho}},$$

and we conclude that $R = \sqrt{\varrho}$.

(b) Since $\sqrt[n]{n^n} = n \to +\infty$ for $n \to +\infty$, and $\dfrac{1}{\varrho} > 0$, we conclude that

$$\limsup \sqrt[n]{n^n\,|c_n|} = \lim n \cdot \frac{1}{\varrho} = +\infty,$$

and the radius of convergence is $R = 0$ in this case.

Remark 3.20 The claim is not correct, if we allow $\varrho = +\infty$. It is possible to construct series $\sum_{n=0}^{+\infty} c_n z^n$ of radius of convergence $\varrho = +\infty$, such that the radius of convergence of $\sum_{n=0}^{+\infty} n^n c_n z^n$ is

(1) $R = 0$, (2) $R \in \mathbb{R}_+$, (3) $R = +\infty$.

An example is

(1) $\sum_{n=0}^{+\infty} c_n z^n = \sum_{n=1}^{+\infty} \dfrac{1}{\sqrt{n^n}}\, z^n$, $\varrho = +\infty$, $R = 0$,

(2) $\sum_{n=0}^{+\infty} c_n z^n = \sum_{n=1}^{+\infty} \dfrac{1}{n^n} \cdot \dfrac{1}{R_0^n}\, z^n$, $\varrho = +\infty$, $R = R_0 \in \mathbb{R}_+$,

(3) $\sum_{n=0}^{+\infty} c_n z^n = \sum_{n=1}^{+\infty} \dfrac{1}{n^{2n}}\, z^n$, $\varrho = +\infty$, $R = +\infty$. $\Diamond$

(c) A straigth computation gives

$$\limsup \sqrt[n]{(2^n-1)\,|c_n|} = \limsup \sqrt[n]{2^n-1} \cdot \sqrt[n]{|c_n|} = \frac{2}{\varrho},$$

and the radius of convergence is $\dfrac{\varrho}{2}$.

Example 3.17 *Given a series $\sum_{n=0}^{+\infty} c_n z^n$ of radius of convergence $\varrho \in \mathbb{R}_+$. Find the radius of convergence for each of the series*

$$(a)\ \sum_{n=0}^{+\infty} \frac{c_n}{n!}\, z^n, \qquad (b)\ \sum_{n=1}^{+\infty} n^k c_n z^n, \qquad (c)\ \sum_{n=0}^{+\infty} c_n^k z^n,$$

where $k \in \mathbb{N}$ denotes some constant.

We shall use that $\limsup \sqrt[n]{|c_n|} = \dfrac{1}{\varrho}$, where $0 < \varrho < +\infty$.

(a) We see that $a_n = \frac{c_n}{n!}$. It is well-known that

$$\sum_{n=0}^{+\infty} \frac{1}{n!}\, z^n$$

has the radius of convergence $+\infty$, so

$$\lim \sqrt[n]{\frac{1}{n!}} = 0, \qquad \text{i.e.} \qquad \limsup \sqrt[n]{\left|\frac{c_n}{n!}\right|} = 0.$$

In fact, choose $N \in \mathbb{N}$, such that

$$\sqrt[n]{|c_n|} \le \frac{1}{\varrho} \qquad \text{for every } n \ge N.$$

Then

$$\limsup \sqrt[n]{\left|\frac{c_n}{n!}\right|} \le \frac{2}{\varrho} \lim \sqrt[n]{\frac{1}{n!}} = 0,$$

and it follows that the radius of convergence is $R = +\infty$.

(b) A straight computation gives

$$\limsup \sqrt[n]{|n^k c_n|} = 1 \cdot \limsup \sqrt[n]{|c_n|} = \frac{1}{\varrho},$$

and it follows that the radius of convergence is $R = \varrho$.

(c) By a straight computation,

$$\limsup \sqrt[n]{|c_n^k|} = \left(\limsup \sqrt[n]{|c_n|}\right)^k = \frac{1}{\varrho^k},$$

and it follows that the radius of convergence is $R = \varrho^k$.

Example 3.18 *Construct a series $\sum_{n=0}^{+\infty} c_n z^n$ of finite radius of convergence ϱ (possibly $\varrho = 0$), such that the series*

$$\sum_{n=0}^{+\infty} (1 + z_0^n)\, c_n z^n$$

for some $z_0 \in \mathbb{C}$ has radius of convergence $+\infty$.

If we choose $z_0 = -1$, then

$$1 + z_0^n = \begin{cases} 2 & \text{for } n \text{ lige,} \\ 0 & \text{for } n \text{ ulige.} \end{cases}$$

Thus it is possible to obtain infinitely many zeros among the coefficients $(1 + z_0^n)$. We can exploit this by putting

$$c_n = \begin{cases} 0 & \text{for } n \text{ even,} \\ n^n & \text{for } n \text{ odd.} \end{cases}$$

In fact,

$$\sum_{n=0}^{+\infty} c_n z^n = \sum_{n=0}^{+\infty} (2n+1)^{2n+1} z^{2n+1}$$

has radius of convergence 0, while

$$\sum_{n=0}^{+\infty} (1 + z_0^n)\, c_n z^n = \sum_{n=0}^{+\infty} 0 \cdot z^n \equiv 0$$

is convergent for every $z \in \mathbb{C}$.

Example 3.19 *1) Find the radius of convergence R for the power series*

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}\, z^n.$$

Prove that the series is absolutely and uniformly convergent in the closed disc

$$\{z \in \mathbb{C} \mid |z| \le R\}.$$

2) Find the set $K \subset \mathbb{C}$, for which the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}\, e^{nz}$$

is convergent for $z \in K$ and divergent for $z \notin K$.

1) If $|z| > 1$, then $\left|\frac{1}{n^2}\, z^n\right| \to +\infty$ for $n \to +\infty$ due to the order of magnitudes. Thus the necessary condition of convergence is not satisfies. We therefore conclude that $R \le 1$.
If instead $|z| \le 1$, then we get the estimate

$$\left|\sum_{n=1}^{+\infty} \frac{1}{n^2}\, z^n\right| \le \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The right hand side is finite and independent of $|z| \le 1$, hence the convergence is absolute and uniform in the closed disc

$$\{z \in \mathbb{C} \mid |z| \le 1\},$$

and the radius of convergence is $R = 1$.

2) By **(1)** the series is convergent, if and only if

$$|e^{nz}| = e^{nx} \le 1,$$

i.e. if and only if $x \le 0$. It follows that

$$K = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \le 0\},$$

and we have convergence in the closed left half plane.

4 Analytic functions described as power series

Example 4.1 *Prove that for* $|z-1|<1$,

$$\frac{1}{z}=\sum_{n=0}^{+\infty}(1-z)^n.$$

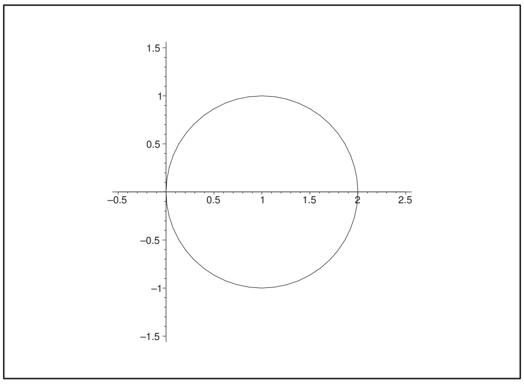


Figure 1: The domain of the series for $\dfrac{1}{z}$ from the expansion point 1.

Put $w=z-1$. Then

$$\frac{1}{z}=\frac{1}{1+w}=\sum_{n=0}^{+\infty}(-1)^n w^n=\sum_{n=0}^{+\infty}(.1)^n(z-1)^n=\sum_{n=0}^{+\infty}(1-z)^n,$$

which holds for $|w|<1$, i.e. for $|z-1|<1$.

Example 4.2 *Find the Taylor series for* e^z *with the expansion point* $z=1$.

By some elementary manipulations,

$$e^z=e\cdot e^{z-1}=e\sum_{n=0}^{+\infty}\frac{1}{n!}\,(z-1)^n=\sum_{n=0}^{+\infty}\frac{e}{n!}\,(z-1)^n.$$

Alternatively we use the *standard method.* Putting $f(z)=e^z$ we get

$$f^{n)}(z)=e^z,\qquad \text{hence}\qquad f^{(n)}(1)=e,$$

and thus

$$e^z=f(z)=\sum_{n=0}^{+\infty}\frac{f^{(n)}(1)}{n!}\,(z-1)^n=\sum_{n=0}^{+\infty}\frac{e}{n!}\,(z-1)^n,$$

which is convergent for every $z\in\mathbb{C}$.

Example 4.3 *Find the power series expansion of*

$$f(z)=\frac{1}{1+z^2}$$

from the point $z=1$, and indicate its domain.

It follows by a *decomposition* that

$$f(z)=\frac{1}{1+z^2}=-\frac{i}{2}\cdot\frac{1}{z-i}+\frac{i}{2}\cdot\frac{1}{z+i}=\frac{i}{2}\left\{\frac{1}{z+i}-\frac{1}{z-i}\right\},$$

hence

$$f^{(n)}(z)=\frac{i}{2}\cdot(-1)^n n!\left\{\frac{1}{(z+i)^{n+1}}-\frac{1}{(z-i)^{n+1}}\right\},\qquad n\in\mathbb{N}_0,$$

and thus

$$\begin{aligned}
a_n &= \frac{1}{n!}\, f^{(n)}(1) = \frac{i}{2} \cdot (-1)^n \cdot \frac{(1-i)^{n+1} - (1+i)^{n+1}}{(1+1)^{n+1}} \\
&= \frac{i}{2} \cdot \frac{(-1)^n}{\left(\sqrt{2}\right)^{n+1}} \cdot \left\{ \exp\left(-i(n+1)\,\frac{\pi}{4}\right) - \exp\left(i(n+1)\,\frac{\pi}{4}\right)\right\} \\
&= \frac{(-1)^n}{\left(\sqrt{2}\right)^{n+1}} \cdot \frac{i}{2} \left\{-2i\, \sin\left((n+1)\,\frac{\pi}{4}\right)\right) \\
&= \frac{(-1)^n}{\left(\sqrt{2}\right)^{n+1}} \cdot \sin\left((n+1)\frac{\pi}{4}\right).
\end{aligned}$$

The series is then

$$\frac{1}{1+z^2} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{\left(\sqrt{2}\right)^{n+1}}\, \sin\left((n+1)\,\frac{\pi}{4}\right) \cdot (z-1)^n.$$

Now,

$$|a_n| \le \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}}\right)^n,$$

where the equality occurs infinitely often, so it follows from the above that the radius of convergence is $\sqrt{2}$. This can also be seen geometrically, because $|1 \pm i| = \sqrt{2}$ is the distance from the point of expansion 1 to the two singularities $\pm i$, where the denominator is 0.

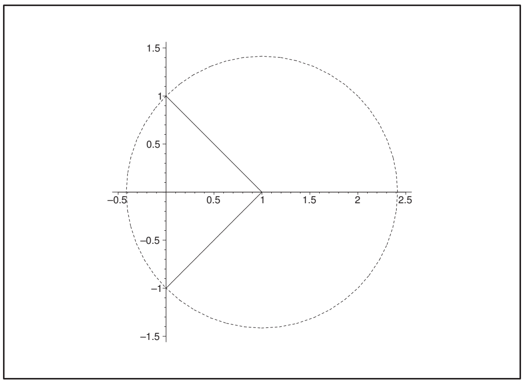


Figure 2: The distance from the point of expansion 1 to the singularities $\pm i$ and the corresponding circle of convergence.

Remark 4.1 One can also set up another expression by computing all the coefficients.
If $n = 8p$, then

$$a_{8p} = \frac{1}{2^{4p}\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2^{4p+1}}.$$

If $n = 8p+1$, then

$$a_{8p+1} = -\frac{1}{2^{4p+1}}.$$

If $n = 8p+2$, then

$$a_{8p+2} = \frac{1}{2^{4p+1}\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2^{4p+2}}.$$

If $n = 8p+3$, then

$$a_{8p+3} = 0.$$

If $n = 8p+4$, then

$$a_{8p+4} = \frac{1}{2^{4p+2}} \cdot \frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2^{4p+3}}.$$

If $n = 8p+5$, then

$$a_{8p+5} = \frac{-1}{2^{4p+3}} \cdot (-1) = \frac{1}{2^{4p+3}}.$$

If $n = 8p+6$, then

$$a_{8p+6} = \frac{1}{2^{4p+3}} \cdot \frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2^{4p+4}}.$$

If $n = 8p+7$, then

$$a_{8p+7} = 0.$$

Summing up we get for $|z-1| < \sqrt{2}$,

$$\frac{1}{1+z^2} = \sum_{n=0}^{+\infty} \frac{1}{2\cdot 16^n} \left\{1-(z-1)+\frac{(z-1)^2}{2}-\frac{(z-1)^4}{4}+\frac{(z-1)^5}{4}-\frac{(z-1)^6}{8}\right\}(z-1)^{8n}.$$

Example 4.4 *Find the Taylor series from $z_0 = 1$ for each of the following functions, and indicate the radius of convergence of the series:*

$$(a)\quad \frac{1}{z-2}, \qquad (b)\quad \frac{1}{z(z-2)}.$$

(a) By a straightforward computation,

$$\frac{1}{z-2} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{+\infty}(z-1)^n \qquad \text{for } |z-1| < 1.$$

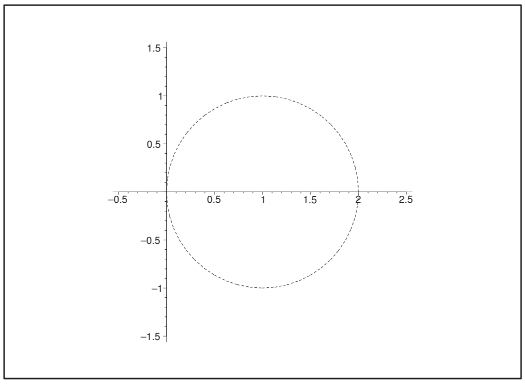


Figure 3: The domain of convergence in both cases.

(b) By using the same trick we get

$$\frac{1}{z(z-2)} = \frac{-1}{\{1+(z-1)\}\{1-(z-1)\}} = -\frac{1}{1-(z-1)^2} = -\sum_{n=0}^{+\infty}(z-1)^{2n},$$

for $|z-1|<1$.

We can also obtain this result by a decomposition and by using **(a)**, because

$$\begin{aligned}\frac{1}{z(z-2)} &= -\frac{1}{2}\cdot\frac{1}{z}+\frac{1}{2}\cdot\frac{1}{z-2} = -\frac{1}{2}\cdot\frac{1}{1+(z-1)}-\frac{1}{2}\cdot\frac{1}{1-(z-1)}\\ &= -\frac{1}{2}\left\{\sum_{n=0}^{+\infty}(-1)^n(z-1)^n+\sum_{n=0}^{+\infty}(z-1)^n\right\} = -\sum_{n=0}^{+\infty}(z-1)^{2n}.\end{aligned}$$

Example 4.5 *Find the Taylor series for the following functions from $z_0 = i$:*

$(a)\ \dfrac{1}{z}, \qquad (b)\ \dfrac{2z-1}{z^2-z}, \qquad (c)\ \operatorname{Log}(1-z).$

(a) Putting $f(z) = z^{-1}$ we get

$$f^{(n)} = (-1)^n n! z^{-n-1},$$

and thus

$$\frac{f^{(n)}(i)}{n!} = \frac{(-1)^n}{i^{n+1} = -i^{n+1}},$$

hence

$$\frac{1}{z} = -\sum_{n=0}^{+\infty} i^{n+1}(z-i)^n = \sum_{n=0}^{+\infty} i^{n-1}(z-i)^n, \qquad \text{for } |z-i|<1.$$

(b) By a decomposition,

$$f(z) = \frac{2z-1}{z^2-z} = \frac{z+z-1}{z(z-1)} = \frac{1}{z}+\frac{1}{z-1},$$

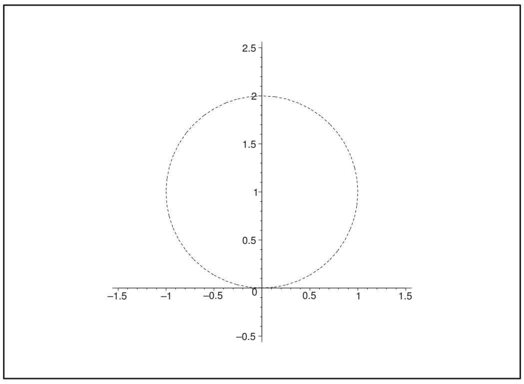


Figure 4: The domain in **(a)** and **(b)**.

and thus (cf. **(a)**),

$$\frac{f^{(n)}(z)}{n!} = (-1)^n \left\{ z^{-n-1} + (z-i)^{-n-1} \right\},$$

i.e.

$$\frac{f^{(n)}(i)}{n!} = -i^{n+1} - \frac{1}{(1-i)^{n+1}} = -i^{n+1} - \left(\frac{(1+i)}{2} \right)^{n+1},$$

hence

$$f(z) = \frac{2z-1}{z^2-z} = -\sum_{n=0}^{+\infty} \left\{ i^{n+1} + \left(\frac{1+i}{2}\right)^{n+1} \right\} (z-i)^n, \qquad |z-i| < 1.$$

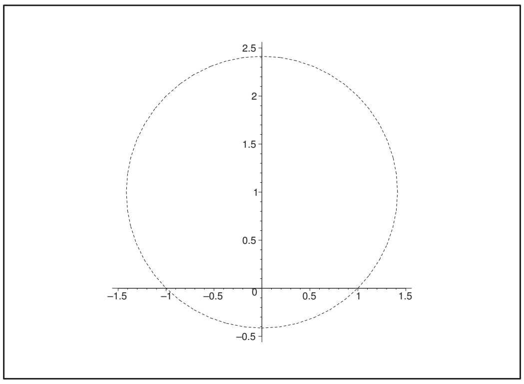


Figure 5: The domain of **(c)**.

(c) If we put $f(z) = \mathrm{Log}(1-z)$, then

$$f^{(n)}(z) = -\frac{(n-1)!}{(1-z)^n}, \qquad n \in \mathbb{N},$$

hence

$$\frac{f^{(n)}(i)}{n!} = -\frac{1}{n} \cdot \frac{1}{(1-i)^n} = -\frac{(1+i)^n}{n \cdot 2^n},$$

and the series is given by

$$f(z) = \mathrm{Log}(1-i) - \sum_{n=1}^{+\infty} \frac{(1+i)^n}{n \cdot 2^n} (z-i)^n = \frac{1}{2} \ln 2 - i\,\frac{\pi}{4} - \sum_{n=1}^{+\infty} \frac{(1+i)^n}{n \cdot 2^n} (z-i)^n, \qquad |z-i| < \sqrt{2}.$$

Example 4.6 *Find the Taylor series from $z_0 = -1+i$ for* Log z*. Determine the radius of convergence of the series as well as the radius of the largest disc of centrum $z_0 = -1+i$, in which the series converges towards* Log z*.*

A Taylor series is of the form

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n .$$

If we put $f(z) = \mathrm{Log}\, z$ (which is analytic in a neighbourhood of $z_0 = -1+i$), then

$$f^{(n)}(z) = (-1)^{n-1}(n-1)! z^{-n}, \qquad n \in \mathbb{N}.$$

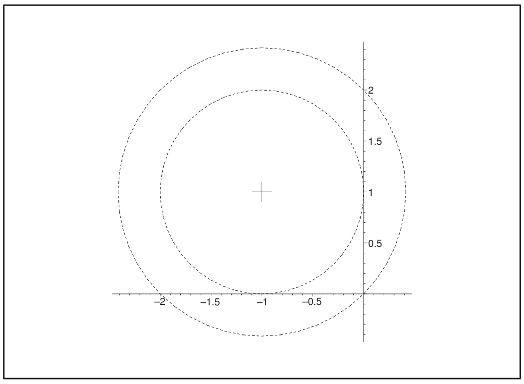


Figure 6: The series is convergent in the larger disc, and it converges towards Log z in the smaller disc, actually in that part of the larger disc which lies in the second quadrant.

It follows from

$$z_0^{-1} = \frac{1}{-1+i} = -\frac{1+i}{2} = -\frac{1}{\sqrt{2}}\,\exp\left(i\,\frac{\pi}{2}\right),$$

that

$$f^{(n)}\left(z_0\right) = -(n-1)!2^{-n}\cdot(1+i)^n = -(n-1)!\left(\frac{1}{\sqrt{2}}\right)^n \exp\left(i\,n\,\frac{\pi}{4}\right),$$

and we get the Taylor series from $z_0 = -1+i$,

$$\begin{aligned}\operatorname{Log} z &= \operatorname{Log}(-1+i) - \sum_{n=1}^{+\infty}\frac{(n-1)!}{n!}\left(\frac{1}{\sqrt{2}}\right)^n \exp\left(i\,n\,\frac{\pi}{4}\right)\,(z+1-i)^n \\ &= \frac{1}{2}\,\ln 2 + i\,\frac{3\pi}{4} - \sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{1}{\sqrt{2}}\right)^n \exp\left(i\,n\,\frac{\pi}{4}\right)\,(z+1-i)^n.\end{aligned}$$

The radius of convergence r is determined by

$$\frac{1}{r} = \limsup_{n\to+\infty}\sqrt[n]{|c_n|} = \lim_{n\to+\infty}\sqrt[n]{\frac{1}{n}\left(\frac{1}{\sqrt{2}}\right)^n} = \frac{1}{\sqrt{2}},$$

thus $r = \sqrt{2}$, which corresponds to the distance $|-1+i-0|$ between the point of expansion $-1+i$ and the branching point 0.

Since Log z has its branch cut along the negative real axis with a discontinuity when we cross over it, the series found does *not* converge towards Log z in that part of the disc of convergence which lies below the x-axis, i.e. in the third quadrant. The radius of the largest (open) *disc* of centrum $z_0 = -1+i$, in which the series is convergent towards Log z, is therefore the distance between the point of expansion $-1+i$ and the negative real axis, thus $1 < \sqrt{2}$.

Example 4.7 *Find the first five terms of the power series expansion in z from $z_0 = 0$ for the following functions:*

$$(a)\ e^{z\,\sin z}, \qquad (b)\ e^{z\,\mathrm{Log}(1+z)}, \qquad (c)\ e^{z}\,\mathrm{Log}(1+z).$$

Remark 4.2 Even if it is very easy to solve the task in MAPLE by using the command `taylor`, it is nevertheless a good exercise to try the more old-fashioned o-technique.

(a) We get by inserting

$$w = z \cdot \sin z = z^2 - \frac{1}{6}\,z^4 + o\left(z^5\right)$$

into the series expansion of e^w,

$$\begin{aligned} e^{z\cdot\sin z} &= 1 + \frac{1}{1!}\left\{z^2 - \frac{1}{6}\,z^4 + o\left(z^5\right)\right\} + \frac{1}{2!}\left\{z^4 + o\left(z^5\right)\right\} + o\left(z^5\right) \\ &= 1 + z^2 + \left(\frac{1}{2} - \frac{1}{6}\right) z^4 + o\left(z^5\right) = 1 + z^2 + \frac{1}{3}\,z^4 + o\left(z^5\right). \end{aligned}$$

(b) By a Taylor expansion,

$$w = z\,\mathrm{Log}(1+z) = z^2 - \frac{z^3}{2} + \frac{z^4}{3} - \frac{z^5}{4} + o\left(z^5\right).$$

Then

$$\begin{aligned} e^{z\,\mathrm{Log}(1+z)} &= e^w = 1 + \left\{z^2 - \frac{z^3}{2} + \frac{z^4}{3} - \frac{z^5}{4}\right\} + \frac{1}{2}\left\{z^4 - z^5\right\} + o\left(z^5\right) \\ &= 1 + z^2 - \frac{1}{2}\,z^3 + \frac{5}{6}\,z^4 - \frac{3}{4}\,z^5 + o\left(z^5\right). \end{aligned}$$

If we expand each factor separately, we get

$$\begin{aligned} e^z\,\mathrm{Log}(1+z) &= \left\{1 + z + \frac{1}{2}\,z^2 + \frac{1}{6}\,z^3 + \frac{1}{24}\,z^4 + \frac{1}{120}\,z^5 + o\left(z^5\right)\right\} \times \\ &\qquad \times \left\{z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} + o\left(z^5\right)\right\} \\ &= z + \left(-\frac{1}{2} + 1\right) z^2 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{2}\right) z^3 + \left(-\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{6}\right) z^4 \\ &\qquad + \left(\frac{1}{5} - \frac{1}{4} + \frac{1}{6} - \frac{1}{12} + \frac{1}{24}\right) z^5 + o\left(z^5\right) \\ &= z + \frac{1}{2}\,z^2 + \frac{1}{3}\,z^3 + \frac{3}{40}\,z^5 + o\left(z^5\right). \end{aligned}$$

Example 4.8 *Determine the terms of order* ≤ 3 *in the power series from 0 for*

$$(a)\quad e^z \sin z, \qquad (b)\quad \sin z\,\cos z, \qquad (c)\quad \frac{e^z-1}{z}.$$

(a) We get by a straightforward computation that

$$\begin{aligned} e^z\cdot\sin z &= \left\{1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right\}\cdot\left\{z-\frac{z^3}{6}+\cdots\right\} = z-\frac{z^3}{6}+z^2+\frac{z^3}{2}+\cdots \\ &= z+z^2+\frac{z^3}{3}+\cdots. \end{aligned}$$

(b) Here we first apply a known trigonometric formula,

$$\sin z\cdot\cos z = \frac{1}{2}\sin 2z = \frac{1}{2}\left\{2z-\frac{8z^3}{3!}+\cdots\right\} = z-\frac{2}{3}\,z^3+\cdots.$$

(c) By a series expansion and a reduction,

$$\frac{e^z-1}{z} = \sum_{n=1}^{+\infty}\frac{1}{n!}\,z^{n-1} = 1+\frac{1}{2}\,z+\frac{1}{6}\,z^2+\frac{1}{24}\,z^3+\cdots.$$

Example 4.9 *Determine the terms of order* ≤ 3 *in the power series from 0 for*

$$(a)\ \frac{e^z-\cos z}{z}, \qquad (b)\ \frac{1}{\cos z}, \qquad (c)\ \frac{\sin z}{\cos z}.$$

HINT: *Let*

$$\frac{1}{\cos z} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots .$$

Multiply **(b)**, *and possibly also* **(c)** *by the power series expansion of* $\cos z$, *and then find the coefficients* a_0, a_1, a_2, a_3.

(a) By a series expansion and a reduction,

$$\frac{e^z-\cos z}{z} = \frac{1}{z}\left\{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots-1+\frac{z^2}{2!}-\frac{z^4}{4!}+\cdots\right\} = 1+z+\frac{z^2}{6}+0\cdot z^3+\cdots .$$

(b) Here $\dfrac{1}{\cos z}$ is analytic for $|z|<\dfrac{\pi}{2}$, so

$$\frac{1}{\cos z} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots .$$

Since $\cos(-z) = \cos z$ is an even function, it follows immediately that $a_1 = 0$ and $a_3 = 0$, and the series expansion is reduced to

$$\frac{1}{\cos z} = a_0 + a_2 z^2 + o\left(z^3\right).$$

Therefore, if $|z| < \dfrac{\pi}{2}$, then

$$\begin{aligned} 1 &= \frac{1}{\cos z}\cdot\cos z = \left(a_0+a_2z^2+a_4z^4+\cdots\right)\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right) \\ &= a_0 + \left(-\frac{a_0}{2}+a_2\right)z^2 + o\left(z^3\right), \end{aligned}$$

and it follows from the identity theorem that

$$a_0 = 1 \qquad \text{and} \qquad a_2 = \frac{a_0}{2} = \frac{1}{2},$$

thus

$$\frac{1}{\cos z} = 1 + \frac{1}{2}\,z^2 + 0\cdot z^3 + \cdots, \qquad |z| < \frac{\pi}{2}.$$

ALTERNATIVELY we find $f^{(n)}(0)$ for $n = 0, 1, 2, 3$:

$$f(z) = \frac{1}{\cos z}, \qquad f'(z) = \frac{\sin z}{\cos^2 z}, \qquad f''(z) = \frac{1}{\cos z} + 2\cdot\frac{\sin^2 z}{\cos^3 z}, \qquad f^{(3)}(z) = \sin z\cdot\{\cdots\},$$

from which follows that

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = 1, \quad f^{(3)}(0) = 0,$$

so by insertion,

$$\frac{1}{\cos z} = 1 + \frac{1}{2!}\,z^2 + 0\cdot z^3 = \cdots = 1 + \frac{1}{2}\,z^2 + 0\cdot z^3 + \cdots .$$

(c) Using the result of **(b)** we get

$$\begin{aligned}\tan z &= \frac{\sin z}{\cos z} = \left(z - \frac{z^3}{3!} + \cdots\right)\left(1 + \frac{1}{2}\,z^2 + 0\cdot z^3 + \cdots\right)\\ &= z + \left(\frac{1}{2} - \frac{1}{6}\right) z^3 + \cdots = z + \frac{1}{3}\,z^3 \cdots .\end{aligned}$$

ALTERNATIVELY,

$$f(z) = \tan z, \qquad f'(z) = 1 + \tan^2 z, \qquad f''(z) = 2\,\tan z + 2\tan^3 z,$$

$$f^{(3)}(z) = 2\left(1 + \tan^2\right) + \tan z \cdot \{\cdots\},$$

hence

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f^{(3)}(0) = 2,$$

and we get

$$\tan z = \frac{1}{1!}\,z + \frac{2}{3!}\,z^3 + \cdots = z + \frac{1}{3}\,z^3 + \cdots, \qquad \text{for } |z| < \frac{\pi}{2}.$$

Example 4.10 *Find the radius of convergence for the Taylor expansion from $z_0 = i$ for*

$$f(z) = \frac{e^z}{(z-1)(z+1)(z-2)(z-3)}.$$

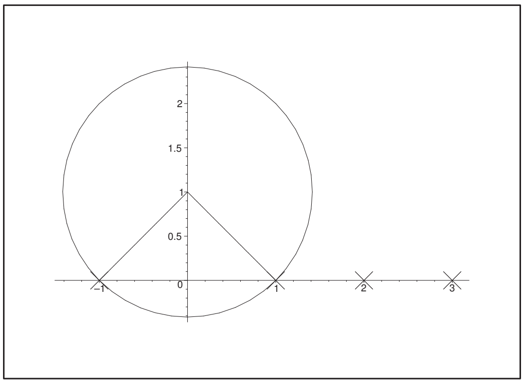


Figure 7: The four singularities and the disc of convergence.

The radius of convergence is the smallest distance from the point of expansion $z_0 = i$ ti the poles $\{-1, 1, 2, 3\}$, hence $\varrho = \sqrt{2}$.

Remark 4.3 Note that one does not want the full Taylor expansion for a very good reason. It will be a very difficult task to find the coefficients using decomposition and multiplication of series. $\Diamond$

Example 4.11 *Find the sum function $f(z)$ in $|z| < 1$ for each of the following series:*

$$(a)\ \sum_{n=1}^{+\infty} n\, z^n, \qquad (b)\ \sum_{n=0}^{+\infty} \frac{z^{2n+1}}{2n+1}, \qquad (c)\ \sum_{n=1}^{+\infty} (-1)^{n+1}\, \frac{z^{2n}}{n}.$$

We shall use various variants of the *geometric series*

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n \qquad \text{for } |z| < 1.$$

(a) It follows by termwise differentiation of the geometric series that

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{+\infty} n\, z^{n-1},$$

hence

$$f(z) = \sum_{n=1}^{+\infty} n\, z^n = \frac{z}{(1-z)^2}.$$

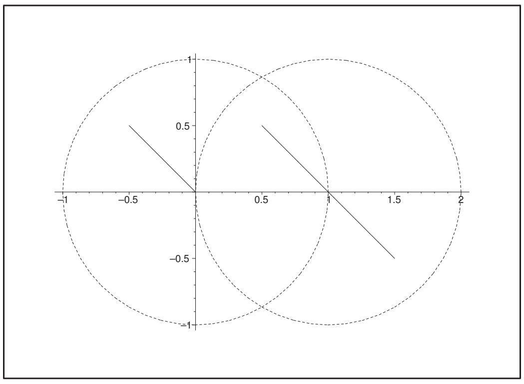


Figure 8: If z lies in the unit disc, then both $1+z$ and $1-z$ lie in the right half plane.

(b) If we differentiate the given series

$$f(z) = \sum_{n=0}^{+\infty} \frac{z^{2n+1}}{2n+1}$$

then

$$f'(z) = \sum_{n=0}^{+\infty} z^{2n} = \sum_{n=0}^{+\infty} \left(z^2\right)^n = \frac{1}{1-z^2},$$

so $f(z)$ is a primitive of

$$\frac{1}{1-z^2} = \frac{1}{2} \cdot \frac{1}{1+z} + \frac{1}{2} \cdot \frac{1}{1-z}.$$

We conclude from

$$\frac{d}{dz}\,\operatorname{Log}(1+z)=\frac{1}{1+z}\quad\text{og}\quad\frac{d}{dz}\operatorname{Log}(1-z)=-\frac{1}{1-z}\qquad\text{for } |z|<1,$$

that

$$f(z)=\frac{1}{2}\left\{\operatorname{Log}(1+z)-\operatorname{Log}(1-z)\right\}+C,\qquad |z|<1.$$

If we put $z=0$, then $C=f(0)=0$, and since both $1+z$ and $1-z$ lie in the right half plane (cf. the figure), their principal arguments lie in $\left]-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right[$, thus

$$\operatorname{Arg}(1+z)-\operatorname{Arg}(1-z)\in\,]-\pi\,,\,\pi[.$$

Then we conclude (and only at this point) that

$$\begin{aligned}f(z)&=\frac{1}{2}\left\{\operatorname{Log}(1+z)-\operatorname{Log}(1-z)\right\}=\frac{1}{2}\left\{\ln\left|\frac{1+z}{1-z}\right|+i\,\operatorname{Arg}\left(\frac{1+z}{1-z}\right)\right\}\\&=\frac{1}{2}\operatorname{Log}\left(\frac{1+z}{1-z}\right),\qquad |z|<1.\end{aligned}$$

(c) We get by the *substitution* $w = z^2$ that

$$f(z) = \sum_{n=1}^{+\infty} (-1)^{n+1} \cdot \frac{z^{2n}}{n} = \sum_{n=1}^{+\infty} (-1)^{n+1} \cdot \frac{w^n}{n} = \operatorname{Log}(1+w) = \operatorname{Log}\left(1+z^2\right), \qquad |z| < 1.$$

ALTERNATIVELY it follows by termwise differentiation,

$$f'(z) = \sum_{n=1}^{+\infty} (-1)^{n+1} \cdot 2z^{2n-1} = 2z \sum_{n=1}^{+\infty} (-1)^{n-1} z^{2(n-1)} = 2z \sum_{n=0}^{+\infty} \left(-z^2\right)^n = \frac{2z}{1+z^2}, \qquad |z| < 1,$$

so $f(z)$ is a primitive of $\dfrac{2z}{1+z^2}$. Since $|z| < 1$, anyone of these primitives is given by

$$f(z) = \operatorname{Log}\left(1+z^2\right) + C.$$

Finally, we put $z = 0$ to obtain $C = f(0) = 0$, thus

$$f(z) = \operatorname{Log}\left(1+z^2\right), \qquad |z| < 1.$$

Example 4.12 *Prove that if* $\dfrac{\sin z}{z}$, $\dfrac{e^z - 1}{z}$ *and* $\dfrac{\operatorname{Log}(1+z)}{z}$ *are all extended by the value 1 to* $z = 0$, *then these functions are analytic in a neighbourhood of 0.*
Then find the Taylor series of

$$S(z) = \int_0^z \frac{\sin \zeta}{\zeta}\, d\zeta, \qquad E(z) = \int_0^z \frac{e^\zeta - 1}{\zeta}\, d\zeta, \qquad L(z) = \int_0^z \frac{\operatorname{Log}(1+\zeta)}{\zeta}\, d\zeta.$$

(a) If $z \neq 0$, then

$$\frac{\sin z}{z} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!}\, z^{2n},$$

which quite naturally is extended analytically to $z = 0$ with the value 1. The radius of convergence is $+\infty$, and it follows by termwise integration,

$$S(z) = \int_0^z \frac{\sin \zeta}{\zeta}\, d\zeta = \int_0^z \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!}\, \zeta^{2n}\, d\zeta = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{z^{2n+1}}{2n+1}.$$

(b) In the same way it follows from

$$e^z - 1 = \sum_{n=1}^{+\infty} \frac{z^n}{n!} = z \sum_{n=0}^{+\infty} \frac{z^n}{(n+1)!}, \qquad z \in \mathbb{C},$$

that

$$\frac{e^z - 1}{z} = \sum_{n=0}^{+\infty} \frac{z^n}{(n+1)!} \qquad \text{for } z \neq 0,$$

where the series has the value 1 for $z = 0$, so we can extend the function analytically by the value 1 at $z = 0$.
Then by a termwise integration for $|z| < +\infty$,

$$E(z) = \int_0^z \frac{e^\zeta - 1}{\zeta}\, d\zeta = \int_0^z \sum_{n=0}^{+\infty} \frac{\zeta^n}{(n+1)!}\, d\zeta = \sum_{n=0}^{+\infty} \frac{1}{(n+1)!}\, \frac{z^{n+1}}{n+1} = \sum_{n=1}^{+\infty} \frac{1}{n \cdot n!}\, z^n.$$

(c) Analogously, it follows from

$$\mathrm{Log}(1+z) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\, z^n = z \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1}\, z^n, \qquad |z| < 1,$$

that

$$\frac{\mathrm{Log}(1+z)}{z} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1}\, z^n, \qquad 0 < |z| < 1,$$

where the series is also defined for $z = 0$ with the value 1. If $|z| < 1$, we get by termwise integration

$$L(z) = \int_0^z \frac{\mathrm{Log}(1+\zeta)}{\zeta}\, d\zeta = \int_0^z \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1}\, \zeta^n\, d\zeta = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)^2}\, z^{n+1} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2}\, z^n.$$

Example 4.13 *Given a piecewise continuous function $f(t)$ for $t \in [0, a]$. Prove that*

$$F(z) = \int_0^a e^{-z\,t}\, f(t)\, dt$$

is an analytic function in $\mathbb{C}$, and find its power series expansion.

The series

$$e^{-zt} = 1 - zt + \frac{(-zt)^2}{2!} + \cdots + \frac{(-zt)^n}{n!} + \cdots$$

is for any fixed value of z uniformly convergent in $t \in [0, a]$. We can therefore multiply by the *bounded* function $f(t)$ and then perform termwise integration. This gives

$$F(z) = \int_0^a f(t)\, dt - z \int_0^a t\, f(t)\, dt + \cdots + \frac{(-z)^n}{n!} \int_0^a t^n f(t)\, dt + \cdots,$$

which is the wanted power series expansion.

Now, f is bounded (because f is piecewise continuous over a closed bounded interval), so $|f(t)| \le M$, and we get

$$\left| \frac{(-z)^n}{n!} \int_0^a t^n f(t)\, dt \right| \le \frac{|z|^n}{n!} \int_0^a t^n M\, dt = \frac{|z|^n a^{n+1} M}{(n+1)!} = c_n |z|^n.$$

According to the criterion of majoring series it suffices to prove that $\sum c_n|z|^n$ is convergent for every $z \in \mathbb{C}$. This is obvious for $z = 0$. If $z \neq 0$, then

$$\sum_{n=0}^{+\infty} \frac{a^{n+1}M}{(n+1)!}\,|z|^n = \frac{M}{|z|}\sum_{n=1}^{+\infty}\frac{(a|z|)^n}{n!} = \frac{M}{|z|}\left(e^{a|z|}-1\right),$$

hence $\sum c_n|z|^n$ is convergent for every $z \in \mathbb{C}$, and the domain of convergence is $\mathbb{C}$, i.e. $F(z)$ is analytic in $\mathbb{C}$.

Remark 4.4 This example indicates a method of determining the Laplace transformed of a piecewise continuous function, which is only $\neq 0$ on a closed bounded interval. $\Diamond$

Example 4.14 *Assume that $g(z)$ is analytic in $|z| < R$, and that $g(0) = 0$. Apply Weierstraß's double series theorem in order to find a power series of $\dfrac{1}{1-g(z)}$.*
Find in particular the first three terms in the power series expansion (from 0) for $\dfrac{1}{\cos z}$.

Since

$$\frac{1}{1-g(z)} = \sum_{n=0}^{+\infty}\{g(z)\}^n = \sum_{n=0}^{+\infty} g_n(z) \qquad \text{for } |g(z)| < 1,$$

and since $g(0) = 0$ implies that $|g(z)| < 1$ in a neighbourhood of 0 (where the convergence is uniform for $|g(z)| \leq k < 1$), and $c_0 = 0$, it follows that

$$\frac{1}{1-g(z)} = \sum_{n=0}^{+\infty}\left\{\sum_{p=1}^{+\infty} c_p z^p\right\}^n = \sum_{n=0}^{+\infty} z^n\left\{\sum_{p=0}^{+\infty} c_{p+1}z^p\right\}^n.$$

Since

$$\frac{1}{\cos z} = \frac{1}{1-(1-\cos z)},$$

we get

$$g(z) = 1-\cos z = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{(2n)!}\,z^{2n} = \frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\cdots, \qquad z \in \mathbb{C},$$

hence (if $|1-\cos z| < 1$)

$$\begin{aligned}\frac{1}{\cos z} &= 1+\left\{\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\cdots\right\}+\left\{\frac{z^2\cdot z^2}{2!2!}-2\cdot\frac{z^2}{2!}\cdot\frac{z^4}{4!}+\cdots\right\}\left\{\frac{z^2\cdot z^2\cdot z^2}{2!2!2!}+\cdots\right\}+\cdots \\ &= 1+\frac{z^2}{2}+\frac{5}{24}\,z^4+\frac{61}{720}\,z^6+\cdots,\end{aligned}$$

when $|1-\cos z| < 1$. Note that because the power series expansion is unique in the larger domain $|z| < \dfrac{\pi}{2}$, the same expansion holds here.

Example 4.15 *Find the first three terms of the power series expansion from* $z_0 = 0$ *for the solution* w *of the transcendental equation*

$$w\,e^{-w} = z.$$

HINT: *Find* $w(0)$, $w'(0)$, $w''(0)$, $\dots$, *and then apply Taylor's formula.*

When we differentiate the equation (with respect to z)

$$z = w\,e^{-w},$$

then

$$1 = \left(e^{-w} - w\,e^{-w}\right)\frac{dw}{dz} = e^{-w}\,(1-w)\,\frac{dw}{dz}.$$

Here we put $z = 0$ and $w = 0$, and then get by a reduction that

$$\left.\frac{dw}{dz}\right|_{z=0} = 1.$$

By another differentiation,

$$e^{-w}\,(1-w)\,\frac{d^2w}{dz^2}-e^{-w}\,(2-w)\left(\frac{dw}{dz}\right)^2=0,$$

and we obtain in the same manner,

$$\left.\frac{d^2w}{dz^2}\right|_{z=0}=2.$$

Since $e^{-w}\neq 0$, this equation is equivalent to the simpler equation,

$$(1-w)\,\frac{d^2w}{dz^2}-(2-w)\left(\frac{dw}{dz}\right)^2=0.$$

When this equation is differentiated we get

$$(1-w)\,\frac{d^3w}{dz^3}-\frac{dw}{dz}\cdot\frac{d^2w}{dz^2}-2(2-w)\,\frac{dw}{dz}\cdot\frac{d^2w}{dz^2}+\left(\frac{dw}{dz}\right)^3=0,$$

hence by insertion of the previous results,

$$\left.\frac{d^3w}{dz^3}\right|_{z=0}=1\cdot 2+2\cdot 2\cdot 1\cdot 2-1=9.$$

Finally, by insertion into Taylor's formula we obtain in a neighbourbood of $z_0=0$ that

$$w(z)=0+\frac{1}{1!}\,z+\frac{2}{2!}\,x^2+\frac{9}{3!}\,z^3\cdots=z+z^2+\frac{3}{2}\,z^3+\cdots.$$

Example 4.16 *Lad $M\in\mathbb{R}$. Prove that if $f(z)$ is analytic in $\mathbb{C}$, and*

$$Re(f(z))\leq M$$

for every z, then $f(z)$ is constant.
HINT: *Apply Liouville's theorem on $exp(f(z))$.*

When we split f into its real and imaginary part, $f=u+i\,v$, it follows from the assumption that

$$u(x,y)\leq M \qquad \text{for every } (x,y).$$

Since $\exp(f(z))$ is analytic and

$$\exp(f(z))=e^u\cdot(\cos v+i\,\sin v),$$

we conclude that

$$|\exp(f(z))\leq e^u\leq e^M \qquad \text{for every } (x,y),$$

thus $\exp(f(z))$ is a bounded analytic function. Then it follows from *Liouville's theorem* that $\exp(f(z))$ is constant, and since f is continuous, we conclude that $f(z)$ is also constant.

Example 4.17 *Define the* Bernouilli numbers B_n *by the power series*

$$\frac{z}{e^z-1}=\sum_{n=0}^{+\infty}\frac{B_n}{n!}\,z^n,$$

where $\lim_{z\to 0}\dfrac{z}{e^z-1}$ *exists.*
First determine this limit.
Then multiply by e^z-1 *to prove that the Bernoulli numbers satisfy the recursion formula*

$$\sum_{j=0}^{n-1}\frac{1}{j!(n-j)!}\,B_j=0 \qquad \textit{for } n>1.$$

Find $B_0,\,B_1,\,\dots,\,B_4$.
Prove that $B_n=0$ *for* $n\neq 1$ *odd.*
Determine the radius of convergence of the series.

Since

$$e^z-1=z+\frac{1}{2!}\,z^2+\cdots$$

is different from zero for $z\neq 2i\,p\,\pi$, $p\in\mathbb{Z}$, we conclude that

$$\lim_{z\to 0}\frac{z}{e^z-1}=\lim_{z\to 0}\frac{z}{z+\frac{1}{2}\,z^2+o\left(z^2\right)}=\lim_{z\to 0}\frac{1}{1+\frac{1}{2}\,z+o(z)}=1=B_0.$$

It follows that $B_0=1$. Furthermore, the power series is convergent in the largest open disc of centrum at 0 which does not contain any number of the form $2i\,p\,\pi$, $p\in\mathbb{Z}\setminus\{0\}$. The two closest singularities of the point of expansion $z_0=0$ are $\pm 2i\,\pi$, so we conclude that the radius of convergence is $|\pm 2i\,\pi|=2\pi$.

Assume that $|z|<2\pi$. If we multiply the equation by

$$e^z-1=\sum_{n=1}^{+\infty}\frac{1}{n!}\,z^n,$$

it follows by a Cauchy multiplication that

$$z=\sum_{j=0}^{+\infty}\frac{B_j}{j!}\,z^j\cdot\sum_{k=1}^{+\infty}\frac{1}{k!}\,z^k=B_0\,z+\sum_{\substack{j+k=n\\ j\geq 0,\,k\geq 1}}\frac{1}{j!k!}\,B_j z^n.$$

When $n>1$, it follows from the uniqueness theorem that

$$0=\sum_{\substack{j+k=n\\ j\geq 0,\,k\geq 1}}\frac{1}{j!k!}\,B_j=\sum_{j=0}^{n-1}\frac{1}{j!(n-j)!}\,B_j.$$

If $n=1$, we again obtain (in accordance with the previous result) that

$$\sum_{j=0}^{0}\frac{1}{j!(1-j)!}\,B_j=B_0=0.$$

Replacing n by $n+1$, it follows from the above for $n \geq 1$,

$$0 = \sum_{j=0}^{n} \frac{1}{j!(n+1-j)!} B_j = \frac{1}{n!1!} B_n + \sum_{j=0}^{n-1} \frac{1}{j!(n+1-j)!} B_j,$$

hence

$$(3)\quad B_n = -\sum_{j=0}^{n-1} \frac{n!}{j!(n+1-j)!} B_j = -\frac{1}{n+1} \sum_{j=0}^{n-1} \binom{n+1}{j} B_j.$$

We have already proved that $B_0 = 1$. Then successively by (3),

$$\begin{aligned}
B_1 &= -\frac{1}{2}\sum_{j=0}^{0}\binom{2}{j}B_j = -\frac{1}{2}\,B_0 = -\frac{1}{2},\\
B_2 &= -\frac{1}{3}\sum_{j=0}^{1}\binom{3}{j}B_j = -\frac{1}{3}\left\{\binom{3}{0}B_0+\binom{3}{1}B_1\right\} = -\frac{1}{3}\left\{1+3\cdot\left(-\frac{1}{2}\right)\right\} = \frac{1}{6},\\
B_3 &= -\frac{1}{4}\sum_{j=0}^{2}\binom{4}{j}B_j = -\frac{1}{4}\left\{\binom{4}{0}B_0+\binom{4}{1}B_1+\binom{4}{2}B_2\right\} = -\frac{1}{4}\left\{1-\frac{4}{2}+\frac{6}{6}\right\} = 0,\\
B_4 &= -\frac{1}{5}\sum_{j=0}^{3}\binom{5}{j}B_j = -\frac{1}{5}\left\{\binom{5}{0}B_0+\binom{5}{1}B_1+\binom{5}{2}B_2+\binom{5}{3}B_3\right\}\\
&= -\frac{1}{5}\left\{1-\frac{5}{2}+\frac{10}{6}+0\right\} = -\frac{1}{30}\left\{6-15+10\right\} = -\frac{1}{30}.
\end{aligned}$$

Summing up we have found the first five Bernouilli numbers,

$$B_0 = 1,\quad B_1 = -\frac{1}{2},\quad B_2 = \frac{1}{6},\quad B_3 = 0,\quad B_4 = -\frac{1}{30}.l$$

If

$$\begin{aligned}
\varphi(z) &= \frac{z}{e^z-1} - B_0 - \frac{B_1}{1!}\,z = \frac{z}{e^z-1} - 1 + \frac{1}{2}\,z\\
&= \sum_{n=2}^{+\infty}\frac{B_n}{n!}\,z^n, \qquad \text{for } 0<|z|<2\pi.
\end{aligned}$$

then for $0<|z|<2\pi$,

$$\begin{aligned}
\sum_{n=2}^{+\infty}\frac{B_n}{n!}\,(-1)^n z^n &= \varphi(z) = \frac{-z}{e^{-z}-1} - 1 - \frac{1}{2}\,z = \frac{-z\,e^z}{1-e^z} - 1 - \frac{1}{2}\,z\\
&= \frac{z\,e^z}{e^z-1} - 1 - \frac{1}{2}\,z = \frac{z\,(e^z-1)+z}{e^z-1} - 1 - \frac{1}{2}\,z\\
&= z + \frac{z}{e^z-1} - 1 - \frac{1}{2}\,z = \frac{z}{e^z-1} - 1 + \frac{1}{2}\,z = \varphi(z) = \sum_{n=2}^{+\infty}\frac{B_n}{n!}\,z^n.
\end{aligned}$$

Since

$$\sum_{n=2}^{+\infty}(-1)^n\frac{B_n}{n!}\,z^n = \sum_{n=2}^{+\infty}\frac{B_n}{n!}\,z^n \qquad \text{for } |z|<2\pi,$$

it follows by a reduction that

$$0 = 2\sum_{n=1}^{+\infty}\frac{B_{2n+1}}{(2n+1)!}\,z^{2n+1}, \qquad \text{for } |z|<2\pi,$$

and we conclude by the identity theorem that $B_{2n+1} = 0$ for every $n\in\mathbb{N}$.

Example 4.18 *Applying the Bernouilli numbers introduced in Example 4.17, prove that*

$$\frac{z}{2}\cdot\frac{\exp\left(\frac{z}{2}\right)+\exp\left(-\frac{z}{2}\right)}{\exp\left(\frac{z}{2}\right)-\exp\left(-\frac{z}{2}\right)}=\sum_{n=0}^{+\infty}\frac{1}{(2n)!}\,B_{2n}\,z^{2n}.$$

Then replace z by $2i\pi z$ to prove that

$$\pi z\cdot\cot\pi z=\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{1}{(2n)!}\cdot(2\pi)^{2n}\,B_{2n}z^{2n}.$$

It follows by a simple computation that

$$\begin{aligned}\frac{z}{2}\cdot\frac{\exp\left(\frac{z}{2}\right)+\exp\left(-\frac{z}{2}\right)}{\exp\left(\frac{z}{2}\right)} &= \frac{z}{2}\cdot\cot\frac{z}{2}=\frac{z}{2}\cdot\frac{e^z+1}{e^z-1}=\frac{z}{2}\cdot\frac{e^z-1+2}{e^z-1}=\frac{z}{2}+\frac{z}{e^z-1}\\ &= \frac{z}{2}+\sum_{n=0}^{+\infty}\frac{B_n}{n!}\,z^n=\frac{z}{2}+B_1z+\sum_{n=0}^{+\infty}\frac{B_{2n}}{(2n)!}\,z^{2n}=\sum_{n=0}^{+\infty}\frac{B_{2n}}{(2n)!}\,z^{2n},\end{aligned}$$

using that $B_1=-\frac{1}{2}$ and $B_{2n+1}=0$ for $n\in\mathbb{N}$ by Example 4.9.

If we replace z by $2i\pi z$, then

$$\begin{aligned}\frac{2i\pi z}{2}\cdot\frac{\exp(i\pi z)+exp(-i\pi z)}{\exp(i\pi z)-\exp(-i\pi z)} &= \pi\,z\cdot\cot(\pi\,z)=\sum_{n=0}^{+\infty}\frac{B_{2n}}{(2n)!}\,(2i\pi z)^{2n}\\ &= \sum_{n=0}^{+\infty}(-1)^n\cdot\frac{1}{(2n)!}\cdot(2\pi)^{2n}\,B_{2n}\,z^{2n}.\end{aligned}$$

Example 4.19 *1) Denote the roots of the polynomial of second order $z^2 + z - 1$ by a and b (where $|b| > |a|$). Prove that the function*

$$f(z) = \frac{1}{1 - z - z^2}$$

has the Taylor expansion

$$\sum_{n=0}^{+\infty} a_n z^n, \qquad |z| < |a|,$$

where the sequence of coefficients (a_n) is determined by the recursion formula

$$a_0 = a_1 = 1, \qquad a_{n+2} = a_{n+1} + a_n.$$

2) Decompose f and then expand termwise to prove the formula

$$a_n = (-1)^n \frac{b^{n+1} - a^{n+1}}{b - a}.$$

3) Find the Laurent expansion of f in the circular annulus $|a| < |z| < |b|$.

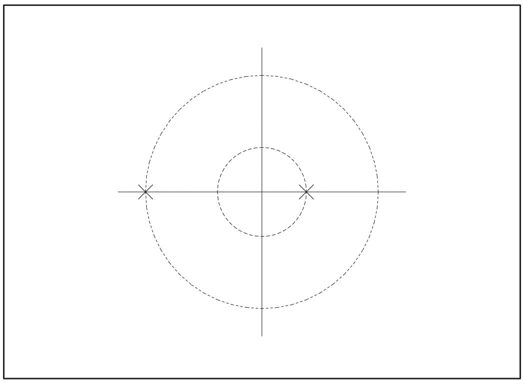


Figure 9: The disc and the annulus defined by a (to the right) and b (to the left). The disc is considered in **(1)** and **(2)**, while the annulus is considered in **(3)**.

Remark 4.5 Since $z^2 + z - 1 = 0$ for

$$z = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2},$$

it follows that

$$a = \frac{\sqrt{5} - 1}{2} \qquad \text{and} b = -\frac{\sqrt{5} + 1}{2},$$

which gives us some feeling of where a and b are lying, cf. the figure. The purpose of the example is, however, that it is possible *not* to apply the exact values of a and b. $\Diamond$

1) The function $f(z)$ has a Taylor expansion for $|z| < |a|$,

$$f(z) = \frac{1}{1-z-z^2} = \sum_{n=0}^{+\infty} a_n z^n, \qquad |z| < |a|.$$

Thus by a multiplication by $a - z - z^2$ for $|z| < |a|$,

$$\begin{aligned}
1 &= \left(1-z-z^2\right)\sum_{n=0}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty} a_n z^n - \sum_{n=0}^{+\infty} a_n z^{n+1} - \sum_{n=0}^{+\infty} a_n z^{n+2} \\
&= a_0 + a_1 z + \sum_{n=2}^{+\infty} a_n z^n - a_0 z - \sum_{n=1}^{+\infty} a_n z^{n+1} - \sum_{n=0}^{+\infty} a_n z^{n+2} \\
&= a_0 + (a_1 - a_0)\, z + \sum_{n=0}^{+\infty} \left\{a_{n+2} - a_{n+1} - a_n\right\} z^{n+2}.
\end{aligned}$$

Then it follows from the identity theorem that

$$\begin{cases} a_0 = 1, \\ a_1 - a_0 = 0, \\ a_{n+2} - a_{n+1} - a_n = 0, \qquad n \in \mathbb{N}_0, \end{cases}$$

thus

$$a_0 = a_1 = 1, \qquad\qquad a_{n+2} = a_{n+1} + a_n, \qquad n \in \mathbb{N}_0.$$

2) The product of the roots is equal to the constant term, so

$$\frac{1}{a} = -b, \qquad \text{and} \qquad \frac{1}{b} = -a,$$

which we shall use in the following. Then we get in the disc $|z| < |a|$ by a decomposition and by the geometric series,

$$\begin{aligned} f(z) &= \frac{1}{1-z-z^2} = -\frac{1}{z^2+z-1} = -\frac{1}{(z-a)(z-b)} = -\frac{1}{a-b}\cdot\frac{1}{z-a} - \frac{1}{b-a}\cdot\frac{1}{z-b} \\ &= -\frac{1}{b-a}\cdot\frac{1}{a}\cdot\frac{1}{1-\frac{z}{a}} + \frac{1}{b-a}\cdot\frac{1}{b}\cdot\frac{1}{1-\frac{z}{b}} = \frac{1}{b-a}\cdot b\cdot\frac{1}{1+bz} - \frac{1}{b-a}\cdot a\cdot\frac{1}{1+az} \\ &= \frac{1}{b-a}\left\{ b\sum_{n=0}^{+\infty}(-1)^n b^n z^n - a\sum_{n=0}^{+\infty}(-1)^n a^n z^n \right\} = \sum_{n=0}^{+\infty}(-1)^n\,\frac{b^{n+1}-a^{n+1}}{b-a}\,z^n, \end{aligned}$$

because e.g. $|bz| = \left|\frac{z}{a}\right| < 1$ in the disc given by $|z| < |a| < |b|$, and analogously $|az| \le |bz| < 1$ in the same disc.
Since this series expansion is the same as the given series expansion,

$$\sum_{n=0}^{+\infty} a_n z^n, \qquad |z| < |a| < |b|,$$

we conclude that

$$a_n = (-1)^n\,\frac{b^{n+1}-a^{n+1}}{b-a}, \qquad n \in \mathbb{N}_0.$$

3) Assume that z lies in the annulus $|a| < |z| < |b|$. Then by the decomposition above,

$$\begin{aligned} f(z) &= \frac{1}{1-z-z^2} = -\frac{1}{a-b}\cdot\frac{1}{z-a} - \frac{1}{b-a}\cdot\frac{1}{z-b} = \frac{1}{b-a}\cdot\frac{1}{z}\cdot\frac{1}{a-\frac{a}{z}} + \frac{1}{b-a}\cdot\frac{1}{b}\cdot\frac{1}{1-\frac{z}{b}} \\ &= \frac{1}{b-a}\cdot\frac{1}{z}\sum_{n=0}^{+\infty}\frac{a^n}{z^n} - \frac{1}{b-a}\cdot a\cdot\frac{1}{1+az} = \frac{1}{b-a}\sum_{n=1}^{+\infty}a^{n-1}\cdot\frac{1}{z^n} - \frac{a}{b-a}\sum_{n=0}^{+\infty}(-1)^n a^n z^n \\ &= \sum_{n=1}^{+\infty}\frac{a^{n-1}}{b-a}\,\frac{1}{z^n} + \sum_{n=0}^{+\infty}(-1)^n\,\frac{a^{n-1}}{b-a}\,z^n, \end{aligned}$$

where the estimates

$$\left|\frac{a}{z}\right| < 1 \qquad \text{and} \qquad \left|\frac{z}{b}\right| = |az| < 1,$$

secure the convergence.

Example 4.20 *Given the sequence* $a_0, a_1, \dots, a_n, \dots,$ *by the recursion formula*

$$a_0 = 1, \qquad a_1 = -1, \qquad a_{n+2} = -a_{n+1} - 2a_n, \quad n \geq 0.$$

1) Prove that the function

$$f(z) = \frac{1}{1+z+2z^2}$$

has the power series expansion

$$\sum_{n=0}^{+\infty} a_n z^n,$$

and determine the radius of convergence.

2) Denote the roots of the polynomial $1+z+2z^2$ *by* a *and* b *(where* $Im(a) > 0$*). Prove the formula*

$$a_n = 2^n \, \frac{a^{n+1} - b^{n+1}}{a-b}$$

by decomposing $f(z)$ *and then expanding every term in some series.*

3) Prove by putting $a = r\,e^{i\,v}$ *the formula*

$$a_n = \sqrt{2^n \cdot \frac{8}{7}} \cdot \sin(n+1)v,$$

where v *is defined by*

$$\cos v = -\frac{1}{2\sqrt{2}}, \qquad 0 < v < \pi.$$

1) Since $2z^2 + z + 1 = 0$ has the roots

$$z = \frac{-1 \pm \sqrt{1-8}}{4} = \frac{-1 \pm i\sqrt{7}}{4},$$

the function $f(z)$ is analytic in the disc

$$|z| < \left| \frac{-1 \pm i\sqrt{7}}{4} \right| = \frac{\sqrt{8}}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

so $f(z)$ has the Taylor expansion

$$f(z) = \frac{1}{1+z+2z^2} = \sum_{n=0}^{+\infty} b_n z^n, \qquad |z| < \frac{1}{\sqrt{2}},$$

where we shall prove that $b_n = a_n$.
When we multiply by $1+z+2z^2 \neq 0$ we get

$$\begin{aligned}
1 &= \left(1+z+2z^2\right) f(z) = \left(1+z+2z^2\right) \sum_{n=0}^{+\infty} b_n z^n = \sum_{n=0}^{+\infty} b_n z^n + \sum_{n=0}^{+\infty} b_n z^{n+1} + \sum_{n=0}^{+\infty} 2b_n z^{n+2} \\
&= b_0 + b_1 z + \sum_{n=2}^{+\infty} b_n z^n + b_0 z + \sum_{n=1}^{+\infty} b_n z^{n+1} + \sum_{n=0}^{+\infty} 2b_n z^{n+2} \\
&= b_0 + (b_0+b_1)\, z + \sum_{n=0}^{+\infty} b_{n+2} z^{n+2} + \sum_{n=0}^{+\infty} b_{n+1} z^{n+2} + \sum_{n=0}^{+\infty} 2b_n z^{n+2} \\
&= b_0 + (b_0+b_1)\, z + \sum_{n=0}^{+\infty} \{b_{n+2} + b_{n+1} + 2b_n\}\, z^{n+2}.
\end{aligned}$$

It follows by the identity theorem that

$$b_0 = 1, \quad b_1 = -b_0 = -1 \quad \text{and} \quad b_{n+2} = -b_{n+1} - 2b_n, \quad n \in \mathbb{N}_0,$$

so the two sequences (a_n) and (b_n) fulfil the same difference equation. The solution of this is unique, so we conclude that $b_n = a_n$, $n \in \mathbb{N}_0$. It also follows from the above that the radius of convergence is $r = \dfrac{1}{\sqrt{2}}$.

2) Then

$$a = \frac{-1+i\sqrt{7}}{4} \qquad \text{and} \qquad b = \frac{-1-i\sqrt{7}}{4}$$

are the roots of the polynomial. Since $a \cdot b = \dfrac{1}{2}$, we have $\dfrac{1}{a} = 2b$ and $\dfrac{1}{b} = 2a$.

Then by a decomposition in the disc $|z| < \dfrac{1}{\sqrt{2}}$,

$$\begin{aligned}
f(z) &= \frac{1}{2z^2+z+1} = \frac{1}{2(z-a)(z-b)} = \frac{1}{2} \cdot \frac{1}{b-a} \cdot \frac{1}{z-a} + \frac{1}{2} \cdot \frac{1}{a-b} \cdot \frac{1}{z-b} \\
&= \frac{1}{2} \cdot \frac{-1}{(b-a)a} \cdot \frac{1}{1-\frac{z}{a}} + \frac{1}{2} \cdot \frac{-1}{(a-b)b} \cdot \frac{1}{1-\frac{z}{b}} = \frac{1}{2} \cdot \frac{1}{a-b} \left\{ \frac{1}{a} \cdot \frac{1}{1-\frac{z}{a}} - \frac{1}{b} \cdot \frac{1}{1-\frac{z}{b}} \right\} \\
&= \frac{1}{2} \cdot \frac{1}{a-b} \left\{ 2b \cdot \frac{1}{1-2bz} - 2a \cdot \frac{1}{1-2az} \right\} = \frac{1}{a-b} \left\{ b \sum_{n=0}^{+\infty} (2b)^n z^n - a \sum_{n=0}^{+\infty} (2a)^n z^n \right\} \\
&= \sum_{n=0}^{+\infty} 2^n \cdot \frac{b^{n+1} - a^{n+1}}{b-a}\, z^n = \sum_{n=0}^{+\infty} a_n z^n.
\end{aligned}$$

Hence it follows by the identity theorem that

$$a_n = 2^n \cdot \frac{a^{n+1} - b^{n+1}}{a-b}, \qquad n \in \mathbb{N}_0.$$

3) Then we introduce polar coordinates

$$a=\frac{-1+i\sqrt{7}}{4}=\frac{1}{\sqrt{2}}\cdot\left(-\frac{\sqrt{2}}{4}+i\,\frac{\sqrt{14}}{4}\right)=r\,e^{i\,v},$$

thus $r=\dfrac{1}{\sqrt{2}}$ and $v\in\,]0,\pi[$, where

$$\cos v=-\frac{\sqrt{2}}{4}=-\frac{1}{\sqrt{8}}.$$

Since $b=\overline{a}=r\,e^{-i\,v}$, it follows tht

$$\begin{aligned}
a_n &= 2^n\cdot\frac{a^{n+1}-b^{n+1}}{a-b}=\frac{2^n\cdot r^{n+1}\cdot\left\{e^{i(n+1)v}-e^{-i(n+1)v}\right\}}{-\dfrac{-1+i\sqrt{7}}{4}-\dfrac{-1-i\sqrt{7}}{4}}=\frac{2^n\cdot\left(\dfrac{1}{\sqrt{2}}\right)^{n+1}2i\,\sin(n+1)v}{\dfrac{i\sqrt{7}}{2}}\\
&= \frac{2^{n-\frac{1}{2}(n+1)+1+1}}{\sqrt{7}}\,\sin(n+1)v=\frac{1}{\sqrt{7}}\cdot 2^{\frac{1}{2}\,n+\frac{3}{2}}\,\sin(n+2)v\\
&= \left(\frac{2^{n+3}}{7}\right)^{\frac{1}{2}}\sin(n+1)v=\left(\frac{8\cdot 2^n}{7}\right)^{\frac{1}{2}}\sin(n+1)v.
\end{aligned}$$

Example 4.21 *Put*

$$S(z)=\sum_{n=0}^{+\infty}(-1)^n\,\frac{3^n z^{2n+1}}{2n+1}\qquad \textit{for } |z|<R,$$

where R denotes the radius of convergence. Determine R, and find explicitly for $|z|<R$ the derivative $S'(z)$ as a function of z.

It follows from

$$S(z)=\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{3^n z^{2n+1}}{2n+1}=\frac{1}{\sqrt{3}}\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{\left(\sqrt{3}\,z\right)^{2n+1}}{2n+1},$$

that the condition of convergence is

$$|\sqrt{3}\,z|<1,\qquad\text{thus}\qquad |z|<R=\frac{1}{\sqrt{3}}.$$

The by termwise differentiation for $|z|<\dfrac{1}{\sqrt{3}}$,

$$S'(z)=\sum_{n=0}^{+\infty}(-1)^n 3^n z^{2n}=\sum_{n=0}^{+\infty}(-1)^n\cdot\left(3z^2\right)^n=\frac{1}{1+3z^2}.$$

5 Linear differential equations and the power series method

Example 5.1 *Solve the differential equation*

$$f'(z) - f(z) = 0$$

by insertion of a formal power series $f(z) = \sum_{n=0}^{+\infty} a_n z^n$.

Remark 5.1 In spite of the formulation we shall try all four solution variants. $\Diamond$

First method. *Inspection.* It follows from

$$\frac{d}{dz}\, e^z = e^z,$$

that the function $f(z) = e^z$ trivially satisfies the homogeneous differential equation, so the complete solution is given by

$$f(z) = c \cdot e^z, \qquad c \in \mathbb{C} \text{ arbitrart constant.}$$

Second method. *Integrating factor.* If the differential equation is multiplied by $e^{-z} \neq 0$, then we get by some small manipulation the equivalent differential equation

$$0 = e^{-z}\, f'(z) - e^{-z} f(z) = e^{-z}\, \frac{df}{dz} + f(z)\, \frac{d}{dz}\, e^{-z} = \frac{d}{dz}\left\{e^{-z}\, f(z)\right\}.$$

We then get by integration,

$$e^{-z}\, f(z) = c, \qquad \text{thus} \qquad f(z) = c \cdot e^{z}, \qquad c \in \mathbb{C} \text{ arbitrary constant.}$$

Third method. *Determination of* $f^{(n)}(z_0)$. We have clearly one degree of freedom, so we choose $f(0) = c \in \mathbb{C}$, arbitrary. We get by successive differentiations of the given differential equation and rearrangements

$$f^{(n)}(z) = f^{(n-1)}(z), \qquad \text{for every } n \in \mathbb{N},$$

hence by a simple recursion,

$$f(n)(0) = f^{(n-1)}(0) = \cdots = f(0) = c, \qquad n \in \mathbb{N}.$$

Then the Taylor series from $z_0 = 0$ is formally given by

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, f^{(n)}(0)\, z^n = c \sum_{n=0}^{+\infty} \frac{1}{n!}\, z^n = c \cdot e^{z}, \qquad z \in \mathbb{C},$$

where we immediately recognize the exponential series with its domain $\mathbb{C}$.

Fourth method. *Determination of a recursion formula for a series solution.* We assume that the equation has a power series solution

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad \text{where} \qquad f'(z) = \sum_{n=1}^{+\infty} n\, a_n\, z^{n-1} \qquad \text{for } |z| \leq \varrho,$$

where we also shall find ϱ. We get by insertion into the differential equation,

$$\begin{aligned} 0 &= f'(z) - f(z) = \sum_{n=1}^{+\infty} n\, a_n\, z^{n-1} - \sum_{n=0}^{+\infty} a_n z^n = \sum_{n=1}^{+\infty} n\, a_n\, z^{n-1} - \sum_{n=1}^{+\infty} a_{n-1}\, z^{n-1}, \\ &= \sum_{n=1}^{+\infty} \left\{n\, a_n - a_{n-1}\right\} z^{n-1}, \end{aligned}$$

where these computations are legal, if only $|z| < \varrho$. Hence, we have a power series expansion of the zero function, and since this is unique, we conclude that we have the *recursion formula*

$$n\, a_n - a_{n-1} = 0 \qquad \text{for every } n \in \mathbb{N}.$$

We multiply this formula by $(n-1)! \neq 0$. Then by a rearrangement and recursion,

$$n!\, a_n = (n-1)!\, a_{n-1} = \cdots = j!\, a_j = \cdots = 0!\, a_0 = a_0,$$

thus

$$a_n = a_0 \cdot \frac{1}{n!} \qquad \text{for every } n \in \mathbb{N},$$

and we derive the *formal* series solution,

$$f(z) = \sum_{n=0}^{+\infty} a_n \, z^n = a_0 \sum_{n=0}^{+\infty} \frac{1}{n!} \, z^n.$$

If we recognize this series as the exponential series, then we have finished our task, because we know that the exponential series is convergent in $\mathbb{C}$. Otherwise, we split into the cases $a_0 = 0$ (where we get the not so interesting zero series which is convergent everywhere) and $a_0 \neq 0$, where

$$\varrho = \lim_{n \to +\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to +\infty} (n+1) = +\infty,$$

and the domain of convergence is $\mathbb{C}$.

Alternatively the recursion formula can be solved less elegantly in the following way:

$$a_n = \frac{1}{n} \, a_{n-1} = \frac{1}{n} \cdot \frac{1}{n-1} \, a_{n-2} = \cdots = \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{2} \cdot \frac{1}{1} \cdot a_0 = \frac{1}{n!} \, a_0,$$

and then we proceed as above.

Example 5.2 *Solve the differential equation*

$$(1-z)f'(z) = f(z)$$

by e.g. inserting a formal power series $f(z) = \sum_{n=0}^{+\infty} a_n z^n$.

First method. *Inspection.* We get by a small rearrangement,

$$0 = (1-z)f'(z) - f(z) = \frac{d}{dz} \{(1-z)f(z)\},$$

the primitive of which is $(1-z)f(z) = c \in \mathbb{C}$, and thus

$$f(z) = \frac{c}{1-z}, \qquad \text{for } z \neq 1.$$

Second method. *Determination of* $f^{(n)}(z_0)$. The expansion point is $z_0 = 0$, and the coefficient of $f'(z)$ is only zero for $z = 1$, so the Taylor series is *at least* convergent for $|z| < 1$. When we differentiate the differential equation it follows after a rearrangement that

$$(1-z)f''(z) = 2\, f'(z).$$

This gives us the hint that we possibly in general have

$$(1-z)f^{(n)}(z) = n\, f^{(n-1)}(z), \qquad n \in \mathbb{N}.$$

This is true for $n = 1$ and $n = 2$, and if we differentiate the conjecture, we get after another small rearrangement the same structure of the equation, where only n has been replaced by $n+1$, and the claim follows by induction.

Since $z_0 = 0$, it then follows by recursion that

$$f^{(n)}(0) = n\, f^{(n-1)}(0) = \cdots = n!\, f(0),$$

and the Taylor series is the same one as we found above,

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, f^{(n)}(0)\, z^n = \sum_{n=0}^{+\infty} \frac{1}{n!} \cdot n'!\, f(0)\, z^n = f(0) \sum_{n=0}^{+\infty} z^n,$$

where the radius of convergence again is seen to be $\varrho = 1$. The series is a quotient series of quotient z, where $|z| < 1$, hence

$$f(z) = \frac{f(0)}{1-z}, \qquad \text{for } |z| < 1,$$

though it is obvious that we can extend it to $\mathbb{C} \setminus \{1\}$, because the differential equation is also fulfilled here.

Third method. *The method of power series.* We assume that the solution has the power series expansion

$$f(z) = \sum_{n=0}^{+\infty} a_n\, z^n \qquad \text{where} \qquad f'(z) = \sum_{n=1}^{+\infty} n\, a_n\, z^{n-1} \qquad \text{for } |z| \le \varrho.$$

When these expressions are put into the differential equation and we assume that $|z| < \varrho$, then

$$\begin{aligned} 0 &= (1-z)f'(z) - f(z) = \sum_{n=1}^{+\infty} n\, a_n\, z^{n-1} - \sum_{n=1}^{+\infty} n\, a_n\, z^n - \sum_{n=0}^{+\infty} a_n\, z^n \\ &= \sum_{n=0}^{+\infty} (n+1) a_{n+1}\, z^n - \sum_{n=0}^{+\infty} (n+1) a_n\, z^n = \sum_{n=0}^{+\infty} (n+1)\left\{a_{n+1} - a_n\right\} z^n. \end{aligned}$$

It follows from $n + 1 \neq 0$ for every $n \in \mathbb{N}_0$, that we can divide by $n+1$ in order to obtain the simpler recursion formula

$$a_{n+1} = a_n, \qquad n \in \mathbb{N}_0.$$

It follows by recursion that

$$a_{n+1} = a_n = a_{n-1} = \cdots = a_0, \qquad n \in \mathbb{N}_0,$$

an our formal series is given by

$$f(z) = a_0 \sum_{n=0}^{+\infty} z^n.$$

If $a_0 = 0$, we get the zero series which is convergent everywhere.

If $a_0 \neq 0$, then

$$\varrho = \lim_{n \to +\infty} \left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{a_0}{a_0} \right| = 1.$$

Example 5.3 *Examine the inhomogeneous linear differential equation*

$$z^2 f'(z) - f(z) = -z,$$

and its possible solutions in the neighbourbood of $z_0 = 0$.

First method. *Inspection.* When $z \neq 0$, we multiply the equation by the integrating factor

$$\frac{1}{z^2}\exp\left(\frac{1}{z}\right) \neq 0 \qquad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

Then,

$$-\frac{1}{z}\exp\left(\frac{1}{z}\right) = \exp\left(\frac{1}{z}\right)\cdot f'(z) - \frac{1}{z^2}\exp\left(\frac{1}{z}\right)\cdot f(z) = \frac{d}{dz}\left\{\exp\left(\frac{1}{z}\right) f(z)\right\}.$$

So far, so good, but then everything goes wrong, because it is not possible to find a primitive of the left hand side in any neighbourhood of $z_0 = 0$. The problem is that the Laurent series expansions starts with the term $-1/z$, so we shall consider a (complex) logarithm. These will all have a branch cut to $z_0 = 0$, and it will follow from a later book in this series that it is impossible to obtain a Laurent series around the point $z_0 = 0$.

Second method. *Formal determination of the Taylor series.* If we put $z = 0$ into the differential equation, then $f(0) = 0$. We proceed by differentiating the differential equation,

$$z^2 f''(z) + (2z-1)f'(z) = -1, \qquad f'(0) = 1,$$

$$z^2 f^{(3)}(z) + (4z-1)f''(z) + 2f'(z) = 0, \qquad f''(0) = 2,$$

$$z^2 f^{(4)}(z) + (6z-1)f^{(3)}(z) + 6f''(z) = 0, \qquad f^{(3)}(0) = 12.$$

Then it follows by induction (left to the reader) that

$$z^2 f^{n+1)}(z) + (2n\,z-1)f^{(n)}(z) + n(n-1)\,f^{(n-1)}(z) = 0, \qquad m \geq 2,$$

hence for $z = 0$,

$$f^{(n)}(0) = n(n-1)\,f^{(n-1)}(0), \qquad n \geq 2.$$

When we divide this recursion formula of $f^{(n)}(0)$ by $n!(n-1)!$, then we get by recursion,

$$\frac{f^{(n)}(0)}{n!(n-1)!} = \frac{f^{(n-1)}(0)}{(n-1)!(n-2)!} = \cdots = \frac{f^{(2)}(0)}{2!1!} = \frac{2}{2} = 1,$$

and the Taylor coefficients become

$$\frac{1}{n!}\,f^{(n)}(0) = (n-1)!, \qquad n \in \mathbb{N},$$

so the *formal* Taylor series is

$$\sum_{n=0}^{+\infty} \frac{1}{n!}\,f^{(n)}(0)\,z^n = \sum_{n=1}^{+\infty} (n-1)!\,z^n.$$

Hence, the radius of convergence is

$$\varrho = \lim_{n\to+\infty} \frac{(n-1)!}{n!} = \lim_{n\to+\infty} \frac{1}{n} = 0,$$

so the Taylor series expanded from $z_0 = 0$ is only convergent for $z = 0$, and we cannot use the series expansion to anything.

Third method. *The power series method.* Assume that the solution has a convergent power series expansion,

$$f(z) = \sum_{n=0}^{+\infty} a_n\,z^n \qquad \text{for } |z| < \varrho.$$

Then by insertion into the differential equation,

$$\begin{aligned} z^2 f'(z) - f(z) &= z^2 \sum_{n=1}^{+\infty} n\,a_n\,z^{n-1} - \sum_{n=0}^{+\infty} a_n\,z^n = \sum_{n=1}^{+\infty} n\,a_n\,z^{n+1} - \sum_{n=0}^{+\infty} a_n\,z^n \\ &= \sum_{n=2}^{+\infty} (n-1)a_{n-1}\,z^n - \sum_{n=0}^{+\infty} a_n\,z^n = -a_0 + \sum_{n=1}^{+\infty} \left\{(n-1)a_{n-1} - a_n\right\} z^n. \end{aligned}$$

If this expression is put equal to $-z$, then $-a_0 = 0$, and $0 \cdot a_0 - a_1 = -1$, thus $a_1 = 1$, and

$$a_n = (n-1)a_{n-1}, \qquad n \ge 2.$$

We get by recursion, $a_n = (n-1)!\, a_1 = (n-1)!$, so the formal series becomes

$$\sum_{n=1}^{+\infty} (n-1)!\, z^n.$$

It is immediately seen that this series is divergent, whenever $z \neq 0$, so we cannot use the series expansion to anything.

Remark 5.2 In particular, the example demonstrates that we have *never* finished this method of power series solution, if we have not also found the corresponding open domain of convergence. $\Diamond$

Example 5.4 *Solve the differential equation*

$$f'(z) - z\, f(z) = 0$$

by assuming that its solution can be written as a (formal) power series $f(z) = \sum_{n=0}^{+\infty} a_n z^n$.

Remark 5.3 As usual we shall again demonstrate all three possible solution variants. $\Diamond$

First variant. *Inspection.* In the neighbourhood of any point in which $f(z) \neq 0$, we see that the equation is equivalent to

$$\frac{f'(z)}{f(z)} = z.$$

Since Log $f(z)$ is *locally* defined in the neighbourhood of any such point, the primitive exists and is given by

$$\operatorname{Log} f(z) = \frac{z^2}{2} + c,$$

so we have *locally*,

$$(4) \quad f(z) = C \cdot \exp\left(\frac{z^2}{2}\right), \qquad C \in \mathbb{C}.$$

Then we CHECK the solution. Any solution must have the structure (4). On the other hand, if $f(z)$ is given by (4), then $f(z)$ is clearly analytic in $\mathbb{C}$, and it follows by differentiation that (4) fulfils

$$f'(z) - z\, f(z) = 0,$$

so (4) gives us all solutions of the differential equation.

Second variant. *Determination of the Taylor coefficients.* We get by a rearrangement,

$$f'(z) = z\,f(z).$$

Then a differentiation gives

$$f''(z) = z\,f'(z) + f(z),$$

and thus

$$f^{(3)}(z) = z\,f''(z) + 2\,f'(z).$$

We shall now show by *induction* that

$$(5)\quad f^{(n+1)}(z) = z\,f^{(n)}(z) + n\,f^{(n-1)}(z).$$

It follows from the above that (5) holds for $n = 1, 2$. Assume that (5) holds for some $n \in \mathbb{N}$. Then by another differentiation,

$$f^{(n+2)}(z) = z\,f^{(n+1)}(z) + (n+1)f^{(n)}(z),$$

which has the same structure as (5), only with n replaced by $n+1$. Then the claim follows by induction (the bootstrap principle).

Now, if we put $z = 0$ into (5), then

$$(6)\quad f^{(n+1)}(0) = n\,f^{(n-1)}(0).$$

It follows from the original equation that

$$f'(0) = 0 \cdot f(0) = 0,$$

hence we conclude from (6) that

$$f^{(2n+1)}(0) = 0 \qquad \text{for every } n \in \mathbb{N}_0.$$

We still have to find $f^{(2n)}(0)$. However,

$$f^{(2n)}(0) = (2n-1)f^{(2n-2)}(0),$$

so by recursion,

$$\frac{f^{(2n)}(0)}{(2n)!} = \frac{1}{2n} \cdot \frac{f^{(2n-2)}(0)}{(2n-2)!} = \cdots = \frac{1}{2n} \cdot \frac{1}{2n-2} \cdots \frac{1}{2} \cdot f(0) = \frac{1}{2^n} \cdot \frac{1}{n!}\,f(0).$$

Then by insertion into Taylor's formula we formally obtain,

$$f(z) = f(0) \sum_{n=0}^{+\infty} \frac{1}{n!}\,\frac{1}{2^n}\,z^{2n} = f(0) \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\frac{z^2}{2}\right)^n = f(0) \cdot \exp\left(\frac{z^2}{2}\right),$$

where we have recognized the exponential series of radius of convergence ∞.

Third variant. *The power series method.* Assume that

$$f(z)=\sum_{n=0}^{+\infty}a_nz^n$$

is a power series solution which is convergent for $|z|<R$. Then we have in this domain of convergence,

$$f'(z)=\sum_{n=1}^{+\infty}n\,a_nz^{n-1}.$$

By insertion of these series into the differential equation we get for $|z|<R$ (where we shall find the radius of convergence R later) that

$$\begin{aligned}0 &= f'(z)-z\,f(z)=\sum_{n=0}^{+\infty}(n+1)a_{n+1}z^n-\sum_{n=0}^{+\infty}a_nz^{n+1}=a_1+\sum_{n=1}^{+\infty}(n+1)a_{n+1}z^n-\sum_{n=1}^{+\infty}a_{n-1}z^n\\ &= a_1+\sum_{n=1}^{+\infty}\{(n+1)a_{n+1}-a_{n-1}\}\,z^n.\end{aligned}$$

By using the identity theorem we get $a_1 = 0$ and the recursion formula

(7) $(n+1)a_{n+1} = a_{n-1}, \qquad n \in \mathbb{N}.$

Since $a_1 = 0$, and since there is a leap of 2 in the recursion formula (7), we conclude that

$$a_{2n+1} = 0 \qquad \text{for } n \in \mathbb{N}_0.$$

For even indices we get instead the recursion formula

$$2n\, a_{2n} = a_{2(n-1)},$$

hence by recursion,

$$a_{2n} = \frac{1}{2n}\, a_{2(n-1)} = \frac{1}{2n} \cdot \frac{1}{2(n-1)}\, a_{2(n-2)} = \cdots = \frac{1}{2n \cdot 2(n-1) \cdots 2 \cdot 1}\, a_0 = \frac{1}{2^n} \cdot \frac{1}{n!}\, a_0.$$

We then conclude from

$$\frac{a_{2(n-1)}}{a_{2n}} = 2n \to +\infty \qquad \text{for } n \to +\infty,$$

that the z^2-radius of convergence is $+\infty$, hence the z-radius of convergence is also $+\infty$, and the series can be written as

$$f(z) = a_0 \sum_{n=0}^{+\infty} \frac{1}{2^n} \cdot \frac{1}{n!}\, z^{2n} = a_0 \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\frac{z^2}{2} \right)^n = a_0 \exp\left(\frac{z^2}{2} \right), \qquad z \in \mathbb{C}.$$

Example 5.5 *Solve the differential equation*

$$(1-z)f'(z) - 2\, f(z) = 0$$

by inserting a formal power series.

Remark 5.4 We shall as usual go through all three standard solution variants. $\Diamond$

First variant. *Inspection.* When $z \neq 1$, we multiply by $1 - z \neq 0$ and obtain the equivalent equation

$$\begin{aligned} 0 &= (1-z)^2 f'(z) - 2(1-z)f(z) = (z-1)^2 f'(z) + 2(z-1)f(z) \\ &= (z-1)^2 \frac{d}{dz}\, f(z) + \frac{d}{dz}\,(z-1)^2 \cdot f(z) = \frac{d}{dz}\, \left\{ (z-1)^2 f(z) \right\}. \end{aligned}$$

The primitive is $(z-1)^2 f(z) = c$, hence

$$f(z) = \frac{c}{(z-1)^2}, \qquad c \in \mathbb{C}, \qquad z \in \mathbb{C} \setminus \{1\}.$$

Only the zero solution can be extended to pass the singularity.

Second variant. *Determination of the Taylor coefficients.* It follows by a differentiation of

$$(1-z)f'(z)-2f(z)=0$$

that

$$(1-z)f''(z)-3f'(z)=0.$$

Then we prove by induction that

$$(8)\quad (1-z)f^{(n)}(z)-(n+1)f^{(n-1)}(z)=0.$$

Assume that (8) holds for some $n \in \mathbb{N}$. Then a differentiation gives

$$(1-z)f^{(n+1)}-(n+2)f^{(n)}(z)=0,$$

which has the same structure as (8), only with n replaced by $n+1$. Since (8) holds for $n=1$, the claim follows by induction (the bootstrap principle), so (8) holds in general.

If we put $z=0$ into (8), then

$$f^{(n)}(0)=(n+1)f^{(n-1)}(0).$$

We divide by $(n+1)!$ and then obtain by recursion that

$$\frac{f^{(n)}(0)}{(n+1)!}=\frac{f^{(n-1)}(0)}{n!}=\cdots=\frac{f'(0)}{2!}=\frac{f(0)}{1!}.$$

We conclude that

$$a_n=\frac{f^{(n)}(0)}{n!}=(n+1)f(0).$$

The *formal* series solution is

$$f(z)=a_0\sum_{n=0}^{+\infty}(n+1)z^n.$$

We see that the radius of convergence is

$$r=\lim_{n\to+\infty}\frac{1}{\sqrt[n]{n+1}}=1,$$

and that we have for $|z|<1$,

$$f(z)=a_0\sum_{n=0}^{+\infty}(n+1)z^n=a_0\,\frac{d}{dz}\sum_{n=0}^{+\infty}z^{n+1}=a_0\,\frac{d}{dz}\,\frac{z}{1-z}=a_0\cdot\frac{1}{(1-z)^2}.$$

Third variant. *The power series method.* Assume that the solution has the form

$$f(z)=\sum_{n=0}^{+\infty}a_nz^n \qquad \text{for } |z|<R,$$

where the coefficients a_n and the radius of convergence $R > 0$ are the unknown.

Since $z = 1$ is the only singular point, we may expect that the radius of convergence is either $R = 1$ or $R = +\infty$, if the solution can be extended beyond $z = 1$.

When we put

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \qquad \text{and} \qquad f'(z) = \sum_{n=1}^{+\infty} n\, a_n z^{n-1}$$

into the differential equation, and $|z| < R$, then

$$\begin{aligned} 0 &= \sum_{n=1}^{+\infty} n\, a_n z^{n-1} - \sum_{\substack{n=1\\(n=0)}}^{+\infty} n\, a_n z^n - 2\sum_{n=0}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty} (n+1)a_{n+1} z^n - \sum_{n=0}^{+\infty} (n+2) a_n z^n \\ &= \sum_{n=0}^{+\infty} \left\{(n+1)a_{n+1} - (n+2)a_n\right\} z^n. \end{aligned}$$

We conclude from the identity theorem that we have the*recursion formula*

$$(n+1)a_{n+1} = (n+2)a_n, \qquad n \in \mathbb{N}_0.$$

This is divided by $(n+1)(n+2) \neq 0$ for $n \in \mathbb{N}_0$, and then we immediately get by recursion that

$$\frac{1}{(n+1)+1}\, a_{n+1} = \frac{1}{n+2}\, a_{n+1} = \frac{1}{n+1}\, a_n = \cdots = \frac{1}{0+1}\, a_0 = a_0,$$

and we get immediately, $a_n = (n+1)a_0$. Therefore, the formal series solution is given by (with some obvious manipulations)

$$f(z) = a_0 \sum_{n=0}^{+\infty} (n+1) z^n = a_0 \sum_{n=1}^{+\infty} n\, z^{n-1} = a_0\, \frac{d}{dz}\left(\sum_{n=0}^{+\infty} z^n\right) = a_0\, \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{a_0}{(1-z)^2},$$

and it is trivial that the radius of convergence is 1.
A check is also trivial, and it even follows that

$$\frac{a_0}{(1-z)^2}$$

is a solution of the differential equation in $\mathbb{C} \setminus \{1\}$.

Example 5.6 *Solve the differential equation*

$$(2z^3-z^2)\, f''(z)-(6z^2-2z)\, f'(z)+(6z-2)f(z)=0$$

by inserting a formal power series expansion $f(z)=\sum_{n=0}^{+\infty} a_n z^n$.

Remark 5.5 In this case it is rather difficult directly to find the Taylor coefficients, so we shall only demonstrate the other two solution variants. However, ironically (left to the reader), it can be shown that the determination of the Taylor coefficients actually is the *easiest method* in the actual case, which is far from evident. Hence, one will not always be able in advance to judge which method is the easiest to apply. $\Diamond$

First variant. *Inspection.* This method is also difficult, because one shall divide by the not so obvious polynomial $(2z-1)^2z^3$ (here I have been guided by the coefficient of the term of highest order of differentiation). If we do this, then we get for $z \neq \frac{1}{2}$ and $z \neq 0$,

$$\begin{aligned}
0 &= \frac{(2z-1)z^2}{2z-)^2z^3}\,f''(z) - \frac{2(3z-)z}{(2z-1)^2z^3}\,f'(z) + \frac{2(3z-1)}{(2z-)^2z^3}\,f(z) \\
&= \frac{1}{(2z-1)z}\,f''(z) - \frac{4z-1}{(2z-1)^2z^2}\,f'(z) + \left\{\frac{4z-1}{(2z-1)^2z^2} - \frac{6z-2}{(2z-1)^2z^2}\right\} f'(z) + \frac{6z-2}{2z-1)^2z^3}\,f(z) \\
&= \frac{d}{dz}\left\{\frac{f'(z)}{(2z-1)z}\right\} - \frac{2z-1}{(2z-1)^2z^2}\,f'(z) + \frac{6z-2}{(2z-1)^2z^3}\,f(z) \\
&= \frac{d}{dz}\left\{\frac{f'(z)}{(2z-1)z}\right\} - \frac{f'(z)}{(2z-1)z^2} + \left\{\frac{6z^2-2z}{(2z-1)^2z^4}\,f(z) - \frac{6z-2}{(2z-1)^2z^3}\,f(z)\right\} + \frac{6z-2}{(2z-1)^2z^3}\,f(z) \\
&= \frac{d}{dz}\left\{\frac{f'(z)}{(2z-1)z}\right\} - \frac{d}{dz}\left\{\frac{f(z)}{(2z-1)z^2}\right\} = \frac{d}{dz}\left\{\frac{f'(z)}{(2z-1)z} - \frac{f(z)}{(2z-1)z^2}\right\} \\
&= \frac{d}{dz}\left\{\frac{1}{2z-1}\,\frac{d}{dz}\left(\frac{f(z)}{z}\right)\right\}.
\end{aligned}$$

We have proved that

$$\frac{d}{dz}\left\{\frac{1}{2z-1}\,\frac{d}{dz}\left(\frac{f(z)}{z}\right)\right\} = 0.$$

A primitive is given by

$$\frac{1}{2z-1}\,\frac{d}{dz}\left(\frac{f(z)}{z}\right) = C_1,$$

thus

$$\frac{d}{dz}\left(\frac{f(z)}{z}\right) = C_1\cdot(2z-1).$$

Another primitive is

$$\frac{f(z)}{z} = C_1\cdot\left(z^2-z\right) + C_2,$$

hence

$$f(z) = C_1\cdot\left(z^3-z^2\right) + C_2 z.$$

A check shows that this is the complete solution in $\mathbb{C}$ for any choice of the constants C_1, $C_2 \in \mathbb{C}$.

Second variant. *The power series method.* The method of inspection relies on a rather nasty trick, and the method of determination of the Taylor coefficients does not look promising (however, cf. the remark in the beginning of the example). Therefore, one would usually start with the *power series method*, in particular because the assumptions of the existence theorem are *not* fulfilled at the singular point z_0. And even by the power series method one must be very careful, because the radius of convergence could be $R = 0$. In fact, the roots of the polynomial coefficient $2z^3 - z^2$ are 0 and $\frac{1}{2}$, so the possible radii of convergence are 0, $\frac{1}{2}$ and $+\infty$.

If we put the formal series

$$f(z)=\sum_{n=0}^{+\infty}, \quad f'(z)=\sum_{n=1}^{+\infty} n\, a_n z^{n-1}, \quad f''(z)=\sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2},$$

(where we later shall find the radius of convergence R) into the equation, we get

$$\begin{aligned}
0 &= \left(2z^3-z^2\right) f''(z)-\left(6z^2-2z\right) f'(z)+(6z-2)f(z) \\
&= \sum_{\substack{n=2\\(n=0)}}^{+\infty} 2n(n-1)a_n z^{n+1} - \sum_{\substack{n=2\\(n=0)}}^{+\infty} n(n-1)a_n z^n - \sum_{\substack{n=1\\(n=0)}}^{+\infty} 6na_n z^{n+1} \\
&\qquad + \sum_{\substack{n=1\\(n=0)}}^{+\infty} 2na_n z^n + \sum_{n=0}^{+\infty} 6a_n z^{n+1} - \sum_{n=0}^{+\infty} 2a_n z^n \\
&= \sum_{n=0}^{+\infty} 2\left\{n^2-n-3n+3\right\} a_n z^{n+1} - \sum_{n=0}^{+\infty} \left\{n^2-n-2n+2\right\} a_n z^n \\
&= \sum_{n=0}^{+\infty} 2\left(n^2-4n+3\right) a_n z^{n+1} - \sum_{n=0}^{+\infty} \left(n^2-3n+2\right) a_n z^n \\
&= \sum_{n=0}^{+\infty} 2(n-3)(n-1)a_n z^{n+1} - \sum_{n=0}^{+\infty} (n-2)(n-1)a_n z^n,
\end{aligned}$$

thus

$$\begin{aligned}
0 &= \sum_{n=0}^{+\infty} 2(n-3)(n-1)a_n z^{n+1} - \sum_{n=0}^{+\infty} (n-2)(n-1)a_n z^n \\
&= \sum_{n=1}^{+\infty} 2(n-4)(n-2)a_{n-1} z^n - \sum_{n=0}^{+\infty} (n-2)(n-1)a_n z^n \\
&= -2a_0 + \sum_{n=1}^{+\infty} (n-2)\left\{2(n-4)a_{n-1}-(n-1)a_n\right\} z^n.
\end{aligned}$$

It follows from the identity theorem that we have $a_0 = 0$ and the *recursion formula*

(9) $(n-2)\left\{2(n-4)a_{n-1}-(n-1)a_n\right\} = 0, \qquad n \in \mathbb{N}.$

Remark 5.6 Notice that we here have kept the common factor. The reason is that if $n-2$ is removed, then we latently divide by 0, when $n = 2$. This is one of the pitfalls of the power series method. $\Diamond$

We have proved that $a_0 = 1$. Then put $n = 1$ into (9) in order to get

$$(-1)\cdot\{2\cdot(-3)\cdot a_0 - 0\cdot a_1\} = 6\cdot 0 + 0\cdot a_1 = 0,$$

which clearly holds *no matter the choice of* a_1, thus a_1 is an *arbitrary constant*.

Then put $n = 2$ into (9). In this case we get the trivial identity

(10) $0 \cdot \{-4a_1 - a_2\} = 0,$

so a_2 is also an arbitrary constant. This shows why we shall keep the factor $n-2$ in (9), because we otherwise would obtain a *false* solution of (10).

If $n > 2$, then $n - 2 \neq 0$, and $n - 1 \neq 0$, hence we get by solution of (9),

(11) $a_n = \dfrac{2(n-4)}{n-1}\, a_{n-1} \qquad \text{for } n \geq 3.$

If $n = 4$, then $a_4 = 0$, and then we get by induction of (11) that $a_n = 0$ for every $n \geq 4$.

We still have to consider the case $n = 3$. Here,

$$a_3 = \frac{2(3-4)}{3-1}\, a_2 = -a_2,$$

and the complete solution then becomes

$$f(z) = a_1 z + a_2\left(z^2 - z^3\right).$$

This solution is a very trivial power series with the domain of convergence equal to $\mathbb{C}$. A check of the solution shows that it is indeed the complete solution in $\mathbb{C}$.

Remark 5.7 Here the reader should pay attention to another *pitfall* in the computation. *If* $a_n \neq 0$, then it follows from (11) that

$$\frac{a_{n-1}}{a_n} = \frac{1}{2}\,\frac{n-1}{n-4}.$$

Therefore, one may be misled to believe that the radius of convergence is

$$\text{“}\lim_{n\to+\infty}\left|\frac{a_{n-1}}{a_n}\right|\text{”} = \lim_{n\to+\infty}\left|\frac{1}{2}\cdot\frac{n-1}{n-4}\right| = \frac{1}{2}.$$

However, this computation is only correct, if $a_n \neq 0$ for every n, and this was not the case here. We have

$$R = +\infty > \frac{1}{2} = \lim_{n\to+\infty}\frac{1}{2}\,\frac{n-1}{n-4}. \qquad \Diamond$$

Example 5.7 *Solve the differential equation*

$$f''(z) - z\,f(z) = 0$$

by inserting a formal power series $f(z) = \sum_{n=0}^{+\infty} a_n z^n$.

Remark 5.8 In case of linear differential equations of order ≥ 2 it is very rare that the method of inspection is successful, and the same can be said about the method of determining the Taylor coefficients. In the present case we even end up with a series expression which cannot be expressed by an elementary function. Thus we are only left with the power series method. $\Diamond$

If we put the formal power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad f'(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1}, \quad f''(z) = \sum_{n=2}^{+\infty} n(n-1) a_n z^{n-2}$$

into the differential equation, we get

$$\begin{aligned} 0 &= f''(z) - z\,f(z) = \sum_{n=2}^{+\infty} n(n-1) a_n z^{n-2} - \sum_{n=0}^{+\infty} a_n z^{n+1} = \sum_{n=-1}^{+\infty} (n+3)(n+2) a_{n+3} z^{n+1} - \sum_{n=0}^{+\infty} a_n z^{n+1} \\ &= 2a_2 + \sum_{n=0}^{+\infty} \left\{(n+3)(n+2)a_{n+3} - a_n\right\} z^{n+1}. \end{aligned}$$

Then it follows by the identity theorem that $a_2 = 0$ and we have the *recursion formula*

$$(12)\quad a_{n+3} = \frac{1}{(n+3)(n+2)}\, a_n, \qquad n \in \mathbb{N}_0,$$

thus $a_{3n+2} = 0$ for $n \in \mathbb{N}_0$, and

$$a_{3n} = \frac{1}{3n(3n-1)} \cdot \frac{1}{(3n-3)(3n-4)} \cdot \frac{1}{(3n-6)(3n-7)} \cdots \frac{1}{2} \cdot \frac{1}{2} \cdot 1_0,$$

$$a_{3n+1} = \frac{1}{(3n+1)\cdot 3n} \cdot \frac{1}{(3n-2)(3n-3)} \cdot \frac{1}{(3n-5)(3n-6)} \cdots \frac{1}{4} \cdot \frac{1}{3} \cdot a_1,$$

and the complete solution (which is not nice) is formally given by

$$\begin{aligned} f(z) \quad &= \quad a_0 \sum_{n=0}^{+\infty} \frac{1}{3n(3n-1)} \cdot \frac{1}{(3n-3)(3n-4)} \cdots \frac{1}{3} \cdot \frac{1}{2} \cdot z^{3n} \\ &\quad\; +a_1 \sum_{n=0}^{+\infty} \frac{1}{(3n+1)3n} \cdot \frac{1}{(3n-2)(3n-3)} \cdots \frac{1}{4} \cdot \frac{1}{3} \cdot z^{3n+1}. \end{aligned}$$

The easiest way to find the radius of convergence is by using the recursion formula (12). We conclude that the x^3-radius of convergence for each of the two series of solution is given by

$$\varrho_{z^3} = \lim_{n \to +\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to +\infty} (n+3)(n+2) = +\infty.$$

Hence we conclude that the domain of convergence is $\mathbb{C}$.

Example 5.8 *Solve the differential equation*

$$\left(1 - z^2\right) f''(z) - 2z\, f'(z) + 2f(z) = 0$$

by inserting a formal power series $f(z) = \sum_{n=0}^{+\infty} a_n z^n$.

Remark 5.9 Here, the power series method is the safe method, However, we shall later show that it is even here not too difficult to find the Taylor coefficients. On the other hand one should not waste time on the inspection method. $\Diamond$

First variant. *The power series method.* The singular points are the roots of $1 - z^2$, hence ± 1. We may therefore expect that the radius of convergence is either 1 or $+\infty$.

If we put the formal power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \quad f'(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1}, \quad f''(z) = \sum_{n=2}^{+\infty} n(n-1) a_n z^{n-2},$$

into the differential equation, we get

$$\begin{aligned}
0 &= \left(1-z^2\right) f''(z) - 2zf'(z) + 2f(z) \\
&= \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2} - \sum_{\substack{n=2 \\ (n=0)}}^{+\infty} n(n-1)a_n z^n - \sum_{\substack{n=1 \\ (n=0)}}^{+\infty} 2na_n z^n + \sum_{n=0}^{+\infty} 2a_n z^n \\
&= \sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2} z^n - \sum_{n=0}^{+\infty} \left\{n^2+n-2\right\} a_n z^n \\
&= \sum_{n=0}^{+\infty} (n+2)\left\{(n+1)a_{n+2} - (n-1)a_n\right\} z^n.
\end{aligned}$$

Since $n+2 \neq 0$ for $n \in \mathbb{N}_0$, it follows by the identity theorem that we have the *recursion formula*

$$(n+1)a_{n+2} = (n-1)a_n, \qquad n \in \mathbb{N}_0.$$

Thus, the structure is given by

$$(\{n+2\}-1)a_{n+2} = (n-1)a_n, \qquad n \in \mathbb{N}_0.$$

If we split into $n = 2p$ even and $n = 2p+1$ odd, it follows by recursion that

$$(2p-1)a_{2p} = \cdots = (0-1)a_0 \quad \text{and} \quad 2pa_{2p+1} = \cdots = (1-1)a_1 = 0,$$

thus $a_{2p+1} = 0$ for $p \in \mathbb{N}$, and a_1 is an arbitrary constant, and

$$a_{2p} = -\frac{1}{2p-1}\, a_0,$$

where a_0 is also an arbitrary constant. Therefore, the formal power series is

$$f(z) = a_1 z - a_0 \sum_{n=0}^{+\infty} \frac{1}{2n-1}\, z^{2n},$$

which of course has 1 as radius of convergence, if $a_0 \neq 0$.
If $a_0 = 0$, then $f(z) = a_1 z$ is convergent in $\mathbb{C}$.

Since

$$\frac{d}{dz} \sum_{n=1}^{+\infty} \frac{1}{2n-1}\, z^{2n-1} = \sum_{n=0}^{+\infty} z^{2n} = \frac{1}{1-z^2} = \frac{1}{2}\,\frac{1}{1-z} + \frac{1}{2}\,\frac{1}{1+z},$$

a primitive in $|z| < 1$ is given by

$$\sum_{n=1}^{+\infty} \frac{1}{2n-1}\, z^{2n-1} = -\frac{1}{2}\operatorname{Log}(1-z) + \frac{1}{2}\operatorname{Log}(1+z) = \frac{1}{2}\operatorname{Log}\left(\frac{1+z}{1-z}\right),$$

because both $1-z$ and $1+z$ lie in the right half plane, so their principal arguments lie in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.
Then the solution can be written

$$f(z) = a_0 \left\{1 - \frac{1}{2}\, z \operatorname{Log}\left(\frac{1+z}{1-z}\right)\right\} + a_1 z \qquad \text{for } |z| < 1.$$

Second variant. *Determination of the Taylor coefficients.* When we differentiate the equation

$$\left(1-z^2\right) f''(z) - 2z\, f'(z) + 2\, f(z) = 0$$

a couple of times, then

$$\left(1-z^2\right) f^{(3)}(z) - 4z\, f''(z) - 0 \cdot f'(z) = 0,$$

$$\left(1-z^2\right) f^{(4)}(z) - 6z\, f^{(3)}(z) - 4 \cdot f^{(2)}(z) = 0,$$

$$\left(1-z^2\right) f^{(5)}(z) - 8z\, f^{(4)}(z) - 10 \cdot f^{(3)}(z) = 0,$$

which are special cases of the formula

$$(13)\quad \left(1-z^2\right) f^{(n)}(z) - 2(n-1)z f^{(n-1)}(z) - n(n-3) f^{(n-2)}(z) = 0$$

for $n = 2, 3, 4, 5$.

Assume that (13) holds for some $n \in \mathbb{N} \setminus \{1\}$. Then by differentiation,

$$\left(1-z^2\right) f^{(n+1)}(z) - 2(n+1-1)z f^{(n+1-1)}(z) - \{n(n-3)+2(n-1)\} f^{(n-1)}(z) = 0,$$

where

$$\begin{aligned} n(n-3)+2(n-1) &= n^2-3n+2n-2 = n^2-n-2 \\ &= (n+1)(n-2) = (n+1)(\{n+1\}-3), \end{aligned}$$

and it follows by induction that (13) holds for every $n \in \mathbb{N} \setminus \{1\}$.

If we put $z = 0$ into (13), then

$$f^{(n)}(0) - n(n-3) f^{(n-2)}(0) = 0, \qquad \text{for } n \geq 2,$$

and the Taylor coefficients are given by the recursion formula

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{n(n-3)}{n!}\, f^{(n-2)}(0) = \frac{n-3}{n-1} \cdot \frac{f^{(n-2)}(0)}{(n-2)!} = \frac{n-3}{n-1} \cdot a_{n-2},$$

thus

$$(n-1)a_n = (n-3)a_{n-2} = (\{n-2\}-1)a_{n-2}.$$

If $n = 2p+1$, $p \in \mathbb{N}$, is odd and > 1, we get by recursion,

$$(2p+1-1)a_{2p+1} = \cdots = (1-1)a_1 = 0,$$

where a_1 is an arbitrary constant.

If $n = 2p$, $p \in \mathbb{N}$, is even, then

$$(2p-1)a_{2p} = \cdots = (0-1)a_0 = -a_0,$$

hence

$$a_{2p} = -\frac{1}{2p-1}\, a_0+, \qquad \text{for } p \in \mathbb{N},$$

where a_0 is an arbitrary constant. We get as above the formal series

$$f(z) = a_1 z - a_0 \sum_{p=0}^{+\infty} \frac{1}{2p-1}\, z^{2p},$$

and it is again obvious that the domain of convergence is $\{z \in \mathbb{C} \mid |z| < 1\}$, if $a_0 \neq 0$, and $\mathbb{C}$, if $a_0 = 0$.

The determination of the sum function is the same as in the first variant, so it shall not be repeated here.

Example 5.9 *Solve the differential equation*

$$\left(1-z^2\right) f''(z)+6\,f(z)=0$$

by insertion of a formal power series $f(z)=\sum_{n=0}^{+\infty} a_n z^n$.

The singular points are ± 1, so we may expect that the radius of convergence is either 1 or $+\infty$.

If we put the formal power series

$$f(z)=\sum_{n=0}^{+\infty} a_n z^n \qquad \text{and} \qquad f''(z)=\sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2}$$

into the differential equation, then

$$\begin{aligned} 0 &= \left(1-z^2\right) f''(z)+6\,f(z)=\sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2}-\sum_{\substack{n=2\\(n=0)}}^{+\infty} n(n-1)a_n z^n+\sum_{n=0}^{+\infty} 6a_n z^n \\ &= \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}z^n-\sum_{n=0}^{+\infty}\left\{n^2-n-6\right\}a_n z^n \\ &= \sum_{n=0}^{+\infty}(n+2)\left\{(n+1)a_{n+2}-(n-3)a_n\right\}z^n. \end{aligned}$$

Since $n+2\neq 0$ for $n\in\mathbb{N}_0$, we derive by the identity theorem the *recursion formula*

$$(n+1)a_{n+2}=(n-3)a_n, \qquad n\in\mathbb{N}_0.$$

If $n=3$, then $a_5=0$, hence $a_{2n+1}=0$ for $n\geq 2$ by induction.

If $n=1$, then $2a_3=-2a_1$, hence $a_3=-a_1$, and one of the two independent solutions is given by

$$a_1\left(z-z^3\right),$$

and it is of course convergent in $\mathbb{C}$.

If $n=2p$, then the recursion formula is written

$$(2p+1)a_{2p+2}=(2p-3)a_{2p}, \qquad p\in\mathbb{N}_0.$$

When this is multiplied by $2p-1\neq 0$, it follows by recursion that

$$\begin{aligned} (2p+1)(2p-1)a_{2p+2} &= (\{2p+2\}-1)(\{2p+2\}-3)a_{2p+2} \\ &= (2p-1)(2p-3)a_{2p}=\cdots=(-1)(-3)a_0 \\ &= 3a_0, \end{aligned}$$

thus

$$a_{2p+2}=a_{2(p+1)}=\frac{3}{4p^2-1}\,a_0,$$

and hence

$$a_{2n} = \frac{3}{4(n-1)^2-1}\, a_0 = \frac{3}{(2n-1)(2n-3)}\, a_0.$$

The radius of convergence is 1 for $a_0 \neq 0$, and $+\infty$ for $a_0 = 0$, and the general solution is

$$f(z) = a_1\left(z - z^3\right) + a_0 \sum_{n=0}^{+\infty} \frac{3}{(2n-1)(2n-3)}\, z^{2n}, \qquad \text{for } |z| < 1.$$

Remark 5.10 It is here possible – though not worth the trouble – to express the corresponding analytic function by means of Log. $\Diamond$

Example 5.10 *Find all power series with expansion point 0, which are a solution of the differential equation*

$$z\, f''(z) - 2\, f'(z) + 4z^3 f(z) = 0.$$

Since $z = 0$ is a singular point, we cannot immediately conclude that there exists a solution.

If we put the formal power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad f'(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1}, \quad f''(z) = \sum_{n=2}^{+\infty} n(n-1) a_n z^{n-2},$$

into the differential equation, we get

$$\begin{aligned} 0 &= z f''(z) - 2f(z) + 4z^3 f(z) = \sum_{\substack{n=2 \\ (n=1)}}^{+\infty} n(n-1) a_n z^{n-1} - \sum_{n=1}^{+\infty} 2n a_n z^{n-1} + \sum_{n=0}^{+\infty} 4 a_n z^{n+3} \\ &= \sum_{n=1}^{+\infty} n(n-3) a_n z^{n-1} + \sum_{n=4}^{+\infty} 4 a_{n-4} z^{n-1} \\ &= 1 \cdot (-2) a_1 + 2(-1) a_2 z + 0 + \sum_{n=4}^{+\infty} \left\{ n(n-3) a_n + 4 a_{n-4} \right\} z^{n-1}. \end{aligned}$$

Hence by the identity theorem, $a_1 = 0$, $a_2 = 0$, and the *recursion formula*

$$n(n-3) a_n = -4 a_{n-4} \qquad \text{for } n \geq 4.$$

It follows immediately by induction that

$$a_{4n+1} = 0 \qquad \text{og} \qquad a_{4n+2} = 0.$$

If $n = 4p$, we get by recursion,

$$\begin{aligned} a_{4p} &= \frac{-4}{4p(4p-3)}\, a_{4(p-1)} = \cdots = \frac{(-4)^p}{4p(4p-3)4(p-1)(4p-7)\cdots 4 \cdot 1}\, a_0 \\ &= \frac{(-1)^p a_0}{p!(4p-3)(4p-7)\cdots 1}. \end{aligned}$$

where the corresponding series is easily shown to have the radius of convergence $+\infty$.

If $n = 4p+3$, $p \in \mathbb{N}$, then we also get by recursion that

$$a_{4p+3} = \frac{-4}{(4p+3)\cdot 4p}\, a_{4p-1} = \frac{-1}{(4p+3)p}\, a_{4(p-1)+1} = \cdots = \frac{(-1)^p}{p!(4p+3)(4p-1)\cdots 3}\, a_3,$$

and the corresponding series has also the radius of convergence $+\infty$.

Summing up, the complete solution is

$$f(z) = a_0 \left\{ 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!(4n-3)(4n-7)\cdots 1}\, z^{4n} \right\} + a_3 \left\{ z^3 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!(4n+3)(4n-1)\cdots 3}\, z^{4n+3} \right\}$$

for $z \in \mathbb{C}$.

Example 5.11 *Consider the series*

$$f(z) = \sum_{n=0}^{+\infty} \frac{z^{2n}}{(n!)^2}.$$

Find its radius of convergence and prove that $f(z)$ satisfies in the domain of convergence the differential equation

$$z^2 f''(z) + z\, f'(z) = 4z^2 f(z).$$

It follows from the estimate

$$|f(z)| \le \sum_{n=0}^{+\infty} \frac{1}{(n!)^2}\, \left|z^2\right|^n \le \sum_{n=0}^{+\infty} \frac{1}{n!}\, \left|z^2\right|^n = \exp\left(\left|z^2\right|\right),$$

that the domain of convergence is $\mathbb{C}$.

It follows from the definition of $f(z)$ by a differentiations that

$$f'(z) = \sum_{n=1}^{+\infty} \frac{2n}{(n!)^2}\, z^{2n-1} \qquad \text{og} \qquad f''(z) = \sum_{n=1}^{+\infty} \frac{2n(2n-1)}{(n!)^2}\, z^{2n-2},$$

hence by insertion into the left hand side of the differential equation,

$$\begin{aligned} z^2 f''(z) + z\, f'(z) &= \sum_{n=1}^{+\infty} \frac{2n(2n-1)}{(n!)^2}\, z^{2n} + \sum_{n=1}^{+\infty} \frac{2n}{(n!)^2}\, z^{2n} = \sum_{n=1}^{+\infty} \frac{(2n)^2}{(n!)^2}\, z^{2n} = 4\sum_{n=1}^{+\infty} \frac{1}{(\{n-1\}!)^2}\, z^{2n} \\ &= 4z^2 \sum_{n=0}^{+\infty} \frac{1}{(n!)^2}\, z^{2n} = 4z^2 f(z). \end{aligned}$$

6 The classical differential equations

Example 6.1 *By the* Bessel equation of order 0 *we shall understand the differential equation*

$$z^2 f''(z) + z\, f'(z) + z^2 f(z) = 0.$$

Find a power series solution

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

of this equation, for which $f(0) = 1$ *and* $f'(0) = 0$, *and then determine its domain of convergence. This solution is called the* Bessel function af order 0 *and it is denoted by* $J_0(z)$.

We get by termwise differentiation in the domain of convergence $|z| < R$,

$$f'(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1} \qquad \text{and} \qquad f''(z) = \sum_{n=2}^{+\infty} n(n-1) a_n z^{n-2}.$$

When these formal power series are put into the differential equation, we get

$$\begin{aligned} 0 &= z^2 f''(z) + z\, f'(z) + z^2 f(z) = \sum_{\substack{n=2\\(n=0)}}^{+\infty} n(n-1)a_n z^n + \sum_{\substack{n=1\\(n=0)}}^{+\infty} n a_n z^n + \sum_{n=0}^{+\infty} a_n z^{n+2} \\ &= \sum_{n=0}^{+\infty} n^2 a_n z^n + \sum_{n=2}^{+\infty} a_{n-2} z^n = 0^2 a_0 + 1^2 a_1 z + \sum_{n=2}^{+\infty} \left(a_{n-2} + n^2 a_n\right) z^n. \end{aligned}$$

Then it follows from the identity theorem that $a_1 = 0$, and

$$a_{n-2} + n^2 a_n = 0, \quad \text{dvs.} \quad a_n = -\frac{1}{n^2}\, a_{n-2}, \qquad n \geq 2.$$

Remark 6.1 Strictly speaking the example is over-determined, because we again derive that $a_1 = 0$ without any assumption at all, and yet it is assumed that $f'(0) = a_1 = 0$. We note in particular, that if our request had been $f'(0) \neq 0$, then this problem would not have a solution. $\Diamond$

It follows by recursion from $a_1 = 0$ that

$$a_{2n+1} = 0 \qquad \text{for } n \in \mathbb{N}_0.$$

Since $f(0) = 1$, we get $a_0 = 1$, hence by recursion,

$$a_{2n} = \frac{(-1)^n}{(n! 2^n)^2} = \frac{(-1)^n}{4^n (n!)^2},$$

and the formal series solution is given by

$$f(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{4^n (n!)^2}\, z^{2n} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}.$$

We have trivially the estimate

$$|f(z)| \leq \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} \left|\left(\frac{z}{2}\right)^2\right|^n \leq \sum_{n=0}^{+\infty} \frac{1}{n!} \left|\left(\frac{z}{2}\right)^2\right|^n = \exp\left(\left|\frac{z}{2}\right|^2\right) < +\infty$$

for every $z \in \mathbb{C}$, so the series is convergent everywhere in $\mathbb{C}$, and the domain of convergence for $f(z) = J_0(z)$ is $\mathbb{C}$.

Example 6.2 *We define the* Bessel function of order m *by*

$$J_m(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(m+n)!} \left(\frac{z}{2}\right)^{2n+m}, \qquad m \in \mathbb{N}_0.$$

Prove that $J_m(z)$ is analytic in $\mathbb{C}$, thus its radius of convergence is $+\infty$.

If we put $\left(\frac{z}{2}\right)^m$ outside the summation and change variable to $w = \left(\frac{z}{2}\right)^2$ in the sum, then it follows that it suffices to prove that

$$\sum_{n=0}^{+\infty} \frac{(-1)w^n}{n!(m+n)!}$$

is convergent for every $w \in \mathbb{C}$. This follows immediately from the estimate

$$\left|\sum_{n=0}^{+\infty} \frac{(-1)^n w^m}{n!(m+n)!}\right| \le \sum_{n=0}^{+\infty} \frac{1}{n!}\,|w|^n = \exp(|w|) = \exp\left(\left|\frac{z}{2}\right|^2\right) < +\infty,$$

so the domain of convergence is $\mathbb{C}$, and $J_m(z)$ is analytic in $\mathbb{C}$.

Example 6.3 *We define the* Hermite differential equation *by*

$$f''(z) + 2m\,f(z) = 2z\,f'(z),$$

where m is a complex constant.

(a) *Find the power series solution which satisfies $f(0) = 1$, $f'(0) = 0$.*

(b) *Find the power series solution which satisfies $f(0) = 0$, $f'(0) = 1$.*

When we put the formal power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, m \quad f'(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1}, \quad f''(z) = \sum_{n=2}^{+\infty} n(n-1) a_n z^{n-2},$$

into the differential equation, we get

$$\begin{aligned} 0 &= f''(z) - 2z\,f'(z) + 2m\,f(z) = \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2} - \sum_{\substack{n=1\\(n=0)}}^{+\infty} 2na_n z^n + \sum_{n=0}^{+\infty} 2m\,a_n z^n \\ &= \sum_{n=0}^{+\infty} \{(n+2)(n+1)a_{n+2} - 2(n-m)a_n\}\,z^n. \end{aligned}$$

Since $(n+2)(n+1) \neq 0$ for $n \ge 0$, we get the following *recursion formula* by the identity theorem,

$$a_{n+2} = \frac{2(n-m)}{(n+2)(n+1)}\,a_n.$$

We conclude from

$$\left|\frac{2(n-m)}{(n+2)(n+1)}\right| \to 0 \qquad \text{for } n \to +\infty,$$

that the z^2-radius of convergence – and hence also the z-radius of convergence – is $+\infty$.

(a) We have in this case $a_0 = 1$ and $a_1 = 0$, so it follows immediately by induction that $a_{2n+1} = 0$. Then by recursion for even indices,

$$\begin{aligned} a_{2n} &= \frac{2(2n-2-m)}{2n \cdot (2n-1)} \cdot a_{2n-2} = \frac{4\left(n-1-\frac{m}{2}\right)}{2n(2n-1)}\,a_{2(n-1)} = \cdots \\ &= \frac{4^n\left(n-1-\frac{m}{2}\right)\left(n-2-\frac{m}{2}\right)\cdots\left(-\frac{m}{2}\right)}{(2n)!}, \end{aligned}$$

and the series becomes

$$\sum_{n=0}^{+\infty} \frac{4^n}{(2n)!}\left(n-1-\frac{m}{2}\right)\left(n-2-\frac{m}{2}\right)\cdots\left(-\frac{m}{2}\right)z^{2n}.$$

Note that if $\frac{m}{2} \in \mathbb{N}$, then the series only contains a finite number of terms, and the solution is a polynomial in this case.

(b) Here, $a_0 = 0$ and $a_1 = 1$, hence $a_{2n} = 0$ by induction. Then we get by recursion,

$$\begin{aligned} a_{2n+1} &= \frac{2(2n-1-m)}{(2n+1)\cdot 2n}\, a_{2n-1} = \frac{4^n\left(n-1-\dfrac{m-1}{2}\right)}{(2n+1)\cdot 2n}\cdot a_{2n-1} = \cdots \\ &= \frac{4^n\left(n-1-\dfrac{m-1}{2}\right)\left(n-2-\dfrac{m-1}{2}\right)\cdots\left(-\dfrac{m-1}{2}\right)}{(2n+1)!}, \end{aligned}$$

and the series becomes

$$\sum_{n=0}^{+\infty} \frac{4^n}{(2n+1)!}\left(n-1-\frac{m-1}{2}\right)\left(n-2-\frac{m-1}{2}\right)\cdots\left(-\frac{m-1}{2}\right) z^{2n+1}.$$

If $\dfrac{m-1}{2} \in \mathbb{N}$, then the series is reduced to a polynomial. $\Diamond$

Example 6.4 *We define the* Chébyshev differential equation *by*

$$\left(1-z^2\right) f''(z)+m^2 f(z)=z\, f'(z),$$

where m is a complex constant.

(a) *Find the power series solution, for which $f(0)=1$, $f'(0)=0$.*

(b) *Find the power series solution, for which $f(0)=0$, $f'(0)=1$.*

When we put the formal power series

$$f(z)=\sum_{n=0}^{+\infty} a_n z^n, \quad f'(z)=\sum_{n=1}^{+\infty} n a_n z^{n-1}, \quad f''(z)=\sum_{n=2}^{+\infty} n(n-1) a_n z^{n-2},$$

into the differential equation, we get

$$\begin{aligned} 0 &= \left(1-z^2\right) f''(z)-z\, f'(z)+m^2 f(z) \\ &= \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2}-\sum_{\substack{n=2\\(n=0)}}^{+\infty} n(n-1)a_n z^n-\sum_{\substack{n=1\\(n=0)}}^{+\infty} n a_n z^n+\sum_{n=0}^{+\infty} m^2 a_n z^n \\ &= \sum_{n=0}^{+\infty}\left\{(n+2)(n+1)a_{n+2}-\left(n^2-m^2\right)a_n\right\} z^n. \end{aligned}$$

Since $(n+2)(n+2) \neq 0$ for n in the summation domain $\mathbb{N}_0$, we derive the following recursion formula by the identity theorem,

$$(14) \quad a_{n+2}=\frac{n^2-m^2}{(n+2)(n+1)}\, a_n=\frac{(n+m)(n-m)}{(n+2)(n+1)}\, a_n.$$

If $m \in \mathbb{Z}$ we sometimes get a polynomial, which of course is convergent in $\mathbb{C}$. If the power series solution is not a polynomial, then (14) implies that the radius of convergence is 1.

(a) If $f(0)=1$ and $f'(0)=0$, then $a_0=1$ and $a_1=0$. Then we conclude by induction that $a_{2n+1}=0$ for every odd index.
For the even indices it follows by recursion that

$$a_{2n+2}=\frac{(2n+m)(2n-m)}{(2n+2)(2n+1)}\, a_{2n}=\frac{4\left(n+\dfrac{m}{2}\right)\left(n-\dfrac{m}{2}\right)}{(2n+2)(2n+1)}\, a_{2n}=\frac{4\left(n^2-\dfrac{m^2}{4}\right)}{(2n+2)(2n+1)}\, a_{2n},$$

thus since $a_0 = 1$,

$$a_{2n} = \frac{4^n \left\{(n-1)^2 - \frac{m^2}{4}\right\}\left\{(n-2)^2 - \frac{m^2}{4}\right\}\cdots\left\{-\frac{m^2}{4}\right\}}{(2n)!},$$

and the series becomes

$$\sum_{n=0}^{+\infty} \frac{4^n \left\{(n-1)^2 - \frac{m^2}{4}\right\}\left\{(n-2)^2 - \frac{m^2}{4}\right\}\cdots\left\{-\frac{m^2}{4}\right\}}{(2n)!} z^{2n}.$$

If $\frac{m}{2} \in \mathbb{Z}$, then we obtain a polynomial with the domain of convergence $\mathbb{C}$.

If $\frac{m}{2} \notin \mathbb{Z}$, the domain of convergence is $\{z \in \mathbb{C} \mid |z| < 1\}$.

(b) If $f(0) = 0$ and $f'(0) = 1$, then $a_0 = 0$ and $a_1 = 1$. We conclude by induction that $a_{2n} = 0$ for even indices.
For the odd indices we get by recursion,

$$\begin{aligned} a_{2n+1} &= \frac{(2n+m-1)(2n-m-1)}{(2n+1)\cdot 2n}\, a_{2n-1} = \frac{4\left\{\left(n-\frac{1}{2}\right)+\frac{m}{2}\right\}\left\{\left(n-\frac{1}{2}\right)-\frac{m}{2}\right\}}{(2n+1)\cdot 2n}\, a_{2n-1} = \cdots \\ &= \frac{4^n}{(2n+1)!}\prod_{j=0}^{n-1}\left\{\left(n-j-\frac{1}{2}\right)^2 - \left(\frac{m}{2}\right)^2\right\}. \end{aligned}$$

The series is then

$$z + \sum_{n=1}^{+\infty} \frac{4^n}{(2n+1)!}\prod_{j=0}^{n-1}\left\{\left(n-j-\frac{1}{2}\right)^2 - \left(\frac{m}{2}\right)^2\right\}\cdot z^{2n+1},$$

where we notice that we formally must isolate the term corresponding to $n = 0$.

For $\frac{m+1}{2} \in \mathbb{Z}$ we get a polynomial of domain of convergence $\mathbb{C}$.

If $\frac{m+1}{2} \notin \mathbb{Z}$, then the domain of convergence is the open disc

$$\{z \in \mathbb{C} \mid |z| < 1\}.$$

Example 6.5 *We define the* Legendre differential equation *by*

$$\left(1-z^2\right) f''(z)+m(m+1)f(z)=2z\,f'(z),$$

where m is a complex constant.

(a) *Find the power series solution, for which $f(0)=1$, $f'(0)=0$.*

(b) *Find the power series solution, for which $f(0)=0$, $f'(0)=1$.*

When we put the formal series

$$f(z)=\sum_{n=0}^{+\infty} a_n z^n, \quad f'(z)=\sum_{n=1}^{+\infty} n a_n z^{n-1}, \quad f''(z)=\sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2},$$

into the differential equation, we get

$$\begin{aligned}
0 &= \left(1-z^2\right) f''(z)-z\,f'(z)+m(m+1)f(z)\\
&= \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2}-\sum_{n=0}^{+\infty}\{n(n-1)a_n+2na_n-m(m+1)a_n\}\,z^n\\
&= \sum_{n=0}^{+\infty}\{(n+2)(n+1)a_{n+2}-\{n(n+1)-m(m+1)\}\,a_n\}\,z^n.
\end{aligned}$$

Since $(n+2)(n+1)\neq 0$ in the summation domain $\mathbb{N}_0$, we get the following recursion formula by the identity theorem,

$$(15)\quad a_{n+2}=\frac{n(n+1)-m(m+1)}{(n+2)(n+1)}\,a_n=\frac{n^2-m^2+n-m}{(n+2)(n+1)}\,a_n=\frac{(n-m)(n+m-1)}{(n+2)(n+1)}\,a_n.$$

It is easily seen that one in general has the radius of convergence 1, and that the function in some cases becomes a polynomial of domain of convergence $\mathbb{C}$, when $m\in\mathbb{Z}$.

(a) If $f(0)=1$ and $f'(0)=0$, then $a_0=1$ and $a_1=0$. Then it follows by induction that $a_{2n+1}=0$ for odd indices.
For even indices it follows from the recursion formula (15) that

$$\begin{aligned}
a_{2n} &= \frac{2n-2-m)(2n-3+m)}{2n\cdot(2n-1)}\,a_{2n-2}=\frac{4\left(n-1-\dfrac{m}{2}\right)\left(n-1+\dfrac{m-1}{2}\right)}{2n\cdot(2n-1)}\,a_{2n-2}\\
&= \cdots\\
&= \frac{4^n}{(2n)!}\left\{n-1-\frac{m}{2}\right\}\left\{n-2-\frac{m}{2}\right)\cdots\left\{-\frac{m}{2}\right\}\cdot\\
&\qquad\cdot\left\{n-1+\frac{m-1}{2}\right\}\left\{n-2+\frac{m-1}{2}\right\}\cdots\left\{\frac{m-1}{2}\right\},
\end{aligned}$$

and the series becomes

$$\begin{aligned}
1+\sum_{n=1}^{+\infty}&\frac{4_n}{(2n)!}\left\{n-1-\frac{m}{2}\right\}\left\{n-2-\frac{m}{2}\right)\cdots\left\{-\frac{m}{2}\right\}\times\\
&\times\left\{n-1+\frac{m-1}{2}\right\}\left\{n-2+\frac{m-1}{2}\right\}\cdots\left\{\frac{m-1}{2}\right\}\,z^{2n}.
\end{aligned}$$

This expression becomes a polynomial, if either $\dfrac{m}{2} \in \mathbb{N}_0$ or $\dfrac{1-m}{2} \in \mathbb{N}_0$.

(b) If $f(0) = 0$ and $f'(0) = 1$, then $a_0 = 0$ and $a_1 = 1$, hence $a_{2n} = 0$ by induction over the even indices. For odd indices we get by recursion

$$
\begin{aligned}
a_{2n+1} &= \frac{(2n-1-m)(2n+2+m)}{(2n+1)\cdot 2n}\, a_{2n-1} = \frac{4\left(n-\dfrac{m+1}{2}\right)\left(n-1+\dfrac{m}{2}\right)}{(2n+1)\cdot 2n}\, a_{2n-1} \\
&= \frac{4^n}{(2n+1)!}\left(n-\frac{m+1}{2}\right)\left(n-1-\frac{m+1}{2}\right)\cdots\left(1-\frac{m+1}{2}\right)\cdot \\
&\qquad \cdot\left(n-1+\frac{m}{2}\right)\left(n-2+\frac{m}{2}\right)\cdots\left(\frac{m}{2}\right)\cdot a_1 \\
&= \frac{4^n}{(2n+1)!}\left(n-1-\frac{m-1}{2}\right)\left(n-2-\frac{m-1}{2}\right)\cdots\left(-\frac{m-1}{2}\right)\cdot \\
&\qquad \cdot\left(n-1+\frac{m}{2}\right)\left(n-2+\frac{m}{2}\right)\cdots\left(\frac{m}{2}\right),
\end{aligned}
$$

and the series becomes

$$\begin{aligned} z+\sum_{n=1}^{+\infty}\frac{4^n}{(2n+1)!}\left(n-1-\frac{m-1}{2}\right)\left(n-2-\frac{m-1}{2}\right)\cdots\left(-\frac{m-1}{2}\right)\times \\ \times\left(n-1+\frac{m}{2}\right)\left(n-2+\frac{m}{2}\right)\cdots\left(\frac{m}{2}\right)z^{2n+1} \\ = \; z+\sum_{n=1}^{+\infty}\frac{4^n}{(2n+1)!}\cdot\left\{\prod_{j=0}^{n-1}\left(j-\frac{m-1}{2}\right)\left(j+\frac{m}{2}\right)\right\}z^{2n+1}. \end{aligned}$$

This expression becomes a polynomial, if either $\dfrac{m-1}{2}\in\mathbb{N}_0$ or $-\dfrac{m}{2}\in\mathbb{N}_0$.

Example 6.6 *We define the* Laguerre differential equation *by*

$$z\,f''(z)+m\,f(z)=(z-1)f'(z),$$

where m is a complex constant.

(a) *Assume that $m\neq 0$. Prove that there does not exist any solution of the equation, such that*

$$(i)\;\; f(0)=1,\;\; f'(0)=0,\qquad (ii)\;\; f(0)=0,\;\; f'(0)=1.$$

(b) *Prove that the equation has a power series solution, which satisfies the conditions*

$$f(0)=1,\qquad f'(0)=-m.$$

When we put the formal series

$$f(z)=\sum_{n=0}^{+\infty}a_nz^n,\quad f'(z)=\sum_{n=1}^{+\infty}na_nz^{n-1},\quad f''(z)=\sum_{n=2}^{+\infty}n(n-1)a_nz^{n-2},$$

into the differential equation, we get

$$\begin{aligned} 0 &= z\,f''(z)+(1-z)f'(z)+m\,f(z) \\ &= \sum_{\substack{n=2\\(n=1)}}^{+\infty}n(n-1)a_nz^{n-1}+\sum_{n=1}^{+\infty}na_nz^{n-1}-\sum_{\substack{n=1\\(n=0)}}^{+\infty}na_nz^n+\sum_{n=0}^{+\infty}ma_nz^n \\ &= \sum_{n=0}^{+\infty}\left\{(n+1)^2a_{n+1}-(n-m)a_n\right\}z^n. \end{aligned}$$

Since $n+1\neq 0$ for n in the summation domain $\mathbb{N}_0$, we get the following recursion formula by the identity theorem,

$$(16)\quad a_{n+1}=\frac{n-m}{(n+1)^2}\,a_n,\qquad n\in\mathbb{N}_0.$$

The leap of the indices in this recursion formula is only 1, so we conclude that (apart from a constant factor) there can at most be one power series solution. Now assume that $m \notin \mathbb{N}_0$, so the series has not degenerated into a polynomial. Then

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{(n+1)^2}{n-m} \to +\infty \qquad \text{for } n \to +\infty,$$

and we conclude that (apart from a constant factor) there will always be a power series solution and that its domain of convergence is $\mathbb{C}$. This is of course also true for $m \in \mathbb{N}_0$, when the solution becomes a polynomial.

(a) If we put $n = 0$ into the recursion formula (16), then

$$a_1 = -m\, a_0.$$

Hence we see for $m \neq 0$, that either a_0 and a_1 are both zero, or none of them are zero. Now, the two given initial conditions are characterized by one being 0, while the other is $\neq 0$, so we conclude that no power series solution can fulfil these initial conditions.

(b) Here we have $a_1 = -m \cdot a_0$, so the initial conditions are fulfilled, and we get by recursion,

$$a_n = \frac{n-m-1}{n^2}\, a_{n-1} = \cdots = \frac{(n-m-1)(n-m-2)\cdots(1-m)(-m)}{(n!)^2},$$

so the series is given by

$$1 + \sum_{n=1}^{+\infty} \frac{(n-m-1)(n-m-2)\cdots(1-m)(-m)}{(n!)^2}\, z^n, \qquad z \in \mathbb{C}.$$

This series becomes a polynomial, when $m \in \mathbb{N}_0$.

7 Some more difficult differential equations

Example 7.1 *Given the differential equation*

$$f'(z) = \frac{1}{2} f\left(\frac{z}{4}\right).$$

Assuming that $f(z)$ can be expressed by a power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \qquad \text{for } |z| < \varrho,$$

we shall find a recursion formula for a_n expressed by a_{n-1}.
Find the radius of convergence ϱ of the series.
Then express a_n by means of a_{n-2}, and in general a_n by a_0.
HINT: *Here we have a couple of variants. In some of them, though not all, we may benefit from the formula*

$$1 + 2 + \cdots + n = \frac{1}{2} n(n+1).$$

Let $a_0 = f(0) = 1$. find the power series expansion of $f(z)$ and the corresponding domain of convergence.
HINT: *The function cannot be expressed by elementary functions.*

Remark 7.1 This is an non-typical example, because the variable on the left hand side is z, while it is $\frac{z}{4}$ on the right hand side, i.e. the derivative at a point $z \neq 0$ is expressed by the value of the function at another point $\frac{z}{4} \neq z$. $\Diamond$

Assume that

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \qquad |z| < \varrho,$$

is a power series expansion of a solution. Then

$$f\left(\frac{z}{4}\right) = \sum_{n=0}^{+\infty} a_n \left(\frac{z}{4}\right)^n = \sum n = 0^{+\infty} \frac{a^n}{4^n} z^n \qquad \text{for } \left|\frac{z}{4}\right| < \varrho,$$

where the condition $\left|\frac{z}{4}\right| < \varrho$ of course is fulfilled, when $|z| < \varrho$.

Furthermore,

$$f'(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} z^n \qquad \text{for } |z| < \varrho.$$

When these series are put into the equation, we get

$$0 = f'(z) - \frac{1}{2} f\left(\frac{z}{4}\right) = \sum_{n=0}^{+\infty} (n+1) a_{n+1} z^n - \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1}{4^n} a_n z^n = \sum_{n=0}^{+\infty} \left\{ (n+1) a_{n+1} - \frac{a_n}{2 \cdot 4^n} \right\} z^n.$$

Since $n+1 \neq 0$ in the summation domain $\mathbb{N}_0$, we get the following recursion formula from the identity theorem,

$$(17) \quad a_{n+1} = \frac{1}{(n+1) \cdot 2 \cdot 4^n} \, a_n, \qquad n \in \mathbb{N}_0.$$

It follows immediately from (17) that if $a_0 \neq 0$, then all $a_n \neq 0$ for $n \in \mathbb{N}_0$. Assuming this, it follows from

$$\frac{a_n}{a_{n+1}} = (n+1) \cdot 2 \cdot 4^n \to +\infty \qquad \text{for } n \to +\infty,$$

that the radius of convergence is $\varrho = +\infty$.

We can now find the solution of the recursion formula (17) in many ways. Here we shall give some of them:

First method. The recursion formula (17) is easiest solved by a *trick*, in which we multiply (17) by $(n+1)!2^{(n+1)^2}$. Then

$$(n+1)!2^{(n+1)^2}a_{n+1} = \frac{(n+1)!2^{(n+1)^2}}{(n+1)\cdot 2\cdot 4^n}\,a_n = n!\,2^{(n+1)^2-1-2n}a_n = n!\,2^{n^2}a_n,$$

and we see that we obtain the right hand side from the left hand side by replacing n everywhere by $n-1$. We therefore get by recursion,

$$(n+1)!\,2^{(n+1)^2}a_{n+1} = n!\,2^{n^2}a_n = \cdots = 1!\,2^1a_1 = 0!2^0a_0 = a_0,$$

thus

$$a_n = \frac{1}{n!\,2^{n^2}}\,a_0,$$

and the solution is

$$f(z) = a_0\sum_{n=0}^{+\infty}\frac{1}{n!\,2^{n^2}}\,z^n, \qquad z\in\mathbb{C}.$$

Second method. ALTERNATIVELY it follows from (17) that

$$a_n = \frac{1}{n\cdot 2\cdot 4^{n-1}}\,a_{n-1} = \frac{2}{n\cdot 4^n}\,a_{n-1}, \qquad n\in\mathbb{N},$$

hence by recursion,

$$a_n = \frac{2}{n\cdot 4^n}\,a_{n-1} = \frac{2}{n\cdot 4^n}\cdot\frac{2}{(n-1)4^{n-1}}\,a_{n-2}, \qquad n\geq 2,$$

and in general,

$$\begin{aligned} a_n &= \frac{2}{n!\,4^n}\cdot\frac{2}{(n-1)\,4^{n-1}}\cdots\frac{2}{2\cdot 4^2}\cdot\frac{2}{1\cdot 4^1}\,a_0 = \frac{2^n}{n!\,4^{1+2+\cdots+n}}\,a_0 \\ &= \frac{2^n}{n!\,4^{\frac{1}{2}\,n(n+1)}}\,a_0 = \frac{2^n}{n!\,2^{n^2+n}}\,a_0 = \frac{1}{n!\,2^{n^2}}\,a_0, \qquad n\in\mathbb{N}_0. \end{aligned}$$

We derive once more that the radius of convergence is $+\infty$. Note that we have applied the hint. The general solution is then given by

$$(18)\quad f(z) = a_0\sum_{n=0}^{+\infty}\frac{1}{n!}\,\frac{1}{2^{n^2}}\,z^n, \qquad a_0\in\mathbb{C}, \quad z\in\mathbb{C},$$

and when $a_0 = f(0) = 1$, we of course get

$$f(z) = \sum_{n=0}^{+\infty}\frac{1}{n!}\,\frac{1}{2^{n^2}}\,z^n, \qquad z\in\mathbb{C},$$

which cannot be further reduced.

Third method. ALTERNATIVELY we may early in the process apply that $4^n = 2^{2n}$. Then the recursion formula (17) can be written

$$a_n = \frac{1}{n \cdot 2^{2n-1}}\, a_{n-1}.$$

Then note that $n^2 - (n-1)^2 = 2n-1$, so we get a telescopic sum by insertion,

$$\sum_{j=1}^{n}(2j-1) = \sum_{j=1}^{n}\left\{j^2-(j-1)^2\right\} = n^2 - 0^2 = n^2.$$

This means that

$$a_n = \frac{1}{n \cdot 2^{2n-1}}\, a_{n-1} = \cdots = \frac{1}{n \cdot 2^{2n-1}} \cdot \frac{1}{(n-1)\cdot 2^{2n-3}} \cdots \frac{a_0}{1 \cdot 2^1} = \frac{a_0}{n!\, 2^{1+3+\cdots+(2n-1)}} = \frac{a_0}{n!\, 2^{n^2}}.$$

Fourth method. DETERMINATION OF THE TAYLOR COEFFICIENTS. It is not necessary to apply the recursion formula (17). In fact, if we differentiate the differential equation

$$f'(z) = \frac{1}{2}\, f\left(\frac{z}{4}\right)$$

then it follows by the chain rule that

$$f''(z) = \frac{1}{2}\cdot\frac{1}{4}\, f'\left(\frac{z}{4}\right) = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\, f\left(\frac{z}{4^2}\right) = \frac{1}{2^2}\cdot\frac{1}{4}\, f\left(\frac{z}{4^2}\right),$$

and furthermore,

$$f^{(3)}(z) = \frac{1}{2^2}\cdot\frac{1}{4}\cdot\frac{1}{4^2}\, f'\left(\frac{z}{4^2}\right) = \frac{1}{2^2}\cdot\frac{1}{4}\cdot\frac{1}{4^2}\cdot\frac{1}{2}\, f\left(\frac{z}{4^3}\right) = \frac{1}{2^3}\cdot\frac{1}{4}\cdot\frac{1}{4^2}\, f\left(\frac{z}{4^3}\right),$$

and then by induction,

$$f^{(n)}(z) = \frac{1}{2^n}\cdot\frac{1}{4\cdot 4^2 \cdots 4^{n-1}}\, f\left(\frac{z}{4^n}\right), \qquad n \in \mathbb{N}.$$

We therefore get the Taylor series

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!}\, z^n = f(0) \sum_{n=0}^{+\infty} \frac{1}{2^n}\cdot\frac{1}{4^{\frac{1}{2}(n-1)n}}\cdot\frac{1}{n!}\, z^n \\ &= f(0) \sum_{n=0}^{+\infty} \frac{1}{2^{n+(n-1)n}\cdot n!}\, z^n = f(0)\sum_{n=0}^{+\infty} \frac{1}{2^{n^2} n!}\, z^n, \end{aligned}$$

i.e. the same series as above with $\mathbb{C}$ as the domain of convergence.

Example 7.2 *Given the differential equation*

(19) $2z\, f''(z) + (3-2z)f'(z) - f(z) = 0.$

Find every power series solution from 0 of (19) and its radius of convergence.

Prove that (19) also has a power function as a s solution in the plane with a branch cut

$$\mathbb{C} \setminus (\mathbb{R}_- \cup \{0\}).$$

1) When the formal power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \qquad f'(z) = \sum_{n=1}^{+\infty} n a_n z^{n-1}, \qquad f''(z) = \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2},$$

are put into (19), we get

$$\begin{aligned} 0 &= 2z\, f''(z) + (3-2z)f'(z) - f(z) \\ &= \sum_{\substack{n=2 \\ (n=1)}}^{+\infty} 2n(n-1)a_n z^{n-1} + \sum_{n=1}^{+\infty} 3n a_n z^{n-1} - \sum_{\substack{n=1 \\ (n=0)}}^{+\infty} 2n a_n z^n - \sum_{n=0}^{+\infty} a_n z^n \\ &= \sum_{n=1}^{+\infty} n(2n-2+3)a_n z^{n-1} - \sum_{n=0}^{+\infty}(2n+1)a_n z^n \\ &= \sum_{n=1}^{+\infty} n(2n+1)a_n z^{n-1} - \sum_{n=0}^{+\infty} a_n z^n \\ &= \sum_{n=0}^{+\infty} \{(n+1)(2n+3)a_{n+1} - (2n+1)a_n\}\, z^n. \end{aligned}$$

Then by the identity theorem,

$$(n+1)(2n+3)a_{n+1} - (2n+1)a_n = 0, \qquad n \in \mathbb{N}_0.$$

Hence we obtain the *recursion formula*

$$\begin{aligned} a_{n+1} &= \frac{1}{n+1} \cdot \frac{2n+1}{2n+3}\, a_n = \cdots = \frac{1}{(n+1)!} \cdot \frac{2n+1}{2n+3} \cdot \frac{2n-1}{2n+1} \cdots \frac{1}{3}\, a_0 \\ &= \frac{1}{2n+3} \cdot \frac{1}{(n+1)!}\, a_0, \end{aligned}$$

so by a change of index,

$$a_n = \frac{1}{2n+1} \cdot \frac{1}{n!}\, a_0,$$

and the series is given by

$$f(z) = a_0 \sum_{n=0}^{+\infty} \frac{1}{2n+1} \cdot \frac{1}{n!}\, z^n,$$

The domain of convergence is obviously $\mathbb{C}$.

2) This question may look strange, until one realizes that the domain

$$\mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$$

is natural for a power series expansion of the form

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^{n+\frac{1}{2}}.$$

We get from this by a formal differentiation in the domain,

$$f'(z) = \sum_{n=-\infty}^{+\infty} \left(n+\frac{1}{2}\right) a_n z^{n-\frac{1}{2}},$$

and

$$f''(z) = \sum_{n=-\infty}^{+\infty} \left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right) a_n z^{n-\frac{3}{2}},$$

hence by insertion

$$\begin{aligned}
0 &= 2z\,f''(z)+(3-2z)f'(z)-f(z)\\
&= \sum_{n=-\infty}^{+\infty} 2\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)a_n z^{n-\frac{1}{2}}+\sum_{n=-\infty}^{+\infty} 3\left(n+\frac{1}{2}\right)a_n z^{n-\frac{1}{2}}\\
&\qquad -\sum_{n=-\infty}^{+\infty} 2\left(n+\frac{1}{2}\right)a_n z^{n+\frac{1}{2}}-\sum_{n=-\infty}^{+\infty} a_n z^{n+\frac{1}{2}}\\
&= \sum_{n=-\infty}^{+\infty}\left(n+\frac{1}{2}\right)\left\{2\left(n-\frac{1}{2}\right)+3\right\}a_n z^{n-\frac{1}{2}}-\sum_{n=-\infty}^{+\infty} 2(n+1)a_n z^{n+\frac{1}{2}}\\
&= \sum_{-\infty}^{+\infty}\frac{1}{2}\,(2n+1)(2n-1+3)a_n z^{n-\frac{1}{2}}-\sum_{n=-\infty}^{+\infty} 2(n+1)a_n z^{n+\frac{1}{2}}\\
&= \sum_{n=-\infty}^{+\infty}(2n+1)(n+1)a_n z^{n-\frac{1}{2}}-\sum_{-\infty}^{+\infty}2(n+1)a_n z^{n+\frac{1}{2}}\\
&= \sum_{-\infty}^{+\infty}(2n+3)(n+2)a_{n+1}z^{n+\frac{1}{2}}-\sum_{-\infty}^{+\infty}2(n+1)a_n z^{n+\frac{1}{2}}\\
&= \sum_{n=-\infty}^{+\infty}\left\{(2n+3)(n+2)a_{n+1}-2(n+1)a_n\right\}z^{n+\frac{1}{2}}.
\end{aligned}$$

Then we get the following *recursion formula* by the identity theorem,

$$(2n+3)(n+2)a_{n+1}=2(n+1)a_n,\qquad n\in\mathbb{Z}.$$

If we put $n=-1$, we get $a_0=0$, while a_{-1} is indefinite.
If we instead put $n=-2$, then $a_{-2}=0$, and again a_{-1} is indefinitet.
We conclude that another power solution is

$$g(z)=\frac{1}{\sqrt{z}},\qquad z\in\mathbb{C}\setminus(\mathbb{R}_-\cup\{0\}).$$

Remark 7.2 ALTERNATIVELY, the equation can be solved by using the change of variable,

$$w=\sqrt{z},\qquad z=w^2,\qquad\text{and}\qquad f(z)=g(w).$$

Then by the chain rule,

$$f'(z)=g'(w)\cdot\frac{dw}{dz}=g'(w)\cdot\frac{1}{\frac{dz}{dw}}=\frac{1}{2w}\,g'(w),$$

and

$$f''(z)=\frac{d}{dw}\left\{\frac{1}{2w}\,g'(w)\right\}\cdot\frac{dw}{dz}=\frac{1}{2w}\left\{\frac{1}{2w}\,g''(w)-\frac{1}{w^2}\,g'(w)\right\}.$$

By insertion and reduction and some computation we finally get

$$\frac{1}{2}\,g''(w)-\left\{w-\frac{1}{w}\right\}g'(w)-g(w).$$

Then introduce another function by

$$g(w)=\frac{1}{w}\,h(w),\qquad h(w)=w\,g(w).$$

This function satisfies the equation

$$h''(w)-2w\,h'(w)=0,$$

which has the trivial solution $h(w)=c$ constant, hence

$$g(w)=\frac{c}{w}=\frac{c}{\sqrt{z}}=f(z).\qquad\Diamond$$

Example 7.3 *Solve the differential equation*

$$z(1-z)f''(z)+(4z-2)f'(z)-4f(z)=0$$

by insertion of a formal power series of the form

$$\sum_{n=0}^{+\infty}a_nz^n.$$

When we insert the series and the termwise differentiated series, we formally get

$$\begin{aligned}
0 &= \left(z-z^2\right)\sum_{n=2}^{+\infty}n(n-1)a_nz^{n-2}+(4z-2)\sum_{n=1}^{+\infty}na_nz^{n-1}-4\sum_{n=0}^{+\infty}a_nz^n\\
&= \sum_{n=2}^{+\infty}n(n-1)a_nz^{n-1}-\sum_{n=2}^{+\infty}n(n-1)a_nz^n+4\sum_{n=1}^{+\infty}na_nz^n-2\sum_{n=1}^{+\infty}na_nz^{n-1}-4\sum_{n=0}^{+\infty}a_nz^n\\
&= \sum_{n=1}^{+\infty}\{n(n-1)-2n\}a_nz^{n-1}+\sum_{n=0}^{+\infty}\{-n(n-1)+4n-4\}a_nz^n\\
&= \sum_{n=1}^{+\infty}n(n-3)a_nz^{n-1}+\sum_{n=0}^{+\infty}(4-n)(n-1)a_nz^n\\
&= \sum_{n=0}^{+\infty}\left\{(n+1)(n-2)a_{n+1}-(n-4)(n-1)a_n\right\}z^n.
\end{aligned}$$

Here we have added some zero terms and changed the summation index.
Then we get the following *recursion formula* from the identity theorem,

$$(n+1)(n-2)a_{n+1}=(n-4)(n-1)a_n,\qquad n\in\mathbb{N}_0.$$

This recursion formula contains the zeros $n = 1, 2, 4$, so we shall first check the values $n = 0, 1, \dots, 4$, separately. Then

$$\begin{array}{lll} n=0 & \text{gives} & -2a_1 = 4a_0, \\ n=1 & \text{giver} & a_2 = 0, \\ n=2 & \text{gives} & 0 = -2a_2, \\ n=3 & \text{gives} & 4a_4 = -2a_3, \\ n=4 & \text{gives} & a_5 = 0, \\ n \geq 5 & \text{gives} & a_n = 0. \end{array}$$

This gives us the solution

$$f(z) = a_0(1-2z) + a_3\left(1 - \frac{1}{2}\,z^4\right), \qquad z \in \mathbb{C},$$

where a_0 and a_3 are arbitrary complex constants.
Since f is a polynomial, the domain of convergence is trivially $\mathbb{C}$.

Example 7.4 *Given the differential equation*

(20) $\left(2z^2 - 3z + 1\right) f''(z) + (8z-6)f'(z) + 4f(z)$.

1) Prove that if $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ is a solution of (20), then the coefficients satisfy the recursion formula

$$a_{n+2} = 3a_{n+1} - 2a_n, \qquad n \in \mathbb{N}.$$

2) Prove that

$$a_n = \left(2^n - 1\right) a_1 - \left(2^n - 2\right) a_0, \qquad n \in \mathbb{N}_0.$$

3) Find the domain of convergence of the solution series.

4) Express the solution series by elementary functions.

5) Prove that the solutions in Ω can be extended to $\mathbb{C}$, with the exception of a few points in $\mathbb{C}$.

Remark 7.3 It is actually possible to solve the equation by inspection and some manipulation. Here we shall only sketch this method, leaving the details to the reader. We get by some small rearrangements

$$\begin{aligned} 0 &= \left(2z^2-3z+1\right) f''(z) + (8z-6)f'(z) + 4f(z) \\ &= \left\{\left(2z^2-3z+1\right) f''(z) + (4z-3)f'(z)\right\} + \{(4z-3)f'(z) + 4f(z)\} \\ &= \frac{d}{dz}\left\{\left(2z^2-3z+1\right) f'(z) + (4z-3)f(z)\right\} \\ &= \frac{d^2}{dz^2}\left\{\left(2z^2-3z+1\right) f(z)\right\}. \end{aligned}$$

Hence by two integrations,

$$\left(2z^2-3z+1\right) f(z) = c_1 z + c_0,$$

where c_0 and c_1 are arbitrary constants. Then it is easy to find $f(z)$. $\Diamond$

1) When we put the formal power series

$$f(z)=\sum_{n=0}^{+\infty} a_n z^n, \qquad f'(z)=\sum_{n=1}^{+\infty} n a_n z^{n-1}, \qquad f''(z)=\sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2},$$

into the differential equation and add some zero terms, we get

$$\begin{aligned}
0 &= 2\sum_{n=0}^{+\infty} n(n-1)a_n z^n - 3\sum_{n=1}^{+\infty} n(n-1)a_n z^{n-1} + \sum_{n=2}^{+\infty} n(n-1)a_n z^{n-2} \\
&\qquad +8\sum_{n=0}^{+\infty} n a_n z^n - 6\sum_{n=1}^{\infty} n a_n z^{n-1} + 4\sum_{n=0}^{+\infty} a_n z^n \\
&= \sum_{n=0}^{+\infty}\{2n(n-1)a_n - 3(n+1)n a_{n+1} + (n+2)(n+1)a_{n+2} + 8n a_n - 6(n+1)a_{n+1} + 4a_n\}\, z^n \\
&= \sum_{n=0}^{+\infty}\left\{\left(2n^2+6n+4\right)a_n - 3(n+1)(n+2)a_{n+1} + (n+2)(n+1)a_{n+2}\right\} z^n \\
&= \sum_{n=0}^{+\infty}(n+1)(n+2)\{2a_n - 3a_{n+1} + a_{n+2}\}\, z^n.
\end{aligned}$$

Since $(n+1)(n+2) \neq 0$ for $n \in \mathbb{N}_0$, we obtain the following *recursion formula* by the identity theorem,

$$a_{n+2} = 3a_{n+1} - 2a_n, \qquad n \in \mathbb{N}_0.$$

2) If we put $b_n = a_{n+1} - a_n$, then it follows by the recursion formula above,

$$b_{n+1} = a_{n+2} - a_{n+1} = 2\{a_{n+1} - a_n\} = 2b_n, \qquad n \in \mathbb{N}_0,$$

hence by recursion with respect to b_n,

$$a_{n+1} - a_n = b_n = 2^n b_0 = 2^n \{a_1 - a_0\}.$$

Then

$$a_n = \sum_{j=0}^{n-1} (a_{j+1} - a_j) + a_0 = \sum_{j=0}^{n-1} 2^j (a_1 - a_0) + a_0 = (2^n - 1)\, a_1 + (2^n - 2)\, a_0,$$

and we have proved the formula.

ALTERNATIVELY, we see that the claimed formula,

$$a_n = (2^n - 1)\, a_1 - (2^n - 2)\, a_0, \qquad n \in \mathbb{N}_0,$$

holds for $n = 0$ and for $n = 1$. Then we prove it by induction, assuming that it holds for n and $n+1$. Then by insertion,

$$\begin{aligned} -2a_n + 3a_{n+1} &= \left(2 - 2^{n+1}\right) a_1 + \left(2^{n+1} - 4\right) a_0 + \left(3 \cdot 2^{n+1} - 3\right) a_1 + \left(6 - 3 \cdot 2^{n+1}\right) a_0 \\ &= \left(2^{n+2} - 1\right) a_1 + \left(2 - 2^{n+2}\right) a_0 = a_{n+2}, \end{aligned}$$

and the formula is proved.

3) If $a_1 = a_0$, then it follows from the formula above that $a_n = a_0$ for every n, so

$$\Omega = \{z \in \mathbb{C} \mid |z| < 1\}.$$

If instead $a_1 \neq a_0$, then

$$a_n = 2^n (a_1 - a_0) + 2a_0 - a_1 = 2^n \left\{a_1 - a_0 + 2^{-n} (2a_0 - a_1)\right\}.$$

It follows that $\sqrt[n]{a_n} \to 2$ for $n \to +\infty$, so

$$\Omega = \left\{ z \in \mathbb{C} \;\middle|\; |z| < \frac{1}{2} \right\}.$$

4) Then compare with the geometric series to obtain

$$f(z) = (a_1 - a_0) \sum_{n=0}^{+\infty} (2z)^2 + (2a_0 - a_1) \sum_{n=0}^{+\infty} z^n = \frac{a_1 - a_0}{1 - 2z} + \frac{2a_0 - a_1}{1 - z}.$$

5) If $a_1 = 0$, then $M = \mathbb{C} \setminus \{1\}$. If $a_1 = 2a_0$, then $M = \mathbb{C} \setminus \left\{\frac{1}{2}\right\}$. In any other case we get

$$M = \mathbb{C} \setminus \left\{\frac{1}{2}, 1\right\}.$$

Clearly, f is analytic in M, and it follows by a differentiation that both $(1-2z)^{-1}$ and $(1-z)^{-1}$ fulfil (20);

$$(2z-1)(z-1) \cdot \frac{8}{(1-2z)^3} + \frac{(8z-6)\cdot 2}{(1-2z)^2} + \frac{4}{1-2z} = \frac{-8(z-1)+2(8z-6)+4(1-2z)}{(1-2z)^2} = 0,$$

$$(2z-1)(z-1) \cdot \frac{2}{(1-z)^3} + \frac{8z-6}{(1-z)^2} \frac{4}{1-z} = \frac{2(1-2z)+8z-6+4(1-z)}{(1-z)^2} = 0.$$

8 Zeros of analytic functions

Example 8.1 *Find the order of the zero at $z = 0$ for each of the functions*

$$(a)\ \ z^2\left\{\exp\left(z^2\right)-1\right\}, \quad (b)\ \ 6\,\sin z^3+z^3\left(z^6-6\right), \quad (c)\ \ e^{\sin z}-e^{\tan z}.$$

Remark 8.1 It will be demonstrated by the variants of solutions that one should be very careful here by choosing the most convenient method. It is of course possible in all three cases (cf. the definition) to differentiate, until one reach the smallest number n, for which $f^{(n)}(0) \neq 0$, but it will usually be more easy to insert known series expansions for the given functions. In particular, **(b)** becomes very difficult to solve by the method of differentiation. $\Diamond$

(a) First method. *Insertion of Taylor series.* It follows from

$$f(z)=z^2\left(e^{z^2}-1\right)=z^2\left\{1+\frac{1}{1!}\,z^2+o\left(z^2\right)-1\right\}=z^4+o\left(z^4\right),$$

that the order is 4.

Second method. *The method of differentiation.* We get by successive differentiation,

$$\begin{array}{ll}
f(z)=z^2e^{z^2}-z^2, & f(0)=0,\\
f'(z)=2z^3e^{z^2}+2z\,e^{z^2}-2z, & f'(0),\\
f''(z)=4z^4e^{z^2}+10z^2e^{z^2}+2e^{z^2}-2, & f''(0)=0,\\
f^{(3)}(z)=8z^5e^{z^2}+36z^3e^{z^2}+24z\,e^{z^2}, & f^{(3)}(0)=0,\\
f^{(4)}(z)=16z^6e^{z^2}+112z^4e^{z^2}+156z^2e^{z^2}+24e^{z^2}, & f^{(4)}(0)=24,
\end{array}$$

so we conclude that the order is 4.

(b) First method. *Insertion of Taylor series.* It follows from

$$\begin{aligned}
f(z) &= 6\sin\left(z^3\right)+z^3\left(z^6-6\right)=6\left\{z^3-\frac{1}{3!}\left(z^3\right)^3+\frac{1}{5!}\left(z^3\right)^5 0o\left(z^{15}\right)\right\}+z^9-6z^3\\
&= \frac{1}{20}\,z^{15}+o\left(z^{15}\right)
\end{aligned}$$

that the order is 15.

Second method. *Differentiation with respect to $w = z^3$.* First note that the function is actually a function of $w = z^3$. If we change variable to w, the differentiation method becomes reasonable, thought still bigger that the **first method**. In fact,

$$f(z)=6\sin z^3+z^9-6z^3=6\sin w+w^3-6w=g(w),$$

and then by differentiation with respect to w,

$$\begin{array}{ll}
g(w)=w^3-6w+6\sin w, & g(0)=0,\\
g'(w)=3w^2-6-6\cos w, & g'(0)=0,\\
g''(w)=6w-6\sin w, & g''(0)=0,\\
g^{(3)}(w)=6-6\cos w, & g^{(3)}(0)=0,\\
g^{(4)}(w)=6\sin w, & g^{(4)}(0)=0,\\
g^{(5)}(w)=6\cos w, & g^{(5)}(0)=6\neq 0,
\end{array}$$

and we conclude that

$$g(w) = \frac{6}{5!}\,w^5 + o\left(w^5\right) = \frac{1}{20}\,w^5 + o\left(w^5\right).$$

Then

$$f(z) = g\left(z^3\right) = \frac{1}{20}\left(z^3\right)^5 + o\left(\left(z^3\right)^5\right) = \frac{1}{20}\,z^{15} + o\left(z^{15}\right),$$

so the order of the zero is 15.

Third method. The difference of time consumption of the two methods of **(a)** was not very big. However, in the present case, the differentiations really grows wild. We get

$$\begin{array}{ll}
f(z) = 6\,\sin\left(z^3\right) + z^9 - 6z^3, & f(0) = 0,\\
f'(z) = 18z^2\cos\left(z^3\right) + 9z^8 - 18z^2, & f'(0) = 0,\\
f''(z) = 36z\,\cos\left(z^3\right) - 54z^4\sin\left(z^3\right) + 72z^7 - 36z, & f''(0) = 0,\\
f^{(3)}(z) = 36\cos\left(z^3\right) - 324z^3\sin\left(z^3\right) & \\
\qquad -162z^6\cos\left(z^3\right) + 504z^6 - 36, & f^{(3)}(0) = 0,\\
f^{(4)}(z) = -1080z^2\sin\left(z^3\right) - 1944z^5\cos\left(z^3\right) & \\
\qquad +486z^8\sin\left(z^3\right) + 3024z^5, & f^{(4)}(0) = 0,\\
f^{(5)}(z) = -2160z\sin\left(z^3\right) - 12\,960z^4\cos\left(z^3\right) & \\
\qquad +9720z^7\sin\left(z^3\right) + 1458z^{10}\cos\left(z^3\right) & \\
\qquad +15\,120z^4, & f^{(5)}(0) = 0,\\
f^{(6)}(z) = -2160\sin\left(z^3\right) - 58\,320z^3\cos\left(z^3\right) & \\
\qquad +106\,920z^6\sin\left(z^3\right) + 43\,740z^9\cos\left(z^3\right) & \\
\qquad -4374z^{12}\sin\left(z^3\right) + 60\,480z^3, & f^{(6)}(0) = 0,\\
f^{(7)}(z) = -181\,440z^2\cos\left(z^3\right) + 816\,480z^5\sin\left(z^3\right) & \\
\qquad +714\,420z^8\cos\left(z^3\right) - 183\,708z^{11}\sin\left(z^3\right) & \\
\qquad -13\,122z^{14}\cos\left(z^3\right) + 181\,440z^2, & f^{(7)}(0) = 0,\\
f^{(8)}(z) = -362\,880z\cos\left(z^3\right) + 4\,626\,720z^4\sin\left(z^3\right) & \\
\qquad 8\,164\,800z^7\cos\left(z^3\right) - 4\,164\,048z^{10}\sin\left(z^3\right) & \\
\qquad -734\,832z^{13}\cos\left(z^3\right) + 39\,366z^{16}\sin\left(z^3\right) & \\
\qquad +362\,880z, & f^{(8)}(0) = 0,\\
f^{(9)}(z) = -362\,880\cos\left(z^3\right) + 19\,595\,520z^3\sin\left(z^3\right) & \\
\qquad +71\,033\,760z^6\cos\left(z^3\right) - 66\,1134\,880z^9\sin\left(z^3\right) & \\
\qquad -22\,044\,960z^{12}\cos\left(x^3\right) + 2\,834\,352z^{15}\sin\left(z^3\right) & \\
\qquad +118\,098z^{18}\cos\left(z^3\right) + 362\,880, & f^{(9)}(0) = 0,\\
f^{(10)}(z) = 59\,875\,200z^2\sin\left(z^3\right) + 484\,989\,120z^5\cos\left(z^3\right) & \\
\qquad -808\,315\,200z^8\sin\left(z^3\right) - 462\,944\,160z^{11}\cos\left(z^3\right) & \\
\qquad +108\,650\,160z^{14}\sin\left(z^3\right) + 10\,628\,820z^{17}\cos\left(z^3\right) & \\
\qquad -354\,294z^{20}\sin\left(z^3\right), & f^{(10)}(0) = 0,\\
f^{(11)}(z) = 119\,750\,400z\sin z^3 + 2\,604\,571\,200z^4\cos z^3 & \\
\qquad -7\,921\,488\,960z^7\sin z^3 - 7\,517\,331\,360z^{10}\cos z^3 & \\
\qquad 2\,909\,934\,720z^{13}\sin z^3 + 506\,640\,420z^{16}\cos z^3 & \\
\qquad -38\,972\,340z^{19}\sin z^3 - 1\,062\,882z^{22}\cos z^3, & f^{(11)}(0) = 0,
\end{array}$$

$$\begin{aligned}f^{(12)}(z) = {} & 119\,750\,400 \sin z^3 + 10\,777\,536\,000 z^3 \cos z^3 \\ & -63\,264\,136\,320 z^6 \sin z^3 + 98\,937\,780\,480 z^9 \cos z^3 \\ & +60\,381\,145\,440 z^{12} \sin z^3 + 16\,836\,050\,880 z^{15} \cos z^3 \\ & +2\,260\,395\,720 z^{18} \sin z^3 - 140\,300\,424 z^{21} \cos z^3 \\ & +3\,188\,646 z^{24} \sin z^3,\end{aligned}$$

$$f^{(12)}(0) = 0,$$

$$\begin{aligned}f^{(13)}(z) = {} & 32\,691\,859\,200 z^2 \cos z^3 - 411\,917\,425\,920 z^5 \sin z^3 \\ & -1\,080\,232\,433\,280 z^8 \cos z^3 + 1\,021\,387\,086\,720 z^{11} \sin z^3 \\ & +433\,684\,199\,520 z^{14} \cos z^3 - 91\,195\,275\,600 z^{17} \sin z^3 \\ & -9\,727\,496\,064 z^{20} \cos z^3 + 497\,428\,776 z^{23} \sin z^3 \\ & +9\,565\,938 z^{26} \cos z^3,\end{aligned}$$

$$f^{(13)}(0) = 0,$$

$$\begin{aligned}f^{(14)}(z) = {} & 65\,383\,718\,400 z \cos z^3 - 2\,157\,662\,707\,200 z^4 \sin z^3 \\ & -9\,877\,611\,744\,000 z^7 \cos z^3 + 14\,475\,955\,253\,760 z^{10} \sin z^3 \\ & +9\,135\,740\,053\,440 z^{13} \cos z^3 - 2\,851\,372\,283\,760 z^{16} \sin z^3 \\ & -468\,135\,748\,080 z^{19} \cos z^3 + 40\,623\,350\,040 z^{22} \sin z^3 \\ & +1\,741\,000\,716 z^{25} \cos z^3 - 28\,697\,814 z^{28} \sin z^3,\end{aligned}$$

$$f^{(14)}(0) = 0,$$

$$\begin{aligned}f^{(15)}(z) = {} & 65\,383\,718\,400 \cos z^3 - 8\,826\,801\,984\,000 z^3 \sin z^3 \\ & -75\,616\,270\,329\,600 z^6 \cos z^3 + 174\,392\,387\,769\,610 z^9 \sin z^3 \\ & +162\,192\,486\,456\,000 z^{12} \cos z^3 - 73\,029\,176\,700\,480 z^{15} \sin z^3 \\ & -17\,448\,696\,064\,800 z^{18} \cos z^3 + 2\,298\,120\,945\,120 z^{21} \sin z^3 \\ & +165\,395\,068\,020 z^{24} \cos z^3 - 6\,026\,540\,940 z^{27} \sin z^3 \\ & -86\,093\,442 z^{30} \cos z^3,\end{aligned}$$

where finally

$$f^{(15)}(0) = 65\,383\,718\,400 = \frac{15!}{20} \neq 0,$$

so we conclude that the order of the zero is 15, and

$$f(z) = \frac{f^{(15)}(0)}{15!}\, z^{15} + \cdots = \frac{1}{20}\, z^{15} + \cdots .$$

(c) It is here difficult – though not quite impossible – to insert the power series expansions, so we prefer here the method of differentiations. It should, however, be mentioned that there is also here an alternative, which requires some intuition. We show here three solution variants, which is far from being exhaustive.

First method. *Intuition.* Since $\cos z \neq 1$ and $\sin z \neq 0$ in a neighbourhood of 0, excluding 0, it follows that

$$u = \sin z \qquad \text{and} \qquad v = \tan z = \frac{\sin z}{\cos z}$$

are different in the same neighbourhood, excluding 0. Then

$$e^{\sin z} - e^{\tan z} = e^u - e^v = \frac{e^u - e^v}{u - v} \cdot (u - v),$$

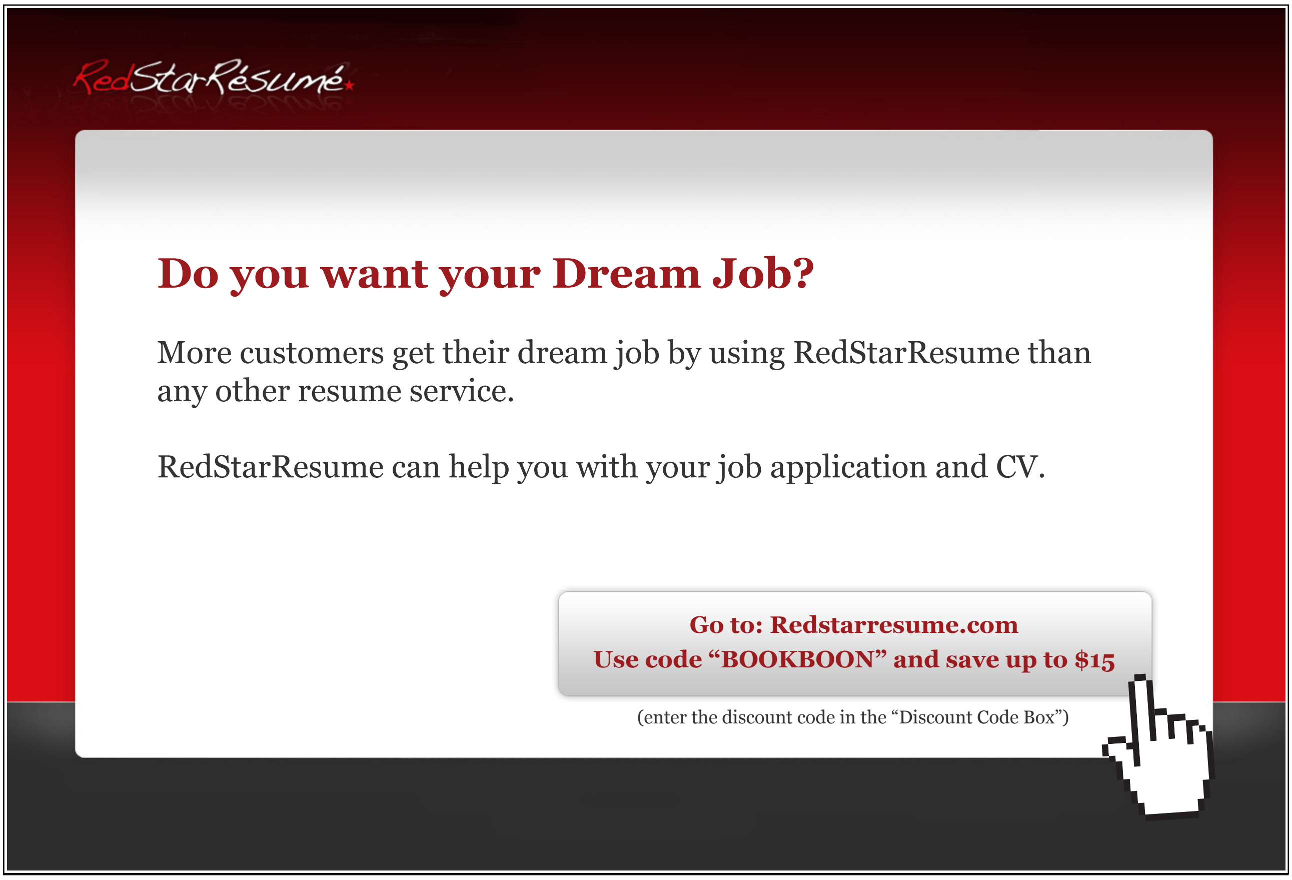
where

$$\lim_{z \to 0} \frac{e^u - e^v}{u - v} = \lim_{w \to 0} \frac{d}{dw}\, e^w = e^0 = 1 \neq 0,$$

and

$$
\begin{aligned}
u - v &= \sin z - \tan z = \sin z - \frac{\sin z}{\cos z} = \frac{\sin z(\cos z - 1)}{\cos z} \\
&= \frac{\left\{z - \dfrac{z^3}{3!} + \cdots\right\}\left\{1 - \dfrac{z^2}{2!} + \cdots - 1\right\}}{1 - \dfrac{z^2}{2!} + \cdots} = -\frac{z^3}{2} + o\left(z^3\right),
\end{aligned}
$$

and we conclude that the zero has order 3.



Second method. *The method of differentiation.* We get by successive differentiations,

$$f(z) = e^{\sin z} - e^{\tan z},$$

$$f(0) = 0,$$

$$f'(z) = \cos z \cdot e^{\sin z} - \left(1 + \tan^2\right) e^{\tan z},$$

$$f'(0) = 0,$$

$$\begin{aligned} f''(z) &= \left(\cos^2 z - \sin z\right) e^{\sin z} - \left(1 + \tan^2 z\right) e^{\tan z} \\ &\quad -2 \tan z \cdot \left(1 + \tan^2 z\right) e^{\tan z} \\ &= \left(\cos^2 z - \sin z\right) e^{\sin z} - \left(1 + \tan^2 z\right) (1 + \tan z)^2 e^{\tan z}, \end{aligned}$$

$$f''(0) = 0,$$

$$\begin{aligned} f^{(3)}(z) &= \left(\cos^3 z - 3 \sin z \cdot \cos z - \cos z\right) e^{\sin z} \\ &\quad -2 \tan z \left(1 + \tan^2 z\right) (1 + \tan z)^2 e^{\tan z} \\ &\quad -2 \left(1 + \tan^2 z\right)^2 (1 + \tan z) e^{\tan z} \\ &\quad - \left(1 + \tan^2 z\right)^2 (1 + \tan z)^2 e^{\tan z}, \end{aligned}$$

$$f^{(3)}(0) = -3,$$

so we conclude that the order is 3.

Third method. *A hybrid of the two solutions above.* Since

$$e^{\sin z} - e^{\tan z} = e^{\sin z} \left\{1 - e^{\tan z - \sin z}\right\},$$

and

$$\lim_{z \to 0} e^{\sin z} = 1 \neq 0,$$

the task is reduced to finding the order of the zero $z_0 = 0$ of the function

$$g(z) = e^{\tan z - \sin z} - 1, \qquad g(0) = 0.$$

Here we get by successive differentiations,

$$g'(z) = \left(1 + \tan^2 z - \cos z\right) e^{\tan z - \sin z},$$

$$g'(0) = 0,$$

$$\begin{aligned} g''(z) &= \left(1 + \tan^2 z - \cos z\right)^2 e^{\tan z - \sin z} \\ &\quad + \left\{2 \tan z \left(1 + \tan^2 z\right) + \sin z\right\} e^{\tan z - \sin z}, \end{aligned}$$

$$g''(0) = 0,$$

$$\begin{aligned} g^{(3)}(z) &= \left(1 + \tan^2 z - \cos z\right)^3 e^{\tan z - \sin z} \\ &\quad +3 \left\{2 \tan z \left(1 + \tan^2 z\right) + \sin z\right\} e^{\tan z - \sin z} \\ &\quad +2 \left\{2 \left(1 + \tan^2 z\right)^2 + 4 \tan^2 z \left(1 + \tan^2 z\right) + \cos z\right\} e^{\tan z - \sin z}, \end{aligned}$$

$$g^{(3)}(0) = 3,$$

so $n = 3$ is the first order of differentiation for which the result is $\neq 0$. This means that the order of the zero is 3.

Example 8.2 *Find the order of the zero $z = 0$ of*

$(\sin z + \sinh z - 2z)^2$.

First method. *Series expansion.* We get

$$\begin{aligned}(\sin z + \sinh z - 2z)^2 &= \left\{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) + \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) - 2z\right\}^2 \\ &= \left\{\frac{2}{5!}\,z^5 + \cdots\right\}^2 = \frac{1}{3600}\,z^{10} + \cdots ,\end{aligned}$$

and we conclude that the order of the zero is 10.

Second method. *The method of differentiation.* It suffices to find the order or the zero of

$$f(z) = \sin z + \sinh z - 2z,$$

because $\{f(z)\}^2 = (\sin z + \sinh z - 2z)^2$ then has twice as many. Note that it is not a good idea just to differentiate the expression $\{f(z)\}^2$ itself.
We get

$$\begin{aligned}
f(z) &= \sin z + \sinh z - 2z, & f(0) &= 0,\\
f'(z) &= \cos z + \cosh z - 2, & f'(0) &= 0,\\
f''(z) &= -\sin z + \sinh z, & f''(0) &= 0,\\
f^{(3)}(z) &= -\cos z + \cosh z, & f^{(3)}(0) &= 0,\\
f^{(4)}(z) &= \sin z + \sinh z, & f^{(4)}(0) &= 0,\\
f^{(5)}(z) &= \cos z + \cosh z, & f^{(5)}(0) &= 2,
\end{aligned}$$

from which we conclude that

$$f(z) = \frac{2}{5!}\,z^5 + o\left(z^5\right),$$

hence

$$(\sin z + \sinh z - 2z)^2 = \frac{1}{60^2}\,z^{10} + o\left(z^{10}\right),$$

and the zero has order 10.

Example 8.3 *Find the order of the zero at $z = 0$ of $3\sin z - z(2+\cos z)$.*

First method. *Series expansion.* We get

$$\begin{aligned}3\sin z - z(2+\cos z) &= 3\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right\} - z\left\{2 + 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right\}\\ &= 3z - \frac{z^3}{2} + \frac{1}{40}\,z^5 - \cdots - 3z + \frac{z^3}{2} - \frac{1}{24}\,z^5 + \cdots = \left(\frac{1}{40} - \frac{1}{24}\right) z^5 + \cdots = -\frac{1}{60}\,z^5 + \cdots,\end{aligned}$$

so the zero has order 5.

Second method. *Method of differentiation.* We get by successive differentiation,

$$\begin{aligned}
f(z) &= 3\sin z - 2z - z\cos z, & f(0) &= 0,\\
f'(z) &= 2\cos z - 2 + z\sin z, & f'(0) &= 0,\\
f''(z) &= -\sin z + z\cos z, & f''(0) &= 0,\\
f^{(3)}(z) &= -z\sin z, & f^{(3)}(0) &= 0,\\
f^{(4)}(z) &= -z\cos z - \sin z, & f^{(4)}(0) &= 0,\\
f^{(5)}(z) &= -2\cos z + z\sin z, & f^{(5)}(0) &= -2,
\end{aligned}$$

from which we conclude that

$$f(z) = -\frac{2}{5!}\,z^5 + \cdots = -\frac{1}{60}\,z^5 + \cdots,$$

and the order of the zero is 5.

Example 8.4 *Find the order of the zero at $z = 0$ of the function*

$$f(z) = 3\sinh\left(z^2\right) - 3\sin\left(z^2\right) - z^6.$$

First method. *Taylor expansion.* Clearly, $f(z)$ is analytic in $\mathbb{C}$. Then by a Taylor expansion from $z_0 = 0$,

$$\begin{aligned}
]f(z) &= 3\sinh\left(z^2\right) - 3\sin\left(z^2\right) - z^6 \\
&= \sum_{n=0}^{+\infty} \frac{3}{(2n+1)!}\left(z^2\right)^{2n+1} - \sum_{n=0}^{+\infty} \frac{3\cdot(-1)^n}{(2n+1)!}\left(z^2\right)^{2n+1} - z^6 \\
&= \sum_{n=0}^{+\infty} \frac{3}{(2n+1)!}\left\{1-(-1)^n\right\} z^{4n+2} - z^6 \\
&= \sum_{p=0}^{+\infty} \frac{2\cdot 3}{(2\{2p+1\}+1)!} z^{4(2p+1)+2} - z^6 \\
&= \sum_{p=0}^{+\infty} \frac{6}{(4p+3)!} z^{8p+6} - z^6 = \sum_{p=1}^{+\infty} \frac{6}{(4p+3)!} z^{8p+6} \\
&= \sum_{p=0}^{+\infty} \frac{6}{(4p+7)!} z^{8p+14}.
\end{aligned}$$

It follows immediately that the order is 14 corresponding to $p = 0$.

Second method. *o-technique*. A shorter variant is to use o-technique:

$$\begin{aligned} f(z) &= 3\sinh\left(z^2\right) - 3\sin\left(z^2\right) - z^6 \\ &= 3\left\{\left(z^2\right) + \frac{1}{3!}\left(z^2\right)^3 + \frac{1}{5!}\left(z^2\right)^5 + \frac{1}{7!}\left(z^2\right)^7 + o\left(z^{14}\right)\right\} \\ &\quad -3\left\{\left(z^2\right) - \frac{1}{3!}\left(z^2\right)^3 + \frac{1}{5!}\left(z^2\right)^5 - \frac{1}{7!}\left(z^2\right)^7 + o\left(z^{14}\right)\right\} - z^6 \\ &= \frac{2\cdot 3}{7!}\, z^{14} + o\left(z^{14}\right), \end{aligned}$$

from which we conclude that the order of the zero is 14.

Third method. *Method of differentiation, first variant.* If we immediately see that $f(z)$ can be considered as a function of $w = z^2$,

$$\begin{aligned} f(z) &= 3\sinh\left(z^2\right) - 3\sin\left(z^2\right) - z^6 \\ &= 3\sinh w - 3\sin w - w^3, \end{aligned}$$

then the example is reduced to find the order of the zero of the function

$$g(w) = 3\sinh w - 3\sin w - w^3, \qquad g(0) = 0,$$

at $w_0 = 0$. It follows by differentiation that

$$\begin{aligned} g'(w) &= 3\cosh w - 3\ cosw - 3w^2, && g'(0) = 0, \\ g''(w) &= 3\sinh w + 3\sin w - 6w, && g''(0) = 0, \\ g^{(3)}(w) &= 3\cosh w + 3\cos w - 6, && g^{(3)}(0) = 0, \\ g^{(4)}(w) &= 3\sinh w - 3\sin w, && g^{(4)}(0) = 0, \\ g^{(5)}(w) &= 3\cosh w - 3\cos w, && g^{(5)}(0) = 0, \\ g^{(6)}(w) &= 3\sinh w + 3\sin w, && g^{(6)}(0) = 0, \\ g^{(7)}(w) &= 3\cosh w + 3\cos w, && g^{(7)}(0) = 6 \neq 0. \end{aligned}$$

Hence, the function $g(w)$ has a zero of order 7 at $w_0 = 0$, thus

$$g(w) = w^7 \cdot g_1(w), \qquad g_1(0) \neq 0.$$

When we put $w = z^2$, we get

$$f(z) = g\left(z^2\right) = \left(z^2\right)^7 g_1\left(z^2\right) = z^{14} g_1\left(z^2\right), \qquad g_1(0) \neq 0,$$

and it follows immediately that f has a zero of order 14 at $z_0 = 0$.

Fourth method. *Method of differentiation, second variant.* If we do not use any of the shortcuts above, we have to go through the following computations

$$f(z) = 3\sinh z^2 - 3\sin z^2 - z^6, \qquad f(0) = 0,$$

$$f'(z) = 6z\cosh z^2 - 6z\cos z^2 - 6z^5, \qquad f'(0) = 0,$$

$$\begin{aligned} f''(z) &= 6\cosh z^2 + 12z^2\sinh z^2 - 6\cos z^2 \\ &\quad +12z^2\sin z^2 - 30z^4, \end{aligned} \qquad f''(0) = 0,$$

$$\begin{aligned} f^{(3)}(z) &= 36z\sinh z^2 + 24z^3\cosh z^2 + 36z\sin z^2 \\ &\quad +24z^3\cos z^2 - 120z^3, \end{aligned} \qquad f^{(3)}(0) = 0,$$

$$\begin{aligned} f^{(4)}(z) &= 36\sinh z^2 + 144z^2\cosh z^2 + 48z^4\sinh z^2 \\ &\quad +36\sin z^2 + 144z^2\cos z^2 - 48z^4\sin z^2 - 360z^2, \end{aligned} \qquad f^{(4)}(0) = 0,$$

$$\begin{aligned} f^{(5)}(z) &= 360z\cosh z^2 + 480z^3\sinh z^2 + 96z^5\cosh z^2 \\ &\quad +360z\cos z^2 - 480z^3\sin z^2 - 96z^5\cos z^2 - 720z, \end{aligned} \qquad f^{(5)}(0) = 0,$$

$$\begin{aligned} f^{(6)}(z) &= 360\cosh z^2 + 2160z^2\sinh z^2 + 1440z^4\cosh z^2 \\ &\quad +192z^6\sinh z^2 + 360\cos z^2 - 2160z^2\sin z^2 \\ &\quad -1440z^4\cos z^2 + 192z^6\sin z^2 - 720, \end{aligned} \qquad f^{(6)}(0) = 0,$$

$$\begin{aligned} f^{(7)}(z) &= 5040z\sinh z^2 + 10\,080z^3\cosh z^2 + 4032z^5\sinh z^2 \\ &\quad +384z^7\cosh z^2 - 5040z\sin z^2 - 10\,080z^3\cos z^2 \\ &\quad +4032z^5\sin z^2 + 384z^7\cos z^2, \end{aligned} \qquad f^{(7)}(0) = 0,$$

$$\begin{aligned} f^{(8)}(z) &= 5040\sinh z^2 + 40\,320z^2\cosh z^2 + 40\,320z^4\sinh z^2 \\ &\quad +10\,752z^6\cosh z^2 + 768z^8\sinh z^2 - 5040\sin z^2 \\ &\quad -40\,320z^2\cos z^2 + 40\,320z^4\sin z^2 \\ &\quad +10\,752z^6\cos z^2 - 768z^8\sinh z^2, \end{aligned} \qquad f^{(8)}(0) = 0,$$

$$\begin{aligned} f^{(9)}(z) &= 90\,720z\cosh z^2 + 241\,929z^3\sinh z^2 + 145\,152z^5\cosh z^2 \\ &\quad +27\,648z^7\sinh z^2 + 1536z^9\cosh z^2 - 90\,720z\cos z^2 \\ &\quad +241\,920z^3\sin z^2 + 145\,152z^5\cos z^2 \\ &\quad -27\,648z^7\sin z^2 - 1536z^9\cos z^2, \end{aligned} \qquad f^{(9)}(0) = 0,$$

$$\begin{aligned} f^{(10)}(z) &= 90\,720\cosh z^2 + 907\,200z^2\sinh z^2 \\ &\quad +1\,209\,600z^4\cosh z^2 + 483\,840z^6\sinh z^2 \\ &\quad +69\,120z^8\cosh z^2 + 3072z^{10}\sinh z^2 \\ &\quad -90\,720\cos z^2 + 907\,200z^2\sin z^2 \\ &\quad +1\,209\,600z^4\cos z^2 - 483\,840z^6\sin z^2 \\ &\quad -69\,120z^8\cos z^2 + 3072z^{10}\sin z^2, \end{aligned} \qquad f^{(10)}(0) = 0,$$

$$\begin{aligned}
f^{(11)}(z) = {} & 1\,995\,840z\sinh z^2 + 6\,652\,800z^3\cosh z^2 \\
& +5\,322\,240z^5\sinh z^2 + 1\,520\,640z^7\cosh z^2 \\
& +168\,960z^9\sinh z^2 + 6144z^{11}\cosh z^2 \\
& +1\,995\,840z\sin z^2 + 6\,652\,800z^3\cos z^2 \\
& -5\,322\,240z^5\sin z^2 - 1\,520\,640z^7\cos z^2 \\
& +168\,960z^9\sin z^2 + 6144z^{11}\cos z^2,
\end{aligned}$$

$$f^{(11)}(0) = 0,$$

$$\begin{aligned}
f^{(12)}(z) = {} & 1\,995\,840\sinh z^2 + 23\,950\,080z^2\cosh z^2 \\
& +39\,916\,800z^4\sinh z^2 + 21\,288\,960z^6\cosh z^2 \\
& +4\,561\,920z^8\sinh z^2 + 405\,504z^{10}\cosh z^2 \\
& +12\,288z^{12}\sinh z^2 + 1\,995\,840\sin z^2 \\
& +23\,950\,080z^2\cos z^2 - 39\,916\,800z^4\sin z^2 \\
& -21\,288\,960z^6\cos z^2 + 4\,561\,920z^8\sin z^2 \\
& +405\,504z^{10}\cos z^2 - 12\,288z^{12}\sin z^2,
\end{aligned}$$

$$f^{(12)}(0) = 0,$$

$$\begin{aligned}
f^{(13)}(z) = {} & 51\,891\,840z\cosh z^2 + 207\,567\,360z^3\sinh z^2 \\
& +207\,567\,360z^5\cosh z^2 + 79\,073\,280z^7\sinh z^2 \\
& +13\,178\,880z^9\cosh z^2 + 958\,464z^{11}\sinh z^2 \\
& +24\,576z^{13}\cosh z^2 + 51\,891\,840z\cos z^2 \\
& -207\,567\,360z^3\sin z^2 - 207\,567\,360z^5\cos z^2 \\
& +79\,073\,280z^7\sin z^2 + 13\,178\,880z^9\cos z^2 \\
& -958\,464z^{11}\sin z^2 - 24\,576z^{13}\cos z^3,
\end{aligned}$$

$$f^{(13)}(0) = 0,$$

$$\begin{aligned}
f^{(14)}(z) = {} & 51\,891\,840\cosh z^2 + 726\,485\,760z^2\sinh z^2 \\
& +1\,452\,971\,520z^4\cosh z^2 + 968\,647\,680z^6\sinh z^2 \\
& +176\,756\,480z^8\cosh z^2 + 36\,900\,864z^{10}\sinh z^2 \\
& +2\,236\,416z^{12}\cosh z^2 + 49\,152z^{14}\sin z^2 \\
& +51\,891\,840\cos z^2 - 726\,485\,760z^2\sin z^2 \\
& -1\,452\,971\,520z^4\cos z^2 + 968\,647\,680z^6\sin z^2 \\
& +276\,756\,480z^8\cos z^2 - 36\,900\,864z^{10}\sin z^2 \\
& -2\,236\,416z^{12}\cos z^2 + 49\,152z^{14}\sin z^2.
\end{aligned}$$

Since

$$f^{(14)}(0) = 103\,783\,680 = \frac{6}{7!}\cdot 14! \neq 0$$

is the first derivative of $f(z)$ at $z = 0$, which is different from 0, we conclude that the order is 14.

Remark 8.2 Note that

$$a_{14} = \frac{f^{(14)}(0)}{14!} = \frac{6}{7!},$$

which is in agreement with the result from the first method. $\Diamond$

Example 8.5 *1) Explain why the function*

$$f(z) = \operatorname{Log}\left(1+z^2\right) - \sin^2 z$$

is analytic in the point set

$$\mathbb{C} \setminus \{z \in \mathbb{C} \mid \operatorname{Re}(z) = 0 \wedge |\operatorname{Im}(z)| \geq 1\}.$$

2) Find the order of the zero at $z = 0$ of f.

3) Denote by

$$\sum_{n=0}^{+\infty} a_n z^n$$

the Taylor series of f. Find the radius of convergence of the series.
(One shall not give an explicit expression of the general term.)

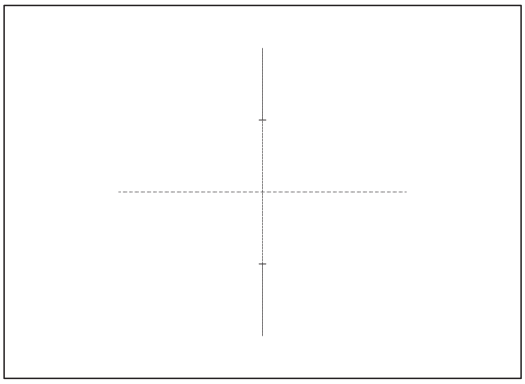


Figure 10: The domain with the branch cuts from $\pm i$.

1) The principal logarithm is analytic in the plane with the branch cuts. Hence, the function $1+z^2$ must not be real negative or zero. The exception set is then

$$1+z^2 = -t, \qquad t \geq 0,$$

i.e.

$$z = \pm i\sqrt{1+t}, \qquad t \geq 0,$$

thus

$$\operatorname{Re}(z) = 0 \qquad \text{and} \qquad |\operatorname{Im}(z)| \geq 1.$$

We have proved that $\operatorname{Log}\left(1+z^2\right)$ is analytic in the given point set. Since Da sine is analytic in the complex plane, the claim follows.

2) Then we have a couple of solution variants.

a) We get from known Taylor series,

$$\begin{aligned} f(z) &= \operatorname{Log}\left(1+z^2\right)-\frac{1}{2}(1-\cos 2z) \\ &= \left\{z^2-\frac{1}{2}z^4+\cdots\right\}-\frac{1}{2}\left\{\frac{1}{2!}(2z)^2-\frac{1}{4!}(2z)^4+\cdots\right\} \\ &= \left\{z^2-\frac{1}{2}z^4+\cdots\right\}-\left\{z^2-\frac{1}{3}z^4+\cdots\right\} \\ &= -\frac{1}{6}z^4+\cdots . \end{aligned}$$

We conclude that the zero at $z=0$ has order 4.

b) *The differentiation method.* Here we get:

$$\begin{aligned}
f'(z) &= \frac{2z}{1+z^2} - 2\,\sin z\,\cos z \\
&= \frac{1}{z+i} + \frac{1}{z-i} - \sin 2z, && f'(0), \\
f''(z) &= -\frac{1}{(z+i)^2} - \frac{1}{(z-i)^2} - 2\,\cos 2z, && f''(0) = 1+1-2 = 0, \\
f^{(3)}(x) &= \frac{2}{(z+i)^3} + \frac{2}{(z-i)^3} + 4\,\sin 2z, && f^{(3)}(0) = 2i - 2i + 0 = 0, \\
f^{(4)}(z) &= \frac{-6}{(z+i)^4} + \frac{-6}{(z-i)^4} + 8\,\cos 2z, && f^{(4)} = -6-6+8 = -4.
\end{aligned}$$

It follows that the order of the zero is 4 and that the first term is

$$\frac{-4}{4!}\,z^4 = -\frac{1}{6}\,z^4.$$

c) If one does not start with a decomposition, the differentiations become more difficult:

$$\begin{aligned}
f'(z) &= \frac{2z}{1+z^2} - \sin 2z, \\
f''(z) &= \frac{2\left(z^2+1\right) - 2z\cdot 2z}{\left(z^2+1\right)^2} - 2\cos 2z = \frac{2\left(1-z^2\right)}{\left(1+z^2\right)^2} - 2\,\cos 2z, \\
f^{(3)}(z) &= \frac{4\left(z^3+3zi^2\right)}{\left(z^2+1\right)^3} + 4\sin 2z = \frac{4z\left(z^2-3\right)}{\left(1+z^2\right)^3} + 4\sin 2z, \\
f^{(4)}(z) &= \frac{-12\left(z^4+6z^2i^2+i^4\right)}{\left(z^2+1\right)^4} + 8\cos 2z = \frac{-12\left(z^4-6z^2+1\right)}{\left(z^2+1\right)^4} + 8\cos 2z,
\end{aligned}$$

followed by putting $z = 0$.

3) If we write

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n,$$

then the domain of convergence is the largest open disc of centrum 0, in which f is analytic. It follows from the figure that the radius of convergence is 1.

Remark 8.3 Even though it is not requested, it is not difficult to find a_n,

$$f(z) = \operatorname{Log}\left(1+z^2\right) + \frac{1}{2}\{\cos(2z) - 1\} = \sum_{n=1}^{+\infty} \frac{1}{n}\,(-1)^{n+1} z^{2n} + \frac{1}{2}\sum_{n=1}^{+\infty} \frac{1}{(2n)!}\,(-1)^n z^{2n}.$$

In particular, $a_{2n+1} = 0$ (odd indices), and

$$a_{2n} = (-1)^{n+1}\left\{\frac{1}{n} - \frac{1}{2\cdot(2n)!}\right\},$$

thus

$$|a_{2n}| \le \frac{1}{n} + \frac{1}{2\cdot(2n)!} = \frac{1}{n} + \frac{1}{2\cdot 2n(2n-1)!} = \frac{1}{n}\left\{1 + \frac{1}{4(2n-1)!}\right\}, \qquad n \in \mathbb{N}.$$

It follows that

$$\sqrt[2n]{|a_{2n}|} = \frac{1}{\sqrt[2n]{n}} \sqrt[2n]{1 + \frac{1}{4(2n-1)!}}.$$

Since $a_{2n+1} = 0$ and

$$\sqrt[2n]{n} = \exp\left(\frac{\ln n}{2n}\right) \to \exp 0 = 1 \qquad \text{for } n \to +\infty,$$

and

$$\sqrt[2n]{1 + \frac{1}{4(2n-1)!}} \to 1 \qquad \text{for } n \to +\infty,$$

we get $\limsup_{n\to+\infty} \sqrt[n]{|a_n|} = 1$, hence the radius of convergence is 1. $\Diamond$

Example 8.6 *Find the order of the zero at $z = 0$ of the function*

$$f(z) = 3\,\sinh z - 3\sin z + \exp\left(z^3\right) - 1.$$

Using known Taylor expansions,

$$\begin{aligned} 3\sinh z &= 3z + \frac{1}{2}\,z^3 + \cdots\,, \\ -3\sin z &= -3z + \frac{1}{2}\,z^3 + \cdots\,, \\ \exp\left(z^3\right) - 1 &= z^3 + \cdots\,, \end{aligned}$$

so

$$f(z) = 2z^3 + \cdots\,,$$

proving that the zero $z = 0$ of $f(z)$ has order 3.

9 Fourier series

Example 9.1 *Put $z = r\,e^{i\,\theta}$ into the exponential series and then derive some new Fourier series.*

It follows from

$$e^z = e^x\,\cos y + i\cdot e^x\,\sin t, \qquad \text{and} \qquad x = r\cdot\cos\theta, \quad y = r\cdot\sin\theta,$$

that

$$\begin{aligned} e^z &= e^x(\cos y + i\,\sin y) = e^{r\,\cos\theta}\cos(r\,\sin\theta) + i\,e^{r\,\cos\theta}\sin(r\,\sin\theta) \\ &= \sum_{n=0}^{+\infty}\frac{1}{n!}\,z^n = \sum_{n=0}^{+\infty}\frac{1}{n!}\,r^n\,e^{in\theta} = \sum_{n=0}^{+\infty}\frac{r^n}{n!}\,\cos n\theta + i\sum_{n=0}^{+\infty}\frac{r^n}{n!}\,\sin n\theta. \end{aligned}$$

By separating into the real and the imaginary parts we see that for every $r \geq 0$ and every $\theta \in \mathbb{R}$,

$$e^{r\,\cos\theta}\cos(r\,\sin\theta) = \sum_{n=0}^{+\infty}\frac{r^n}{n!}\,\cos n\theta, \qquad \text{og} \qquad e^{r\,\sin\theta}\cos(r\,\sin\theta) = \sum_{n=0}^{+\infty}\frac{r^n}{n!}\,\sin n\theta.$$

Example 9.2 *Put $z = e^{i\theta}$. Prove for $m,\, n \in \mathbb{N}_0$ that*

$$\frac{1}{2\pi}\int_0^{2\pi} z^m\overline{z}^n\,d\theta = \begin{cases} 0 & \text{for } m \neq n, \\ 1 & \text{for } m = n. \end{cases}$$

One says that the functions 1, z, z^2, z^3, ..., form an orthogonal system on the unit circle.

The example is trivial since we get by insertion

$$\frac{1}{2\pi}\int_0^{2\pi} z^m\overline{z}^n\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{im\theta}e^{-in\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta}\,d\theta.$$

If $m \neq n$, then

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta}\,d\theta = 0,$$

and if $m = n$, then

$$\frac{1}{2\pi}\int_0^{2\pi} 1\,d\theta = 1.$$

Example 9.3 *Apply the power series expansion of* $\operatorname{Log}\left(\dfrac{1}{1-z}\right)$ *in order to get the Fourier series of*

$$(a)\quad \ln\left(1+r^2-2r\,\cos\theta\right), \qquad (b)\quad \operatorname{Arctan}\left(\frac{r\,\sin\theta}{1-r\,\cos\theta}\right), \qquad r\in[0,1[.$$

Put $z = r\,e^{i\theta}$, where $0 \le r < 1$. Then

$$\operatorname{Log}\left(\frac{1}{1-z}\right) = \sum_{n=1}^{+\infty}\frac{z^n}{n} = \sum_{n=1}^{+\infty}\frac{r^n}{n}\cos n\theta + i\sum_{n=1}^{+\infty}\frac{r^n}{n}\sin n\theta.$$

On the other hand,

$$\frac{1}{1-z} = \frac{1}{1-r\,\cos\theta - i\,r\,\sin\theta} = \frac{1-r\,\cos\theta+i\,r\,\sin\theta}{(1-r\,\cos\theta)^2+r^2\sin^2\theta} = \frac{1-r\,\cos\theta+i\,r\,\sin\theta}{1+r^2-2r\,\cos\theta},$$

thus

$$\operatorname{Log}\left(\frac{1}{1-z}\right) = -\frac{1}{2}\ln\left(1+r^2-2r\,\cos\theta\right) + i\operatorname{Arctan}\left(\frac{r\,\sin\theta}{1-r\,\cos\theta}\right),$$

because $1-z$, and hence also $\dfrac{1}{1-z}$, lies in the right half plane. When we identify the real and the imaginary parts, we obtain the Fourier series

(a)

$$\ln\left(1+r^2-2r\,\cos\theta\right) = -2\sum_{n=1}^{+\infty}\frac{r^n}{n}\,\cos n\theta, \qquad 0 \le r < 1,$$

(b)

$$\operatorname{Arctan}\left(\frac{r\,\sin\theta}{1-r\,\cos\theta}\right) = \sum_{n=1}^{+\infty}\frac{r^2}{n}\,\sin n\theta, \qquad 0 \le r < 1.$$

Remark 9.1 If instead $r > 1$, then $R = \dfrac{1}{r} < 1$, and we get

(a')

$$\begin{aligned}\ln\left(1+r^2-2r\,\cos\theta\right) &= \ln\left(r^2\left\{1+R^2-2R\,\cos\theta\right\}\right) = 2\,\ln r + \ln\left(1+R^2-2R\,\cos\theta\right)\\ &= 2\,\ln r - 2\sum_{n=1}^{+\infty}\frac{R^2}{n}\,\cos n\theta = 2\,\ln r - 2\sum_{n=1}^{+\infty}\frac{1}{n\,r^n}\,\cos n\theta, \qquad r > 1.\end{aligned}$$

We cannot find a similar result in **(b)**, because the denominator in

$$\frac{r\,\sin\theta}{1-r\,\cos\theta}$$

is zero, when $\cos\theta = \dfrac{1}{r} = R \in \,]0,1[$. $\lozenge$

Example 9.4 *Assume without proof that*

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

and that

(21) $\operatorname{Log}\left(1-e^{i\theta}\right) = \ln\left(2\,\sin\dfrac{\theta}{2}\right) - \dfrac{i}{2}\,(\pi-\theta), \qquad \theta \in \,]0,\pi].$

(a) *Let $r \in \,]0,1[$ and $\theta \in \mathbb{R}$. Find the coefficients $a_n(r)$ of the Fourier series expansion*

$$\operatorname{Log}\left(1-r\,e^{i\theta}\right) = \sum_n a_n(r)\,e^{i\,n\,\theta}.$$

(b) *Compute the integral*

$$\int_{-\pi}^{\pi} \left|\operatorname{Log}\left(1-r\,e^{i\,\theta}\right)\right|^2 d\theta, \qquad r \in \,]0,1[,$$

expressed by $a_n(r)$.
We assume without proof that

$$\int_{-\pi}^{\pi} \left|\operatorname{Log}\left(1-e^{i\,\theta}\right)\right|^2 d\theta = \lim_{r\to 1-} \int_{-\pi}^{\pi} \left|\operatorname{Log}\left(1-r\,e^{i\,\theta}\right)\right|^2 d\theta.$$

Find the value of

$$\int_{-\pi}^{\pi} \left|\operatorname{Log}\left(2-e^{i\,\theta}\right)\right|^2 d\theta.$$

(c) *Finally, prove by using (21) and* **(b)** *that*

$$\int_0^{\frac{\pi}{2}} \{\ln(2\sin t)\}^2\,dt = \frac{\pi^3}{24}.$$

(a) Since $\left|-r\,e^{i\theta}\right| < 1$, we get by insertion of $z = -r\,e^{i\,\theta}$ into the logarithmic series that

$$\operatorname{Log}\left(1-r\,e^{i\,\theta}\right) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\,(-r)^n e^{i\,n\,\theta} = -\sum_{n=1}^{+\infty} \frac{r^n}{n}\,e^{i\,n\,\theta},$$

and we conclude that

$$a_n(r) = -\frac{1}{n}\,r^n \quad \text{for } n \in \mathbb{N}; \qquad a_n(r) = 0 \quad \text{ellers.}$$

(b) Then by Parseval's formula,

$$\int_{-\pi}^{\pi} \left|\text{Log}\left(1 - r\, e^{i\,\theta}\right)\right|^2 \, d\theta = 2\pi \sum_{n=1}^{+\infty} \frac{1}{n^2}\, r^{2n}, \qquad r \in]0,1[.$$

By using that the limit process $r \to 1-$ will give the correct result, we get

$$\begin{aligned}\int_{-\pi}^{\pi} \left|\text{Log}\left(1 - e^{i\,\theta}\right)\right|^2 \, d\theta &= \lim_{r\to 1-} \int_{-\pi}^{\pi} \left|\text{Log}\left(1 - r\, e^{i\,\theta}\right)\right|^2 \, d\theta \\ &= \lim_{r\to 1-} 2\pi \sum_{n=1}^{+\infty} \frac{1}{n^2}\, r^{2n} = 2\pi \sum_{n=1}^{+\infty} \frac{1}{n^2} = 2\pi \cdot \frac{\pi^2}{6} = \frac{\pi^3}{3}.\end{aligned}$$

(c) Finally, it follows from **(b)** and (21),

$$\begin{aligned}\frac{\pi^3}{3} &= \int_{-\pi}^{\pi} \left|\text{Log}\left(1 - e^{i\,\theta}\right)\right|^2 \, d\theta = 2\int_0^{\pi} \left|\text{Log}\left(1 - e^{i\,\theta}\right)\right|^2 \, d\theta \\ &= 2\int_0^{\pi} \left\{ \left(\ln\left(2\,\sin\frac{\theta}{2}\right)\right)^2 + \left(\frac{1}{2}\,(\pi-\theta)\right)^2 \right\} d\theta \\ &= 2\int_0^{\pi} \left\{ \ln\left(2\,\sin\frac{\theta}{2}\right)\right\}^2 d\theta + \frac{1}{2}\int_0^{\pi} (\pi-\theta)^2\, d\theta \\ &= 4\int_0^{\frac{\pi}{2}} \{\ln(2\sin t)\}^2\, dt + \frac{1}{2}\int_0^{\pi} t^2\, dt \\ &= 4\int_0^{\frac{\pi}{2}} \{\ln(2\sin t)\}^2 dt + \frac{\pi^3}{6},\end{aligned}$$

hence by a rearrangement,

$$\int_0^{\frac{\pi}{2}} \{\ln(2\sin t)\}^2\, dt = \frac{1}{4}\left\{\frac{\pi^3}{3} - \frac{\pi^3}{6}\right\} = \frac{\pi^3}{24}.$$

10 The maximum principle

Example 10.1 *Given $f(z) = (z+1)^2$. Find the maximum and the minimum of $|f/z)|$ in the set A, where A is the closed triangle of the corners $z=0$, $z=2$ and $z=i$.*

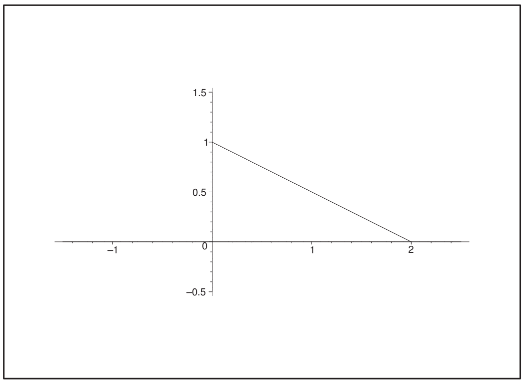


Figure 11: The triangle A.

This example was originally constructed in order to illustrate the maximum principle. However, it is easily seen that a geometric argument is much easier to apply, because $|f(z)|$ indicates the square of the distance from -1 to z.
Clearly, the minimum is obtained at $z=0$, corresponding to $|f(0)| = 1$, and the maximum is obtained at $z=2$, corresponding to $|f(z)| = 9$.

Example 10.2 *Find the maximum of $|\sin z|$ on the set $[0, 2\pi] \times [0.2\pi]$.*

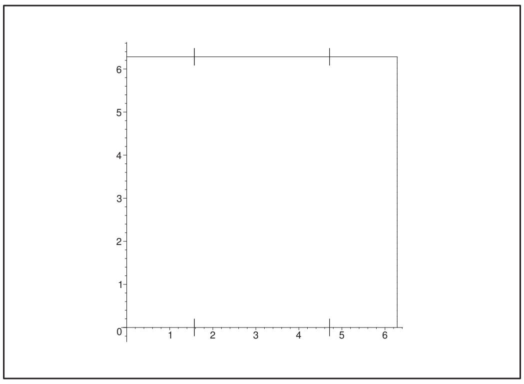


Figure 12: The domain Ω.

It follows from the maximum principle that the maximum is attained at the boundary of the domain. We find

1) On the line $z = x + i \cdot 0$, $x \in [0, 2\pi]$, we get

$$\max_{x \in [0,2\pi]} |\sin x| = \sin \frac{\pi}{2} = \left| \sin \frac{3\pi}{2} \right| = 1.$$

2) On the line $z = i\, y$, $y \in [0, 2\pi]$, we get

$$\max_{y \in [0,2\pi]} |\sin(i\, y)| = \max_{y \in [0,2\pi]} \sinh y = \sinh(2\pi).$$

3) On the line $z = x + 2i\pi$, $x \in [0, 2\pi]$, we get

$$\begin{aligned}
\max_{x \in [0,2\pi]} |\sin(x + 2i\pi)| &= \max_{x \in [0,2\pi]} |\sin x \cdot \cosh 2\pi + i \cdot \cos x \cdot \sinh 2\pi| \\
&= \max_{x \in [0,2\pi]} \sqrt{\sin^2 x \cdot \cosh^2 2\pi + \cos^2 \cdot \sinh^2 2\pi} = \max_{x \in [0,2\pi]} \sqrt{\cosh^2 2\pi - \cos^2 x} \\
&= \sqrt{\cosh^2 2\pi} = \cosh 2\pi.
\end{aligned}$$

4) On the line $z = 2\pi + i\,y$, $y \in [0, 2\pi]$, we get

$$\max_{y\in[0,2\pi]} |\sin(2\pi + i\,y)| = \max_{y\in[0,2\pi]} |\sin(i\,y)| = \max_{y\in[0,2\pi]} \sinh y = \sinh 2\pi.$$

By comparing these four results it follows that

$$\max_{\Omega} |\sin z| = \cosh 2\pi,$$

so the maximum is obtained for

$$z = \frac{\pi}{2} + 2i\pi \qquad \text{and} \qquad z = \frac{3\pi}{2} + 2i\,\pi.$$

Example 10.3 *Find the maximum of* $\left|\exp\left(z^2\right)\right|$ *on* $\{z \in \mathbb{C} \mid |z| \le 1\}$.

It follows from the maximum principle that the maximum is attained on the boundary $|z| = 1$, where we put $z = e^{i\,\theta}$, so

$$\left|\exp\left(z^2\right)\right| = \left|\exp\left(2^{2i\,\theta}\right)\right| = \exp(\cos 2\theta), \qquad \theta \in [0, 2\pi].$$

Obviously, we obtain the maximum when $\cos 2\theta = 1$, hence the maximum is $e^1 = e$.

Example 10.4 *Prove that the transformation*

$$T(z) = \frac{R\,(z - z_0)}{R^2 - \overline{z}_0 z}, \qquad |z_0| < R,$$

maps the open disc of radius R and centrum 0 into the unit disc with $T\,(z_0) = 0$.
Hint: *Apply the maximum principle, and prove that $|z| = R$ implies that $|T(z)| = 1$.*

Clearly, $T\,(z_0) = 0$. If $|z| = R$, then

$$|T(z)| = R\left|\frac{z - z_0}{R^2 - \overline{z}_0 z}\right| = R\left|\frac{z - z_0}{z\,\overline{z} - \overline{z}_0 z}\right| = \frac{R}{|z|}\cdot\left|\frac{z - z_0}{\overline{z} - \overline{z}_0}\right| = \frac{R}{R}\cdot 1 = 1.$$

Then it follows from the maximum principle that $|T(z)| \le 1$ for $|z| < R$, and since $T\,(z_0) = 0$, we cannot have equality. so $|T(z)| < 1$ for $|z| < R$.