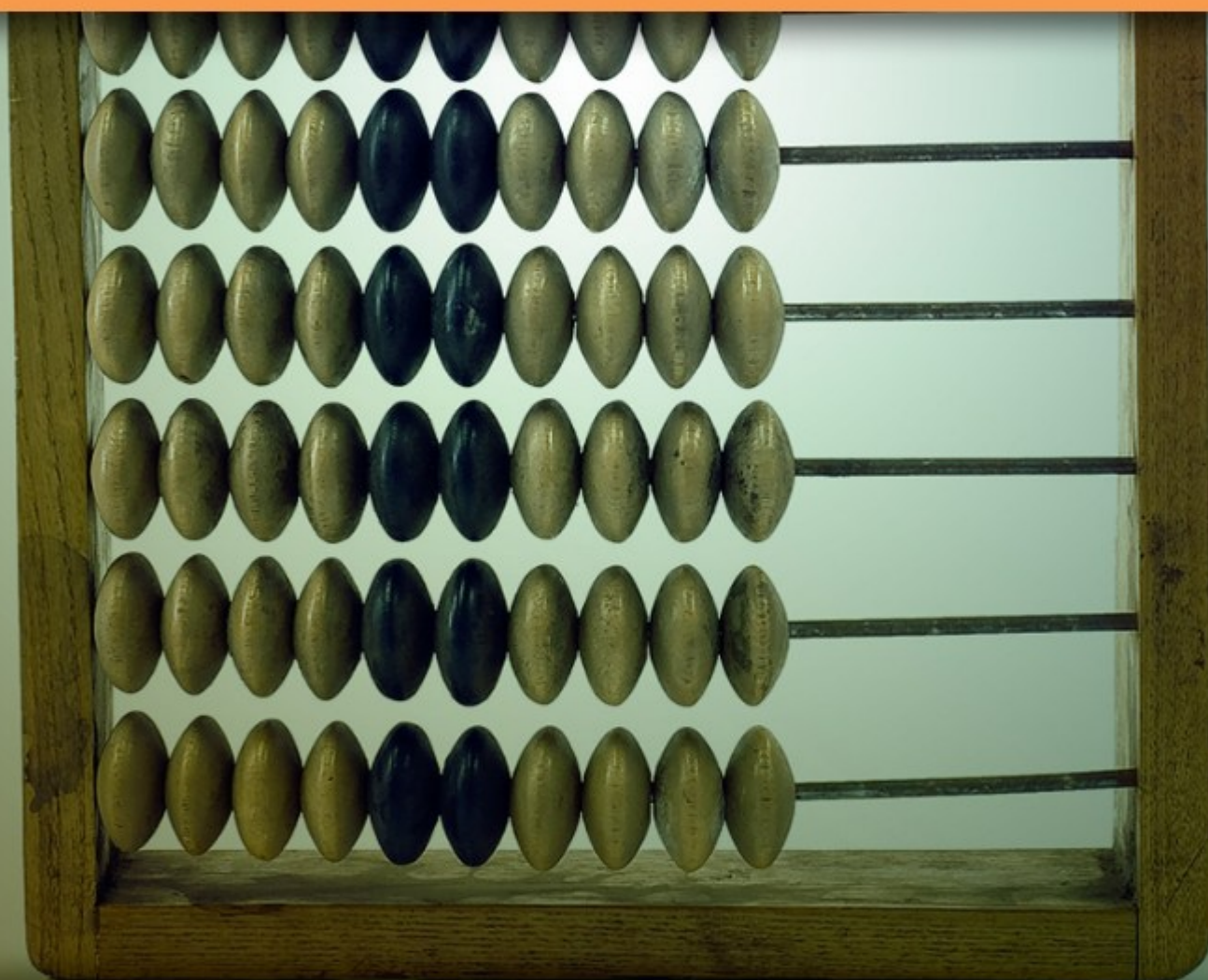


Real Functions of Several Variables - Max. and Min

Leif Mejlbro



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Real Functions of Several Variables

Examples of Maximum and Minimum
Calculus 2c-4

Real Functions of Several Variables - Examples of Maximum and Minimum

– Calculus 2c-4

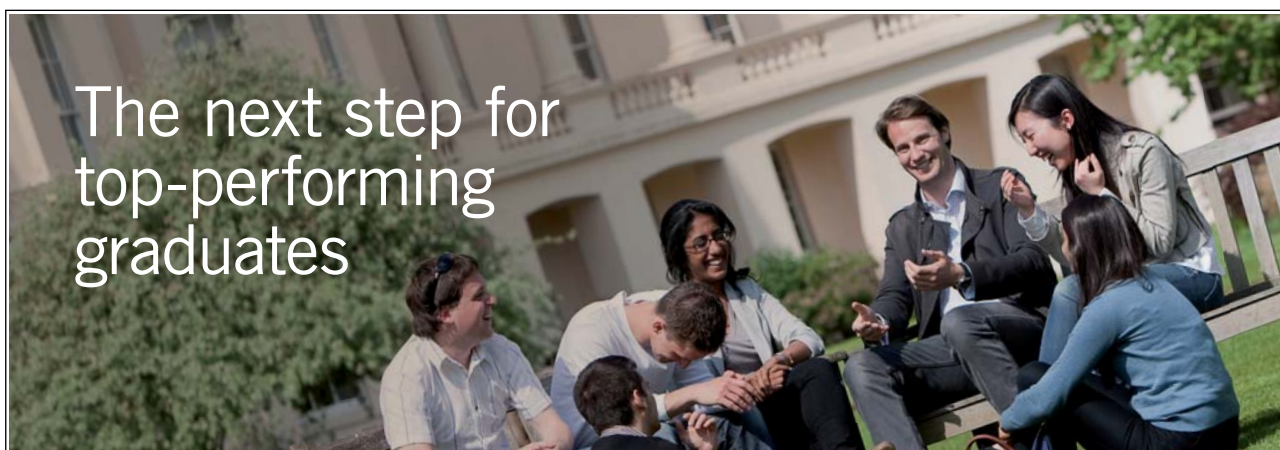
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Preface

In this volume I present some examples of *maximum and minimum*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I. Implementation**. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C Control**, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of \wedge I shall either write "and", or a comma, and instead of \vee I shall write "or". The arrows \Rightarrow and \Leftrightarrow are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
17th October 2007

1 Maximum and Minimum

Example 1.1 Explain in each of the following cases why the indicated function has a maximum and a minimum, and find these values.

- 1) $f(x, y) = 8\sqrt{x^2 + 3y^2} - 5x - y^2$ for $x^2 + 3y^2 \leq 4$.
- 2) $f(x, y) = x^2 - 3y^2 - 3xy$ for $x^2 + y^2 \leq 4$.
- 3) $f(x, y) = xy + y^2 - 5y - 3\ln x$ for $x \geq 1$ and $0 \leq y \leq 5 - x$.
- 4) $f(x, y) = (x^2 + y^2 - 2y)(x^2 + y^2 - 6y)$ for $x^2 + y^2 \leq 36$.
- 5) $f(x, y) = xy + \frac{64}{x} + \frac{64}{y}$ for $x \geq 1$, $y \geq 1$ and $xy \leq 32$.
- 6) $f(x, y) = 3x^2 + 3y^2 - 2xy - 2x^2y^2$ for $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 4$.
- 7) $f(x, y) = e^{-2y} + e^{-y} \sin x$ for $(x, y) \in [0, 2\pi] \times [0, 1]$.
- 8) $f(x, y) = x^4 + y^4 - x^2 + 2xy - y^2$ for $x^2 + y^2 \leq 4$.
- 9) $f(x, y) = 8xy^2 - xy^3 - x^3y$ for $(x, y) \in [0, 4] \times [0, 8]$.

- A** Maximum and minimum for continuous functions on closed and bounded (i.e. compact) sets.
- D** Apply the second main theorem. Sketch the domain. Apply that the maximum and the minimum are either attained at an exception point or in a stationary point of on the boundary.
- I** All functions are continuous on a closed and bounded set, so by the second main theorem for continuous functions the function has both a maximum and a minimum on the set.

With 1) as the only exception, all the rest of the functions are of class C^∞ in the interior of their respective domains. We shall therefore in all these cases only find the stationary points and examine the boundary.

In 1) the function is continuous everywhere, and not differentiable at $(0, 0)$, so this is an exceptional point.

- 1) The domain is an ellipsoidal disc given by

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\frac{2}{\sqrt{3}}}\right)^2 \leq 1$$

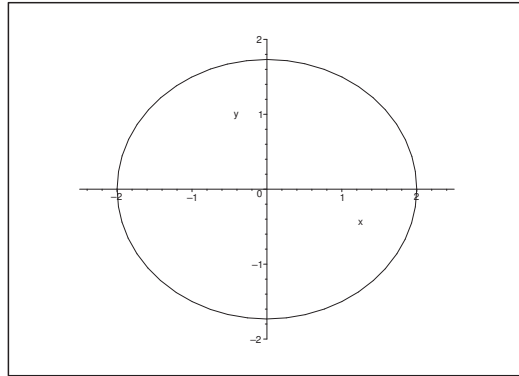
of centrum $(0, 0)$ and half axes 2 and $\frac{2}{\sqrt{3}}$.

- a) **THE EXCEPTION POINT.** The function is of class C^∞ everywhere inside the ellipse, except for $(0, 0)$, which is an *exception point*. The value of the function is here

$$f(0, 0) = 0.$$

- b) **STATIONARY POINTS.** In the domain given by $0 < x^2 + 3y^2 < 4$, the equations of the stationary points are

$$\frac{\partial f}{\partial x} = \frac{8x}{\sqrt{x^2 + 3y^2}} - 5 = 0, \quad \frac{\partial f}{\partial y} = \frac{24y}{\sqrt{x^2 + 3y^2}} - 2y = 0.$$



The latter equation gives the following two possibilities

$$y = 0 \quad \text{or} \quad \sqrt{x^2 + 3y^2} = 12.$$

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The latter possibility is rejected, because $\sqrt{x^2 + 3y^2} \leq 2$ in the given domain. We conclude that we necessarily have $y = 0$, whenever we have a stationary point, if any. When $y = 0$ is put into the former equation of the stationary points we get

$$8 \frac{x}{|x|} = 5,$$

which is never fulfilled. We conclude that there is no stationary point in the domain.

c) EXAMINATION OF THE BOUNDARY. On the boundary, $x^2 + 3y^2 = 4$, hence

$$y^2 = \frac{1}{3}(4 - x^2), \quad x \in [-2, 2].$$

When $x \in [-2, 2]$ is used as a parameter, we get along the boundary

$$\begin{aligned} f\left(x, \pm\sqrt{\frac{4-x^2}{3}}\right) &= 8 \cdot 2 - 5x - \frac{1}{3}(4 - x^2) = \frac{1}{3}(x^2 - 15x + 44) \\ &= \frac{1}{3} \left\{ \left(x - \frac{15}{2}\right)^2 - \frac{49}{4} \right\} = \frac{1}{3}(x - y)(x - 11). \end{aligned}$$

The maximum in the interval $[-2, 2]$ is obtained for $x = -2$, corresponding to

$$f(-2, 0) = 16 + 10 = 26,$$

and the minimum for $x = 2$, corresponding to

$$f(2, 0) = 16 - 10 = 6.$$

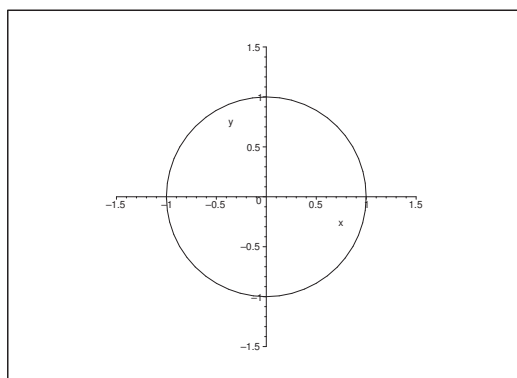
d) NUMERICAL COMPARISON. Summarizing the maximum and the minimum are included in the values

$$f(-2, 0) = 26, \quad f(0, 0) = 0, \quad f(x, 0) = 6.$$

It follows that

$$S = f(-2, 0) = 26 \quad (\text{maximum}) \quad \text{and} \quad M = f(0, 0) = 0 \quad (\text{minimum}).$$

2) The domain is here the unit disc.



- a) EXCEPTION POINTS. The function is everywhere of class C^∞ , so there are no exception points.
- b) STATIONARY POINTS. The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 2x - 3y = 0, \quad \frac{\partial f}{\partial y} = -6y - 3x = 0,$$

and it follows that $(0,0)$ is the only stationary point in the domain. The value of the function is here

$$f(0,0) = 0.$$

- c) EXAMINATION OF THE BOUNDARY. We shall use the following parametric description of the boundary,

$$x = \cos \varphi, \quad y = \sin \varphi, \quad \varphi \in [0, 2\pi[,$$

hence by insertion,

$$\begin{aligned} g(\varphi) &= f(x, y) = x^2 - 3y^2 - 3xy = \cos^2 \varphi - 3 \sin^2 \varphi - 3 \cos \varphi \cdot \sin \varphi \\ &= \frac{1}{2} \{1 + \cos 2\varphi - 3(1 - \cos 2\varphi) - 3 \sin 2\varphi\} = \frac{1}{2} \{4 \cos 2\varphi - 3 \sin 2\varphi - 2\} \\ &= \frac{5}{2} \left\{ \frac{4}{5} \cos 2\varphi - \frac{3}{5} \sin 2\varphi \right\} - 1 = \frac{5}{2} \cos(2\varphi + \varphi_0) - 1, \end{aligned}$$

where

$$\cos \varphi_0 = \frac{4}{5} \quad \text{and} \quad \sin \varphi_0 = \frac{3}{5}.$$

Now $\cos(2\varphi + \varphi_0)$ goes twice through the interval $[-1, 1]$, when φ goes through $[0, 2\pi[$, and we find the maximum

$$\frac{5}{2} - 1 = \frac{3}{2},$$

and the minimum

$$-\frac{5}{2} - 1 = -\frac{7}{2}$$

on the boundary.

- d) NUMERICAL COMPARISON. By comparison we see that the value $f(0,0) = 0$ from the stationary point lies between these two values on the boundary. We therefore conclude that

$$S = \frac{3}{2} \quad \text{og} M = -\frac{7}{2}.$$

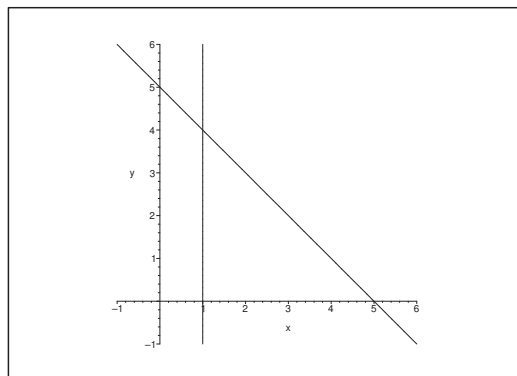
- 3) The domain is the closed triangle between the lines $x = 1$, $y = 0$ and $y = 5 - x$.

- a) STATIONARY POINTS. The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = y - \frac{3}{x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = x + 2y - 5 = 0,$$

from which $x + \frac{6}{x} - 5 = 0$. Since $x \geq 1$, we get

$$x^2 - 5x + 6 = (x - 2)(x - 3) = 0,$$



the roots of which are $x = 2$, corresponding to $y = \frac{3}{2}$, and $x = 3$, corresponding to $y = 1$.
We get the two stationary points

$$\left(2, \frac{3}{2}\right) \quad \text{og} \quad (3, 1).$$

The values of the function are here

$$f\left(2, \frac{3}{2}\right) = 2 \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 - 5 \cdot \frac{3}{2} - 3 \ln 2 = -\frac{9}{4} - 3 \ln 2,$$

and

$$f(3, 1) = 3 \cdot 1 + 1^2 - 5 \cdot 1 - 3 \ln 3 = -1 - 3 \ln 3.$$

b) EXAMINATION OF THE BOUNDARY.

i) If $y = 0$, then the restriction

$$f(x, 0) = -3 \ln x, \quad x \in [1, 5],$$

is monotonous with its maximum $f(1, 0) = 0$ and its minimum $f(5, 0) = -3 \ln 5$.

ii) If $x = 1$, then the restriction

$$f(1, y) = y^2 - 4y, \quad y \in [1, 4]$$

has the maximum $f(1, 0) = f(1, 4) = 0$ and the minimum $f(1, 2) = -4$.

iii) If $y = 5 - x$, then

$$\begin{aligned} f(x, 5 - x) &= x(5 - x) + (5 - x)^2 - 5(5 - x) - 3 \ln x \\ &= -3 \ln x \end{aligned} \quad \text{for } x \in [1, 5],$$

where the maximum is $f(1, 4) = 0$ and the minimum is $f(5, 0) = -3 \ln 5$.

c) NUMERICAL COMPARISON. We shall compare

$$f\left(2, \frac{3}{2}\right) = -\frac{9}{4} - 3 \ln 2, \quad f(3, 1) = -1 - 3 \ln 3,$$

$$f(1, 0) = 0, \quad f(5, 0) = -3 \ln 5,$$

$$\begin{aligned}f(1,0) = f(1,4) = 0, & \quad f(1,2) = -4, \\f(1,4) = 0, & \quad f(5,0) = -3 \ln 5.\end{aligned}$$

The maximum is clearly

$$f(1,0) = f(1,4) = 0.$$

By using a pocket calculator we then get approximately,

$$-\frac{9}{4} - 3 \ln 2 \approx -4,33, \quad -1 - 3 \ln 3 \approx -4,30, \quad -3 \ln 5 \approx -4,83.$$

We conclude that the minimum is

$$f(5,0) = -3 \ln 5 \approx -4,83.$$

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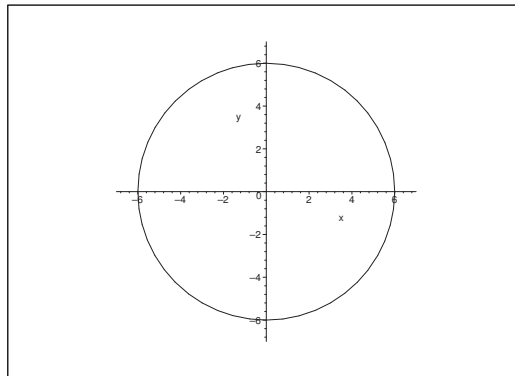
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- 4) The domain is the closed disc of centrum $(0, 0)$ and radius 6.



It is easy to find the three stationary points

$$(0, 0), \quad (0, 3 + \sqrt{3}) \quad \text{and} \quad (0, 3 - \sqrt{3}),$$

where $(0, 0)$ is not an extremum, while $(0, 3 + \sqrt{3})$ is a minimum point, and $(0, 3 - \sqrt{3})$ is a maximum point. The function values are then

$$f(0, 3 + \sqrt{3}) = -36 - 24\sqrt{3}, \quad f(0, 3 - \sqrt{3}) = -36 + 24\sqrt{3}.$$

STATIONARY POINTS. We get by differentiation

$$\frac{\partial f}{\partial x} = 2x(x^2 + y^2 - 6y) + 2x(x^2 + y^2 - 2y) = 4x(x^2 + y^2 - 4y)$$

and

$$\begin{aligned} \frac{\partial f}{\partial y} &= (2y - 2)(x^2 + y^2 - 6y) + (2y - 6)(x^2 + y^2 - 2y) \\ &= (2y - 4)(x^2 + y^2 - 6y) + 2(x^2 + y^2 - 6y) \\ &\quad + (2y - 4)(x^2 + y^2 - 2y) - 2(x^2 + y^2 - 2y) \\ &= 4(y - 2)(x^2 + y^2 - 4y) - 8y \\ &= 4 \{ (y - 2)(x^2 + y^2 - 4y) - 2y \}. \end{aligned}$$

The two equations for the stationary points are

$$(1) \quad \begin{cases} x(x^2 + y^2 - 4y) = 0, \\ (y - 2)(x^2 + y^2 - 4y) = 2y. \end{cases}$$

It follows from the former equation that the stationary points either lie on the line $x = 0$ or on the circle $x^2 + (y - 2)^2 = 2^2$

a) If $x = 0$, it follows from the latter equation that

$$0 = (y - 2)(y^2 - 4y) - 2y = y\{(y - 2)(y - 4) - 2\} = y\{y^2 - 6y + 6\},$$

which has the two solutions $y = 0$ and $y = 3 \pm \sqrt{3}$. Thus we get in this case three stationary points

$$(0, 0), \quad (0, 3 + \sqrt{3}) \quad \text{and} \quad (0, 3 - \sqrt{3}).$$

- b) If $x^2 + y^2 - 4y = 0$, it follows from the latter equation of (1) that $y = 0$, and thus $x = 0$, so we find again $(0, 0)$.

Summarizing we get the three stationary points

$$(0, 0), \quad (0, 3 + \sqrt{3}) \quad \text{and} \quad (0, 3 - \sqrt{3}).$$

REMARK. Since we are searching the *maximum* and the *minimum* (and not the extrema), it is sufficient to calculate the value of the function in these points, and e.g. the (r, s, t) -investigation is totally superfluous (and a waste of time). \diamond

By computation we get $f(0, 0) = 0$, and

$$\begin{aligned} f(0, 3 + \sqrt{3}) &= \left\{ (3 + \sqrt{3})^2 - 2(3 + \sqrt{3}) \right\} \left\{ (3 + \sqrt{3})^2 - 6(3 + \sqrt{3}) \right\} \\ &= (12 + 6\sqrt{3} - 6 - 2\sqrt{3})(12 + 6\sqrt{3} - 18 - 6\sqrt{3}) \\ &= (6 + 4\sqrt{3})(-6) \\ &= -36 - 24\sqrt{3}, \end{aligned}$$

and analogously

$$f(0, 3 - \sqrt{3}) = -36 + 24\sqrt{3}.$$

EXAMINATION OF THE BOUNDARY. We have on the boundary $x^2 + y^2 = 36$, so

$$\begin{aligned} f(x, y) &= (x^2 + y^2 - 2y)(x^2 + y^2 - 6y) \\ &= (36 - 2y)(36 - 6y) \\ &= 12(18 - y)(6 - y), \quad \text{for } y \in [-6, 6], \end{aligned}$$

where the maximum is

$$f(0, -6) = 12 \cdot 24 \cdot 12 = 3456,$$

and the minimum is $f(0, 6) = 0$.

NUMERICAL COMPARISON. Since $f(0, 3 + \sqrt{3}) < 0$, the maximum and the minimum of the function in the domain are respectively,

$$S = f(0, -6) = 3456 \quad \text{and} \quad M = f(0, 3 + \sqrt{3}) = -36 - 24\sqrt{3}.$$

ALTERNATIVE SOLUTION. The argument of the second main theorem concerning the existence of the maximum and the minimum is the same as above.

Write the function f in the following way:

$$\begin{aligned} f(x, y) &= (x^2 + y^2 - 2y)(x^2 + y^2 - 6y) \\ &= \{x^2 + (y - 1)^2 - 1\} \{x^2 + (y - 3)^2 - 3^2\}, \end{aligned}$$

and discuss the sign of f in the domain, i.e. sketch the zero sets (the circles of respectively centrum $(0, 1)$ and radius 1, and of centrum $(0, 3)$ and radius 3) inside the domain and find the signs in each of the thus defined subregions.

The function is positive inside the two sets of zero circles and also outside the same two sets of circles, while it is negative between the two zero circles.

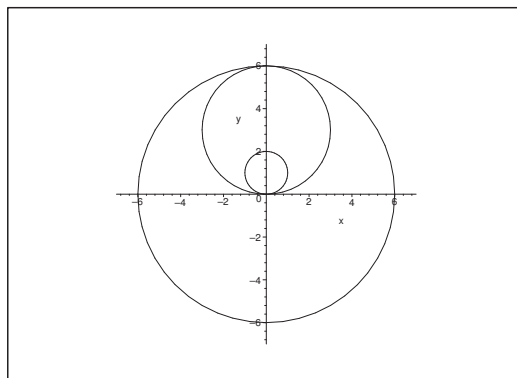


Figure 1: The zero curves inside the domain.

It follows immediately that the minimum must lie between the two sets of zero circles, i.e. in the set

$$K((0,3);3) \setminus \overline{K}((0,1);1) = A_1,$$

and also that the minimum point must be a stationary point, because A_1 is open. Because $3 - \sqrt{3} < 2 = 1 + 1$, we see that $(0, 3 + \sqrt{3})$ is the only stationary point in $A >_1$, so the *minimum* is (originally only a local minimum)

$$f(0, 3 + \sqrt{3}) = -36 - 24\sqrt{3}.$$

It follows in the same way that $(0, 3 - \sqrt{3}) \in A_2 = K((0,1);1)$ must be a local maximum

$$f(0, 3 - \sqrt{3}) = -36 + 24\sqrt{3}.$$

Finally by using polar coordinates in the plane,

$$\begin{aligned} f(x, y) &= (r^2 - 2r \sin \varphi)(r^2 - 6r \sin \varphi) \\ &= r^2(r - 2 \sin \varphi)(r - 6 \sin \varphi), \quad 0 \leq r \leq 6. \end{aligned}$$

In the remaining region A_3 both factors are positive, so

$$r - 2 \sin \varphi > 0 \quad \text{og} \quad r - 6 \sin \varphi > 0.$$

The product is largest when $\sin \varphi$ is smallest, i.e. when $\varphi = -\frac{\pi}{2}$, thus $\sin \varphi = -1$, corresponding to

$$f(x, y) = r^2(r + 2)(r + 6), \quad 0 \leq r \leq 6.$$

This product is largest when r is largest, i.e. when $r = 6$ (the boundary), corresponding to $(x, y) = (0, -6)$, and

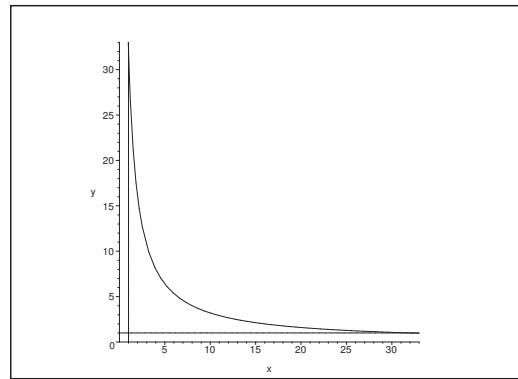
$$f(0, -6) = 3456.$$

When we compare with the other candidate above we conclude that the *maximum* in A is

$$f(0, -6) = 3456.$$

- 5) In this case it is difficult to sketch the domain because the hyperbola $xy = 32$ is very steep in the neighbourhood of $(1, 32)$, and very flat in the neighbourhood of $(32, 1)$. It is demonstrated on the figure what MAPLE does in this case.

The domain is bounded by the hyperbola $xy = 32$ and the lines $x = 1$ and $y = 1$.



a) STATIONARY POINTS. The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = y - \frac{64}{x^2} = 0 \quad \text{og} \quad \frac{\partial f}{\partial y} = x - \frac{64}{y^2} = 0.$$

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From these we get

$$\frac{64}{x} = xy = \frac{64}{y},$$

so the stationary points lie on the line $y = x$. Then by insertion $x^3 = 64 = 4^3$, and thus $x = y = 4$. We conclude from

$$x = 4 \geq 1, \quad y = 4 \geq 1 \quad \text{and} \quad xy = 16 \leq 32,$$

that we have the stationary point $(4, 4)$ in the domain. The value of the function is here

$$f(4, 4) = 16 + 16 + 16 = 48.$$

NOTICE that we shall not check if $(4, 4)$ is an extremum.

b) EXAMINATION OF THE BOUNDARY.

- i) We get along the boundary curve $xy = 32$, $x \in [1, 32]$, that $y = \frac{32}{x}$, and the corresponding restriction is

$$f(x) = f\left(x, \frac{32}{x}\right) = 32 + \frac{64}{x} + 2x, \quad \text{for } x \in [1, 32],$$

where

$$g'(x) = -\frac{64}{x^2} + 2 = 0 \quad \text{for } x = \sqrt{32},$$

corresponding to a minimum. The y -value is $y = \frac{32}{\sqrt{32}} = \sqrt{32}$, and the value of the function at the point $(\sqrt{32}, \sqrt{32})$ is

$$f(\sqrt{32}, \sqrt{32}) = 32 + \frac{64}{\sqrt{32}} + \frac{64}{\sqrt{32}} = 32 + 4\sqrt{32} = 32 + 16\sqrt{2}.$$

At the end points of $xy = 32$, $x \in [1, 32]$, we get the values

$$f(1, 32) = f(32, 1) = 32 + 64 + \frac{64}{32} = 98.$$

- ii) We get along the boundary curve $y = 1$, $x \in [1, 32]$, the following restriction

$$h(x) = f(x, 1) = x + \frac{64}{x} + 64,$$

where

$$h'(x) = 1 - \frac{64}{x^2} = 0 \quad \text{for } x = 8,$$

corresponding to a minimum

$$f(8, 1) = 8 + \frac{64}{8} + 64 = 80.$$

At the end points we get

$$f(1, 1) = 1 + 64 + 64 = 129 \quad \text{and} \quad f(32, 1) = 32 + 2 + 64 = 98.$$

iii) Due to the symmetry we get along the boundary curve $x = 1$ that the minimum is

$$f(1, 8) = 80,$$

and that the values at the end points are

$$f(1, 1) = 129 \quad \text{and} \quad f(1, 32) = 98.$$

iv) NUMERICAL COMPARISON. Summarizing the minimum is one of the values

$$f(4, 4) = 48, \quad f(\sqrt{32}, \sqrt{32}) = 32 + 16\sqrt{2}, \quad f(1, 8) = f(8, 1) = 80.$$

From $32 + 16\sqrt{2} > 32 + 16 = 48$ follows that the minimum is

$$M = f(4, 4) = 48.$$

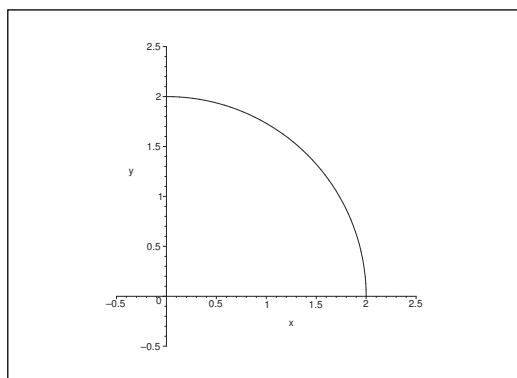
Analogously the maximum is one of the numbers

$$f(1, 1) = 129 \quad \text{and} \quad f(1, 32) = f(32, 1) = 98,$$

hence the maximum is

$$S = f(1, 1) = 129.$$

6) The domain is the quarter of a disc in the first quadrant of centrum $(0, 0)$ and radius 2.



a) STATIONARY POINTS. the stationary points are the solutions of the equations

$$(2) \quad \begin{cases} \frac{\partial f}{\partial x} = 6x - 2y - 4xy^2 = 0, \\ \frac{\partial f}{\partial y} = 6y - 2x - 4x^2y = 0. \end{cases}$$

When we add the two equations we get the following necessary condition

$$\begin{aligned} 0 &= 4x + 4y - 4xy^2 - 4x^2y \\ &= 4\{(x + y) - xy(x + y)\} \\ &= 4(x + y)(1 - xy), \end{aligned}$$

so either $x + y = 0$ (not possible in this domain) or $xy = 1$,

Analogously it follows from (2) that

$$4x^2y^2 = 6x^2 - 2xy = 6y^2 - 2xy,$$

hence $x^2 = y^2$, which together with $xy = 1$ and $x > 0$ give $x = y = 1$. The only possibility is $(1, 1)$, and by insertion into (2) we get that $(1, 1)$ is a stationary point. Furthermore, we see that $(1, 1)$ belongs to the domain. The value of the function at the point is

$$f(1, 1) = 3 + 3 - 2 - 2 = 2.$$

- b) **EXAMINATION OF THE BOUNDARY.** On the boundary curve $x^2 + y^2 = 4$ we shall use the parametric description

$$x = 2 \cos \varphi, \quad y = 2 \sin \varphi, \quad \varphi \in \left[0, \frac{\pi}{2}\right].$$

Then

$$\begin{aligned} g(\varphi) &= f(x, y) = 3x^2 + 3y^2 - 2xy(1 + xy) \\ &= 3 \cdot 4 - 2 \cdot 4 \cos \varphi \cdot \sin \varphi (1 + 4 \cos \varphi \cdot \sin \varphi) \\ &= 12 - 4 \sin 2\varphi \cdot (1 + 2 \sin 2\varphi) \\ &= 12 - 4 \sin 2\varphi - 8 \sin^2 2\varphi \\ &= 12 - 8 \left\{ \sin^2 2\varphi + \frac{1}{2} \sin 2\varphi + \frac{1}{16} \right\} + \frac{8}{16} \\ &= \frac{25}{2} - 8 \left(\sin 2\varphi + \frac{1}{4} \right)^2 \quad \text{for } 2\varphi \in [0, \pi]. \end{aligned}$$

Here $\sin 2\varphi \in [0, 1]$ for $2\varphi \in [0, \pi]$, so the maximum of $g(\varphi)$ is obtained for $\sin 2\varphi = 0$, i.e. for either $\varphi = 0$ or $\varphi = \frac{\pi}{2}$, corresponding to

$$g(0) = g\left(\frac{\pi}{2}\right) = f(2, 0) = f(0, 2) = 12.$$

The minimum is obtained for $\sin 2\varphi = 1$, corresponding to $\varphi = \frac{\pi}{4}$, or $x = y = \sqrt{2}$, where

$$g\left(\frac{\pi}{4}\right) = f(\sqrt{2}, \sqrt{2}) = 12 - 2 \cdot 2 \cdot (1 + 2) = 0.$$

ALTERNATIVELY we see that $z = xy$ runs through the interval $[0, 2]$, when (x, y) runs through the arc of the quarter circle. This means that

$$g_1(z) = f(x, y) = 3x^2 + 3y^2 - 2xy(1 + xy) = 12 - 2z(1 + z),$$

which is largest in the interval $[0, 2]$, when $z = 0$, which corresponds to $(x, y) = (2, 0)$ or $(0, 2)$, and smallest when $z = 2$, which corresponds to $(x, y) = (\sqrt{2}, \sqrt{2})$.

For $x = 0$ we get the restriction $h(y) = f(0, y) = 3y^2$ with the minimum $f(0, 0) = 0$ and the maximum

$$f(0, 2) = f(2, 0) = 12,$$

and where we exploit the symmetry of x and y .

c) NUMERICAL COMPARISON. The minimum is one of the values

$$f(1, 1) = 2, \quad f(\sqrt{2}, \sqrt{2}) = 0, \quad f(0, 0) = 0,$$

and the maximum is one of the values

$$f(1, 1) = 2, \quad f(2, 0) = f(0, 2) = 12.$$

It follows that the minimum is

$$M = f(\sqrt{2}, \sqrt{2}) = f(0, 0) = 0,$$

and that the maximum is

$$S = f(2, 0) = f(0, 2) = 12.$$

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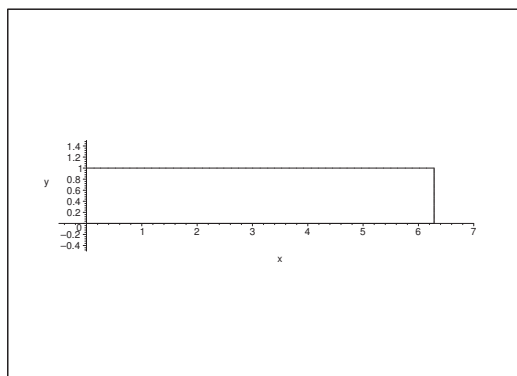
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7) The domain is here the rectangle $[0, 2\pi] \times [0, 1]$.



a) STATIONARY POINTS. The stationary points are given by the equations

$$\begin{cases} \frac{\partial f}{\partial x} = e^{-y} \cos x = 0, \\ \frac{\partial f}{\partial y} = -2e^{-2y} - e^{-y} \sin x = 0. \end{cases}$$

It follows from the former of these that $\cos x = 0$, so either $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$. Then we get from the latter equation,

$$e^{-y}(2e^{-y} + \sin x) = 0.$$

Now $e^{-y} > 0$, hence $\sin x < 0$ and whence $x = \frac{3\pi}{2}$. Accordingly $2e^{-y} = 1$ and thus $y = \ln 2 \in]0, 1[$.

The function has a stationary point in the domain, namely $\left(\frac{3\pi}{2}, \ln 2\right)$. The value of the function is here

$$f\left(\frac{3\pi}{2}, \ln 2\right) = e^{-2\ln 2} + e^{-\ln 2} \sin\left(\frac{3\pi}{2}\right) = -\frac{1}{4}.$$

b) EXAMINATION OF THE BOUNDARY

i) The restriction along the boundary curve $y = 0$, $x \in [0, 2\pi]$ is

$$g(x) = f(x, 0) = 1 + \sin x.$$

It follows immediately that the maximum is $f\left(\frac{\pi}{2}, 0\right) = 2$ and the minimum is $f\left(\frac{3\pi}{2}, 0\right) = 0$.

ii) the restriction along the boundary curve $y = 1$, $x \in [0, 2\pi]$, is

$$h(x) = f(x, 1) = \frac{1}{e^2} + \frac{1}{e} \sin x.$$

This gives the maximum

$$f\left(\frac{\pi}{2}, 1\right) = \frac{1}{e^2} + \frac{1}{2} \quad (< 2)$$

and the minimum

$$f\left(\frac{3\pi}{2}\right) = \frac{1}{e^2} - \frac{1}{e} = -\frac{e-1}{e^2} \quad (< 0).$$

- iii) We get the same restriction $r(y) = e^{-2y}$, $y \in [0, 1]$, along the boundary curves $x = 0$ and $x = 2\pi$ which corresponds to the maximum $f(0, 0) = f(2\pi, 0) = 1$ and the minimum $f(0, 1) = f(2\pi, 1) = \frac{1}{e^2}$.

- c) NUMERICAL COMPARISON. The maximum is one of the values

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -\frac{1}{4}, \quad f\left(\frac{\pi}{2}, 0\right) = 2,$$

$$f\left(\frac{\pi}{2}, 1\right) = \frac{1}{e^2} + \frac{1}{e}, \quad f(0, 0) = f(2\pi, 0) = 1.$$

The minimum is one of the values

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -\frac{1}{4}, \quad f\left(\frac{3\pi}{2}, 0\right) = 0,$$

$$f\left(\frac{3\pi}{2}, 1\right) = -\frac{e-1}{e^2}, \quad f(0, 1) = f(2\pi, 1) = \frac{1}{e^2}.$$

We conclude that the maximum is

$$S = f\left(\frac{\pi}{2}, 0\right) = 2.$$

From $(e-2)^2 = e^2 - 4e + 4 > 0$ follows by a rearrangement that $e^2 > 4(e-1) > 0$, hence $0 < \frac{e-1}{e^2} < \frac{1}{4}$, and the minimum is

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -\frac{1}{4}.$$

- 8) The domain is the disc of centrum $(0, 0)$ and radius 2.

- a) STATIONARY POINTS. The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 4x^3 - 2x + 2y = 0, \quad \frac{\partial f}{\partial y} = 4y^3 + 2x - 2y = 0,$$

hence $4x^3 = 2x - 2y = -4y^3$, i.e. $y = -x$.

We get by insertion

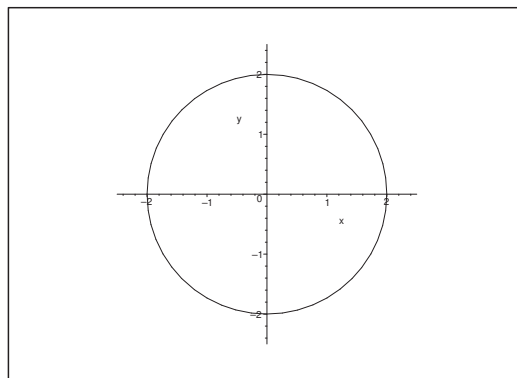
$$0 = 4x^3 - 4x = 4x(x^2 - 1),$$

so we have the possibilities $x = 0, 1, -1$, corresponding to the candidates

$$(0, 0), \quad (1, -1), \quad \text{and} \quad (-1, 1)$$

of the stationary points. It follows by insertion (i.e. by testing) that they are actually all stationary points. The values of the function are here

$$f(0, 0) = 0 \quad \text{og} \quad f(1, -1) = f(-1, 1) = 1 + 1 - 1 - 2 - 1 = -2.$$



b) EXAMINATION OF THE BOUNDARY. We have on the boundary $x^2 + y^2 = 4$,

$$\begin{aligned} f(x, y) &= x^4 + y^4 - x^2 + 2xy - y^2 \\ &= (x^2 + y^2)^2 - 2x^2y^2 - (x^2 + y^2) + 2xy \\ &= 12 + 2xy(1 - xy). \end{aligned}$$

i) FIRST ALTERNATIVE. We get by the parametric description

$$x = 2 \cos \varphi, \quad y = 2 \sin \varphi, \quad \varphi \in [0, 2\pi[,$$

that

$$\begin{aligned} g_1(\varphi) &= f(x, y) = 12 + 2xy(1 - xy) \\ &= 12 + 2 \cdot 4 \cos \varphi \cdot \sin \varphi (1 - 4 \cos \varphi \cdot \sin \varphi) \\ &= 12 + 4 \sin 2\varphi (1 - 2 \sin 2\varphi) \\ &= 12 - 8 \left(\sin^2 2\varphi - \frac{1}{2} \sin 2\varphi + \frac{1}{16} \right) + \frac{1}{2} \\ &= \frac{25}{2} - 8 \left(\sin 2\varphi - \frac{1}{4} \right)^2. \end{aligned}$$

The function $\left(\sin 2\varphi - \frac{1}{4} \right)^2$ is smallest when $\sin 2\varphi_0 = \frac{1}{4}$, corresponding to the maximum $g_1(\varphi) = \frac{25}{2}$.

The function $\left(\sin 2\varphi - \frac{1}{4} \right)^2$ is largest when $\sin 2\varphi_1 = -1$, i.e. when $(x, y) = (-\sqrt{2}, \sqrt{2})$ or $(\sqrt{2}, -\sqrt{2})$, corresponding to the value of the function

$$f(-\sqrt{2}, \sqrt{2}) = f(\sqrt{2}, -\sqrt{2}) = 0.$$

ii) SECOND ALTERNATIVE. We see that $z = xy$ runs through $[-2, 2]$, when (x, y) runs through the circle of the equation $x^2 + y^2 = 4$. This means that it suffices to check

$$g_2(z) = 12 + 2xy(1 - xy) = 12 + 2z - 2z^2 = \frac{25}{2} - 2 \left(z - \frac{1}{2} \right)^2$$

for $z \in [-2, 2]$. We get the maximum for $z = \frac{1}{2}$, corresponding to the value $\frac{25}{2} - 0 = \frac{25}{2}$.

REMARK. As a curiosity we mention that the corresponding (x, y) -values are

$$(3) \quad \left(\frac{\sqrt{5} + \sqrt{3}}{2}, \frac{\sqrt{5} - \sqrt{3}}{2} \right), \quad \left(\frac{\sqrt{5} - \sqrt{3}}{2}, \frac{\sqrt{5} + \sqrt{3}}{2} \right), \\ \left(-\frac{\sqrt{5} + \sqrt{3}}{2}, -\frac{\sqrt{5} - \sqrt{3}}{2} \right), \quad \left(-\frac{\sqrt{5} - \sqrt{3}}{2}, -\frac{\sqrt{5} + \sqrt{3}}{2} \right),$$

but there is no need at all to find these point exactly to find the maximum and the minimum. \diamond

The minimum is obtained for $z = -2$, corresponding to $g_2(-2) = 0$ and $(x, y) = (-\sqrt{2}, \sqrt{2})$ or $(\sqrt{2}, -\sqrt{2})$.

c) MAXIMUM AND MINIMUM. Summarizing we see that the maximum is

$$S = \max \left\{ 0, \frac{25}{2} \right\} = \frac{25}{2}.$$

One is not asked of where the maximum is attained, so here is added that it is the value of the function at the points given by (3).

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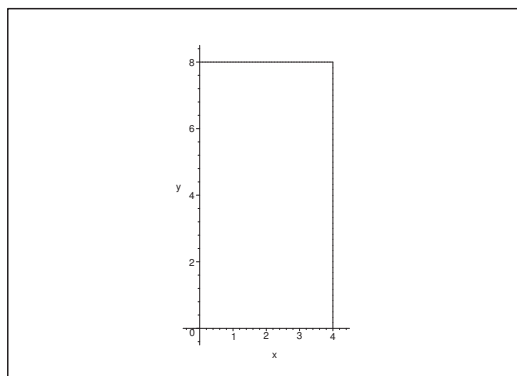


The minimum is

$$M = \min\{-2, 0\} = -2.$$

It is the value of the functions in the points $(-1, 1)$ and $(1, -1)$.

9) The domain is here the rectangle $[0, 4] \times [0, 8]$.



a) STATIONARY POINTS. The stationary points are the solutions of the equations

$$(4) \quad \begin{cases} \frac{\partial f}{\partial x} = 8y^2 - y^3 - 3x^2y = y(8y - y^2 - 3x^2) = 0, \\ \frac{\partial f}{\partial y} = 16xy - 3xy^2 - x^3 = x(16y - 3y^2 - x^2) = 0. \end{cases}$$

We have $xy > 0$ in the interior of the rectangle, so the equations are reduced to

$$y^2 - 8y + 3x^2 = 0 \quad \text{and} \quad 3y^2 - 16y + x^2 = 0,$$

hence

$$-3x^2 = y^2 - 8y = 9y^2 - 48y.$$

This is again reduced to

$$8y^2 - 40y = 8y(y - 5) = 0.$$

Now $y > 0$, thus $y = 5 \in]0, 8[$, hence $x^2 = 16y - 3y^2 = 80 - 75 = 5$, or (since $x > 0$), $x = \sqrt{5} \in]0, 4[$. When we put $(x, y) = (\sqrt{5}, 5)$ into (4) we get

$$\frac{\partial f}{\partial x} = 5(40 - 25 - 15) = 0, \quad \frac{\partial f}{\partial y} = \sqrt{5}(80 - 75 - 5) = 0,$$

and we conclude that $(\sqrt{5}, 5)$ is a stationary point in the domain. The corresponding value of the function is

$$f(\sqrt{5}, 5) = 8\sqrt{5} \cdot 25 - \sqrt{5} \cdot 125 - 5\sqrt{5} \cdot 5 = \sqrt{5}\{200 - 125 - 25\} = 50\sqrt{5}.$$

b) EXAMINATION OF THE BOUNDARY. It follows from

$$f(x, y) = xy(8y - y^2 - x^2),$$

that $f(0, y) = f(x, 0) = 0$.

The restriction on the boundary curve $x = 4$, $y \in [0, 8]$, is

$$g(y) = -4y(16 - 8y + y^2) = -4y(y - 4)^2$$

where

$$g'(y) = -4(y - 4)^2 - 8y(y - 4) = -4(y - 4)\{y - 4 + 2y\} = -12(y - 4) \left(y - \frac{4}{3}\right).$$

Now $g'(y) = 0$ for $y = 4$ and for $y = \frac{4}{3}$, and the values of the function are

$$f\left(4, \frac{4}{3}\right) = -\frac{16}{3} \left(4 - \frac{4}{3}\right)^2 = -\frac{256}{27} \cdot 4 = -\frac{1024}{27}$$

and $f(4, 4) = 0$, and at the end points

$$f(4, 0) = 0, \quad f(4, 8) = -32 \cdot 16 = -512.$$

The restriction on the boundary curve $y = 8$, $x \in [0, 4]$, is

$$h(x) = 8x(64 - 64 - x^2) = -8x^3,$$

which clearly takes its maximum for $x = 0$ and minimum for $x = 4$, corresponding to the maximum $f(0, 8) = 0$ and the minimum $f(4, 8) = -512$.

c) NUMERICAL COMPARISON. The candidates of the minimum are

$$\begin{aligned} f(\sqrt{5}, 5) &= 50\sqrt{5}, & f(0, y) &= f(x, 0) = 0, \\ f\left(4, \frac{4}{3}\right) &= -\frac{1024}{27}, & f(4, 8) &= -256. \end{aligned}$$

The candidates of the maximum are

$$f(\sqrt{5}, 5) = 50\sqrt{5}, \quad f(0, y) = f(x, 0) = f(4, 4) = 0.$$

By comparison we get the minimum

$$M = f(4, 8) = -512$$

and the maximum

$$S = f(\sqrt{5}, 5) = 50\sqrt{5}.$$

Example 1.2 We shall construct a cage for transportation of poultry from a board of the length 6 dm and the breadth b dm. The board is broken at two places and then a net of steel wire is stretched over it. Finally, two additional boards are added so that one gets a cage of a cross section of an equilateral trapeze. We want to break the given board in such a way that the volume of the cage V dm³ becomes as large as possible.

First prove that

$$V(x, y) = (6 - 2x - x \cos y)bx \sin y.$$

Then explain why the function V shall only be considered on the set $[0, 3] \times \left[\frac{\pi}{2}, \pi\right]$. Finally, find the maximum of the volume and the corresponding set (x, y) of coordinates.

A Maximum.

D Analyze the text. Check the model and find the maximum.

By cutting the trapeze (chop off the two triangles, so one gets a rectangle) we get the height (i.e. the breadth of the trapeze)

$$h(x, y) = x \cdot \sin(\pi - y) = x \cdot \sin y.$$

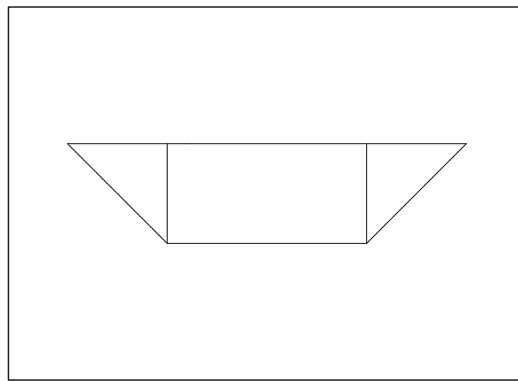


Figure 2: The skew lines are each of the length x , and the two obtuse angles are each of the size y .

Then compute the area,

$$\begin{aligned} A(x, y) &= (6 - 2x)(h(x, y)) + 2 \cdot \frac{1}{2} h(x, y) \cdot x \cos(\pi - y) = (6 - 2x)x \sin y + x \sin y \{-x \cos y\} \\ &= (6x - 2x^2) \sin y - x^2 \sin y \cdot \cos y. \end{aligned}$$

REMARK. It follows from the sign of the latter term that the areas of the corners are counted negatively, when $y \in \left]0, \frac{\pi}{2}\right[$, which is quite reasonable when one sketches the corresponding figure.
◇

It follows from the above that the volume is

$$V(x, y) = b A(x, y) = \{(6x - 2x^2) \sin y - x^2 \sin y \cos y\}b.$$

Since $x \geq 0$ and $6 - 2x \geq 0$, we must have $x \in [0, 3]$.

We can clearly choose y in the interval $[0, \pi]$; but since we shall find a maximum, we must have $-x^2 \sin y \cos y \geq 0$, so $y \in \left[\frac{\pi}{2}, \pi\right]$.

The task is then reduced to finding the maximum of the function

$$V(x, y) = \{(6x - 2x^2) \sin y - x^2 \sin y \cos y\}b$$

in the set $[0, 3] \times \left[\frac{\pi}{2}, \pi\right] = A$.

- 1) STATIONARY POINTS. The stationary points in the interior of A are the solutions of the equations

$$\begin{cases} \frac{\partial V}{\partial x} = b\{(6 - 4x) \sin y - 2x \sin y \cos y\} = 0, \\ \frac{\partial V}{\partial y} = b\{(6x - 2x^2) \cos y - x^2(\cos^2 y - \sin^2 y)\} = 0. \end{cases}$$

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Since $b > 0$, $x > 0$ and $\sin y > 0$ in A° , these equations are reduced to

$$(5) \quad \begin{cases} (6 - 4x) - 2x \cos y = 0, \\ (6 - 2x) \cos y - x(2 \cos^2 y - 1) = 0. \end{cases}$$

It follows from the former equation that

$$-1 < \cos y = \frac{3 - 2x}{x} = \frac{3}{x} - 2 < 0,$$

so $\frac{3}{2} < x < 3$ for possible stationary points. When the value above $\cos y$ is put into the latter equation of (5), then

$$\begin{aligned} 0 &= (6 - 2x) \cdot \frac{3 - 2x}{x} - x \left\{ 2 \left(\frac{3 - 2x}{x} \right)^2 - 1 \right\} \\ &= \frac{1}{x} \{ (6 - 2x)(3 - 2x) - 2(3 - 2x)^2 + x^2 \} \\ &= \frac{1}{x} \{ (3 - 2x)[(6 - 2x) - (6 - 4x)] + x^2 \} \\ &= \frac{1}{x} \{ (3 - 2x) \cdot 2x + x^2 \} = 2(3 - 2x) + x = 6 - 4x + x \\ &= 6 - 3x, \end{aligned}$$

hence $x = 2$, and thus $\cos y = \frac{3 - 4}{2} = -\frac{1}{2}$, corresponding to the candidate $(x, y) = \left(2, \frac{2\pi}{3}\right)$.

TEST. That $\left(2, \frac{2\pi}{3}\right)$ really is a stationary point, follows from the computations

$$\begin{aligned} \frac{\partial V}{\partial x} &= b \left\{ (6 - 8) \sin \frac{2\pi}{3} - 2 \cdot 2 \sin \frac{2\pi}{3} \cos \frac{2\pi}{3} \right\} = b \sin \frac{2\pi}{3} \left(-2 - 2 \cdot 2 \cdot \left(-\frac{1}{2}\right) \right) = 0, \\ \frac{\partial V}{\partial y} &= b \left\{ (12 - 8) \cos \frac{2\pi}{3} - 4 \left(2 \cos^2 \frac{2\pi}{3} - 1 \right) \right\} = 4b \left\{ -\frac{1}{2} - \left(\frac{1}{2} - 1 \right) \right\} = 0, \end{aligned}$$

and we have tested our result. \diamond

The value of the function at $\left(2, \frac{2\pi}{3}\right)$ is

$$\begin{aligned} V\left(2, \frac{2\pi}{3}\right) &= b \left\{ (12 - 8) \sin \frac{2\pi}{3} - 4 \sin \frac{2\pi}{3} \cos \frac{2\pi}{3} \right\} = b \left\{ 4 \cdot \frac{\sqrt{3}}{2} - 4 \cdot \frac{\sqrt{3}}{2} \cdot \left(-\frac{1}{2}\right) \right\} \\ &= b\{2\sqrt{3} + \sqrt{3}\} = 3\sqrt{3}b. \end{aligned}$$

- 2) EXAMINATION OF THE BOUNDARY. We get $V(0, y) \equiv 0$ on the boundary curve $x = 0$, $y \in \left[\frac{\pi}{2}, \pi\right]$.

On the boundary curve $x = 3$, $y \in \left[\frac{\pi}{2}, \pi\right]$, we have the restriction

$$V(3, y) = -9b \sin y \cdot \cos y = -\frac{9}{2}b \sin 2y,$$

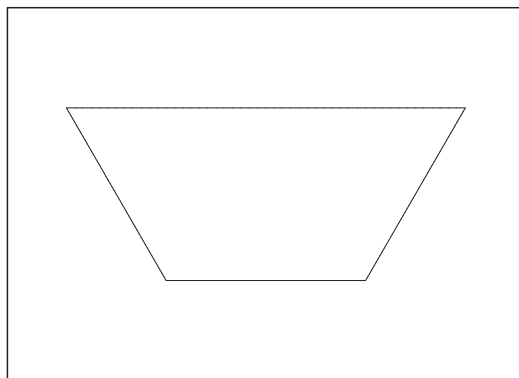


Figure 3: The form of the cage of maximum volume corresponding to the stationary point.

which has its minimum $V\left(3, \frac{\pi}{2}\right) = V(3, \pi) = 0$ and its maximum

$$V\left(3, \frac{3\pi}{4}\right) = \frac{9}{2}b.$$

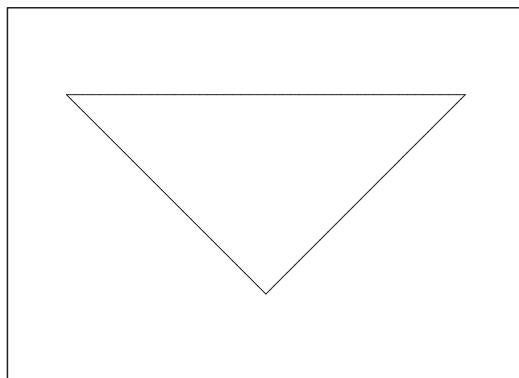


Figure 4: The triangle corresponding to $V\left(3, \frac{3\pi}{4}\right) = \frac{9}{2}b$.

Notice that this case corresponds to a degenerated trapeze, i.e. to a rectangular triangle, cf. the figure.

On the boundary curve $y = \frac{\pi}{2}$, $x \in [0, 3]$, we get the restriction

$$V\left(x, \frac{\pi}{2}\right) = (6x - 2x^2)b = 2b \left\{ \frac{9}{4} - \left(x - \frac{3}{2}\right)^2 \right\},$$

with its minimum $V\left(3, \frac{\pi}{2}\right) = V(3, \pi) = 0$, and its maximum

$$V\left(\frac{3}{2}, \frac{\pi}{2}\right) = \frac{9}{2}b.$$

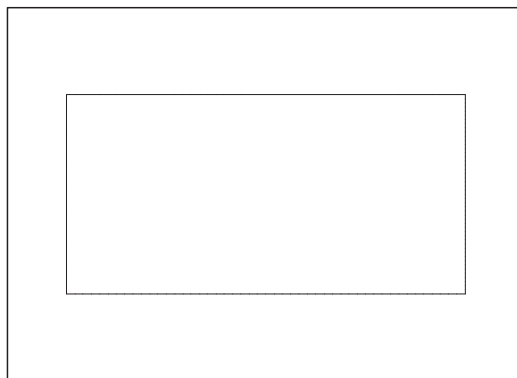


Figure 5: The rectangle corresponding to $V\left(\frac{3}{2}, \frac{\pi}{2}\right) = \frac{9}{2}b$. The Height is $\frac{3}{2}$, and each of the horizontal pieces have the length 3.

Finally, $V(x, \pi) = 0$ on the boundary curve $y = \pi$, $x \in [0, 3]$, which does not contribute to the candidates.

- 3) NUMERICAL COMPARISON. We conclude from $3\sqrt{3} > 3 \cdot \frac{3}{2} = \frac{9}{2}$ that the maximum is attained at the stationary point $(x, y) = \left(2, \frac{2\pi}{3}\right)$ with the value of the function

$$V\left(2, \frac{2\pi}{3}\right) = 3\sqrt{3}b.$$

The form of the corresponding cage is shown of Figure 3 in connection with the stationary point.

Example 1.3 Check in each of the following cases if the given function has a maximum or a minimum or both or none of the kind. If it has a maximum or a minimum give the value of the function at these points.

- 1) $f(x, y) = (x + y) \exp(-x^2 - y^2)$ for $(x, y) \in \mathbb{R}^2$.
- 2) $f(x, y) = xy \exp(-x^2 - y^2)$ for $(x, y) \in \mathbb{R}^2$.
- 3) $f(x, y) = \exp(x^2 - y^2) - x^2 - y^2$ for $(x, y) \in \mathbb{R}^2$.
- 4) $f(x, y) = \exp(x^2 + y^2) - 4xy$ for $(x, y) \in \mathbb{R}^2$. [cf. **Example 1.5**]
- 5) $f(x, y) = \frac{1 + x^2}{x^2 + y^2 - 2y - 3}$ for $x^2 + y^2 < 2y + 3$.
- 6) $f(x, y) = x^3 + 2y^3$ for $(x, y) \in \mathbb{R}^2$.
- 7) $f(x, y) = 3xy + \ln(1 - x^2 - y^2)$ for $x^2 + y^2 < 1$.

8) $f(x, y) = x^2 - 2x + y^2 + 3y + 5$ for $x^2 + y^2 < 5$.

9) $f(x, y) = x + \tanh y$ for $x^2 + y^2 < 2$.

10) $f(x, y) = x^2y - 2x^2 + 4y^2$ for $|x| < 2$ og $|y| < 1$.

11) $f(x, y) = (7x^2 + 4xy) \exp(-y^2)$ for $|x| < 1$.

A Extrema in open domain of C^∞ -functions.

D Find the stationary points, if any. Check if they are extrema. Check also $f(x, y)$, when (x, y) tends towards the boundary or towards ∞ in the sense $x^2 + y^2 \rightarrow +\infty$.

I 1) The domain is \mathbb{R}^2 . It follows from the different magnitudes of the terms that

$$f(x, y) = (x + y) \exp(-x^2 - y^2) \rightarrow 0 \quad \text{for } x^2 + y^2 \rightarrow +\infty.$$

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The stationary points are found by solving the equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= \exp(-x^2 - y^2) - 2x(x + y) \exp(-x^2 - y^2) \\ &= \{1 - 2x(x + y)\} \exp(-x^2 - y^2) = 0, \\ \frac{\partial f}{\partial y} &= \{1 - 2y(x + y)\} \exp(-x^2 - y^2) = 0,\end{aligned}$$

where we get the latter equation by a symmetric argument.

We conclude from the equations above that

$$2x(x + y) = 1 = 2y(x + y),$$

hence $x \neq 0$, $y \neq 0$, $x + y \neq 0$ and $y = x$, so $4x^2 = 1$. This implies that the stationary points are

$$\left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{og} \quad \left(-\frac{1}{2}, -\frac{1}{2}\right).$$

The values of the function are

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\sqrt{e}} \quad \text{and} \quad f\left(-\frac{1}{2}, -\frac{1}{2}\right) = -\frac{1}{\sqrt{e}},$$

which compared with the examination of the boundary gives that the minimum is

$$M = f\left(-\frac{1}{2}, -\frac{1}{2}\right) = -\frac{1}{\sqrt{e}},$$

and the maximum is

$$S = f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\sqrt{e}}.$$

The function is of course 0 on the line $y = -x$.

2) It follows from the rules of magnitudes that

$$f(x, y) = xy \exp(-x^2 - y^2) \rightarrow 0 \quad \text{for } x^2 + y^2 \rightarrow +\infty.$$

The stationary points are the solutions of the equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= (y - 2x^2y) \exp(-x^2 - y^2) \\ &= y(1 - 2x^2) \exp(-x^2 - y^2) = 0, \\ \frac{\partial f}{\partial y} &= x(1 - 2y^2) \exp(-x^2 - y^2) = 0,\end{aligned}$$

where the latter equations follows by the symmetry.

These equations are reduced to the system

$$y(1 - 2x^2) = 0 \quad \text{og} \quad x(1 - 2y^2) = 0.$$

If $x = 0$, then $y = 0$, hence $(0, 0)$ is a stationary point.

If $x = \pm \frac{1}{\sqrt{2}}$, then $y = \pm \frac{1}{\sqrt{2}}$. We find in total five stationary points,

$$(0, 0), \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

The values of the function at these points are

$$f(0, 0) = 0,$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e},$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}.$$

Summarizing the maximum is (by comparison)

$$S = f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e},$$

and the minimum is

$$M = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}.$$

3) If $x = 0$, then

$$f(0, y) = \exp(-y^2) - y^2 \rightarrow -\infty \quad \text{for } y \rightarrow +\infty.$$

If $y = 0$, then

$$f(x, 0) = \exp(x^2) - x^2 \rightarrow +\infty \quad \text{for } x \rightarrow +\infty.$$

We conclude that the function has neither a maximum nor a minimum in the domain \mathbb{R}^2 .

REMARK. Even though it is superfluous, we shall nevertheless for the exercise show how the possible stationary points are found. The corresponding equations are

$$\frac{\partial f}{\partial x} = 2x\{\exp(x^2 - y^2) - 1\} = 0,$$

$$\frac{\partial f}{\partial y} = -2y\{\exp(x^2 - y^2) + 1\} = 0.$$

It follows from the latter equation that $y = 0$, which put into the former one gives

$$2x\{\exp(x^2) - 1\} = 0.$$

This equation is only fulfilled for $x = 0$, hence $(0, 0)$ is the only stationary point. The value of the function is here $f(0, 0) = 0$, and it is obvious that $f(x, y)$ can be both positive and negative in any neighbourhood of $(0, 0)$, so there exists no point in which a maximum or a minimum can be attained. \diamond

4) It follows from the rules of magnitudes that

$$f(x, y) = \exp(x^2 + y^2) - 4xy \rightarrow +\infty \quad \text{for } x^2 + y^2 \rightarrow +\infty,$$

and the function does not have a maximum.

The possible stationary points are the solutions of the equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x \exp(x^2 + y^2) - 4y = 0, \\ \frac{\partial f}{\partial y} &= 2y \exp(x^2 + y^2) - 4x = 0. \end{aligned}$$

These equations are reduced to

$$(6) \quad \begin{cases} x \exp(x^2 + y^2) = 2y, \\ y \exp(x^2 + y^2) = 2x. \end{cases}$$

We get by adding these equations,

$$(x + y) \exp(x^2 + y^2) = 2(x + y).$$

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It follows from (6) that x and y must either be of the same sign or be 0. Therefore, if $x + y = 0$, then $(x, y) = (0, 0)$.

If $x + y \neq 0$, we get $x^2 + y^2 = \ln 2$. It follows in this case from (6) that

$$x^2 \exp(x^2 + y^2) = 2xy = y^2 \exp(x^2 + y^2),$$

so $x^2 = y^2$, or $y = x$. The stationary points $\neq (0, 0)$ are then satisfying

$$x^2 + y^2 = 2x^2 = \ln 2,$$

i.e.

$$x = y = \pm \sqrt{\frac{\ln 2}{2}}.$$

By insertion into (6) it follows that they are indeed stationary points, so we have three stationary points,

$$(0, 0), \quad \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}} \right), \quad \left(-\sqrt{\frac{\ln 2}{2}}, -\sqrt{\frac{\ln 2}{2}} \right).$$

The values of the function here are $f(0, 0) = 1$ and

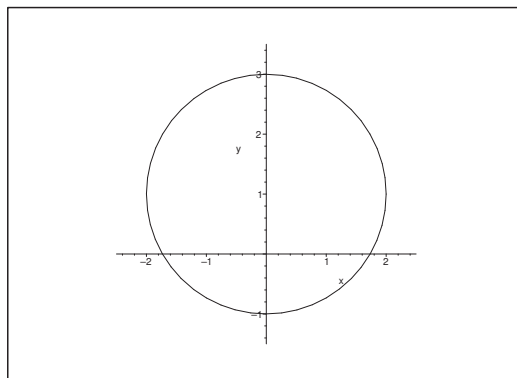
$$f\left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = f\left(-\sqrt{\frac{\ln 2}{2}}, -\sqrt{\frac{\ln 2}{2}}\right) = 2 - 2 \ln 2 = 2(1 - \ln 2).$$

From $\ln 2 > \frac{1}{2}$ follows that the minimum is

$$f\left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = f\left(-\sqrt{\frac{\ln 2}{2}}, -\sqrt{\frac{\ln 2}{2}}\right) = 2(1 - \ln 2).$$

It was mentioned above that the function has no maximum.

- 5) The domain is the open disc of centrum $(0, 1)$ and radius 2.



Clearly, the function

$$f(x, y) = \frac{1 + x^2}{x^2 + y^2 - 2y + 3}, \quad x^2 + y^2 < 2y + 3,$$

tends towards $-\infty$, when (x, y) tends to the circle $x^2 + (y - 1)^2 = 4$ from the inside. Therefore, the minimum does not exist.

The *stationary points* are the solutions of the equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{(x^2 + y^2 - 2y - 3)^2} [2x\{x^2 + y^2 - 2y - 3\} - 2x(1 + x^2)] = 0, \\ \frac{\partial f}{\partial y} &= \frac{1}{(x^2 + y^2 - 2y - 3)^2} \{-(1 + x^2) \cdot 2(y - 1)\} = 0.\end{aligned}$$

Inside the domain, these equations are reduced to

$$x\{(y - 1)^2 - 5\} = 0 \quad \text{and} \quad y - 1 = 0,$$

hence $y = 1$ and $x = 0$. The only stationary point is the centre of the circle $(0, 1)$. This must necessarily be a maximum, because the maximum exists, and it can only be attained at a stationary point (because the function is of class C^∞). Hence the maximum is

$$S = f(0, 1) = -\frac{1}{4}.$$

REMARK 1. One can also find the maximum in the following way without any calculation. Note that the numerator is positive, and the denominator is negative everywhere in the domain. Hence, we shall make the numerator as small as possible (for $x = 0$), and the denominator numerically as big as possible. The denominator can be written $(x - 0)^2 + (y - 1)^2 - 4$, so this situation occurs for $(x, y) = (0, 1)$. Since the optimum possibility for both the numerator and the denominator occurs for at least $x = 0$, we conclude that $(0, 1)$ is a maximum point and that the maximum is

$$S = f(0, 1) = -\frac{1}{4}. \quad \diamond$$

REMARK 2. Another possible solution is the following: If x is kept fixed, then

$$f(x, y) = -\frac{1 + x^2}{4 - x^2 - (y - 1)^2}$$

is largest when $y = 1$, and $(x, 1)$ belongs to the domain, so the only condition on x is $|x| < 2$.

Then the task is reduced to finding the maximum of

$$f(x, 1) = \frac{x^2 + 1}{x^2 - 4} = 1 + \frac{5}{x^2 - 4} \quad \text{for } x^2 \in [0, 4[.$$

This is obtained for $x = 0$, so the maximum is

$$S = f(0, 1) = 1 - \frac{5}{4} = -\frac{1}{4}. \quad \diamond$$

- 6) The function $f(x, y) = x^3 + 2y^3$ has neither a maximum nor a minimum in \mathbb{R}^2 , because

$$f(x, 0) = x^3 \rightarrow \begin{cases} +\infty & \text{for } x \rightarrow +\infty, \\ -\infty & \text{for } x \rightarrow -\infty. \end{cases}$$

We mention – though it is not necessary – that $(0, 0)$ is the only stationary point.

7) Clearly, the function

$$f(x, y) = 3xy + \ln(1 - x^2 - y^2), \quad x^2 + y^2 < 1,$$

tends towards $-\infty$, when (x, y) is approaching the boundary $x^2 + y^2 = 1$ of the unit disc (from the inside). Therefore, the function has no minimum.

Clearly, it has a maximum, because it is continuous on every closed subset of the open unit disc. We may choose this subset such that $f(x, y) = -C$ on the boundary of the subset, where $C > 0$ is any positive constant. According to the second main theorem, f has a maximum on the closed subset, and since $f(0, 0) = 0 > -C$, we cannot have the maximum on the boundary. Since f is of class C^∞ , the maximum must be attained at a stationary point.

The *stationary points* are the solutions of the equations

$$(7) \quad \begin{cases} \frac{\partial f}{\partial x} = 3y - \frac{2x}{1 - x^2 - y^2} = 0, \\ \frac{\partial f}{\partial y} = 3x - \frac{2y}{1 - x^2 - y^2} = 0. \end{cases}$$

Clearly, $(0, 0)$ is a stationary point. It is almost obvious (due to the variation of $3xy$ in a neighbourhood of $(0, 0)$) that the maximum is *not* attained at $(0, 0)$.

We shall now find the stationary points $\neq (0, 0)$, which must exist, cf. the discussion above. According to (7), such stationary points must satisfy

$$3y^2 = \frac{2xy}{1 - x^2 - y^2} = 3x^2,$$

thus $y^2 = x^2$. It follows from $xy > 0$ that x and y must have the same sign, so we conclude that $y = x$. By eliminating y we get

$$0 = 3y(1 - x^2 - y^2) - 2x = x\{3 - 6x^2 - 2\} = x\{1 - 6x^2\}.$$

Since $x \neq 0$, we get $x = y = \pm \frac{1}{\sqrt{6}}$, and the stationary points are

$$(0, 0), \quad \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \quad \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right).$$

The corresponding values are $f(0, 0) = 0$ (found previously, and we have already shown that this cannot be a maximum), and

$$f\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = f\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = \frac{1}{2} - \ln \frac{3}{2},$$

which then necessarily must be the maximum. (This follows also from that $\ln \frac{3}{2} < \frac{1}{2}$).

8) Let

$$f(x, y) = x^2 - 2x + y^2 - 3y + 5 = (x - 1)^2 + \left(y - \frac{3}{2}\right)^2 + \frac{7}{4}$$

for $x^2 + y^2 < (\sqrt{5})^2$.

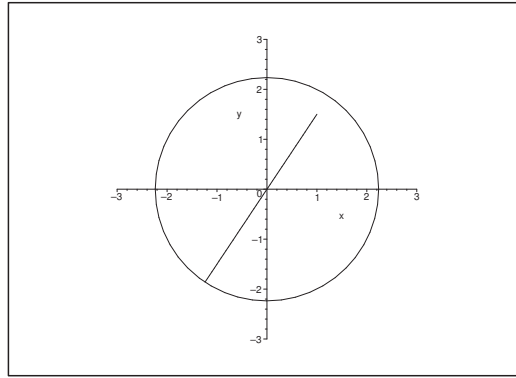


Figure 6: The domain with the longest possible line through $(1, \frac{3}{2})$ inside the domain.

Interpret $(x-1)^2 + \left(y - \frac{3}{2}\right)^2$ as the square of the distance from (x, y) to the point $\left(1, \frac{3}{2}\right)$ in the domain. Clearly, $\left(1, \frac{3}{2}\right)$ is the only stationary point. It follows from the rearrangement that this is a minimum point corresponding to the minimum

$$M = f\left(1, \frac{3}{2}\right) = \frac{7}{4}.$$

The function $f(x, y)$ has a continuous extension to the closure of the domain given by $x^2 + y^2 \leq 5$. It follows from the above that the maximum exists (the second main theorem) and since there are no stationary points at hand, it must be attained at a boundary point. Then it follows from the geometric interpretation above that the maximum is obtained at the intersection of the line of the equation $y = \frac{3}{2}x$ and the boundary given by $x^2 + y^2 = 5$ in the third quadrant. This implies that the function does not have a maximum in the given open domain.

REMARK 1. For completeness it should be mentioned that the intersection point is

$$\left(-2\sqrt{\frac{5}{13}}, -3\sqrt{\frac{5}{13}}\right).$$

The distance from this point to $\left(1, \frac{3}{2}\right)$ is of geometrical reasons $\frac{1}{2}\sqrt{13} + \sqrt{5}$, so the maximum on the boundary (hence also in the *closed* domain is

$$f\left(-2\sqrt{\frac{5}{13}}, -3\sqrt{\frac{5}{13}}\right) = \left\{\frac{1}{2}\sqrt{13} + \sqrt{5}\right\}^2 + \frac{7}{4} = \frac{13}{4} + 5 + \sqrt{65} + \frac{7}{4} = 10 + \sqrt{65}. \quad \diamond$$

REMARK 2. It should be obvious that if one did not use the geometric interpretation above, then the task would give quite unpleasant computations. It is left to the reader to find the maximum on the boundary by inserting the parametric description

$$x = \sqrt{5} \cos t, \quad y = \sqrt{5} \sin t, \quad t \in [0, 2\pi[. \quad \diamond$$

- 9) Clearly, the function $f(x, y) = x + \tanh y$, $x^2 + y^2 < 2$, has no stationary point (e.g. $\frac{\partial f}{\partial x} = 1 \neq 0$), and since the function is of class C^∞ , and the domain is open (i.e. without boundary points) the function has neither a minimum nor a maximum in the domain.

- 10) Clearly, the function

$$f(x, y) = x^2y - 2x^2 + 4y^2, \quad |x| < 2, |y| < 1,$$

is of class C^∞ , so a possible maximum or minimum can only be attained at a stationary point.

The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 2xy - 4x = 2x(y - 2) = 0, \quad \frac{\partial f}{\partial y} = x^2 + 8y = 0.$$

It follows from the former equation that either $x = 0$ or $y = 2$.

- a) If $x = 0$, then we get $y = 0$ from the latter equation and $(0, 0)$ is a stationary point.

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b) If $y = 2x$ is put into the latter equation, we get

$$0 = x^2 + 16x = x(x + 16),$$

so either $x = 0$ (and $y = 0$ again as above) or $x = -16$. The latter is not possible inside the domain.

The only stationary point is $(0, 0)$. The value of the function is here

$$f(0, 0) = 0.$$

We conclude the task in the following way: The two restrictions

$$f(0, y) = 4y^2, \quad |y| < 1,$$

and

$$f(x, 0) = -2x^2, \quad |x| < 2,$$

attain both positive and negative values at any point close to $(0, 0)$, hence $f(0, 0) = 0$ is neither a maximum nor a minimum. Since $(0, 0)$ is the only possibility of extremum, there does not exist any.

ALTERNATIVELY (the standard procedure) the function is extended continuously to the boundary, and then we continue by examining the values on the boundary.

a) If $x = -2$ and $y \in [-1, 1]$ the restriction is

$$g_1(y) = f(-2, y) = 4y - 8 + 4y^2$$

where $g_1'(y) = 4 - 8y = 0$ for $y = \frac{1}{2}$. The value is

$$f\left(-2, \frac{1}{2}\right) = g_1\left(\frac{1}{2}\right) = 4 \cdot \frac{1}{2} - 8 + 4\left(\frac{1}{2}\right)^2 = 2 - 8 + 1 = -5.$$

At the end points of the interval we get the values of the function

$$f(-2, -1) = g_1(-1) = -4 - 8 + 4 = -8,$$

$$f(-2, 1) = g_1(1) = 4 - 8 + 4 = 0.$$

b) If $y = 1$ and $x \in [-2, 2]$ the restriction is

$$g_2(x) = f(x, 1) = x^2 - 2x^2 + 4 = 4 - x^2,$$

where $g_2(x) = -2x = 0$ for $x = 0$. The relevant values of the function are

$$f(-2, 1) = g_2(-2) = 4 - 8 + 4 = 0,$$

$$f(0, 1) = g_2(0) = 4,$$

$$f(2, 1) = g_2(2) = 4 - 8 + 4 = 0.$$

Even if we have not examined the remaining two boundary curves, we can finish the task now, because $(0, 0)$ is the only stationary point with the value of the function $f(0, 0) = 0$. The boundary does *not* belong to the domain, and by continuous extensions we find the values at specially chosen boundary points

$$f(-2, -1) = -8 < f(0, 0) = 0 < f(0, 1) = 4.$$

Accordingly, f has neither a maximum nor a minimum in the open domain. In fact, due to the continuity we can inside the domain obtain values of the function as close to -8 as well to 4 as we wish.

11) The function

$$f(x, y) = (7x^2 + 4xy) \exp(-y^2), \quad |x| < 1,$$

is of class C^∞ , and it can be extended by the same definition to all of \mathbb{R}^2 . This domain is open, so a possible maximum or minimum can only be attained at a stationary point.

Possible stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = (14x + 4y) \exp(-y^2) = 0, \quad \frac{\partial f}{\partial y} = \{4x - 2y(7x^2 + 4xy)\} \exp(-y^2).$$

Now $\exp(-y^2) \neq 0$, so these equations are equivalent to

$$7x = -2y \quad \text{and} \quad x\{2 - 7xy - 4y^2\} = 0.$$

If we only eliminate $7x$ in the term $7xy$ of the latter equation, we get

$$0 = x\{2 + 2y^2 - 4y^2\} = 2x\{1 - y^2\}.$$

Combining this with the equation $7x = -2y$ we get the possibilities

$$x = 0, \quad \text{i.e. } y = 0, \quad \text{hence } (x, y) = (0, 0),$$

$$y = 1, \quad \text{i.e. } x = -\frac{2}{7}, \quad \text{hence } (x, y) = \left(-\frac{2}{7}, 1\right),$$

$$y = -1, \quad \text{i.e. } x = \frac{2}{7}, \quad \text{hence } (x, y) = \left(\frac{2}{7}, -1\right).$$

All three stationary points lie in the open domain, and the values of the function are

$$f(0, 0) = 0,$$

and

$$f\left(-\frac{2}{7}, 1\right) = f\left(\frac{2}{7}, -1\right) = \left(7 \cdot \frac{4}{49} - 4 \cdot \frac{2}{7} \cdot 1\right) \exp(-1) = -\frac{6}{7e}.$$

EXAMINATION OF THE BOUNDARY. As mentioned above the function can be extended to the boundary. As $f(-x, y) = f(x, y)$, it suffices to examine the restriction

$$g(y) = f(1, y) = (7 + 4y) \exp(-y^2), \quad y \in \mathbb{R}.$$

From

$$\begin{aligned} g'(y) &= \{4 - 2y(7 + 4y)\} \exp(-y^2) = (4 - 14y - 8y^2) \exp(-y^2) \\ &= \left\{ \left(y + \frac{7}{8}\right)^2 - \left(\frac{9}{8}\right)^2 \right\} \exp(-y^2), \end{aligned}$$

follows $g'(y) = 0$ for

$$y = -\frac{7}{8} \pm \frac{9}{8} = \begin{cases} \frac{1}{4}, \\ -2, \end{cases}$$

corresponding to the boundary values

$$f\left(1, \frac{1}{4}\right) = g\left(\frac{1}{4}\right) = 8 \exp\left(-\frac{1}{16}\right) > 0,$$

and

$$f(1, -2) = g(-2) = -\exp(-4).$$

Finally, it follows by the different magnitudes of the terms that

$$f(x, y) \rightarrow 0 \quad \text{for } |y| \rightarrow +\infty.$$

Then by a numerical comparison

$$\begin{aligned} -\frac{6}{7e} &= f\left(-\frac{2}{7}, 1\right) = f\left(\frac{2}{7}, -1\right) < -\frac{1}{e^4} = f(1, -2) < 0 = f(0, 0) \\ &< 8 \exp\left(-\frac{1}{16}\right) = f\left(1, \frac{1}{4}\right). \end{aligned}$$

Since $\left(1, \frac{1}{4}\right)$ is a boundary point which is not included in the domain, and $f\left(1, \frac{1}{4}\right)$ is the maximum in the closure, the function has no maximum in the domain given by $|x| < 1$.

On the other hand,

$$M = f\left(-\frac{2}{7}, 1\right) = f\left(\frac{2}{7}, -1\right) = -\frac{6}{7e}$$

is a minimum in both the closed and open set, so the minimum exists.

Example 1.4 Find in each of the following cases the largest volume of a rectangular box for which the indicated condition is fulfilled.

- 1) The sum of the 12 edges is given equal to $12a$.
- 2) The area of the surface of the box is given and equal to $6a^2$.
- 3) The length of the space diagonal of the box is given and equal to a .


A Maximum.

D Put the box into a rectangular coordinate system with one corner at $(0,0,0)$ and where the corresponding edges lie along the axes in the positive sense, i.e. the box can be described as the domain

$$[0, x] \times [0, y] \times [0, z].$$

Find the volume as a function of the edges. Exploit the condition in each sub-question to eliminate one of the variables. Indicate the domain, in which the remaining edges can vary. Finally, find the maximum.

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I When the lengths of the edges are $x, y, z \geq 0$, then the volume is given by

$$V(x, y, z) = xyz.$$

1) The condition that the sum of the 12 edges is equal to $12a$ is written

$$4(x + y + z) = 12a, \quad \text{i.e.} \quad x + y + z = 3a.$$

REMARK. For symmetrical reasons we may expect that the solution is given by $x = y = z = a$. The remaining part of the task is to prove that this hunch in this case is correct. \diamond

First eliminate z ,

$$z = 3a - x - y \geq 0, \quad \text{i.e.} \quad x \geq 0, \quad y \geq 0, \quad x + y \leq 3a.$$

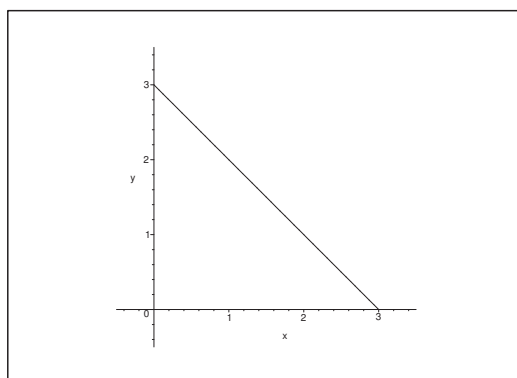


Figure 7: The triangle in which we shall find the maximum of $f(x, y) = xy(3a - x - y)$.

By insertion we see that we shall find the maximum of the function

$$f(x, y) = xy(3a - x - y), \quad x \geq 0, y \geq 0, x + y \leq 3a.$$

We have $f(x, y) > 0$ in the interior of the triangle, and on the boundary we have $f(x, y) = 0$. Now $f(x, y)$ is of class C^∞ , so the maximum exists (second main theorem) and it can only be attained at an interior stationary point.

The equations of the stationary points are

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3ay - 2xy - y^2 = y(3a - 2x - y) = 0, \\ \frac{\partial f}{\partial y} &= x(3a - x - 2y), \end{aligned}$$

where we immediately get the latter expression by the symmetry.

Since $x > 0$ and $y > 0$ in the interior of the triangle A we get the reduced equations

$$2x + y = 3a \quad \text{and} \quad x + 2y = 3a,$$

and we find (as we guessed) $x = y = a$. The only stationary point is (a, a) corresponding to $z = 3a - a - a = a$, and as mentioned above this corresponds to a maximum,

$$S = V(a, a, a) = a^3.$$

2) In this case the condition is

$$2(xy + yz + xz) = 6a^2.$$

REMARK. Due to the strong symmetry we may again expect the solution to be $x = y = z = a$. This satisfies at least the condition. The remaining part of the task is to prove that also this hunch is correct. \diamond .

It follows from the condition that

$$z = \frac{3a^2 - xy}{x + y},$$

where $xy \leq 3a^2$, $x \geq 0$, $y \geq 0$ and $(x, y) \neq (0, 0)$, hence a troublesome expression.

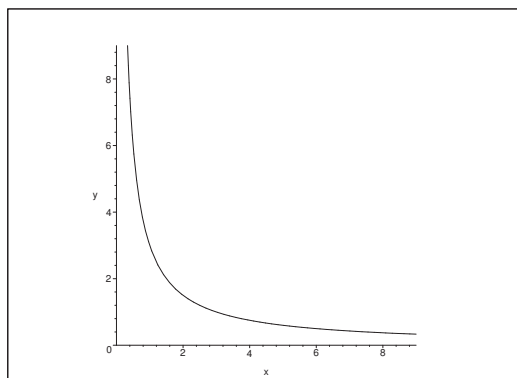


Figure 8: The infinite domain of the function in **Example 1.4.2**.

The volume function is given in this domain by

$$f(x, y) = xy \cdot \frac{3a^2 - xy}{x + y} = \frac{3a^2xy - x^2y^2}{x + y}.$$

We conclude from

$$0 \leq f(x, y) \leq \frac{y}{x + y} \cdot 3a^2 \cdot x \leq x \cdot 3a^2,$$

that $f(x, y)$ can be extended by continuity from the first quadrant to the positive X and Y axes supplied by the point $(0, 0)$ by putting the value of the function equal to 0. We have also the value 0 of the function of the branch of the hyperbola $xy = 3a^2$ which lies in the first quadrant. Finally, $f(x, y) \rightarrow 0$, when either $x \rightarrow +\infty$ or $y \rightarrow +\infty$ inside the open domain. Since $f(x, y) > 0$ is of class C^∞ in the open domain, it follows from the second main theorem that $f(x, y)$ has a maximum, which necessarily must be attained at a stationary point in the interior of the domain.

The equations of the stationary points are

$$\begin{aligned}
 \frac{\partial f}{\partial x} &= \frac{1}{(x^2 + y^2)} \{ (3a^2y - 2xy^2)(x + y) - (3a^2xy - x^2y^2) \} \\
 &= \frac{y}{(x + y)^2} \{ 3a^2(x + y) - 2x^2y - 2xy^2 - 3a^2x + x^2y \} \\
 &= \frac{y}{(x + y)^2} \{ 3a^2y - x^2y - 2xy^2 \} \\
 &= \frac{y^2}{(x + y)^2} \{ 3a^2 - x^2 - 2xy \} = 0,
 \end{aligned}$$

and of symmetrical reasons,

$$\frac{\partial f}{\partial y} = \frac{x^2}{(x + y)^2} \{ 3a^2 - y^2 - 2xy \} = 0.$$

Since $x \neq 0$, $y \neq 0$ and $x + y \neq 0$ in the interior of the domain, these equations are reduced to

$$x^2 + 2xy = 3a^2, \quad y^2 + 2xy = 3a^2,$$

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accordingly $x^2 = y^2$, or $y = x$, because both x and y are positive. This implies that $3x^2 = 3a^2$, so $x = y = a$, and then finally,

$$z = \frac{3a^2 - a^2}{2a} = a.$$

Since (a, a) is the only stationary point, the maximum must be attained here. We find as expected that the maximum is

$$S = V(a, a, a) = a^3.$$

- 3) The condition is here that the length of the space diagonal of the box is equal to a , i.e.

$$\sqrt{x^2 + y^2 + z^2} = a.$$

REMARK. As before the strong symmetry suggests that the solution must satisfy $x = y = z = \frac{a}{\sqrt{3}}$. We shall again prove this hunch. \diamond

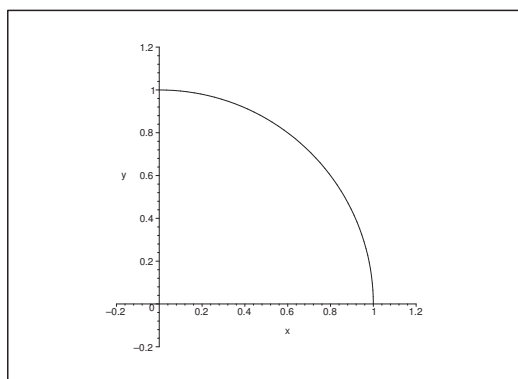


Figure 9: The domain of the function $xy\sqrt{a^2 - x^2 - y^2}$.

It follows that

$$z = \sqrt{a^2 - x^2 - y^2}, \quad x \geq 0, y \geq 0, x^2 + y^2 \leq a^2.$$

By insertion we get that the task is reduced to finding the maximum of the function

$$f(x, y) = xy\sqrt{a^2 - x^2 - y^2}$$

in the closed quarter disc

$$A = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq a^2\}.$$

Clearly, $f(x, y) = 0$ on the boundary ∂A of A . We have $f(x, y) > 0$ in the interior of A . Since f is of class C^∞ in the interior of A , and is continuous on A , it follows from the second main theorem that the maximum (which exists) must be attained at a stationary point in A° .

The stationary points satisfy the equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= y\sqrt{a^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{a^2 - x^2 - y^2}} = \frac{y}{\sqrt{a^2 - x^2 - y^2}}\{a^2 - 2x^2 - y^2\} = 0, \\ \frac{\partial f}{\partial y} &= \frac{x}{\sqrt{a^2 - x^2 - y^2}}\{a^2 - x^2 - 2y^2\} = 0,\end{aligned}$$

where the latter equation follows by the symmetry.

We then derive the equations

$$y = 0 \quad \text{or} \quad 2x^2 + y^2 = a^2,$$

and

$$x = 0 \quad \text{or} \quad x^2 + 2y^2 = a^2.$$

Clearly, one of the coordinates must be 0, so

$$(0, 0), \quad (a, 0), \quad (-a, 0), \quad (0, a), \quad (0, -a),$$

are all the possible stationary points. Unfortunately they all lie outside the interior of the domain, so none of the counts in the following. (Three of them lie on the boundary, because $f(x, y) = 0$, and the remaining two points do not lie at all in the closure of A).

Our only possibility is obtained when

$$2x^2 + y^2 = a^2 \quad \text{and} \quad x^2 + 2y^2 = a^2.$$

We get by a subtraction, $y^2 = x^2$, so $x^2 = y^2 = \frac{a^2}{3}$. Now $x^2 + y^2 = \frac{2}{3}a^2 < a^2$, so $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$ is the only stationary point in A° , corresponding to the maximum

$$S = f\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = \frac{a^3}{3\sqrt{3}} \quad \left[= \left(\frac{a}{\sqrt{3}}\right)^3 \right],$$

and we see that $x = y = z = \frac{a}{\sqrt{3}}$ in agreement with our earlier hunch.

Example 1.5 Show by applying polar coordinates that the function

$$f(x, y) = \exp(x^2 + y^2) - 4xy, \quad y \geq 0, \quad y \geq 0,$$

has a minimum and find this minimum. [cf. **Example 1.3.4**].

A Minimum by polar coordinates.

D Introduce the polar coordinates. Find the stationary points. Examine the boundary and what happens when $\varrho \rightarrow +\infty$.

I The domain is the first quadrant. This is described in polar coordinates by

$$x = \varrho \cos \varphi, \quad y = \varrho \sin \varphi, \quad \varrho \in [0, +\infty[, \quad \varphi \in \left[0, \frac{\pi}{2}\right].$$

The function is now

$$g(\varrho, \varphi) = f(x, y) = \exp(\varrho^2) - 4\varrho^2 \cos \varphi \sin \varphi = \exp(\varrho^2) - 2\varrho^2 \sin 2\varphi.$$

The equations of the stationary points become

$$(8) \quad \begin{cases} \frac{\partial g}{\partial \varrho} = 2\varrho \exp(\varrho^2) - 4\varrho \sin 2\varphi = 2\varrho \{\exp(\varrho^2) - 2 \sin 2\varphi\} = 0, \\ \frac{\partial g}{\partial \varphi} = -4\varrho^2 \cos 2\varphi = 0. \end{cases}$$

It follows from $\varrho > 0$ in the interior of the domain and the latter equation of (8) that the condition is

$$\cos 2\varphi = 0, \quad \text{i.e.} \quad \sin 2\varphi = \pm 1,$$

where the former equation of (8) shows that only $\sin 2\varphi = +1$ can be applied. Then

$$\varphi = \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right].$$

We get by insertion into for former equation of (8),

$$\exp(\varrho^2) = 2, \quad \text{dvs.} \quad \varrho = \sqrt{\ln 2}.$$

Thus, the only stationary point is

$$(\varrho, \varphi) = \left(\sqrt{\ln 2}, \frac{\pi}{4}\right),$$

corresponding in rectangular coordinates to

$$(x, y) = \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right).$$

The value of the function is here

$$g\left(\sqrt{\ln 2}, \frac{\pi}{4}\right) = \exp(\ln 2) - 2 \ln 2 = 2(1 - \ln 2).$$

If $\varrho \rightarrow +\infty$, then obviously $g(\varrho, \varphi) \rightarrow +\infty$.

We have on the boundary curves $\varphi = 0$ and $\varphi = \frac{\pi}{2}$, $\varrho \in [0, +\infty[$, that

$$g(\varrho, 0) = g\left(\varrho, \frac{\pi}{2}\right) = \exp(\varrho^2), \quad \varrho \in [0, +\infty[,$$

so the minimum value is here $g(0, 0) = 1 > 2(1 - \ln 2)$.

Summarizing, $f(x, y) = g(\varrho, \varphi)$ has a minimum in A for

$$(\varrho, \varphi) = \left(\sqrt{\ln 2}, \frac{\pi}{4} \right),$$

corresponding in rectangular coordinates to

$$(x, y) = \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}} \right).$$

The value of the function is here

$$g\left(\sqrt{\ln 2}, \frac{\pi}{4}\right) = f\left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = 2(1 - \ln 2),$$

in accordance with the result of **Example 1.3.4**.

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ALTERNATIVELY it is also possible only to use rectangular coordinates, and this is actually not that difficult. The equations of the stationary points

$$\frac{\partial f}{\partial x} = 2x \exp(x^2 + y^2) - 4y = 0, \quad \frac{\partial f}{\partial y} = 2y \exp(x^2 + y^2) - 4x = 0,$$

i.e.

$$x \exp(x^2 + y^2) = 2y, \quad y \exp(x^2 + y^2) = 2x,$$

so

$$2y^2 = xy \exp(x^2 + y^2) = 2x^2,$$

and hence $y = x$, because we only consider the open first quadrant. Then

$$x \exp(2x^2) = 2x, \quad \text{i.e.} \quad \exp(2x^2) = 2,$$

hence

$$y = x = +\sqrt{\frac{\ln 2}{2}}.$$

The only stationary point in the open first quadrant is

$$(x, y) = \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}} \right).$$

We have on the boundary either $f(x, 0) = \exp(x^2)$, or $f(0, y) = \exp(y^2)$, with the minimum value $f(0, 0) = 1 > 2(1 - \ln 2)$, so the minimum value is attained at the stationary point in the first quadrant.

Example 1.6 Find the maximum and the minimum of the function

$$f(x, y, z) = x^2 + y^2 + z^2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad x + y + z \leq 1.$$

A Maximum and minimum.

D The existence follows from the second main theorem. Then either argue geometrically, or use the standard method.

I The function $f(x, y, z) = x^2 + y^2 + z^2$ is continuous on the given closed and bounded domain. It follows from the second main theorem that f has both a maximum and a minimum.

1) THE ELEGANT GEOMETRICAL SOLUTION. The value

$$f(x, y, z) = x^2 + y^2 + z^2$$

can be interpreted as the square of the distance from $(0, 0, 0)$ to (x, y, z) . This distance is smallest ($= 0$) for $(x, y, z) = (0, 0, 0)$, and largest ($= 1$) at the other corners $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ of the tetrahedron which describes A .

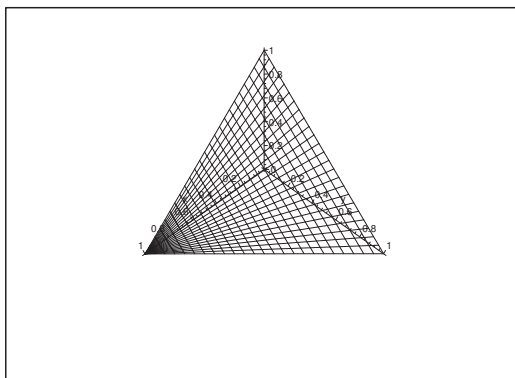
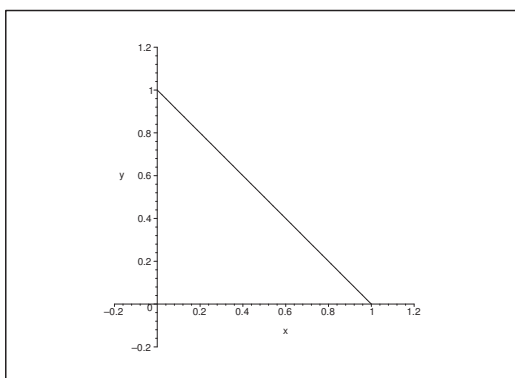


Figure 10: The domain.

Figure 11: The projection onto the XY plane.

- 2) THE STANDARD PROCEDURE. It follows immediately that $(0, 0, 0)$ is the only possible stationary point, but since it lies on the boundary it is a matter of definition if it should be counted as a stationary point or not. It will always be treated as a boundary point, when we examine the boundary.

EXAMINATION OF THE BOUNDARY.

- a) If $z = 0$, then $f(x, y, 0) = x^2 + y^2$, which on the triangle in the XY plane is smallest at $(0, 0)$ and largest at $(1, 0)$ and $(0, 1)$.

This is immediately seen, and it can also be obtained by another examination of the boundary in the XY plane on the triangle shown on the figure.

- b) Due to the symmetry the same is true for the surfaces $x = 0$ and $y = 0$.
 c) If $z = 1 - x - y$, then we get the same parametric domain as the domain above on the figure in the XY plane, and the restriction is given by

$$g(x, y) = f(x, y, 1 - x - y) = x^2 + y^2 + (1 - x - y)^2.$$

The equations of the possible stationary points are

$$\frac{\partial g}{\partial x} = 2x - 2(1 - x - y) = 2(2x + y - 1) = 0,$$

$$\frac{\partial g}{\partial y} = 2(x + 2y - 1) = 0,$$

(the latter equation by symmetry). Accordingly, $y = x$ and $3(x + y) = 2$, i.e. $x = y = \frac{1}{3}$.

Then by insertion,

$$g\left(\frac{1}{3}, \frac{1}{3}\right) = 3 \cdot \frac{1}{3^3} = \frac{1}{3}.$$

i) We get on the boundary $x + y = 1$,

$$g(x, 1 - x) = x^2 + (1 - x)^2, \quad x \in [0, 1],$$

with its minimum

$$g\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \cdot \frac{1}{4} = \frac{1}{2} \quad \text{for } x = \frac{1}{2},$$

and its maximum for $x = 0$ or $x = 1$, corresponding to

$$g(0, 1) = g(1, 0) = 1.$$

ii) We also get the restriction

$$\varphi(x) = x^2 + (1 - x)^2, \quad x \in [0, 1],$$

on the boundary $y = 0$, and on the boundary $x = 0$ we get

$$\psi(y) = y^2 + (1 - y)^2, \quad y \in [0, 1].$$

Anyone of these will lead to precisely the same investigation as above.

By comparison of these values we conclude that the minimum is

$$M = f(0, 0, 0) = 0,$$

and the maximum is

$$S = f(1, 0, 0) = f(0, 1, 0) = f(0, 0, 1) = 1.$$

The latter method is rather troublesome compared to the geometric interpretation.

Example 1.7 *Prove that the function*

$$f(x, y) = x^2 + 2y^2 - 2x, \quad (x, y) \in \mathbb{R}^2,$$

has both a maximum and a minimum on the point set

$$A = \{(x, y) \mid x \geq 0, x^2 + y^2 \leq 2\},$$


and find these values.

A The second main theorem. Maximum and minimum.

D Apply the second main theorem for continuous functions. Then find the maximum and minimum.

I Since $f(x, y)$ is real and continuous on the closed and bounded domain A , it follows from the second main theorem for continuous functions that f has a maximum and a minimum on A . This proves the existence. We shall in the following give two methods for the explicit determination of these expressions.

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1) GEOMETRICAL CONSIDERATION. First rewrite f in the following way,

$$f(x, y) = x^2 + 2y^2 - 2x = (x - 1)^2 + 2y^2 - 1 = 2 \left\{ \left(\frac{x - 1}{\sqrt{2}} \right)^2 + y^2 \right\} - 1.$$

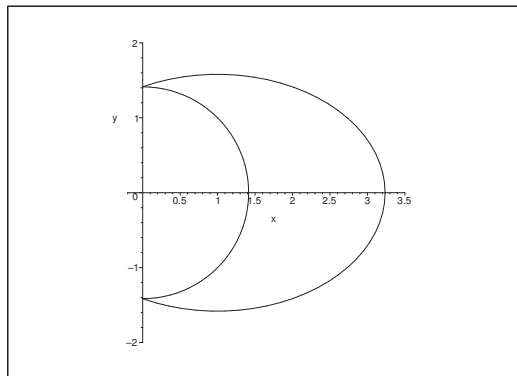


Figure 12: The largest possible ellipse, which intersects (touches) the domain.

It follows immediately from the first rearrangement that the minimum is attained at the point $(1, 0) \in A$ of the value of the function

$$M = f(1, 0) = -1.$$

From the latter rearrangement follows that

$$f(x, y) = 2\alpha^2 - 1$$

is constant for every point on the ellipse of the equation

$$\left(\frac{x - 1}{\sqrt{2}} \right)^2 + y^2 = \alpha^2.$$

Therefore, the maximum must be attained on the largest of these ellipses (characterized by $\alpha > 0$ being largest), which has points in common with A . It follows immediately from the figure that these maximum points must be $(x, y) = (0, \pm\sqrt{2})$ on the Y axis. Hence the maximum value is attained at these points,

$$S = f(0, \pm\sqrt{2}) = 4.$$

2) THE STANDARD PROCEDURE. The function is of class C^∞ . Therefore we shall only find the stationary points in A° followed by an examination of the boundary and numerical comparisons.

a) STATIONARY POINTS. The possible stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 2(x - 1) = 0, \quad \frac{\partial f}{\partial y} = 4y = 0.$$

It follows that $(1, 0)$ is the only stationary point in A° . The value of the function is here

$$f(1, 0) = -1.$$

b) EXAMINATION OF THE BOUNDARY.

i) The restriction on the boundary $x = 0$, $y \in [-\sqrt{2}, \sqrt{2}]$, is

$$f(0, y) = 2y^2,$$

which has its minimum $f(0, 0) = 0$, and its maximum

$$f(0, \sqrt{2}) = f(0, -\sqrt{2}) = 4.$$

ii) The restriction on the boundary curve $x^2 + y^2 = 2$, $x \in [0, \sqrt{2}]$, is

$$\begin{aligned} g(x) &= f(x, y) = x^2 + 2y^2 - 2x = 2(x^2 + y^2) - x^2 - 2x \\ &= 4 + 1 - (x + 1)^2 = 5 - (x + 1)^2, \end{aligned}$$

which has the minimum

$$g(\sqrt{2}) = 5 - (\sqrt{2} + 1)^2 = 2 - 2\sqrt{2} = -2(\sqrt{2} - 1) > -1,$$

and its maximum $g(0) = 4$, corresponding to $y = \pm\sqrt{2}$.

c) NUMERICAL COMPARISON. Summarizing, the minimum in A is

$$M = f(1, 0) = -1,$$

and the maximum is

$$S = f(0, \sqrt{2}) = f(0, -\sqrt{2}) = 4.$$

Example 1.8 Given the function

$$f(x, y) = x^2 + y^2 + e^{xy}, \quad (x, y) \in \mathbb{R}^2.$$

First show that f does not have a maximum in \mathbb{R}^2 . Then prove the following:

1) The function f has a minimum M on the disc $A = \overline{K}((0, 0); 10)$.

2) M is smaller than 100.

3) We have the estimate $f(x, y) \geq 100$ in the point set $\mathbb{R}^2 \setminus A$.

Finally, check if f has a minimum in \mathbb{R}^2 .

A Extrema.

D Prove that $f(x, y) \rightarrow +\infty$ for $x^2 + y^2 \rightarrow +\infty$. Then prove 1) – 3). Finally, argue for a minimum.

I Clearly,

$$(9) \quad f(x, y) = x^2 + y^2 + e^{xy} > x^2 + y^2,$$

thus $f(x, y) \rightarrow +\infty$ for $x^2 + y^2 \rightarrow +\infty$. It follows that $f(x, y)$ has no maximum in \mathbb{R}^2 .

1) Since $f(x, y)$ is continuous and $A = \overline{K}((0, 0); 10)$ is a closed and bounded set, $f(x, y)$ has according to the second main theorem for continuous functions a minimum value M on A .

2) Clearly, $M \leq f(0, 0) = 1 < 100$.

3) If $(x, y) \in \mathbb{R}^2 \setminus A$, then it follows directly from (9) that

$$f(x, y) > 10^2 = 100.$$

Now $\mathbb{R}^2 = A \cup (\mathbb{R}^2 \setminus A)$, and $f(x, y) > 100$ on the entire $\mathbb{R}^2 \setminus A$, while there are points in A , for which $f(x, y) < 100$. Therefore, a possible minimum must lie in A , and since it exists according to 1) and is equal to M , we conclude that f has the minimum M in all of \mathbb{R}^2 .

ADDITIONAL REMARK. Since $f(x, y)$ is of class C^∞ , the minimum M is attained at a stationary point. The equations of the stationary points are

$$(10) \quad \begin{cases} \frac{\partial f}{\partial x} = 2x + ye^{xy} = 0, \\ \frac{\partial f}{\partial y} = 2y + xe^{xy} = 0, \end{cases}$$

which has only $(0, 0)$ as a stationary point, so

$$M = f(0, 0) = 1.$$

That $(0, 0)$ is the only stationary point follows from the following: Clearly, $xy \leq 0$. We have according to (10),

$$2x^2 = -xye^{xy} = 2y^2,$$

so $y = -x$. By insertion into the former equation of (10) we get

$$0 = 2x - x \exp(-x^2) = x(2 - \exp(-x^2)).$$

From $\exp(-x^2) < 2$ follows that $x = 0$, and thus $y = 0$, so $(0, 0)$ is the only stationary point. \diamond

Example 1.9 It is well-known that if a C^1 -function $g : \mathbb{R} \rightarrow \mathbb{R}$ has precisely one stationary point x_0 , which is a local minimum point x_0 , then g has a global minimum at x_0 . Show that no such result exists in general for functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, by considering

$$f(x, y) = x^2 + y^2(1 + x)^3, \quad (x, y) \in \mathbb{R}^2.$$

A Another illustration of the difference between one and several variables.

D Show that the given function f has precisely one stationary point in which there is a local minimum, and that this minimum is *not* a global minimum.

I When

$$f(x, y) = x^2 + y^2(1 + x)^3,$$

we get the following equations of the stationary points,

$$\frac{\partial f}{\partial x} = 2x + 3y^2(1 + x)^2 = 0, \quad \frac{\partial f}{\partial y} = 2y(1 + x)^3 = 0.$$

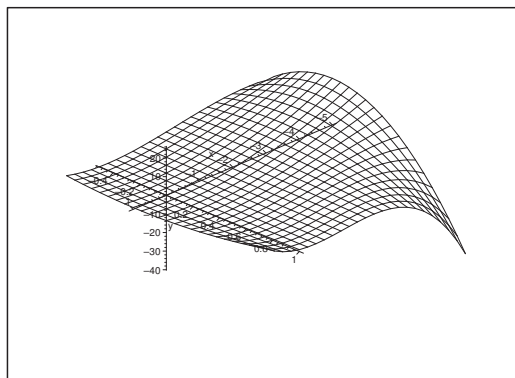


Figure 13: How the surface of the graph might look like for a counterexample.

The latter equation shows that either $y = 0$ or $x = -1$.

If we put $y = 0$ into the former equation, we get $x = 0$, so $(0, 0)$ is a stationary point.

If $x = -1$, we conclude from the former equation that $\frac{\partial f}{\partial x} = -2 \neq 0$.

This shows that $(0, 0)$ is the only stationary point. The value of the function is here $f(0, 0) = 0$.

Clearly, the approximating polynomial of at most second degree from $(0, 0)$ is

$$P_2(x, y) = x^2 + y^2.$$

This structure shows that $(0, 0)$ is a (local) minimum point.

We then get along the restriction $y = 1$ that

$$f(x, 1) = x^2 + (x + 1)^3 \rightarrow -\infty \quad \text{for } x \rightarrow -\infty,$$

so $f(x, y)$ does not have a global minimum.

Example 1.10 Find the range of

$$f(x, y) = -3y + 4y^2 + x^2y + y^3$$

on the open disc $K((0, 0); 1)$, and on \mathbb{R}^2 , resp.

A Ranges.

D Find the possible stationary points; examine the boundary. Apply the main theorems for continuous functions.

I The function is of class C^∞ in \mathbb{R}^2 . Restricted to the closed disc $\overline{K}((0, 0); 1)$ we have according to the second main theorem both a maximum and a minimum, and these are either attained at a stationary point or on the boundary.

Since the domain is connected, it follows from the first main theorem for continuous functions that the range is an interval.

STATIONARY POINTS. The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 2xy = 0, \quad \frac{\partial f}{\partial y} = -3 + 8y + x^2 + 3y^2 = 0.$$

If $y = 0$, then $x = \pm\sqrt{3}$.

If $x = 0$, then $3y^2 + 8y - 3 = 0$, i.e. either $y = -3$ or $y = \frac{1}{3}$.

The stationary points are

$$(\sqrt{3}, 0), \quad (-\sqrt{3}, 0), \quad \left(0, \frac{1}{3}\right), \quad (0, -3).$$

Of these, only $\left(0, \frac{1}{3}\right)$ belongs to the open unit disc. The value of the function is here

$$f\left(0, \frac{1}{3}\right) = -3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{9} + 0 + \frac{1}{27} = -\frac{14}{27}.$$

Try this...



The sequence 2, 4, 6, 8, 10, 12, 14, 16, ... is the sequence of even whole numbers. The 100th place in this sequence is the number...?

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EXAMINATION OF THE BOUNDARY. We have on the boundary $x^2 + y^2 = 1$, which can also be written $x^2 = 1 - y^2$. Hence, we get the restriction

$$\begin{aligned} g(y) &= f(x, y)|_{x^2+y^2=1} = 4y^2 - 2y \\ &= \left(2y - \frac{1}{2}\right)^2 - \frac{1}{4} \\ &= 4\left(y - \frac{1}{4}\right)^2 - \frac{1}{4}, \quad y \in [-1, 1]. \end{aligned}$$

It follows from these rearrangements that the minimum on the boundary is $g\left(\frac{1}{4}\right) = -\frac{1}{4}$, and the maximum is

$$g(-1) = 4\left(-\frac{5}{4}\right)^2 - \frac{1}{4} = \frac{25}{4} - \frac{1}{4} = 6.$$

Since the boundary is connected, the range of the boundary is the interval $\left[-\frac{1}{4}, 6\right]$.

The value of the function at the stationary point $\left(0, \frac{1}{3}\right)$ is smaller than the smallest value of the function on the boundary, because

$$-\frac{14}{27} < -\frac{1}{4}.$$

The range is connected, and the value 6 of the function on the boundary cannot be obtained in the interior, although we may come as close to this value as we wish. Hence

$$f(K((0, 0); 1)) = \left[-\frac{14}{27}, 6\right].$$

The range of f over \mathbb{R}^2 is of course again \mathbb{R} . For instance, the restriction

$$f(0, y) = -3y + 4y^2 + y^3$$

tends towards $+\infty$ for $y \rightarrow +\infty$, and towards $-\infty$ for $y \rightarrow -\infty$, and by the first main theorem the range is an interval.

Example 1.11 When we approximate a function by an approximating polynomial based on Taylor's formula, the error is zero at the point of expansion, and it will usually increase with the distance from the point of expansion. We may get different results by using other polynomials for our approximation. One of the possibilities is to level out the error by demanding that the integral of the square of the error should be as small as possible. As an illustration we consider the function

$$f(t) = \cos t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2},$$

and a polynomial

$$Q(t; x, y) = x - ty^2, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

Find x and y , such that the integral

$$I(x, y) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{f(t) - Q(t; x, y)\}^2 dt$$

becomes as small as possible. Then compute the error of $t = \pm \frac{\pi}{2}$, partly by approximation by the found polynomial Q , and partly by using the Taylor polynomial of at most second degree P_2 .

A Minimizing in L^2 norm.

D Compute $I(x, y)$ and minimize. Alternatively, compute $\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ directly. Compare with the Taylor polynomial.

I Since $f(t) = \cos t$ is an even function, it is quite reasonable to approximate by a polynomial $Q(t; x, y)$ of even degree in t . Since $f(t) - Q(t; x, y)$ is even in t , we get

$$I(x, y) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{f(t) - Q(t; x, y)\}^2 dt = 2 \int_0^{\frac{\pi}{2}} \{\cos t - x + yt^2\}^2 dt.$$

Clearly, $I(x, y) \geq 0$ is continuous, and $I(x, y) \rightarrow +\infty$ for $x^2 + y^2 \rightarrow +\infty$, so $I(x, y)$ has a minimum in \mathbb{R}^2 . Now $I(x, y)$ is of class C^∞ , so this minimum must be attained at a stationary point.

1) FIRST VARIANT. By differentiation under the sign of integration we get the equations of the possible stationary points

$$\begin{aligned} \frac{\partial I}{\partial x} &= 2 \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial x} \{\cos t - x + yt^2\}^2 dt = -4 \int_0^{\frac{\pi}{2}} (\cos t - x + yt^2) dt \\ &= -4 \left[\sin t - xt + \frac{1}{3} yt^3 \right]_0^{\frac{\pi}{2}} = -4 \left(1 - \frac{\pi}{2} x + \frac{\pi^3}{24} y \right) = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial I}{\partial y} &= 2 \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial y} \{\cos t - x + yt^2\}^2 dt = 4 \int_0^{\frac{\pi}{2}} t^2 (\cos t - x + yt^2) dt \\ &= 4 \int_0^{\frac{\pi}{2}} (t^2 \cos t - xt^2 + yt^4) dt = \left[t^2 \sin t + 2t \cos t - 2 \sin t - \frac{x}{3} t^3 + \frac{y}{5} t^5 \right]_0^{\frac{\pi}{2}} \\ &= 4 \left(\frac{\pi^2}{4} - 2 - \frac{\pi^3}{24} x + \frac{\pi^5}{160} y \right) = 0. \end{aligned}$$

These are reduced to

$$x - \frac{\pi^2}{12}y = \frac{2}{\pi} \quad \text{og} \quad x - \frac{3\pi^2}{20}y = \frac{6}{\pi^3}(\pi^2 - 8),$$

hence

$$\left(\frac{3\pi^2}{20} - \frac{\pi^2}{12}\right)y = \frac{2}{\pi} - \frac{6}{\pi} + \frac{48}{\pi^3} = \frac{4}{\pi^3}(12 - \pi^2),$$

and thus

$$y = \frac{15}{\pi^2} \cdot \frac{4}{\pi^3}(12 - \pi^2) = \frac{60}{\pi^5}(12 - \pi^2)$$

and accordingly

$$x = \frac{\pi^2}{12}y + \frac{2}{\pi} = \frac{\pi^2}{12} \cdot \frac{60}{\pi^5}(12 - \pi^2) + \frac{2}{\pi} = \frac{1}{\pi^3}(60 - 5\pi^2) + \frac{2}{\pi} = \frac{3}{\pi^2}(20 - \pi^2).$$

The only stationary point is

$$(x, y) = \left(\frac{3}{\pi^3}(20 - \pi^2), \frac{60}{\pi^5}(12 - \pi^2)\right),$$

and it must correspond to a minimum for $I(x, y)$. Notice that $x > 0$ and $y > 0$.

2) SECOND VARIANT. Alternatively we compute $I(x, y)$:

$$\begin{aligned} I(x, y) &= 2 \int_0^{\frac{\pi}{2}} \{\cos t - x + yt^2\}^2 dt \\ &= 2 \int_0^{\frac{\pi}{2}} \{\cos^2 t + x^2 + y^2 t^2 - 2x \cos t - 2xyt^2 + 2yt^2 \cos t\} dt \\ &= 2 \left[\left\{ \frac{1}{2}t + \frac{1}{2} \sin t \cos t \right\} + x^2 t + \frac{y^2}{5} t^5 - 2x \sin t \right. \\ &\quad \left. - \frac{2}{3}xyt^3 + 2y\{t^2 \sin t + 2t \cos t - 2 \sin t\} \right]_0^{\frac{\pi}{2}} \\ &= 2 \left[\frac{\pi}{4} + \frac{\pi}{2}x^2 + \frac{\pi^5}{160}y^2 - 2x - \frac{\pi^3}{12}xy + 2y \left\{ \frac{\pi^2}{4} - 2 \right\} \right] \\ &= \pi x^2 + \frac{\pi^5}{80}y^2 - \frac{\pi^3}{6}xy - 4x + (\pi^2 - 8)y + \frac{\pi}{2}. \end{aligned}$$

The equations of the stationary points,

$$\begin{aligned} \frac{\partial I}{\partial x} &= 2\pi x - \frac{\pi^3}{6}y - 4 = 0, \\ \frac{\partial I}{\partial y} &= \frac{\pi^5}{40}y - \frac{\pi^3}{6}x + \pi^2 - 8 = 0, \end{aligned}$$

are identical with the equations of the first variant. The unique solution is

$$(x_0, y_0) = \left(\frac{3}{\pi^3}(20 - \pi^2), \frac{60}{\pi^5}(12 - \pi^2)\right) \approx (0.980162; 0.417698),$$

corresponding to a minimum for $I(x, y)$, given approximatively by

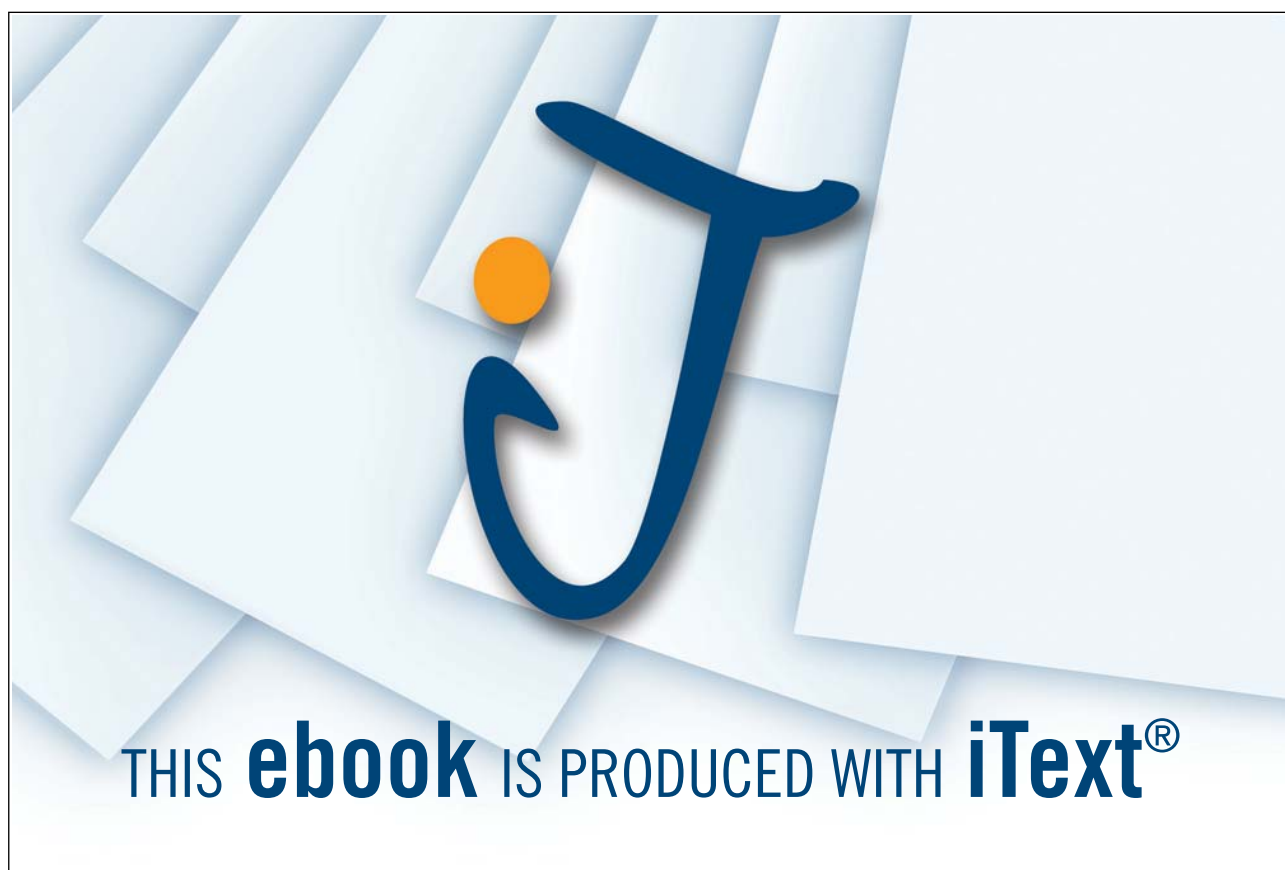
$$I(x_0, y_0) \approx 0.000\,936.$$

REMARK. This approximation is also called the *approximation in energy over the given interval*. It is seen that this is extremely good even for a polynomial of degree two (an error of less than 1 per thousand). This is the right concept of convergence in Communication Systems and other applications in the technical sciences. Unfortunately, most students are at this level still most familiar with the pointwise convergence, in spite of the fact that this concept in practice often is very awkward. It will below be demonstrated that the inaccuracy by the various pointwise approximations are bigger than the approximation in energy. \diamond

We have found the approximation

$$Q(t) := Q(t; x_0, y_0) = \frac{3}{\pi^3}(20 - \pi^2) - \frac{60}{\pi^5}(12 - \pi^2)t^2 \approx 0.980\,162 - 0.417\,698t^2$$

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of $\cos t$. The corresponding approximation by a *Taylor polynomial* is

$$P_2(t) = 1 - \frac{1}{2}t^2.$$

If $t = \pm \frac{\pi}{2}$, then $\cos t = 0$, and

$$P_2\left(\pm \frac{\pi}{2}\right) = 1 - \frac{\pi^2}{8} = \frac{8 - \pi^2}{8} \approx -0.233\,701,$$

and

$$\begin{aligned} Q\left(\pm \frac{\pi}{2}\right) &= \frac{3}{\pi^3}(20 - \pi^2) - \frac{60}{\pi^5}(12 - \pi^2) \cdot \frac{\pi^2}{4} \\ &= \frac{1}{\pi^3}\{60 - 3\pi^2 - 180 + 15\pi^2\} = \frac{12\pi^2 - 120}{\pi^3} \\ &= \frac{12(\pi^2 - 10)}{\pi^3} \approx -0.050\,465. \end{aligned}$$

It is seen by comparison that the approximation in energy also gives a better pointwise result than the Taylor polynomial.

Example 1.12 Consider the function

$$f(x, y) = 3x^3 + 6xy^2 + 4y^3 - 9x^2, \quad (x, y) \in A,$$

where A is given in the following way: We remove from the ellipsoidal disc given by $x^2 + 2y^2 - 3x \leq 0$ those points which also satisfy $y < -\frac{1}{2}x$. Sketch A and then find the maximum and the minimum of the function.

- A** Maximum and minimum for a continuous function in a closed and bounded domain.
- D** Sketch the set A . Then refer to the second main theorem for continuous functions. Find the stationary points and examine the boundary points.
- I** Clearly, $f(x, y)$ is of class C^∞ , even as a function in all of \mathbb{R}^2 . Hence there are no exception points.

We first identify the ellipsoidal disc by the following rearrangement

$$0 \geq x^2 + 2y^2 - 3x = \left\{x^2 - 2 \cdot \frac{3}{2}x + \left(\frac{3}{2}\right)^2\right\} + 2y^2 - \left(\frac{3}{2}\right)^2,$$

hence

$$\left(x - \frac{3}{2}\right)^2 + 2y^2 \leq \left(\frac{3}{2}\right)^2,$$

or in the usual normed form

$$\left\{\frac{x - \frac{3}{2}}{\frac{3}{2}}\right\}^2 + \left\{\frac{y}{\frac{3}{2\sqrt{2}}}\right\}^2 \leq 1.$$

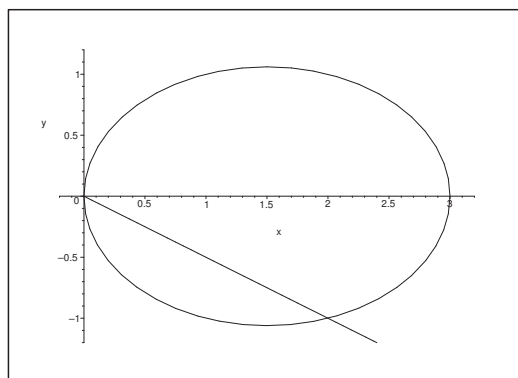


Figure 14: The closed set A lies above the line and inside the ellipse.

It follows that the inequality describes a closed ellipsoidal disc of centrum $\left(\frac{3}{2}, 0\right)$ and of the half axes $a = \frac{3}{2}$ and $b = \frac{3}{2\sqrt{2}}$.

The domain A is the intersection of the closed ellipsoidal disc and the closed half plane $y \geq -\frac{1}{2}x$, i.e. that part of the ellipsoidal disc which lies above the line. It follows that A is closed and bounded.

According to the second main theorem for continuous functions, f has a maximum and a minimum in A . Since f is of class C^∞ , these are among the values of the function at either the stationary points in the interior of A or at the points of the boundary.

First notice that the line $y = -\frac{1}{2}x$ intersects the ellipse in two points given by

$$0 = x^2 + 2y^2 - 3x = x^2 + 2 \cdot \frac{x^2}{4} - 3x = \frac{3}{2}x^2 - 3x = \frac{3}{2}x(x - 2),$$

hence $x = 0$ and $x = 2$, corresponding to the points $(0, 0)$ and $(2, -1)$.

1) STATIONARY POINTS. The equations of the stationary points are

$$\begin{aligned} \frac{\partial f}{\partial x} &= 9x^2 + 6y^2 - 18x = 0, \\ \frac{\partial f}{\partial y} &= 12xy + 12y^2 = 12(x + y)y = 0. \end{aligned}$$

The line $x + y = 0$ does not intersect the interior of A , so we only get the possibility $y = 0$. If we put this into the former equation we get

$$0 = 9x^2 + 0 - 18x = 9x(x - 2),$$

hence $x = 0$ or $x = 2$, corresponding to the stationary points $(0, 0)$ and $(2, 0)$. Only $(2, 0)$ lies in A . Then compute the value of the function at this point,

$$f(2, 0) = 3 \cdot 2^3 + 0 + 0 - 9 \cdot 2^2 = 24 - 36 = -12.$$

2) EXAMINATION OF THE BOUNDARY. The boundary is split into three obvious parts:

- a) $y = -\frac{1}{2}x$, for $0 \leq x \leq 2$ and $y \leq 0$,
- b) $y = -\sqrt{\frac{3x-x^2}{2}}$, for $2 \leq x \leq 3$ and $y \leq 0$,
- c) $y = +\sqrt{\frac{3x-x^2}{2}}$, for $0 \leq x \leq 3$ and $y \geq 0$.

Notice that if (x, y) lies on the (boundary of the) ellipse, then

$$f(x, y) = 3x^3 + 6xy^2 + 4y^3 - 9x^2 = 3x\{x^2 + 2y^2 - 3x\} + 4y^3 = 4y^3,$$

which is a trick that will help us a lot in the following.

a) The restriction of $f(x, y)$ to $y = -\frac{1}{2}x$, $0 \leq x \leq 2$, is

$$\begin{aligned}\varphi(x) &= f\left(x, -\frac{1}{2}x\right) = 3x^3 + 6x \cdot \frac{x^2}{4} - 4 \cdot \frac{x^3}{8} - 9x^2 \\ &= 3x^3 + \frac{3}{2}x^3 - \frac{1}{2}x^3 - 9x^2 = 4x^3 - 9x^2, \quad \text{for } 0 \leq x \leq 2,\end{aligned}$$

where

$$\varphi'(x) = 12x^2 - 18x = 12x\left(x - \frac{3}{2}\right), \quad 0 < x < 2.$$

In the open interval we get $\varphi'(x) = 0$ for $x = \frac{3}{2}$, corresponding to $y = -\frac{3}{4}$, and

$$\varphi\left(\frac{3}{2}\right) = f\left(\frac{3}{2}, -\frac{3}{4}\right) = 4 \cdot \frac{27}{8} - 9 \cdot \frac{9}{4} = \frac{27}{2} - \frac{81}{4} = -\frac{27}{4}.$$

At the end points of the interval,

$$\varphi(0) = f(0, 0) = 0 \text{ og } \varphi(2) = f(2, -1) = 4 \cdot 8 - 9 \cdot 4 = -4.$$

b) On the ellipsoidal boundary of A we reduce $f(x, y)$ to

$$f(x, y) = 4y^3.$$

It follows geometrically (consider the figure) that the maximum on this part of the boundary is

$$f\left(\frac{3}{2}, \frac{3}{2\sqrt{2}}\right) = 4 \cdot \left(\frac{3}{2\sqrt{2}}\right)^3 = \frac{27}{4\sqrt{2}},$$

and the minimum is

$$f(2, -1) = -4.$$

c) NUMERICAL COMPARISON. We shall now find the maximum and the minimum among the values of the function,

$$f(2, 0) = -12, \quad f\left(\frac{3}{2}, -\frac{3}{4}\right) = -\frac{27}{4}, \quad f(0, 0) = 0,$$

$$f(2, -1) = -4, \quad f\left(\frac{3}{2}, \frac{3}{2\sqrt{2}}\right) = \frac{27}{4\sqrt{2}},$$

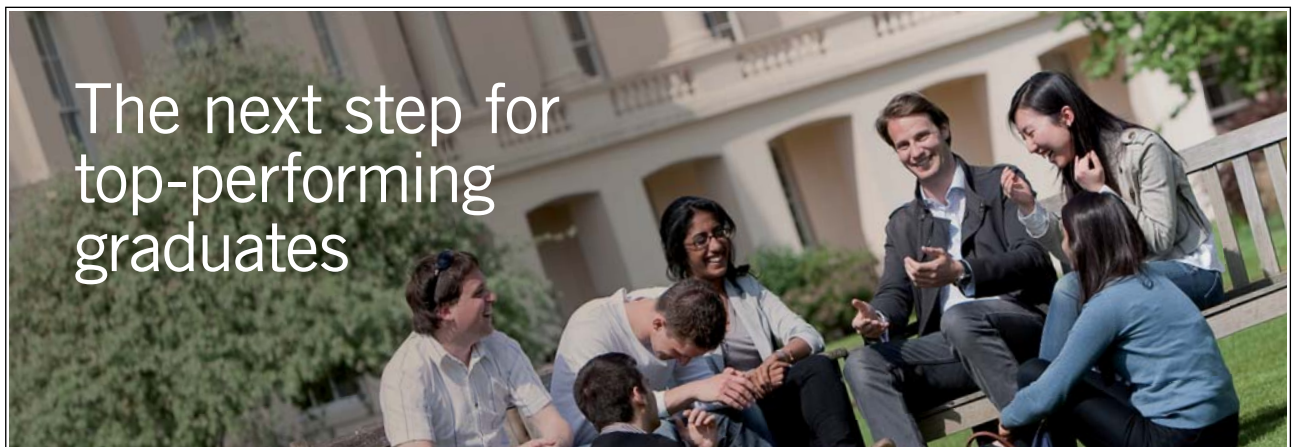
thus the maximum is

$$S = f\left(\frac{3}{2}, \frac{3}{2\sqrt{2}}\right) = \frac{27}{4\sqrt{2}},$$

and the minimum is

$$f(2, 0) = -12.$$

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Example 1.13 Explain why the function given by

$$f(x, y) = xy^2, \quad (x, y) \in \overline{K}((0, 0); 1),$$

has both a maximum and a minimum, and find these values.

A Maximum and minimum.

D A continuous function on a closed, bounded set. Find the possible stationary points and examine the points of the boundary.

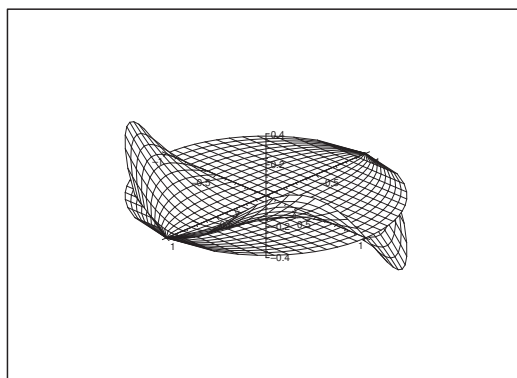


Figure 15: The graph of $f(x, y)$ over $\overline{K}((0, 0); 1)$.

I Since $f(x, y)$ is continuous (even of class C^∞), and $\overline{K}(\mathbf{0}; 1)$ is closed and bounded, it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in $\overline{K}(\mathbf{0}; 1)$. These are either attained at a stationary point or at a boundary point.

Since

$$\frac{\partial f}{\partial x} = y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy$$

are 0 for $y = 0$, i.e. on the X axis, the set of stationary points is $[-1, 1] \times \{0\}$. The value of $f(x, y)$ is here trivially $f(x, 0) = 0$.

We use on the boundary the parametric description $(x, y) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, thus the restriction to the boundary is given by

$$\varphi(t) = f(\cos t, \sin t) = \cos t \cdot \sin^2 t, \quad t \in [0, 2\pi],$$

with the derivative

$$\varphi'(t) = -\sin^3 t + 2 \sin t \cdot \cos^3 t = \sin t \cdot (2 \cos^2 t - \sin^2 t) = \sin t(3 \cos^2 t - 1).$$

This expression is 0 for $t = 0, \pi$, or for $\cos t = \pm \frac{1}{\sqrt{3}}$, corresponding to $\sin^2 t = \frac{2}{3}$. When we put $t = 0, \pi$, we get the value 0 of the function.

If we insert $\cos t = \pm \frac{1}{\sqrt{3}}$, $\sin^2 t = \frac{2}{3}$, we get

$$f(\cos t, \sin t) = \pm \frac{1}{\sqrt{3}} \cdot \frac{2}{3},$$

hence

$$f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = \frac{2}{3\sqrt{3}} \quad \text{is the maximum value,}$$

and

$$f\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}\right) = -\frac{2}{3\sqrt{3}} \quad \text{is the minimum value.}$$

Example 1.14 The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by

$$f(x, y) = (x^2 + y^2 - 1)(x^2 + y^2 - 4).$$

- 1) Find the set of stationary points for f .
- 2) Show that f has a proper extremum at $(0, 0)$, and indicate the type of this extremum.
- 3) Find the largest value which is attained by f in the disc

$$D = \{(x, y) \in \mathbb{R}^2 \mid (x - 2)^2 + y^2 \leq 4\}.$$

A Extremum.

D An alternative solution is to rewrite this problem as a 1-dimensional problem. Exploit this idea to solve as much of the problem as possible. Then compute the problem as indicated in the text.

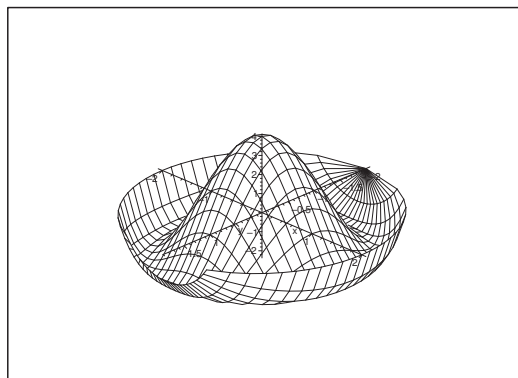


Figure 16: The surface $z = (x^2 + y^2 - 1)(x^2 + y^2 - 4)$ over D .

I The problem is actually 1-dimensional, because by switching to polar coordinates,

$$f(x, y) = f(\varrho \cos \varphi, \varrho \sin \varphi) = (\varrho^2 - 1)(\varrho^2 - 4) = \varrho^4 - 5\varrho^2 + 4 = g(\varrho), \quad \varrho \geq 0,$$

and

$$g'(\varrho) = 4\varrho^3 - 10\varrho = 4\varrho \left(\varrho^2 - \frac{5}{2} \right)$$

is 0 for $\varrho = 0$ and for $\varrho = +\sqrt{\frac{5}{2}}$. Furthermore, $g(\varrho)$ is decreasing for $0 < \varrho < \sqrt{\frac{5}{2}}$ and increasing for $\varrho > \sqrt{\frac{5}{2}}$.

In particular, $(0, 0)$ which corresponds to $\varrho = 0$ is a local maximum point of the value of the function

$$f(0, 0) = 4.$$

We get the minimum for $\varrho = \sqrt{\frac{5}{2}}$, corresponding to the value of the function

$$f\left(\sqrt{\frac{5}{2}} \cos \varphi, \sqrt{\frac{5}{2}} \sin \varphi\right) = g\left(\sqrt{\frac{5}{2}}\right) = \left(\frac{5}{2} - 1\right)\left(\frac{5}{2} - 4\right) = -\frac{9}{4}.$$

Since D contains both $(0, 0)$ and $(4, 0)$ on the X axis, cf. the figure, the maximum value is one of

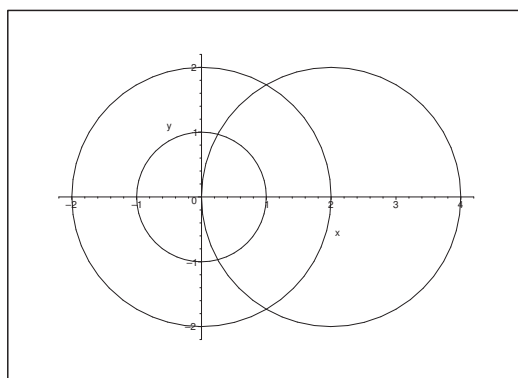


Figure 17: The domain D and the zero curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ for $f(x, y)$.

the values $g(0)$ and $g(4)$. We conclude from

$$f(4, 0) = g(4) = (16 - 1)(16 - 4) = 15 \cdot 12 = 180 > 4 = f(0, 0),$$

that $f(4, 0) = 180$ is the maximum value in D .

Finally, the stationary points are necessarily $(0, 0)$ and $\left\{ (x, y) \mid x^2 + y^2 = \frac{5}{2} \right\}$.

Now, return to the very beginning, and handle the example according to the original intention.

- 1) The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 2x(x^2 + y^2 - 4) + 2x(x^2 + y^2 - 1) = 4x \left(x^2 + y^2 - \frac{5}{2} \right) = 0,$$

$$\frac{\partial f}{\partial y} = 2y(x^2 + y^2 - 4) + 2y(x^2 + y^2 - 1) = 4y \left(x^2 + y^2 - \frac{5}{2} \right) = 0.$$

We conclude that the stationary points are

$$\left\{ (x, y) \mid x^2 + y^2 = \frac{5}{2} \right\} \cup \{(0, 0)\}.$$

- 2) It follows from $f(x, y) = g(\varrho) = (1 - \varrho^2)(4 - \varrho^2)$ that $g(\varrho)$ is increasing when $\varrho \rightarrow 0$, hence $(0, 0)$ is a maximum.

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ALTERNATIVELY,

$$\frac{\partial^2 f}{\partial x^2} = 4 \left(x^2 + y^2 - \frac{5}{2} \right) + 8x^2, \quad r = 4 \cdot \left(-\frac{5}{2} \right) = -10,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 8xy, \quad s = 0,$$

$$\frac{\partial^2 f}{\partial y^2} = 4 \left(x^2 + y^2 - \frac{5}{2} \right) + 8y^2, \quad t = 4 \cdot \left(-\frac{5}{2} \right) = -10,$$

so $rt - s^2 = 100 > 0$, and $r < 0$, $t < 0$, corresponding to that $(0, 0)$ is a maximum point.

3) Since

$$f(x, y) = \left(\frac{5}{2} - 1 \right) \left(\frac{5}{2} - 4 \right) = -\frac{9}{4} \quad \text{for } x^2 + y^2 = \frac{5}{2},$$

only the examination of the boundary remains. A parametric description of the boundary curve of D is

$$(x, y) = (2 + 2 \cos t, 2 \sin t) = 2(1 + \cos t, \sin t), \quad t \in [0, 2\pi],$$

where

$$x^2 + y^2 = 4(1 + 2 \cos t + \cos^2 t + \sin^2 t) = 8(1 + \cos t).$$

The restriction $h(t)$ to the boundary curve is given by

$$\begin{aligned} h(t) &= f(2(1 + \cos t), 2 \sin t) = \{8(1 + \cos t) - 1\}\{8(1 + \cos t) - 4\} \\ &= 4(7 + 8 \cos t)(1 + 2 \cos t) = 4(7 + 22 \cos t + 16 \cos^2 t), \end{aligned}$$

where

$$h'(t) = -4 \sin t(22 + 32 \cos t),$$

which is only 0, when either $t = 0$ or $t = \pi$ or $\cos t = -\frac{11}{16}$. We get by insertion

$$\begin{aligned} h(0) &= f(4, 0) = 4(7 + 22 + 16) = 4 \cdot 45 = 180, \\ h(\pi) &= f(0, 0) = 4(7 - 22 + 16) = 4, \end{aligned}$$

and

$$\begin{aligned} h\left(\arccos\left(-\frac{11}{16}\right)\right) &= 4\left(7 + 22 \cdot \left(-\frac{11}{16}\right) + 16 \cdot \left(-\frac{11}{16}\right)^2\right) \\ &= 4\left(7 - 2 \cdot \frac{11^2}{16} + \frac{11^2}{16}\right) = 4\left(7 - \frac{121}{16}\right) = 4\left(\frac{112 - 121}{16}\right) = -\frac{9}{4}. \end{aligned}$$

A numerical comparison gives that

$$f(4, 0) = 180$$

is the maximum value in D .

Example 1.15 Given the function

$$f(x, y) = xy \exp(y - x^2), \quad (x, y) \in \mathbb{R}^2.$$

- 1) Find the stationary points of f .
- 2) Explain why f has both a maximum and a minimum in

$$A = \{(x, y) \mid x^2 - 3 \leq y \leq 0\},$$

and find the values of the function at these points.

- 3) Check if f has a global maximum and a global minimum in \mathbb{R}^2 .

A Maximum and minimum.

D Find the stationary points and sketch A .

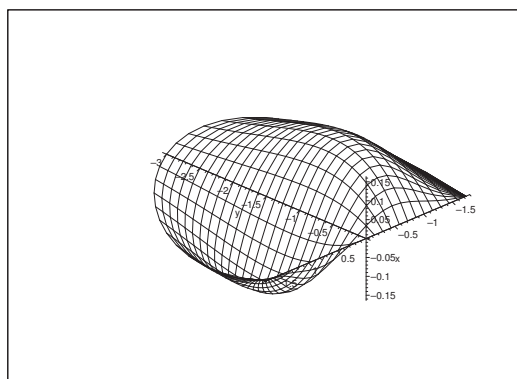


Figure 18: The graph of $f(x, y)$ over A .

I 1) The function is of class C^∞ , and the partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial x} &= y \exp(y - x^2) - 2x^2 y \exp(y - x^2) = y(1 - 2x^2) \exp(y - x^2), \\ \frac{\partial f}{\partial y} &= x \exp(y - x^2) + xy \exp(y - x^2) = x(1 + y) \exp(y - x^2). \end{aligned}$$

The exponential is never zero, so these two expressions are both equal to zero, if and only if

$$y(1 - 2x^2) = 0 \quad \text{and} \quad x(1 + y) = 0.$$

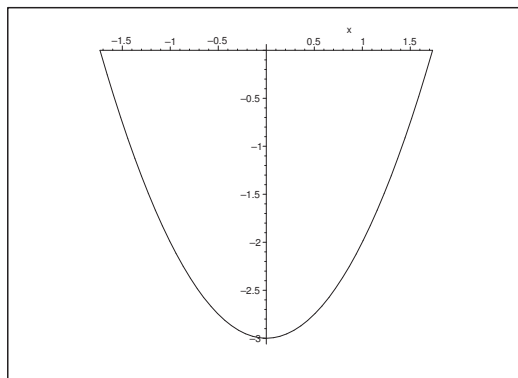
If $x = 0$, the latter equation is fulfilled, and it follows from the former equation that $y = 0$.

If $y = -1$, the latter equation is again satisfied. It follows from the former equation that $x = \pm \frac{1}{\sqrt{2}}$.

Thus the possible stationary points are

$$(0, 0), \quad \left(\frac{1}{\sqrt{2}}, -1\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, -1\right).$$

Here $(0, 0)$ lies on the boundary, so it will also enter the analysis of the boundary points.

Figure 19: The domain A .

- 2) Since f is continuous and A is closed and bounded, it follows from the second main theorem for continuous functions that f has a maximum and a minimum in A . These points are either an inner stationary point or a boundary point. The inner stationary points are

$$\left(\frac{1}{\sqrt{2}}, -1\right) \quad \text{og} \quad \left(-\frac{1}{\sqrt{2}}, -1\right)$$

with the values of the function

$$f\left(\frac{1}{\sqrt{2}}, -1\right) = -\frac{1}{\sqrt{2}} \exp\left(-1 - \frac{1}{2}\right) = -\frac{1}{\sqrt{2}e^3}$$

and

$$f\left(-\frac{1}{\sqrt{2}}, -1\right) = \frac{1}{\sqrt{2}e^3}.$$

On the boundary curve $y = 0$, $-\sqrt{3} \leq x \leq \sqrt{3}$,

$$f(x, 0) = 0.$$

On the boundary curve $y = x^2 - 3$, $-\sqrt{3} \leq x \leq \sqrt{3}$, we have the restriction

$$\varphi(x) = f(x, x^2 - 3) = x(x^2 - 3) \exp(x^2 - 3 - x^2) = (x^3 - 3x)e^{-3}$$

with

$$\varphi'(x) = 3(x^2 - 1)e^{-3}.$$

We have already checked the end points $x = \pm\sqrt{3}$, so it only remains to compute

$$\varphi(1) = (1 - 3)e^{-3} = -2e^{-3} \quad \text{and} \quad \varphi(-1) = 2e^{-3}.$$

The maximum value and the minimum value are among

$$f\left(\frac{1}{\sqrt{2}}, -1\right) = -\frac{1}{\sqrt{2}e^3}, \quad f\left(-\frac{1}{\sqrt{2}}, -1\right) = \frac{1}{\sqrt{2}e^3},$$

$$f(1, -2) = -\frac{2}{e^3}, \quad f(-1, 2) = \frac{2}{e^3},$$

hence

$$S = f\left(-\frac{1}{\sqrt{2}}, -1\right) = \frac{1}{\sqrt{2}e^3} \quad \text{and} \quad M = f\left(\frac{1}{\sqrt{2}}, -1\right) = -\frac{1}{\sqrt{2}e^3}.$$

3) We get along the curve $y = x^2$,

$$\psi(x) = f(x, x^2) = x^3,$$

which clearly has neither a maximum value nor a minimum value in \mathbb{R}^2 , because the range is all of \mathbb{R} .

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Example 1.16 1) Sketch the set

$$B = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, 0 \leq y \leq 1 - 2x\},$$

and explain why the function $f(x, y) = xy$ has both a maximum S and a minimum value M on B .

2) Find S and M .

A Maximum value and minimum value.

D Solve the problem geometrically.

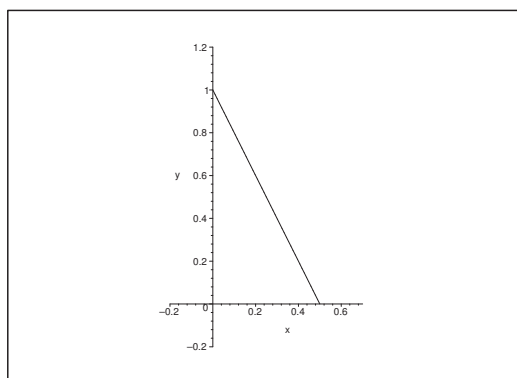


Figure 20: The domain B .

I 1) The set B is closed and bounded, and f is continuous. By the second main theorem for continuous functions, f has a maximum and a minimum on B .

2) Clearly, $f(x, y) \geq 0$ på B , and the minimum value $M = 0$ must be attained on the axes.

Since $f(x, y) = C$ on the hyperbola $xy = C$, we conclude from considering the set of curves that the maximum value must be attained on the line $y = 1 - 2x$. Then consider the restriction

$$g(x) = f(x, 1 - 2x) = x - 2x^2, \quad x \in \left[0, \frac{1}{2}\right],$$

of f to this boundary curve. First we vet $g'(x) = 1 - 4x$, which corresponds to a maximum for $x = \frac{1}{4}$, i.e.

$$S = g\left(\frac{1}{4}\right) = f\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{8}.$$

REMARK. An alternative way is of course to realize that $(0, 0)$ is the only candidate of a stationary point. Since f is 0 on the axes, only the examination of the boundary line $y = 1 - 2x$ remains, and this was done above.

Example 1.17 Given the function

$$f(x, y) = 32x^2y - 20x^3y - xy^3, \quad (x, y) \in \mathbb{R}^2.$$

- 1) Find the stationary points of f .
- 2) Check for each of the points $(1, 2)$, $(1, 1)$ and $(0, 0)$, if f has an extremum at the given point.
- 3) Find the range of the function f .

A Extremum.

D Follow the guidelines.

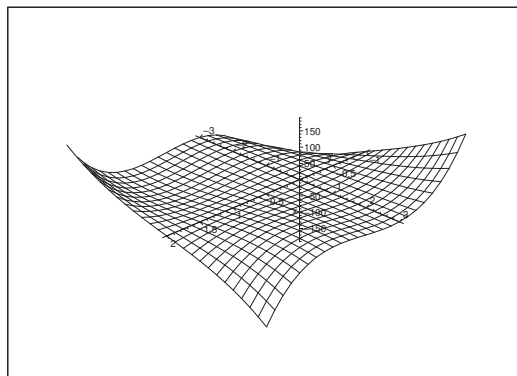


Figure 21: Part of the graph of f .

I 1) The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 64xy - 60x^2y - y^3 = y(64x - 60x^2 - y^2) = 0,$$

$$\frac{\partial f}{\partial y} = 32x^2 - 20x^3 - 3xy^2 = x(32x - 20x^2 - 3y^2) = 0.$$

- a) If $x = 0$, then the latter equation is fulfilled, and we obtain $y = 0$ from the first, so $(0, 0)$ is a stationary point.
- b) If $y = 0$, then the former equation is fulfilled, and the latter equation gives either $x = 0$ or $x = \frac{8}{5}$. Thus we get another stationary point $\left(\frac{8}{5}, 0\right)$.
- c) If both $x \neq 0$ and $y \neq 0$, the equations are reduced to

$$y^2 = 64x - 60x^2 \quad \text{and} \quad 3y^2 = 32x - 20x^2.$$

By elimination of y ,

$$32x - 20x^2 = 3y^2 = 192x - 180x^2,$$

from which we get the *necessary condition*

$$0 = 160x^2 - 160x = 160x(x - 1).$$

Since $x \neq 0$, we only get the *possibility* of $x = 1$.

If we put $x = 1$ into the *original equations* (i.e. we test our possible solution), then

$$y(4 - y^2) = 0 \quad \text{and} \quad 12 - 3y^2 = 3(4 - y^2) = 0,$$

which are both satisfied for $y = \pm 2$.

Summarizing, the stationary points are

$$(0, 0), \quad \left(\frac{8}{5}, 0\right), \quad (1, 2), \quad (1, -2).$$

2) EXAMINATION OF THE EXTREMA.

- a) Since $(1, 1)$ is not a stationary point, it cannot be an extremum.

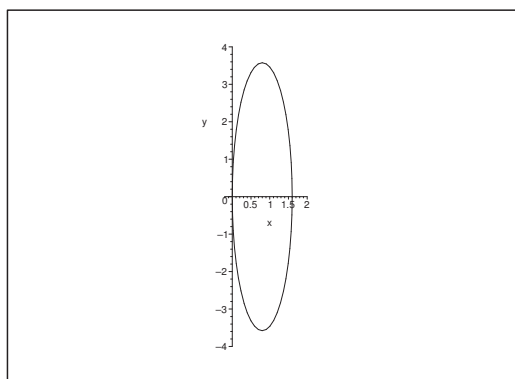


Figure 22: The function $f(x, y)$ changes its sign, whenever one crosses either one of the axes or the ellipse.

- b) The point $(0, 0)$ is not an extremum.

First variant. By an analysis of the sign of the function, cf. the figure,

$$\begin{aligned} f(x, y) &= 32x^2y - 20x^3y - xy^3 = xy \{32x - 20x^2 - y^2\} = \dots \\ &= \frac{64}{5}xy \left\{1 - \frac{25}{16} \left(x - \frac{4}{5}\right)^2 - \frac{5}{64}y^2\right\}, \end{aligned}$$

we see that f is both positive and negative in any neighbourhood of $(0, 0)$, and we have no extremum.

Second variant. If we take the restriction of $f(x, y)$ to the line $y = x$, it follows, that

$$\varphi(x) = f(x, x) = 32x^3 - 21x^4 = x^3(32 - 21x)$$

is both positive and negative in any neighbourhood of $(0, 0)$, and we cannot have an extremum.

- c) When we check $(1, 2)$, we have many methods at hand. We shall here restrict ourselves to two. First note that $(1, 2)$ is in fact a stationary point, so it is possible that we have an extremum at the point.

First variant. Approximating polynomial of at most degree two from $(1, 2)$.

If we put

$$x = x_1 + 1 \quad \text{and} \quad y = y_1 + 2,$$

and neglect terms of higher order than 2 in (x_1, y_1) (symbolized by dots), we get

$$\begin{aligned} f(x, y) &= 32x^2y - 20x^3y - xy^3 \\ &= 32(x_1+1)^2(y_1+2)^2 - 20(x_1+1)^3(y_1+2) - (x_1+1)(y_1+2)^3 \\ &= 32(x_1^2+2x_1+1)(y_1+2) - 20(1+3x_1+3x_1^2+\dots)(y_1+2) \\ &\quad - (x_1+1)(8+12y_1+6y_1^2+\dots) \\ &= 32(2+4x_1+2x_1^2+y_1+2x_1y_1+\dots) - 20(2+6x_1+6x_1^2+y_1+3x_1y_1+\dots) \\ &\quad - (8+12y_1+6y_1^2+8x_1+12x_1y_1+\dots) \\ &= (64-40-8) + (128-120-8)x_1 + (32-20-12)y_1 \\ &\quad + (64-120)x_1^2 + (64-60-12)x_1y_1 - 6y_1^2 + \dots \\ &= 16 - 56x_1^2 - 8x_1y_1 - 6y_1^2 + \dots \\ &= 16 - \frac{160}{3}x_1^2 - 6\left(\frac{2}{3}x_1 + y_1\right)^2 + \dots, \end{aligned}$$

proving that we have a local maximum for $(x_1, y_1) = (0, 0)$, i.e. for $(x, y) = (1, 2)$.

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Second variant. The (r, s, t) -method.

First compute

$$\frac{\partial^2 f}{\partial x^2} = 64y - 120xy, \quad \frac{\partial^2 f}{\partial x \partial y} = 64x - 60x^2 - 3y^2, \quad \frac{\partial^2 f}{\partial y^2} = -6xy.$$

At the point $(1, 2)$,

$$r = 128 - 240 = -112, \quad s = -8, \quad t = 12,$$

hence

$$r < 0, \quad t < 0 \quad \text{og} \quad rt > s^2.$$

It follows by the (r, s, t) -method that there is a proper maximum at the point $(1, 2)$.

3) The range is \mathbb{R} .

We have e.g.

$$f(x, x) = 32x^3 - 21x^4 \rightarrow -\infty \quad \text{for } x \rightarrow +\infty,$$

and

$$f(x, -x) = -32x^3 + 21x^4 \rightarrow +\infty \quad \text{for } x \rightarrow +\infty,$$

and the rest follows from that $f(x, y)$ is continuous.

Example 1.18 Given the function

$$f(x, y) = (x + y^2) \exp(-2x^2), \quad (x, y) \in \mathbb{R}^2.$$

1) Find the stationary points.

2) Explain why f has both a maximum S and a minimum M on the closed triangle with the vertices $(0, -1)$, $(0, 1)$ and $(2, 0)$; then find S and M .

3) Show that f does not have a (global) maximum in \mathbb{R}^2 .

A Stationary points. Maximum and minimum.

D Follow the guidelines.

I 1) The stationary points are the solutions of

$$\frac{\partial f}{\partial x} = \exp(-2x^2) - 4x(x + y^2) \exp(-2x^2) = (1 - 4x^2 - 4xy^2) \exp(-2x^2) = 0,$$

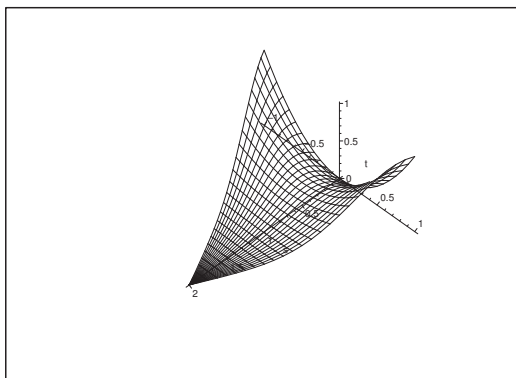
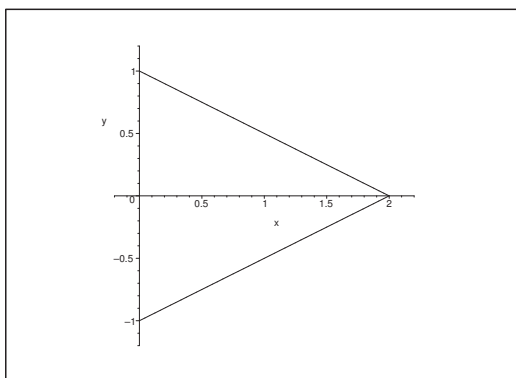
and

$$\frac{\partial f}{\partial y} = 2y \exp(-2x^2) = 0.$$

We get from the latter equation that $y = 0$, which by insertion into the former one gives

$1 - 4x^2 = 0$, i.e. $x = \pm \frac{1}{2}$. The stationary points are

$$\left(\frac{1}{2}, 0\right) \quad \text{and} \quad \left(-\frac{1}{2}, 0\right).$$

Figure 23: The graph of f over the triangle D .Figure 24: The domain D .

- 2) The triangle D is closed and bounded, and f is continuous on D . Hence, by the second main theorem for continuous functions, both S and M exist on D . These values are either attained at the stationary point $\left(\frac{1}{2}, 0\right) \in D$ or on the boundary ∂D .

From $f(x, y) \geq 0$ in D and $f(0, 0) = 0$ with $(0, 0) \in D$ follows that

$$M = f(0, 0) = 0.$$

When x is fixed, we see that $f(x, y)$ is largest in D , when $y = \pm\left(1 - \frac{x}{2}\right)$. This excludes $\left(\frac{1}{2}, 0\right)$, and the maximum must be attained on the line

$$y = \pm\left(1 - \frac{x}{2}\right).$$

The restriction of f to this line is

$$\varphi(x) = f\left(x, \pm\left(1 - \frac{x}{2}\right)\right) = \left(1 + \frac{x^2}{4}\right) \exp(-2x^2), \quad x \in [0, 2],$$

with the derivative

$$\begin{aligned}\varphi'(x) &= \frac{x}{2} \exp(-2x^2) - 4x \left(1 + \frac{x^2}{4}\right) \exp(-2x^2) \\ &= \frac{x}{2} \exp(-2x^2) \cdot \{-7 - 2x^2\} \leq 0, \quad x \in [0, 2].\end{aligned}$$

Hence the maximum value is attained for $x = 0$, corresponding to

$$S = f(0, 1) = f(0, -1) = 1.$$

3) Since $f(0, y) = y^2$, it follows that f does not have a global maximum in \mathbb{R}^2 .

Example 1.19 Given the function

$$f(x, y) = (x + y)^2 + 2 \cos(2x + y), \quad (x, y) \in \mathbb{R}^2.$$

1) Find the stationary points of f ; check if f has proper extrema.

2) Let A be the point set

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \pi, -2x \leq y \leq -x\}.$$

Prove that f has no stationary points in the interior of A .

Explain why f has both a maximum S and a minimum M on A , and find S and M .

3) Find the range $f(\mathbb{R}^2)$ of f .

A Extrema.

D Follow the guidelines

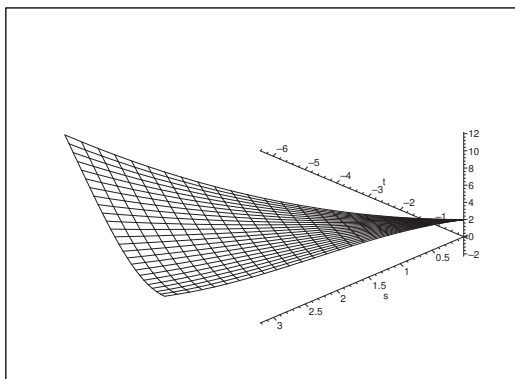


Figure 25: The graph of f the point set A .

I 1) The stationary points are the solutions of the equation $\nabla f = \mathbf{0}$, i.e.

$$\frac{\partial f}{\partial x} = 2(x + y) - 4 \sin(2x + y) = 0,$$

$$\frac{\partial f}{\partial y} = 2(x + y) - 2 \sin(2x + y) = 0.$$

It follows that $\sin(2x + y) = 0$ and $x + y = 0$, so

$$\sin(2x + y) = \sin(x + (x + y)) = \sin x = 0.$$

Accordingly, $x = p\pi$, $p \in \mathbb{Z}$, and $y = -x$. Thus the stationary points are

$$\{(p\pi, -p\pi) \mid p \in \mathbb{Z}\}.$$

The values of the function in these points are

$$f(p\pi, -p\pi) = 2 \cos p\pi = 2 \cdot (-1)^p = \begin{cases} 2 & \text{for } p \text{ even,} \\ -2 & \text{for } p \text{ odd.} \end{cases}$$

The extrema are now found by the (r, s, t) -method.

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From

$$\frac{\partial^2 f}{\partial x^2} = 2 - 8 \cos(2x + y) = 2 - 8 \cos(x + (x + y)),$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2 - 4 \cos(2x + y) = 2 - 4 \cos(x + (x + y)),$$

$$\frac{\partial^2 f}{\partial y^2} = 2 - 2 \cos(2x + y) = 2 - 2 \cos(x + (x + y)),$$

follows that

$$r_p = \frac{\partial^2 f}{\partial x^2}(p\pi, -p\pi) = 2 - 8(-1)^p = \begin{cases} -6 & \text{for } p \text{ even,} \\ 10 & \text{for } p \text{ odd,} \end{cases}$$

$$s_p = \frac{\partial^2 f}{\partial x \partial y}(p\pi, -p\pi) = 2 - 4(-1)^p = \begin{cases} -2 & \text{for } p \text{ even,} \\ 6 & \text{for } p \text{ odd,} \end{cases}$$

$$t_p = \frac{\partial^2 f}{\partial y^2}(p\pi, -p\pi) = 2 - 2(-1)^p = \begin{cases} 0 & \text{for } p \text{ even,} \\ 4 & \text{for } p \text{ odd.} \end{cases}$$

If p is even, then

$$r_p = -6, \quad t_p = 0, \quad r_p \cdot t_p = 0 < (-2)^2 = s_p^2,$$

and we have no extremum.

If p is odd, then

$$r_p = 10, \quad t_p = 4, \quad r_p \cdot t_p = 40 > 36 = s_p^2,$$

and we have a proper minimum for p odd.

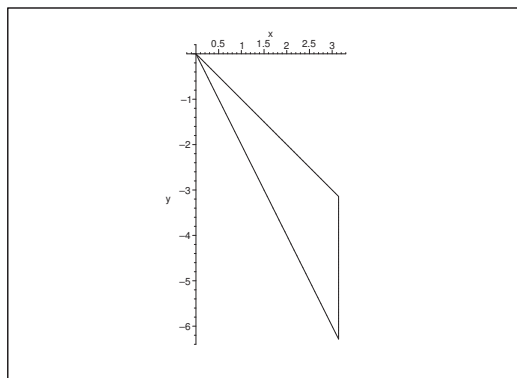


Figure 26: The domain A .

- 2) It follows from the figure that $(0, 0)$ and $(\pi, -\pi)$ are the only stationary points in A , and they both lie on the boundary of A .

Since $f(x, y)$ is continuous on the closed and bounded set A , it follows from the second main theorem for continuous functions that $f(x, y)$ has both a maximum and a minimum in A .

As proved above, the interior A° does not contain any stationary point, so the maximum and the minimum must be attained on the boundary of A .

EXAMINATION OF THE BOUNDARY.

a) If $y = -x$, $0 \leq x \leq \pi$, then the restriction is

$$f(x, -x) = 2 \cos x.$$

This has its maximum $f(0, 0) = 2$ and its minimum $f(\pi, -\pi) = -2$.

b) If $y = -2x$, $0 \leq x \leq \pi$, then the restriction is

$$f(x, -2x) = x^2 + 2 \cos 0 = x^2 + 2.$$

This has its maximum value $f(\pi, -2\pi) = \pi^2 + 2$ and its minimum value $f(0, 0) = 2$.

c) If $x = \pi$, $-2\pi \leq y \leq -\pi$, then the restriction is

$$f(\pi, y) = (\pi + y)^2 + 2 \cos(2\pi + y) = (\pi + y)^2 + 2 \cos y$$

with

$$f'_y(\pi, y) = 2(\pi + y) - 2 \sin y = 2\{\pi + y - \sin y\},$$

$$f''_{yy}(\pi, y) = 2\{1 - \cos y\} \geq 0,$$

so $f'_y(\pi, y)$ is increasing in y . From $f'_y(\pi, -\pi) = 0$ follows that $f'_y(\pi, y) < 0$ for $y \in [-2\pi, -\pi[$.

Since $f(\pi, -\pi) = 0 + 2 \cos \pi = -2$ and $f(\pi, -2\pi) = \pi^2 + 2$, these two values are respectively the minimum and the maximum.

Summarizing,

$$S = f(\pi, -2\pi) = \pi^2 + 2, \quad M = f(\pi, -\pi) = -2.$$

d) Clearly, $f(x, 0) \rightarrow +\infty$ for $x \rightarrow +\infty$, and $f(x, y) \geq -2$, so the range is contained in $[-2, +\infty[$.

Since

$$f((2n+1)\pi, -(2n+1)\pi) = -2, \quad n \in \mathbb{Z},$$

and f is continuous on the connected set \mathbb{R}^2 , it follows from the first main theorem of continuous functions that the range is connected, so

$$f(\mathbb{R}^2) = [-2, +\infty[.$$

Example 1.20 Given the function

$$f(x, y) = \frac{4}{3}x^3 - xy^2 + y, \quad (x, y) \in \mathbb{R}^2.$$

- 1) Find the stationary points. Check for each of them if we have an extremum.
- 2) Find the maximum and the minimum of the function in the square $A = [0, 1] \times [0, 1]$.

A Extrema and maximum and minimum.

D Check the stationary points and the boundary points.

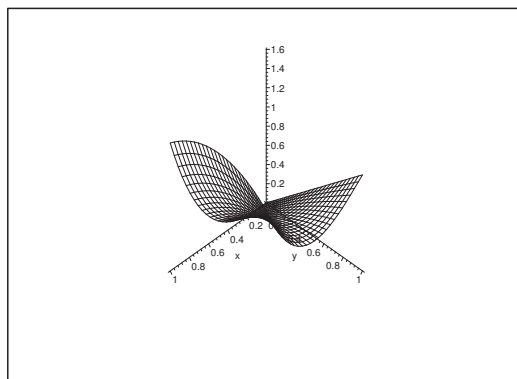


Figure 27: The graph of f over $A = [0, 1] \times [0, 1]$.

I 1) The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = 4x^2 - y^2 = (2x - y)(2x + y) = 0,$$

$$\frac{\partial f}{\partial y} = -2xy + 1 = 0, \quad \text{i.e.} \quad 2xy = 1.$$

- a) If $y = 2x$, then $1 = 2xy = 4x^2$, hence $x = \pm \frac{1}{2}$. In this case the stationary points are

$$\left(\frac{1}{2}, 1\right) \quad \text{og} \quad \left(-\frac{1}{2}, -1\right).$$

- b) If $y = -2x$, then $1 = 2xy = -4x^2$, which is not fulfilled for any real x .

Thus the stationary points are $\left(\frac{1}{2}, 1\right)$ and $\left(-\frac{1}{2}, -1\right)$.

EXAMINATION FOR EXTREMA.

First method. The (r, s, t) -method. It follows from

$$\frac{\partial^2 f}{\partial x^2} = 8x, \quad \frac{\partial^2 f}{\partial x \partial y} = -2y, \quad \frac{\partial^2 f}{\partial y^2} = -2x,$$

that

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = -16x^2 < 0, \quad \text{for } x \neq 0,$$

and we conclude that we have no extremum.

Second method. We shall find the approximating polynomial $P_2(x, y)$ of at most second degree in the neighbourhood of the stationary points.

a) In the neighbourhood of $\left(\frac{1}{2}, 1\right)$ we put $x = \frac{1}{2} + \xi$ and $y = 1 + \eta$. Then by insertion,

$$\begin{aligned} f(x, y) &= \frac{4}{3}x^3 - xy^2 + y = \frac{4}{3}\left(\frac{1}{2} + \xi\right)^3 - \left(\frac{1}{2} + \xi\right)(1 + \eta)^2 + 1 + \eta \\ &= \frac{4}{3}\left(\frac{1}{8} + \frac{3}{4}\xi + \frac{3}{2}\xi^2 + \xi^3\right) - \left(\frac{1}{2} + \xi\right)(1 + 2\eta + \eta^2) + 1 + \eta \\ &= \frac{1}{6} + \xi + 2\xi^2 + \frac{4}{3}\xi^3 - \frac{1}{2} - \eta - \frac{1}{2}\eta^2 - \xi - 2\xi\eta - \xi\eta^2 + 1 + \eta \\ &= \left(1 - \frac{1}{2} + \frac{1}{6}\right) + 2\xi^2 - 2\xi\eta - \frac{1}{2}\eta^2 + \dots \\ &= \frac{2}{3} + 2\left(\xi - \frac{1}{2}\eta\right)^2 - \eta^2 + \dots, \end{aligned}$$

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and we see that $f(x, y)$ in any neighbourhood of $\left(\frac{1}{2}, 1\right)$ attains both values $> f\left(\frac{1}{2}, 1\right) = \frac{2}{3}$ and values $< f\left(\frac{1}{2}, 1\right) = \frac{2}{3}$, so there is no extremum.

- b) In the neighbourhood of $\left(-\frac{1}{2}, -1\right)$ we put instead $x = -\frac{1}{2} - \xi$ and $y = -1 - \eta$. Then by analogous calculations as above,

$$f(x, y) = -\frac{2}{3} - 2\left(\xi - \frac{1}{2}\eta\right)^2 + \eta^2 + \dots,$$

and we conclude that $\left(-\frac{1}{2}, -1\right)$ is not an extremum.

- 2) Now f is continuous on the closed and bounded set A , hence it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in A . Since f is of class C^∞ with no extremum at the stationary points, the maximum and the minimum must be attained at boundary points.

EXAMINATION OF THE BOUNDARY.

- a) If $y = 0$ and $0 \leq x \leq 1$, then the restriction

$$f(x, 0) = \frac{4}{3}x^3, \quad x \in [0, 1],$$

has its minimum $f(0, 0) = 0$ and its maximum $f(1, 0) = \frac{4}{3}$.

- b) If $x = 0$ and $0 \leq y \leq 1$, then the restriction

$$f(0, y) = y, \quad y \in [0, 1],$$

has its minimum $f(0, 0) = 0$ and its maximum $f(0, 1) = 1$.

- c) If $x = 1$ and $0 \leq y \leq 1$, then the restriction

$$f(1, y) = \frac{4}{3} - y^2 + y = \frac{4}{3} + \frac{1}{4} - \left(y - \frac{1}{2}\right)^2 = \frac{19}{12} - \left(y - \frac{1}{2}\right)^2$$

has its minimum $f(1, 0) = f(1, 1) = \frac{4}{3}$, and its maximum $f\left(1, \frac{1}{2}\right) = \frac{19}{12}$.

- d) If $y = 1$ and $0 \leq x \leq 1$, then the restriction is

$$f(x, 1) = \frac{4}{3}x^3 - x + 1,$$

where

$$f'_x(x, 1) = 4x^2 - 1 = 0 \quad \text{for } x = \frac{1}{2} \in [0, 1].$$

Thus we have the possibilities

$$f(0, 1) = 1, \quad f\left(\frac{1}{2}, 1\right) = \frac{2}{3}, \quad f(1, 1) = \frac{4}{3},$$

and we see that this restriction has its minimum $f\left(\frac{1}{2}, 1\right) = \frac{2}{3}$, and its maximum

$$f(1, 1) = \frac{4}{3}.$$

Summarizing, we get by a numerical comparison that $f(0, 0) = 0$ is the minimum, and $f\left(1, \frac{1}{2}\right) = \frac{19}{12}$ is the maximum in A .

Example 1.21 Explain why the function

$$f(x, y) = 2y \sqrt{x(1-x)} - y^2, \quad (x, y) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [0, 3],$$

has both a maximum S and a minimum M .

Find these values as well as the range of the function.

A Maximum and minimum.

D Use the standard method.

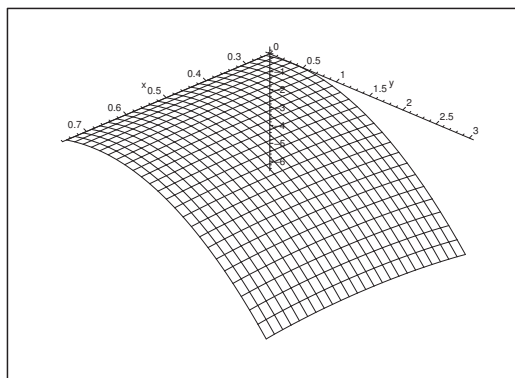


Figure 28: The graph of f . Notice the different scales on the axes.

I Since $f(x, y)$ is continuous and A is closed and bounded, it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in A . Since f is of class C^∞ in the interior of the domain, these values are either obtained at a stationary point or at a boundary point.

The stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = \frac{y(1-2x)}{\sqrt{x(1-x)}} = 0 \quad \text{og} \quad \frac{\partial f}{\partial y} = 2\sqrt{x(1-x)} - 2y = 0.$$

The former equation implies the possibilities of $y = 0$ or $x = \frac{1}{2}$. If $y = 0$, we consider only the boundary of A . Furthermore, $\frac{\partial f}{\partial y} \neq 0$ in A on that line.

If $x = \frac{1}{2}$, then

$$\frac{\partial f}{\partial y} = 2\sqrt{\frac{1}{2} \cdot \frac{1}{2}} - 2y = 1 - 2y = 0,$$

thus $y = \frac{1}{2}$, and the stationary point is $\left(\frac{1}{2}, \frac{1}{2}\right) \in A$. The value of the function is here

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

THE BOUNDARY POINTS

- 1) If $y = 0$ and $\frac{1}{4} \leq x \leq \frac{3}{4}$, then the restriction degenerates to $f(x, 0) = 0$.
- 2) If $y = 3$ and $\frac{1}{4} \leq x \leq \frac{3}{4}$, then the restriction is given by

$$f(x, 3) = 6\sqrt{x(1-x)} - 9 = 6\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2} - 9.$$

We get the maximum for $x = \frac{1}{2}$:

$$f\left(\frac{1}{2}, 3\right) = \frac{6}{2} - 9 = -6, .$$

and the minimum for $x = \frac{1}{4}$ and $x = \frac{3}{4}$, where

$$f\left(\frac{1}{4}, 3\right) = f\left(\frac{3}{4}, 3\right) = 6 \frac{\sqrt{3}}{4} - 9 = \frac{3\sqrt{3}}{2} - 9.$$

- 3) If $x = \frac{1}{4}$ or $x = \frac{3}{4}$, and $0 \leq y \leq 3$, then the restriction is

$$f\left(\frac{1}{4}, y\right) = f\left(\frac{3}{4}, y\right) = 2y\sqrt{\frac{3}{16}} \cdot y^2 = \frac{3}{16} - \left(y - \frac{\sqrt{3}}{4}\right)^2,$$

and we get the maximum for $y = \frac{\sqrt{3}}{4}$:

$$f\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right) = f\left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right) = \frac{3}{16},$$

and the minimum for $y = 3$,

$$f\left(\frac{1}{4}, 3\right) = f\left(\frac{3}{4}, 3\right) = 6 \frac{\sqrt{3}}{4} - 9 = \frac{3\sqrt{3}}{2} - 9.$$

By a numerical comparison of the candidate above we conclude that the maximum is

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4},$$

and similarly the minimum,

$$f\left(\frac{1}{4}, 3\right) = f\left(\frac{3}{4}, 3\right) = \frac{3\sqrt{3}}{2} - 9.$$

Since f is continuous and the domain is connected we conclude from the first main theorem for continuous functions that the range is

$$M, S] = \left[\frac{3\sqrt{3}}{2} - 9, \frac{1}{4} \right].$$

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Example 1.22 Given the function

$$f(x, y) = (x + y)^2 (8 - (x^2 + y^2)), \quad (x, y) \in \mathbb{R}^2.$$

- 1) Find the set of stationary points for f .
- 2) Explain why f has both a maximum and a minimum in the set

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\},$$

and find those values.

- 3) Check if f has a maximum and a minimum on the set

$$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 16\},$$

and find the range $f(B)$.

A Stationary points; maximum and minimum.

D Use the standard method.

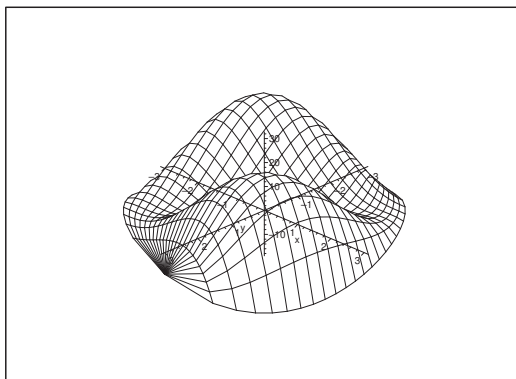


Figure 29: The graph of f over the set A .

- I** 1) The stationary points are the solutions of the equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2(x + y) \{8 - (x^2 + y^2)\} - 2x(x + y)^2 \\ &= 2(x + y) \{8 - x^2 - (x^2 + xy + y^2)\} = 0 \\ \frac{\partial f}{\partial y} &= 2(x + y) \{8 - y^2 - (x^2 + xy + y^2)\} = 0. \end{aligned}$$

The obvious solution is $x + y = 0$. The other possibility is

$$x^2 + (x^2 + xy^2) = 8 = y^2 + (x^2 + xy^2),$$

so $y = \pm x$. We have already found $y = -x$, so let $y = x$. Then $4x^2 = 8$, hence $x = y = \pm\sqrt{2}$.

Summarizing, the stationary points are

$$\{(x, -x) \mid x \in \mathbb{R}\} \cup \{(\sqrt{2}, \sqrt{2})\} \cup \{(-\sqrt{2}, -\sqrt{2})\}.$$

2) The function f is continuous on the closed and bounded set A , so f has a maximum and a minimum according to the main theorem for continuous functions. These are either attained at singular points or at boundary points.

a) If $y = -x$, then $f(x, -x) = 0$.

b) If $(x, y) = \pm(\sqrt{2}, \sqrt{2})$, then

$$f(\pm\sqrt{2}, \pm\sqrt{2}) = 8 \cdot (8 - 4) = 32$$

(same sign at both places).

c) If $x^2 + y^2 = 9$, then $x = 3 \cos t$ and $y = 3 \sin t$, $t \in [-\pi, \pi]$, so the restriction is

$$\varphi(t) = f(x(t), y(t)) = 9(\cos t + \sin t)^2 \cdot (8 - 9) = -18 \sin^2 \left(t + \frac{\pi}{4} \right),$$

and it follows that the maximum is 0 for $t = -\frac{\pi}{4}$ and $t = \frac{3\pi}{4}$, while the minimum is -18 for $t = \frac{\pi}{4}$ and $t = -\frac{3\pi}{4}$, i.e. for

$$(x, y) = \pm \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right).$$

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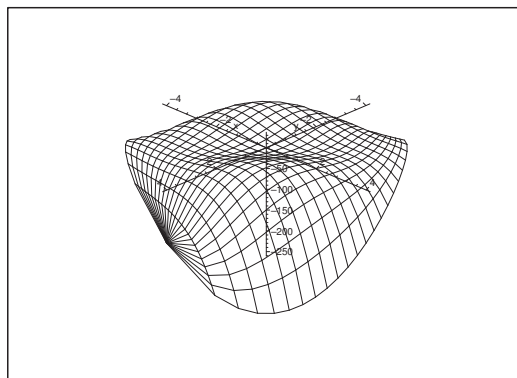
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Figure 30: The graph of f over the set B .

Summarizing, the maximum is

$$f(\pm\sqrt{2}, \pm\sqrt{2}) = 32 \quad (\text{same sign at both places}),$$

and the minimum is

$$f\left(\pm\frac{3}{2}\sqrt{2}, \pm\frac{3}{2}\sqrt{2}\right) = -18 \quad (\text{same sign at both places}).$$

- 3) The set B contains the same stationary points as A . The boundary is here $x^2 + y^2 = 4^2 = 16$. If we put $x = 4 \cos t$ and $y = 4 \sin t$, we get the restriction

$$\varphi(t) = f(x(t), y(t)) = 16(\cos t + \sin t)^2 \cdot (-8) = -256 \sin^2\left(t + \frac{\pi}{4}\right).$$

The maximum on the boundary is 0, and the minimum is -256 . The boundary is disjoint from B , so we shall never obtain the minimum, though the maximum in B is also

$$f(\pm\sqrt{2}, \pm\sqrt{2}) = 32 \quad (\text{same sign in both cases}).$$

The set B is connected, and f is continuous on B . It follows from the first main theorem for continuous functions that the range is also connected. Finally, we conclude from the above that

$$f(B) =] - 256, 32].$$

Example 1.23 Let B be the triangle of the vertices $\left(-\frac{1}{2}, 0\right)$, $(1, 0)$ and $(0, 1)$. Let the function $f : B \rightarrow \mathbb{R}$ be given by

$$f(x, y) = (x + y)(2x + y) + x, \quad (x, y) \in B.$$

- 1) Explain why f has a maximum S and a minimum M .
- 2) Find S and M and the points in which these values are attained.

A Maximum and minimum.

D Apply the second main theorem for continuous functions; find the stationary points and check the boundary points.

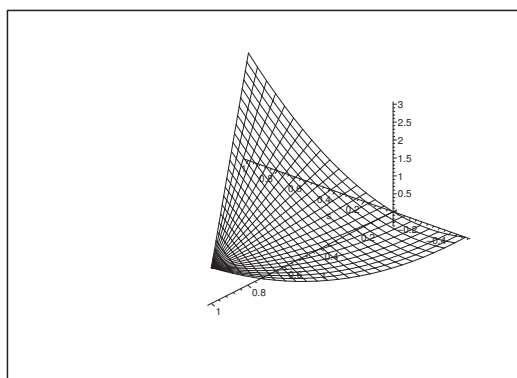


Figure 31: The graph of f over B .

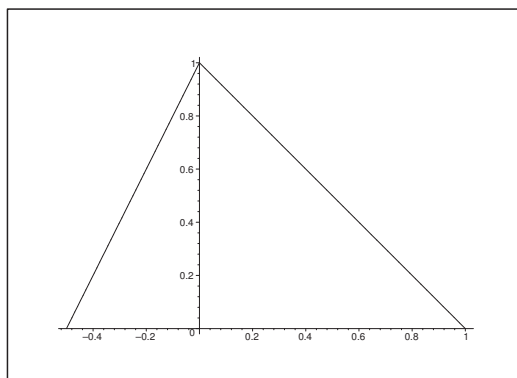


Figure 32: The domain B .

- I** 1) Since f is a real C^∞ -function, and B is a closed and bounded domain, it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in B .

Since f is of class C^∞ , the maximum and the minimum can only be attained at either a stationary point or on the boundary of B .

2) The possible *stationary points* are the solutions of the equations

$$f'_x(x, y) = 2x + y + 2(x + y) + 1 = 4x + 3y + 1 = 0,$$

$$f'_y(x, y) = 2x + y + x + y = 3x + 2y = 0.$$

Clearly, $(2, -3)$ is the only stationary point, and as it is in the fourth quadrant, it lies outside B .

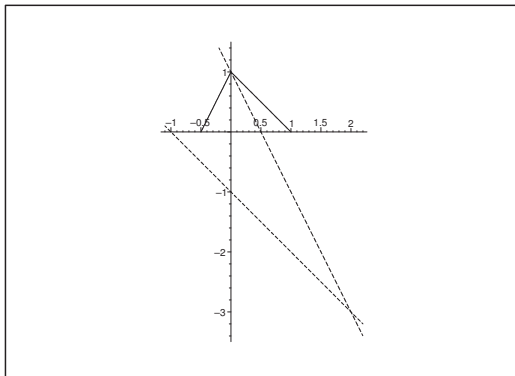


Figure 33: “Analysis of the sign” by means of the stationary point $(2, -3)$. The value of the function is < 1 in the angular spaces over the acute angles, and the value of the function is > 1 in the other two angular spaces.

THE BOUNDARY POINTS.

a) If $y = 0$, $x \in \left[-\frac{1}{2}, 1\right]$, then we get the restriction

$$f(x, 0) = x \cdot 2x + x = 2x^2 + x = 2\left(x + \frac{1}{4}\right)^2 - \frac{1}{8},$$

and it follows that the minimum and the maximum on this part of the boundary are

$$f\left(-\frac{1}{4}, 0\right) = -\frac{1}{8}, \quad \text{and} \quad f(1, 0) = 3, \quad \text{respctively.}$$

b) If $x + y = 1$, i.e. $y = 1 - x$, $x \in [0, 1]$, then the restriction is

$$f(x, 1 - x) = (x + 1) + x = 2x + 1.$$

Clearly, the minimum and the maximum are on this part of the boundary

$$f(0, 1) = 1 \quad \text{og} \quad f(1, 0) = 3.$$

c) If $y = 2x + 1$, $x \in \left[-\frac{1}{2}, 0\right]$, then we get the restriction

$$f(x, 2x + 1) = 12x^2 + 8x + 1 = 12\left(x + \frac{1}{3}\right)^2 - \frac{1}{3},$$

and the minimum and the maximum on this part of the boundary are

$$f\left(-\frac{1}{3}, +\frac{1}{3}\right) = -\frac{1}{3} \quad \text{og} \quad f(0, 1) = 1.$$

Finally, by a numerical comparison,

$$M = f\left(-\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3} \quad \text{og} \quad S = f(1, 0) = 3.$$

REMARK. If we translate the coordinate system to the stationary point, i.e. if we put

$$x_1 = x - 2 \quad \text{og} \quad y_1 = y + 3,$$

then

$$f(x, y) = f_1(x_1, y_1) = (x_1 + y_1)(2x_1 + y_1) + 1.$$

A geometrical consideration shows that the minimum must be attained on the line $y = 2x + 1$, $x \in \left[-\frac{1}{2}, 0\right]$, and the maximum on the line $x + y = 1$, in accordance with the results above. \diamond

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2 Ranges of Functions

Example 2.1 Sketch the domain of the function

$$f(x, y) = \sqrt{2x - x^2 - y^2} + \frac{1}{\sqrt{2y - x^2 - y^2}}.$$

Then find the range of the function.

A Domain and range.

D First find the domain. Then find the possible stationary points and check the values of the function on the boundary.

I The first term is defined (and ≥ 0) for

$$0 \leq 2x^2 - x^2 - y^2 = 1 - (x - 1)^2 - y^2,$$

thus for

$$(x - 1)^2 + y^2 \leq 1.$$

This inequality represents a closed disc of centrum $(1, 0)$ and radius 1.

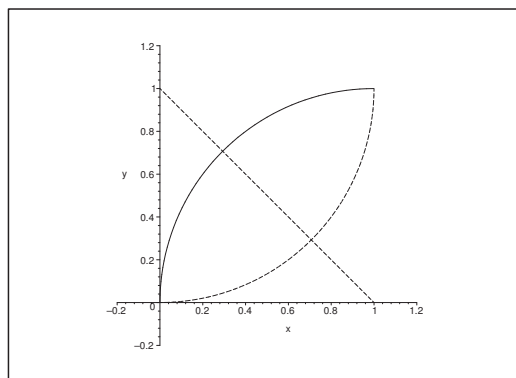


Figure 34: The domain D lies between the two circular arcs.

The second term is defined (and > 0) for

$$0 < 2y - x^2 - y^2 = 1 - x^2 - (y - 1)^2, \text{ dvs. } x^2 + (y - 1)^2 < 1.$$

This inequality describes an open disc of centrum $(0, 1)$ and radius 1.

The domain D is the intersection of these two discs. It is neither open nor closed, because one of the boundary curves is contained in D , while the other is not.

Notice in particular that $0 < x < 1$ and $0 < y < 1$ for $(x, y) \in D$.

STATIONARY POINTS. The function is of class C^∞ in the interior D° of the domain. Therefore, there are no exception points. The equations of the possible stationary points are

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} = \frac{1-x}{\sqrt{2x-x^2-y^2}} + \frac{x}{(\sqrt{2y-x^2-y^2})^3}, \\ 0 &= \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{2x-x^2-y^2}} - \frac{1-y}{(\sqrt{2y-x^2-y^2})^3}. \end{aligned}$$

It was mentioned above that in particular $0 < x < 1$ and $0 < y < 1$ for $(x, y) \in D^\circ$, so $\frac{\partial f}{\partial x} > 0$ and $\frac{\partial f}{\partial y} < 0$ everywhere in D° . The equations of the stationary points are not fulfilled in D° , so there does not exist any of them.

Clearly, $f(x, y) \rightarrow +\infty$, when (x, y) tends to any point on that arc of ∂D , which does *not* belong to D . The set D is connected and f is continuous, so it follows from the first and the second main theorem for continuous functions that $f(D) = [a, +\infty[$, where the value a is attained at a point of the boundary which also lies in D , i.e. in $\partial D \cap D$. Clearly, a is a minimum value.

The restriction of the function to $\partial D \cap D$ is

$$f(x, y) = 0 + \frac{1}{\sqrt{1 - x^2 - (y - 1)^2}}, \quad (x - 1)^2 + y^2 = 1, \quad x > 0, \quad y > 0.$$

This value of the function is smallest, when $1 - x^2 - (y - 1)^2$ is largest. This means that $x^2 + (y - 1)^2$, which can be interpreted as the square of the distance from $(0, 1)$ to (x, y) on the circular arc must be as small as possible. A geometric consideration shows that this point lies on both the line $x + y = 1$ and on the circle $(x - 1)^2 + y^2 = 1$, as well as in the first quadrant, hence the searched point is

$$(x, y) = \left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

The minimum value a is the value of the function at this point.

$$\begin{aligned} a &= f\left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{1 - \left(-\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}} - 1\right)^2}} \\ &= \frac{1}{\sqrt{1 - 2\left(1 - \frac{1}{\sqrt{2}}\right)^2}} = \frac{1}{\sqrt{1 - (\sqrt{2} - 1)^2}} = \frac{1}{\sqrt{1 - 2 - 1 + 2\sqrt{2}}} \\ &= \frac{1}{\sqrt{2\sqrt{2} - 2}} \cdot \frac{\sqrt{2\sqrt{2} + 2}}{\sqrt{2\sqrt{2} + 2}} = \frac{\sqrt{2\sqrt{2} + 2}}{2} = \sqrt{\frac{\sqrt{2} + 1}{2}}. \end{aligned}$$

We conclude that the range is

$$f(D) = \left[\frac{\sqrt{2\sqrt{2} + 2}}{2}, +\infty\right] = \left[\sqrt{\frac{\sqrt{2} + 1}{2}}, +\infty\right].$$

Example 2.2 *Let*

$$f(x, y) = -2xy^2 + 4x^2 + y^2 - 2x, \quad (x, y) \in A.$$

Find the range of the function in the following cases:

1) *The domain A is the closed half ellipsoidal disc given by*

$$4x^2 + y^2 \leq 1 \quad \text{and} \quad x \geq 0.$$

2) *The domain A is the open half ellipsoidal disc given by*

$$4x^2 + y^2 < 1 \quad \text{and} \quad x > 0.$$

3) *The domain A is the whole plane.*

A Maximum and minimum; range.

D From $f \in C^\infty$ follows that there are no exception point. The set A is connected in all three cases and the function is continuous, so the range is again connected, according to the first main theorem for continuous functions, hence an interval. We can in the three cases apply the following methods:

- 1) Since A is closed and bounded, we can apply the second main theorem, hence the maximum and minimum exist and they are attained at either a stationary point or at a boundary point.
- 2) Here A is bounded and open. The closure was treated in 1), so we can derive the range from 1).
- 3) In this case A is unbounded. We argue on the term of highest degree.

I The equations of the possible stationary points in the plane are

$$\begin{aligned} \frac{\partial f}{\partial x} &= -2y^2 + 8x - 2 = 0, \\ \frac{\partial f}{\partial y} &= -4xy + 2y = 2y(1 - 2x) = 0. \end{aligned}$$

From the latter equation we get the possibilities

$$\text{a) } y = 0 \quad \text{and} \quad \text{b) } x = \frac{1}{2}.$$

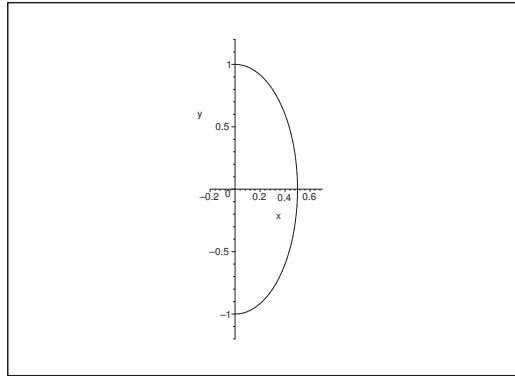
Thus:

- a) If $y = 0$, then $\frac{\partial f}{\partial x} = 0$ for $x = \frac{1}{4}$, so $\left(\frac{1}{4}, 0\right)$ is a stationary point for f .
- b) If $x = \frac{1}{2}$, then $\frac{\partial f}{\partial x} = -2y^2 + 4 - 2 = 0$ for $y = \pm 1$, so $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, -1\right)$ are stationary points for f .

Summarizing, the stationary points for f in the plane are

$$\left(\frac{1}{4}, 0\right), \quad \left(\frac{1}{2}, 1\right), \quad \left(\frac{1}{2}, -1\right).$$

Of these, only the first one lies in the half ellipsoidal disc of the first two questions.



- 1) If $A = \{(x, y) \mid 4x^2 + y^2 \leq 1 \text{ and } x \geq 0\}$, then in the stationary point of A ,

$$f\left(\frac{1}{4}, 0\right) = 0 + 4 \cdot \frac{1}{16} + 0 - 2 \cdot \frac{1}{4} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.$$

The boundary falls naturally into two pieces:

- a) If $x = 0$ and $y \in [-1, 1]$, we get the restriction

$$\varphi(y) = f(0, y) = y^2, \quad y \in [-1, 1],$$

with the minimum value

$$\varphi(0) = f(0, 0) = 0,$$

and the maximum value

$$\varphi(1) = \varphi(-1) = f(0, 1) = f(0, -1) = 1.$$

- b) If $4x^2 + y^2 = 1$ and $x \geq 0$, then $y^2 = 1 - 4x^2$ and $x \in \left[0, \frac{1}{2}\right]$, so we get the restriction

$$\begin{aligned} \psi(x) &= f(x, \pm\sqrt{1-4x^2}) = -2x(y^2 + 1) + (4x^2 + y^2) \\ &= -2x(1 - 4x^2 + 1) + 1 = 8x^3 - 4x + 1, \quad x \in \left[0, \frac{1}{2}\right], \end{aligned}$$

with the derivative

$$\psi'(x) = 24x^2 - 4 = 24\left(x^2 - \frac{1}{6}\right),$$

which in the interval $\left]0, \frac{1}{2}\right[$ is zero for $x = \frac{1}{\sqrt{6}}$. Hence

$$f\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right) = f\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{3}}\right) = -2 \cdot \frac{1}{\sqrt{6}}\left(2 - \frac{4}{6}\right) + 1 = 1 - \frac{8}{3\sqrt{6}} < 0,$$

because $(3\sqrt{6})^2 = 9 \cdot 6 = 54 < 8^2 = 64$. At the end points,

$$f(0, \pm 1) = 1 \quad \text{and} \quad f\left(\frac{1}{2}, 0\right) = \frac{8}{8} - \frac{4}{2} + 1 = 0.$$

The maximum and minimum values shall be found among the value at the stationary point

$$f\left(\frac{1}{4}, 0\right) = -\frac{1}{4},$$

and the values at the boundary points found above,

$$f(0, 0) = 0, \quad f(0, \pm 1) = 1, \quad f\left(\frac{1}{2}, 0\right) = 0, \quad f\left(\frac{1}{\sqrt{6}}, \pm \frac{1}{\sqrt{3}}\right) = 1 - \frac{8}{3\sqrt{6}}.$$

Clearly, the *maximum value* is

$$S = f(0, 1) = f(0, -1) = 1,$$

(both are boundary points). From

$$25 \cdot 27 = 26^2 - 1 > 24^2 - 8^2 = 16 \cdot 32,$$

follows that

$$\frac{25}{16} > \frac{32}{27} = \frac{64}{54},$$

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so

$$\sqrt{\frac{25}{16}} = \frac{5}{4} > \sqrt{\frac{64}{54}} = \frac{8}{3\sqrt{6}},$$

and hence

$$f\left(\frac{1}{\sqrt{6}}, \pm \frac{1}{\sqrt{3}}\right) = 1 - \frac{8}{3\sqrt{6}} > 1 - \frac{5}{4} = -\frac{1}{4} = f\left(\frac{1}{4}, 0\right).$$

This shows that the *minimum value* is attained at the stationary point and that the value is

$$M = f\left(\frac{1}{4}, 0\right) = -\frac{1}{4}.$$

The domain is connected, so it follows from the first main theorem that the range is also connected, i.e.

$$f(A) = \left[-\frac{1}{4}, 1\right].$$

- 2) The closure of A was treated in 1), where the minimum value was attained at a stationary point, in particular in an interior point, and where the maximum value is attained at a boundary point. We therefore conclude that the range is

$$f(A) = \left[-\frac{1}{4}, 1\right].$$

- 3) By restriction to the line $y = x$,

$$g(x) = f(x, x) = -2x^3 + 5x^2 - 2x.$$

For large x the expression is dominated by $-2x^3$, and as $-2x^3 \rightarrow -\infty$ for $x \rightarrow +\infty$, and $-2x^3 \rightarrow +\infty$ for $x \rightarrow -\infty$, and as the range is an interval (the first main theorem again), we conclude that

$$f(A) = \mathbb{R}.$$

Example 2.3 *Let*

$$f(x, y) = 3x^3 + 4y^3 + 6xy^2 - 9x^2, \quad (x, y) \in A.$$

Find the range of the function in the following cases.

- 1) *The domain A is the closed triangle of the vertices $(0, 0)$, $(3, 3)$ and $\left(3, -\frac{3}{2}\right)$. item The domain A is the interior of the point set of 1).*
- 2) *The domain A is the whole plane.*

A Maximum and minimum; range.

D From $f \in C^\infty(\mathbb{R}^2)$ follows that there are no exception points. First find the stationary points in the plane. Since A is connected in all three cases, it follows from the first main theorem that all the ranges are intervals.

- 1) Since A is closed and bounded, and f is continuous, it follows from the second main theorem that f has both a maximum and a minimum in A . These can only be attained at a stationary point or at a boundary point.
- 2) Because A is the interior of the set of 1), we can apply the results from 1).
- 3) Consider e.g. the restriction of f to the line $y = x$.

The possible stationary points in the whole plane must satisfy the equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= 9x^2 + 6y^2 - 18x = 0, \\ \frac{\partial f}{\partial y} &= 12y^2 + 12xy = 12y(y + x) = 0. \end{aligned}$$

It follows from the latter equation that either $y = 0$ or $y = -x$. We therefore get the following possibilities:

- 1) If $y = 0$, then either $x = 0$ or $x = 2$, and the stationary points in this case are

$$(0, 0) \quad \text{and} \quad (2, 0).$$

- 2) If $y = -x$, then

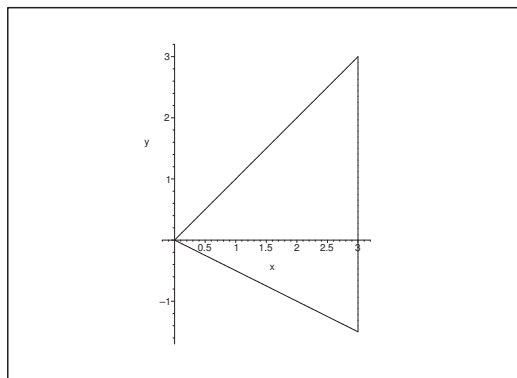
$$0 = 15x^2 - 18x = 15x \left(x - \frac{6}{5}\right),$$

which corresponds to the stationary points

$$(0, 0) \quad \text{and} \quad \left(\frac{6}{5}, -\frac{6}{5}\right).$$

Summarizing, the stationary points in the plane are

$$(0, 0), \quad (2, 0), \quad \left(\frac{6}{5}, -\frac{6}{5}\right).$$



- 1) It follows from the figure that $(2, 0)$ is the only *stationary point* in A . The value is here

$$f(2, 0) = 3 \cdot 8 - 9 \cdot 4 = -12.$$

THE BOUNDARY.

- a) The restriction to $y = x$ is

$$g_1(x) = f(x, x) = 13x^3 - 9x^2, \quad x \in [0, 3],$$

with the derivative

$$g_1'(x) = 39x^2 - 18 = 0 \quad \text{for } x = +\sqrt{\frac{6}{13}} \in [0, 3].$$

The candidates are

$$f\left(\sqrt{\frac{6}{13}}, \sqrt{\frac{6}{13}}\right) = 13\sqrt{\frac{6}{13}} \cdot \frac{6}{13} - 9 \cdot \frac{6}{13} = \frac{6}{13}(\sqrt{6 \cdot 13} - 9) < 0,$$

because $\sqrt{6 \cdot 13} - 9 = \sqrt{78} - 9 < \sqrt{81} - 9 = 0$, and

$$f(0, 0) = 0, \quad f(3, 3) = 13 \cdot 27 - 81 = 270.$$

- b) The restriction to $x = 3$ is

$$g_2(y) = 81 + 41y^3 + 18y^2 - 81 = 4y^3 + 18y^2$$

for $y \in \left[-\frac{3}{2}, 3\right]$, with the derivative

$$g_2'(y) = 12y^2 + 36y = 12y(y + 3) = 0 \quad \text{for } y = 0 \in \left[-\frac{3}{2}, 3\right].$$

The candidates are

$$f\left(3, -\frac{3}{2}\right) = 4\left(-\frac{3}{2}\right)^3 + 18\left(\frac{3}{2}\right)^2 = -4 \cdot \frac{27}{8} + 18 \cdot \frac{9}{4} = \frac{54}{2} = 27,$$

$$f(3, 0) = 0 \quad \text{and} \quad f(3, 3) = 270.$$

c) The restriction to $y = -\frac{1}{2}x$ is

$$\begin{aligned} g_3(x) &= 3x^3 + 4\left(-\frac{1}{2}x\right)^3 + 6x\left(-\frac{1}{2}x\right)^2 - 9x^2 \\ &= 3x^3 - \frac{1}{2}x^3 + \frac{3}{2}x^3 - 9x^2 \\ &= 4x^3 - 9x^2 \quad \text{for } x \in [0, 3], \end{aligned}$$

with the derivative


$$g'_3(x) = 12x^2 - 18x = 12x\left(x - \frac{3}{2}\right) = 0$$

for $(x = 0 \text{ and}) x = \frac{3}{2} \in [0, 3]$. The candidates are

$$f\left(\frac{3}{2}, -\frac{3}{4}\right) = 4\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 = \frac{4 \cdot 27}{8} - \frac{81}{4} = -\frac{27}{4},$$

$$f(0, 0) = 0 \quad \text{og} \quad f\left(3, -\frac{3}{2}\right) = 27.$$

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Summarizing, all the candidates are

$$f(2, 0) = -12,$$

(the stationary point), and

$$f\left(\sqrt{\frac{6}{13}}, \sqrt{\frac{6}{13}}\right) = 6\sqrt{\frac{6}{13}} - \frac{54}{13}, \quad f(3, 0) = 0, \quad f\left(\frac{3}{2}, -\frac{3}{4}\right) = -\frac{27}{4},$$

$$f(0, 0) = 0, \quad f(3, 3) = 270, \quad f\left(3, -\frac{3}{2}\right) = 27,$$

(at boundary points).

By a numerical comparison we find the *minimum value* at a stationary point

$$M = f(2, 0) = -12,$$

and the *maximum value* at a boundary point,

$$S = f(3, 3) = 270.$$

It follows from the first main theorem that the range is the interval

$$f(A) = [-12, 270].$$

- 2) When we remove the boundary of 1), we also remove the maximum value from the range. However, due to the continuity we can in A obtain values as close to 270 as we wish, so the range becomes

$$f(A) = [-12, 270[.$$

- 3) The restriction to the line $y = x$ is

$$g(x) = 13x^3 - 9x^2 = x^2(13x - 9), \quad x \in \mathbb{R}.$$

The range is clearly

$$f(A) = \mathbb{R}.$$

Example 2.4 *Let*

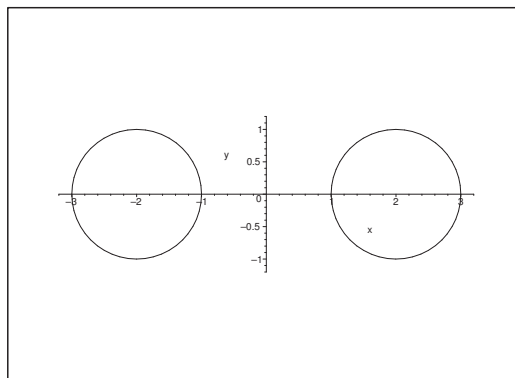
$$f(x, y) = x + y + \sqrt{|4x| - x^2 - y^2 - 3}, \quad (x, y) \in A.$$

1) *Find the domain of the function.*

2) *Find the range $f(A)$.*

A Domain and range.

D Sketch the set A . Consider the two main theorems of continuous functions.



I 1) It follows from the rearrangement

$$f(x, y) = x + y + \sqrt{|4x| - x^2 - y^2 - 3} = x + y + \sqrt{1 - (|x| - 2)^2 - y^2}$$

that the domain of f is the union of two closed discs,

$$A = \overline{K}((2, 0); 1) \cup \overline{K}((-2, 0); 1).$$

This set is closed and bounded, but not connected, so we can only apply the second main theorem. We conclude that f has a maximum and a minimum of both of its connected components.

2) The investigation is now split into the two cases of $x > 0$ and $x < 0$.

a) If $x > 0$, then

$$f(x, y) = x + y + \sqrt{1 - (x - 2)^2 - y^2}, \quad (x - 2)^2 + y^2 \leq 1.$$

The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 1 - \frac{x - 2}{\sqrt{1 - (x - 2)^2 - y^2}} = 0, \quad \frac{\partial f}{\partial y} = 1 - \frac{y}{\sqrt{1 - (x - 2)^2 - y^2}} = 0,$$

hence

$$y = x - 2 = \sqrt{1 - (x - 2)^2 - y^2},$$

so $y \geq 0$ (and $x \geq 2$), and $y = \sqrt{1 - 2y^2}$, thus

$$1 - 2y^2 = y^2, \quad \text{or} \quad y = +\frac{1}{\sqrt{3}} > 0,$$

and $x = 2 + \frac{1}{\sqrt{3}}$.

In this connected component, the only stationary point is $\left(2 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, corresponding to the value

$$f\left(2 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 2 + \frac{2}{\sqrt{3}} + \sqrt{1 - \frac{2}{3}} = 2 + \frac{3}{\sqrt{3}} = 2 + \sqrt{3}.$$

The square root is 0 on the *boundary*, so we shall only find the maximum and the minimum of $x + y$ on the boundary. A geometric analysis shows that the points must lie on the line $y = x - 2$. The radius of the circle is 1, so we get the points

$$\left(2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \text{og} \quad \left(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

corresponding to the values

$$f\left(2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 - \frac{2}{\sqrt{2}} = 2 - \sqrt{2} \quad \text{and} \quad f\left(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2}.$$

It follows by a numerical comparison of

$$f\left(2 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 2 + \sqrt{3}, \quad f\left(2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2},$$

$$f\left(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2},$$

that the range is

$$[2 - \sqrt{2}, 2 + \sqrt{3}].$$

ALTERNATIVELY the boundary is described by

$$x = 2 + \cos \theta, \quad y = \sin \theta, \quad \theta \in [0, 2\pi],$$

so the restriction becomes

$$g_1(\theta) = x + y + 0 = 2 + \cos \theta + \sin \theta$$

with the derivative

$$g_1'(\theta) = -\sin \theta + \cos \theta = -\sqrt{2} \sin\left(\theta - \frac{\pi}{4}\right),$$

which is 0 for $\theta = \frac{\pi}{4}$ or for $\theta = \frac{5\pi}{4}$. This gives us the candidates

$$f\left(2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2} \quad \text{for } \theta = \frac{\pi}{4},$$

$$f\left(2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \quad \text{for } \theta = \frac{5\pi}{4},$$

and formally (though not in reality)

$$g_1(0) = g_1(2\pi) = f(3, 0) = 3 \quad (< 2 + \sqrt{2}) \quad \text{for } \theta = 0 \text{ and } \theta = 2\pi.$$

The range of the connected subregion is

$$[2 - \sqrt{2}, 2 + \sqrt{3}].$$

3) If $x < 0$, then

$$f(x, y) = x + y + \sqrt{1 - (x + 2)^2 - y^2}, \quad (x + 2)^2 + y^2 \leq 1,$$

and the equations of the stationary points are

$$\frac{\partial f}{\partial x} = 1 - \frac{x + 2}{\sqrt{1 - (x + 2)^2 - y^2}} = 0,$$

$$\frac{\partial f}{\partial y} = 1 - \frac{y}{\sqrt{1 - (x + 2)^2 - y^2}} = 0.$$

The possible stationary points satisfy

$$y = x + 2 = \sqrt{1 - (x + 2)^2 - y^2} \geq 0,$$

thus as before

$$y = +\frac{1}{\sqrt{3}} \quad \text{and} \quad x = -2 + \frac{1}{\sqrt{3}},$$

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corresponding to the value

$$f\left(-2 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -2 + \frac{2}{\sqrt{3}} + \sqrt{1 - \frac{2}{3}} = -2 + \sqrt{3}.$$

The investigation of the boundary is similar to the previous one (we have again two variants), so we find the candidates

$$\left(-2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \text{og} \quad \left(-2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

of the corresponding values of the function

$$f\left(-2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -2 - \sqrt{2} \quad \text{og} \quad f\left(-2 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -2 + \sqrt{2}.$$

We conclude by a numerical comparison that the maximum and the minimum values on this connected component are

$$S = f\left(-2 + \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -2 + \sqrt{3}$$

and

$$M = f\left(-2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -2 - \sqrt{2},$$

corresponding to the subinterval

$$[-2 - \sqrt{2}, -2 + \sqrt{3}].$$

Summarizing, the range becomes

$$f(A) = [-2 - \sqrt{2}, -2 + \sqrt{3}] \cup [2 - \sqrt{2}, 2 + \sqrt{3}],$$

which is the union of two closed intervals with no points in common.

Example 2.5 Find the range of the function

$$f(x, y, z) = xy + z^2, \quad x^2 + y^2 + z^2 \leq a^2.$$

- A** the range of a polynomial in three variables over a closed ball, i.e. a closed and bounded and connected set.
- D** According to the second main theorem, f has a maximum value S and a minimum value M on this set. Then the first main theorem implies that the range is connected, hence the interval $[M, S]$. The maximum and the minimum values are either attained at a stationary point or at a boundary point, because a polynomial does not have exceptional points. Therefore, find the possible stationary points, and examine the behaviour of the function on the boundary. Finally, make a numerical comparison.

I The equations of the possible stationary points are

$$\frac{\partial f}{\partial x} = y = 0, \quad \frac{\partial f}{\partial y} = x = 0, \quad \frac{\partial f}{\partial z} = 2z = 0.$$

Clearly, $(0, 0, 0)$ is the only stationary point, and the value is here $f(0, 0, 0) = 0$.

We shall use spherical coordinates by the EXAMINATION OF THE BOUNDARY,

$$\begin{cases} x = a \sin \theta \cos \varphi, \\ y = a \sin \theta \sin \varphi, \\ z = a \cos \theta, \end{cases} \quad \text{where } \theta \in [0, \pi] \text{ and } \varphi \in [0, 2\pi].$$

By insertion we get the following restriction to the boundary

$$\begin{aligned} f(x, y, z) &= xy + z^2 = a^2 \sin^2 \theta \cos \varphi \sin \varphi + a \cos^2 \theta \\ &= \frac{a^2}{2} \{ \sin^2 \theta \sin 2\varphi + 2 \cos^2 \theta \} \\ &= \frac{a^2}{2} \{ 2 + \sin^2 \theta (\sin 2\varphi - 2) \}. \end{aligned}$$

These rearrangements show that $f(x, y, z)$ is largest on the boundary (the sphere), when $\sin^2 \theta = 0$, and the corresponding maximum value is

$$S = \frac{a^2}{2} \{ 2 + 0 \} = a^2 \quad (> 0).$$

Furthermore, $f(x, y, z)$ is smallest on the boundary (the sphere), when $\sin^2 \theta = 1$ and $\sin 2\varphi = -1$, which corresponds to the minimum value

$$M = \frac{a^2}{2} \{ 2 + 1 \cdot (-1 - 2) \} = -\frac{a^2}{2}.$$

Since the ball is connected and the function is continuous, it follows from the first main theorem that the range is

$$f(A) = \left[-\frac{a^2}{2}, a^2 \right].$$

REMARK. Since we shall not explicitly indicate where the maximum and the minimum values are attained, we shall only argue instead of doing some heavy computations.. \diamond

Example 2.6 Given the function

$$f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{2 - x - y - z}, \quad (x, y, z) \in A.$$

- 1) Find the domain A and sketch it.
- 2) Explain why the function has both a maximum value S and a minimum value M .
- 3) Find S and M and the points in which they are attained.
- 4) Find the range $f(A)$ of the function.

A Domain, maximum and minimum and range.

D Use the standard methods and some symmetry arguments.

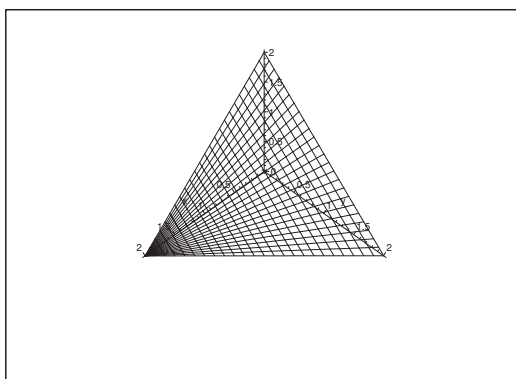


Figure 35: The domain A .

- I** 1) The function is defined (and continuous) for $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x + y + z \leq 2$, i.e. in the closed tetrahedron

$$A = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 2 - x, 0 \leq z \leq 2 - x - y\}$$

shown on the figure.

- 2) Since A is closed and bounded, and f is continuous on A , it follows from the second main theorem for continuous functions that f has both a maximum and a minimum on A .

Since f is of class C^∞ in the interior of A , there are no exception points, so the maximum and the minimum lie either in the interior stationary points or on the boundary.

Since A is connected, also $f(A)$ is connected, due to the first main theorem for continuous functions, and the range is the interval

$$f(A) = [M, S],$$

cf. 4).

3) The equations of the possible stationary points in the interior of A are

$$\frac{\partial f}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x}} - \frac{1}{2} \frac{1}{\sqrt{2-x-y-z}} = 0,$$

$$\frac{\partial f}{\partial y} = \frac{1}{2} \frac{1}{\sqrt{y}} - \frac{1}{2} \frac{1}{\sqrt{2-x-y-z}} = 0.$$

$$\frac{\partial f}{\partial z} = \frac{1}{2} \frac{1}{\sqrt{z}} - \frac{1}{2} \frac{1}{\sqrt{2-x-y-z}} = 0.$$

Since $x, y, z > 0$ and $x + y + z < 2$ in the interior of A we get

$$x = y = z = 2 - x - y - z \quad (-2 - 3x),$$

thus $x = y = z = \frac{1}{2} > 0$. The only stationary point in the interior of A is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, which we intuitively could expect from the symmetry. The value of the function at this stationary point is

$$f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 4\sqrt{\frac{1}{2}} = 2\sqrt{2}.$$

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THE BOUNDARY.

- a) First consider the boundary surface B , which also lies in the (X, Y) -plane, i.e. where $z = 0$. The restriction is given by

$$\varphi(x, y) = f(x, y, 0) = \sqrt{x} + \sqrt{y} + \sqrt{2 - x - y}, \quad (x, y) \in B,$$

where

$$B = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2 - x\}.$$

This restriction φ is of class C^∞ in the interior of B , so it has a maximum and a minimum

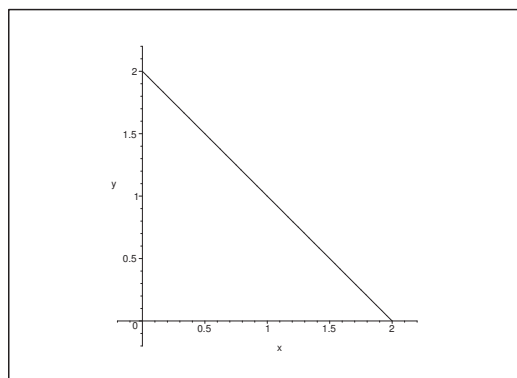


Figure 36: The part of the boundary B in the (X, Y) -plane.

in B according to the second main theorem. It follows from the conditions of a (restricted) stationary point

$$\frac{\partial \varphi}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x}} - \frac{1}{2} \frac{1}{\sqrt{2 - x - y}} = 0,$$

$$\frac{\partial \varphi}{\partial y} = \frac{1}{2} \frac{1}{\sqrt{y}} - \frac{1}{2} \frac{1}{\sqrt{2 - x - y}} = 0,$$

that

$$0 < x = y = 2 - x - y \quad (= 2 - 2x),$$

hence $x = y = \frac{2}{3}$, which corresponds to

$$\varphi\left(\frac{2}{3}, \frac{2}{3}\right) = f\left(\frac{2}{3}, \frac{2}{3}, 0\right) = 3\sqrt{\frac{2}{3}} = \sqrt{6}.$$

This is the “stationary point” on the part of the boundary in the (X, Y) -plane.

There is a similar “examination of the boundary” connected with this part of the boundary. If e.g. $y = 0$, we get the restriction

$$\psi(x) = \varphi(x, 0) = f(x, 0, 0) = \sqrt{x} + \sqrt{2 - x}, \quad x \in [0, 2].$$

Using the symmetry we get $\psi'(x) = 0$ for $x = 1$, which corresponds to the candidate

$$\psi(1) = \varphi(1, 0) = f(1, 0, 0) = 2.$$

If instead $x = 0$, then similarly,

$$f(0, 1, 0) = 2.$$

The remaining boundary curve of the surface in the (X, Y) -plane lies on the line $x + y = 2$, so the restriction becomes

$$\eta(x) = \varphi(x, 2 - x) = \sqrt{x} + \sqrt{2 - x}, \quad x \in [0, 2],$$

which is identical with $\psi(x)$ from above. Therefore we get the candidate

$$\eta(1) = \varphi(1, 1) = f(1, 1, 0) = 2.$$

- b) It follows by the symmetry that we have analogous results in those parts of the boundary surface which lie in the planes $y = 0$ and $x = 0$, respectively. We see that we have at the same time examined the *boundary curves* of the remaining oblique part of the boundary surface, so we are only missing the investigation of the “interior” of this remaining part of the boundary surface.
- c) The restriction to the “interior” of the oblique boundary surface lying in the plane $z = 2 - x - y$ is given by

$$\Theta(x, y) = f(x, y, 2 - x - y) = \sqrt{x} + \sqrt{y} + \sqrt{2 - x - y},$$

where the domain of the parameter is the domain B in the (X, Y) -plane. Since we formally have $\Theta(x, y) = \varphi(x, y)$ over the same domain, we can reuse the results from φ , hence the interesting point is $\left(\frac{2}{3}, \frac{2}{3}\right)$. The value of the function is here

$$\Theta\left(\frac{2}{3}, \frac{2}{3}\right) = f\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \varphi\left(\frac{2}{3}, \frac{2}{3}\right) = \sqrt{6}.$$

As a CONCLUSION we have the following candidates of S and M :

Stationary point:

$$f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2\sqrt{2} \quad (= \sqrt{8}).$$

Interior point on the boundary surfaces in $x = 0$, $y = 0$, $z = 0$, respectively:

$$f\left(0, \frac{2}{3}, \frac{2}{3}\right) = f\left(\frac{2}{3}, 0, \frac{2}{3}\right) = f\left(\frac{2}{3}, \frac{2}{3}, 0\right) = \sqrt{6}.$$

Interior point of the oblique part of the boundary surface lying in $x + y + z = 2$:

$$f\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \sqrt{6}.$$

Interior points on the edges lying on one of the axes:

$$f(1, 0, 0) = f(0, 1, 0) = f(0, 0, 1) = 2 \quad (= \sqrt{4}).$$

Interior points lying on one of the oblique edges:

$$f(1, 1, 0) = f(1, 0, 1) = f(0, 1, 1) = 2 \quad (= \sqrt{4}).$$

The corners:

$$f(0, 0, 0) = f(2, 0, 0) = f(0, 2, 0) = f(0, 0, 2) = \sqrt{2}.$$

It follows by a numerical comparison that the maximum value is

$$S = f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2\sqrt{2},$$

and the minimum value is

$$M = f(0, 0, 0) = f(2, 0, 0) = f(0, 2, 0) = f(0, 0, 2) = \sqrt{2}.$$

REMARK. Notice that the examination of the boundary which is here given in all details may be quite large. In some cases one could have made a shortcut, but for pedagogical reasons we have here only skipped trivial arguments of symmetry. \diamond

- 4) Since A is connected and f is continuous, it follows from the first main theorem for continuous functions that the range is the interval

$$f(A) = [\sqrt{2}, 2\sqrt{2}].$$

Example 2.7 Given the function

$$f(x, y) = \exp(-x^4 - y^2), \quad (x, y) \in \mathbb{R}^2.$$

- 1) Show without differentiating that f has a proper maximum at $(0, 0)$.
- 2) Find the range of the function.
- 3) Show that f has both a maximum and a minimum on the set

$$A = \{(x, y) \in \mathbb{R}^2 \mid 2x^2 + y^2 \leq 6\},$$

and find these values.

A Maximum and minimum without differentiating.

D Analyze $f(x, y)$.

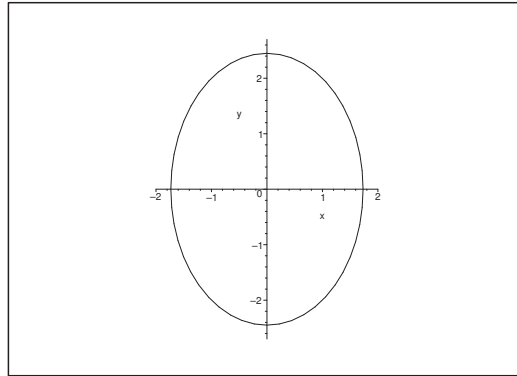
- I** 1) Clearly, $x^4 + y^2 \geq 0$, and we have only equality at $(0, 0)$. Since \exp is strictly increasing, f must have a proper maximum at $(0, 0)$ where $f(0, 0) = 1$.
- 2) Since $x^4 + y^2 \rightarrow +\infty$ for $x^2 + y^2 \rightarrow +\infty$, and since \exp is strictly increasing and continuous where

$$\exp(-(x^4 + y^2)) \rightarrow 0 \quad \text{for } x^2 + y^2 \rightarrow +\infty,$$

it follows that the range is $]0, 1]$.

- 3) Since A is a closed ellipsoidal disc (in particular a closed and bounded set), and since f is continuous, it follows from the second main theorem for continuous functions that f has both a maximum value and a minimum value on A . From $(0, 0) \in A$ and 1) follow that the maximum value is

$$S = f(0, 0) = 1.$$

Figure 37: The domain A .

Clearly, f is constant on the curves

$$x^4 + y^2 = C \geq 0$$

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of the value $f(x, y) = e^{-C}$. We shall therefore only find the largest C , for which we can find an (x, y) on the curve, also lying in A .

Clearly, such a point (x, y) must lie in ∂A , so $y^2 = 6 - 2x^2$. Hence, the constant C is the maximum value of the function

$$\varphi(x) = x^4 + y^2 = x^4 - 2x^2 + 6 = (x^2 - 1)^2 + 5 \quad \text{for } x \in [-\sqrt{3}, \sqrt{3}].$$

If we put $t = x^2 \in [0, 3]$, it follows that we shall find the maximum value of

$$\psi(t) = (t - 1)^2 + 5, \quad t \in [0, 3].$$

The only possibilities are $t = 0$, $t = 1$ and $t = 3$. We get by insertion

$$\psi(0) = 5, \quad \psi(1) = 5,$$

and

$$\psi(3) = \varphi(\pm\sqrt{3}) = 4 + 5 = 9 = C,$$

and we conclude that the minimum value of f is

$$f(\pm\sqrt{3}, 0) = e^{-9}.$$

Example 2.8 1) Find the domain A of the function

$$f(x, y) = \sqrt{x} + \sqrt{y} + \sqrt{2 - x - y},$$

and sketch A .

- 2) Explain why f has both a maximum value and a minimum value on A .
- 3) Then find the maximum value and the minimum value of f as well as the points in which they are attained.
- 4) Finally, find the range $f(A)$.

A Domain; maximum and minimum; range.

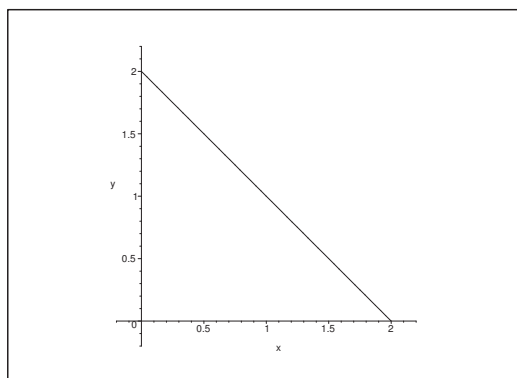
D Standard example.

I 1) The function is defined when

$$x \geq 0, \quad y \geq 0 \quad \text{and} \quad x + y \leq 2,$$

i.e. in the closed triangle A with the corners $(0, 0)$, $(2, 0)$ and $(0, 2)$.

- 2) Since A is closed and bounded, and f is continuous in A , it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in A .

Figure 38: The domain A .

- 3) Since $f \in C^1$ in the interior of A , the maximum and the minimum can only be attained at either a stationary point in A° or in a point on the boundary ∂A .

The stationary points shall satisfy the equations

$$\frac{\partial f}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x}} - \frac{1}{2} \frac{1}{\sqrt{2-x-y}} = 0, \quad \frac{\partial f}{\partial y} = \frac{1}{2} \frac{1}{\sqrt{y}} - \frac{1}{2} \frac{1}{\sqrt{2-x-y}} = 0,$$

hence $\sqrt{x} = \sqrt{y} = \sqrt{2-x-y}$, i.e. $x = y = 2 - 2x$, and the only stationary point is

$$(x, y) = \left(\frac{2}{3}, \frac{2}{3} \right).$$

The value of the function is here

$$f\left(\frac{2}{3}, \frac{2}{3}\right) = 3\sqrt{\frac{2}{3}} = \sqrt{6}.$$

REMARK. Since we shall only find the maximum and the minimum on a closed and bounded set, a numerical comparison is sufficient, and we do not have to go through an elaborated examination of extrema. \diamond

THE BOUNDARY:

- a) We get on the boundary curve $y = 0$, $x \in [0, 2]$, the restriction

$$\varphi(x) = f(x, 0) = \sqrt{x} + \sqrt{2-x},$$

which for symmetric reasons has its maximum for $x = 1$ and its minimum for $x = 0$ and $x = 2$.

ALTERNATIVELY,

$$\varphi'(x) = \frac{1}{2} \frac{1}{\sqrt{x}} - \frac{1}{2} \frac{1}{\sqrt{2-x}} = 0 \quad \text{for } x = 1.$$

the interesting values are

$$f(0, 0) = f(2, 0) = \sqrt{2} \quad \text{and} \quad f(1, 0) = 2.$$

- b) By interchanging the letters $(x, y) \rightarrow (y, x)$ we get on the boundary curve $x = 0, y \in [0, 2]$ the same function as above. Hence the candidates are

$$f(0, 0) = f(0, 2) = \sqrt{2} \quad \text{and} \quad f(0, 1) = 2.$$

- c) We get on the boundary curve $y = 2 - x, x \in [0, 2]$ the restriction

$$\psi(x) = f(x, 2 - x) = \sqrt{x} + \sqrt{2 - x} = \varphi(x),$$

cf. above, so the candidates are

$$f(0, 2) = f(2, 0) = \sqrt{2} \quad \text{and} \quad f(1, 1) = 2.$$

All things considered, we have found the candidates

$$f\left(\frac{2}{3}, \frac{2}{3}\right) = \sqrt{6}, \quad (\text{stationary point}),$$

$$f(1, 0) = f(0, 1) = f(1, 1) = 2,$$

$$f(0, 0) = f(2, 0) = f(0, 2) = \sqrt{2}.$$

Thence by comparison, the minimum value is

$$M = f(0, 0) = f(2, 0) = f(0, 2) = \sqrt{2},$$

and the maximum value is

$$S = f\left(\frac{2}{3}, \frac{2}{3}\right) = \sqrt{6}.$$

- 4) Since f is continuous and the triangle A is connected, it follows from the first main theorem for continuous functions that the range $f(A)$ is also connected, hence an interval. Then it follows from 3) that

$$f(A) = [M, S] = [\sqrt{2}, \sqrt{6}].$$

Example 2.9 The function $f : A \rightarrow \mathbb{R}$ is given by

$$f(x, y) = \sqrt{2 - x - y}, \quad A = \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 1, x^2 \leq y \leq 2 - x\}.$$

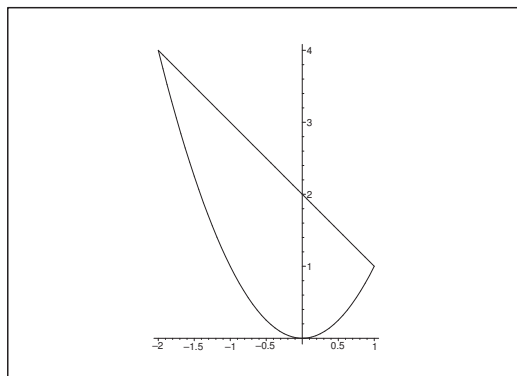
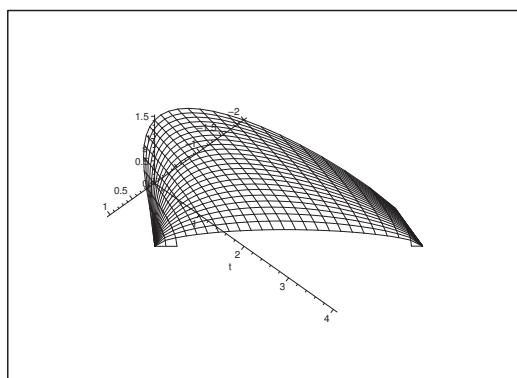
- 1) Explain why f has both a maximum value S and a minimum value M .
- 2) Find S and M and the range $f(A)$ of the function.

A Maximum and minimum and range.

D Apply the standard methods.

I 1) The domain A is indicated on the figure. Clearly, A is closed and bounded.

It follows that f is defined and continuous for $y \leq 2 - x$, thus in particular in A . It follows from the second main theorem for continuous functions that f has both a maximum value S and a minimum value M on A . Since f is of class C^∞ in the interior of A , these values are either attained at an inner stationary point or on the boundary.

Figure 39: The domain A .Figure 40: The graph of f over A .

2) We get in the interior of A

$$\frac{\partial f}{\partial x} = -\frac{1}{2} \frac{1}{\sqrt{2-x-y}}, \quad \frac{\partial f}{\partial y} = -\frac{1}{2} \frac{1}{\sqrt{2-x-y}}.$$

These equations are never 0 in the interior of A , so f has no stationary points. We conclude that S and M are attained at boundary points.

- a) When $y = 2 - x$, the restriction is $f(x, 2 - x) = 0$. As $f(x, y) \geq 0$, this must be the minimum value $M = 0$.
 b) When $y = x^2$, the restriction is

$$\begin{aligned} f(x, x^2) &= \sqrt{2 - x - x^2} = \sqrt{2 + \frac{1}{4} - \left(x^2 + x + \frac{1}{4}\right)} \\ &= \sqrt{\left(\frac{3}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2}, \quad x \in [-2, 1]. \end{aligned}$$

It follows immediately that the maximum value is obtained for $x = -\frac{1}{2}$, i.e.

$$S = f\left(-\frac{1}{2}, \frac{1}{4}\right) = \frac{3}{2} \quad \text{og} \quad M = f(x, 2-x) = 0.$$

Since A is connected, it follows from the first main theorem for continuous functions that the range is

$$f(A) = [M, S] = \left[0, \frac{3}{2}\right].$$

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Example 2.10 1) Sketch the set

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2 \text{ og } y \leq 0\};$$

Indicate in particular the boundary ∂A on the figure. Explain why A is bounded and closed.

2) Explain why

$$f(x, y) = (x + y)\sqrt{2 - x^2 - y^2}, \quad (x, y) \in A,$$

has a minimum value M and a maximum value S . Find these values as well as the points where they are attained.

3) Finally, explain why the range $f(A)$ of the function is an interval, and find $f(A)$.

A Maximum and minimum. The first and second main theorems for continuous functions.

D Use the standard methods.

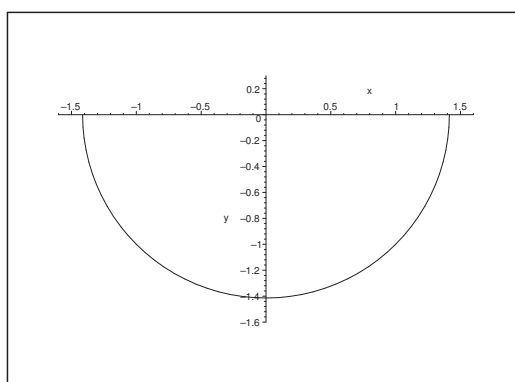


Figure 41: The point set A is the closed half disc in the lower half plane of centrum $(0, 0)$ and radius $\sqrt{2}$.

I 1) The point set A is the intersection of the closed disc of centrum $(0, 0)$ and radius $\sqrt{2}$, and the closed lower half plain. An intersection of two closed sets is also closed, so A is closed. Since A is contained in a disc of finite radius, A is bounded.

2) The function

$$f(x, y) = (x + y)\sqrt{2 - x^2 - y^2}$$

is defined and continuous on A .

Since A is closed and bounded and f is continuous, it follows from the second main theorem for continuous functions that f has a minimum value M and a maximum value S on A .

Since f is of class C^∞ in the interior of A , the values S and M can only be attained at an interior stationary point in A° or on the boundary ∂A .

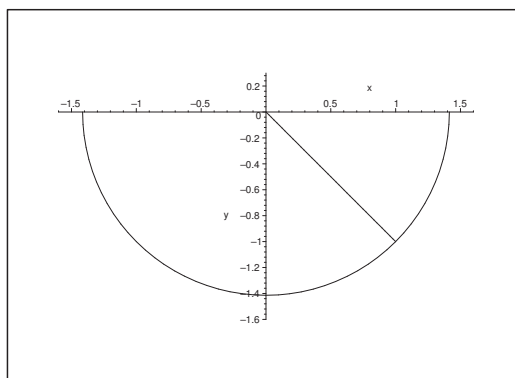


Figure 42: The zero lines of the function $f(x, y)$ in A . There are the union of the half circle and the oblique line $y = -x$. The function is negative to the left of this oblique line, and positive to the right of it.

When we examine the sign of f we see that f can be both positive and negative. When we specify S and M , we can exclude the zero curves, i.e. that part of the boundary which lies on the circular arc, as well as the points of A , which lie on the line $y = -x$.

In particular, the examination of the boundary is reduced to the segment $y = 0$, $x \in [-\sqrt{2}, \sqrt{2}]$, on the X -axis, where we also can exclude the end points because the value is here 0. The restriction of f to this part of the boundary is

$$\varphi(x) = f(x, 0) = x\sqrt{2 - x^2}, \quad x \in [-\sqrt{2}, \sqrt{2}],$$

with the derivative

$$\varphi'(x) = \sqrt{2 - x^2} - \frac{x^2}{\sqrt{2 - x^2}} = \frac{2(1 - x^2)}{\sqrt{2 - x^2}}, \quad \text{for } x \in]-\sqrt{2}, \sqrt{2}[.$$

It follows that $\varphi'(x) = 0$ for $x = \pm 1$. Hence, on the boundary we get the following candidates of S and M (because we have already excluded the end points and the circular arc),

$$f(1, 0) = 1 \quad \text{og} \quad f(-1, 0) = -1.$$

The possible *stationary points* in the interior of A (i.e. where $2 - x^2 - y^2 > 0$ and $y < 0$) are the solutions of the equations

$$\begin{cases} \frac{\partial f}{\partial x} = \sqrt{2 - x^2 - y^2} - \frac{x(x + y)}{\sqrt{2 - x^2 - y^2}} = \frac{2 - 2x^2 - xy - y^2}{\sqrt{2 - x^2 - y^2}} = 0, \\ \frac{\partial f}{\partial y} = \frac{1 - x^2 - xy - 2y^2}{\sqrt{2 - x^2 - y^2}} = 0, \end{cases}$$

which are reduced in the interior of A to

$$(11) \quad \begin{cases} 2x^2 + xy + y^2 = 2, \\ x^2 + xy + 2y^2 = 2. \end{cases}$$

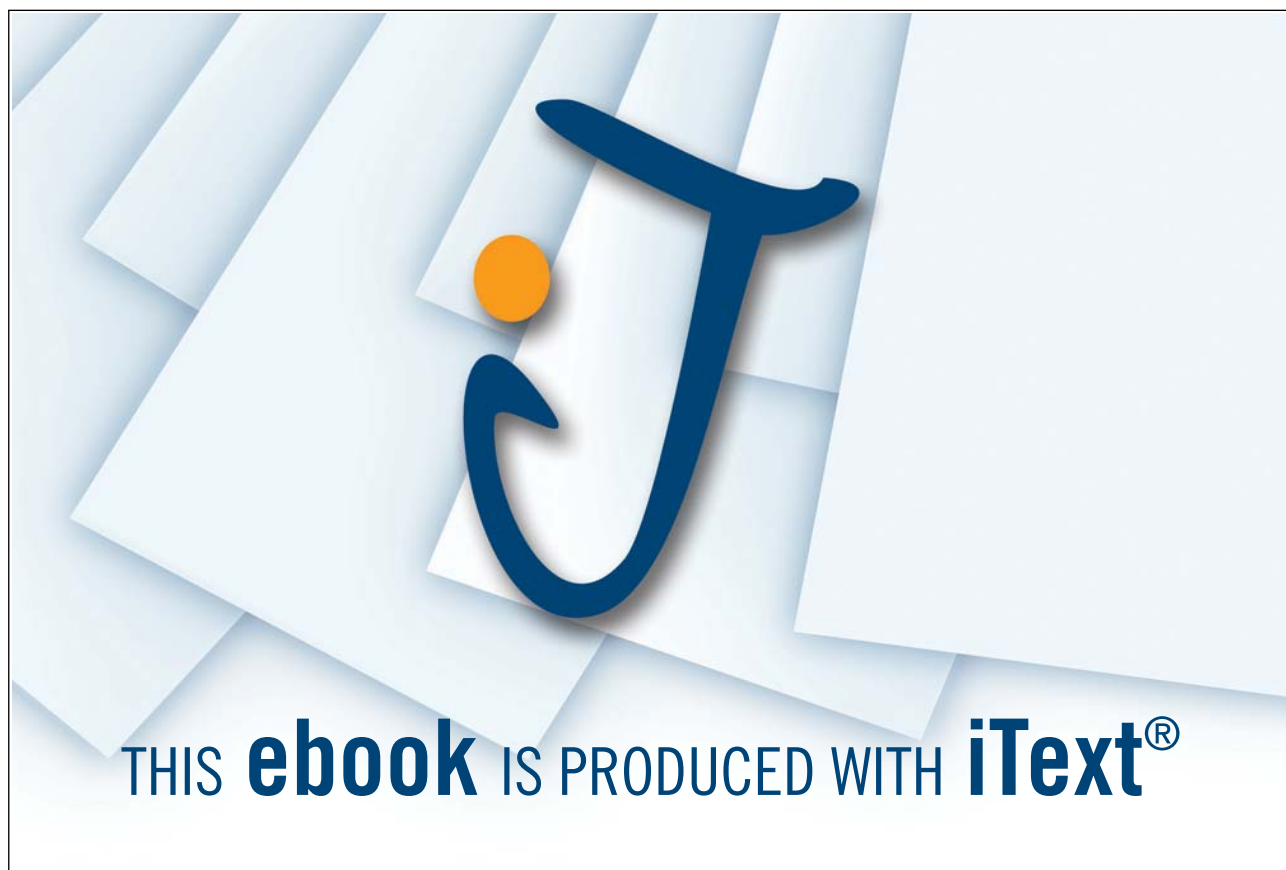
Thus $y^2 = x^2$, i.e. either $y = x$ or $y = -x$. By the analysis of the sign of f , neither of the stationary points on $y = -x$ can be a maximum or a minimum (the value of the function is here zero). In the chase of the candidates the investigation is now reduced to the line segment $y = x$, $x \in]-1, 0[$, (because $y \leq 0$). Then we get by insertion into either of the equations of (11) that

$$2 = 2x^2 + xy + y^2 = 4x^2,$$

hence $x = y = -\frac{1}{\sqrt{2}}$. We note again that f may have stationary points on $y = -x$, but these are of no importance because the only *relevant* stationary point is $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, corresponding to the value of the function

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}.$$

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We have now found three candidates of M and S :

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}, \quad f(-1, 0) = -1, \quad f(1, 0) = 1.$$

It follows by a numerical comparison that

$$M = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2} \quad \text{and} \quad S = f(1, 0) = 1.$$

- 3) Since A is convex, A is in particular connected. Since f is continuous in A , it follows from the first main theorem for continuous functions that the range (here a subset of \mathbb{R}) is connected, hence an interval. Now f has according to 2) a minimum and a maximum, so we finally get

$$f(A) = [M, S] = [-\sqrt{2}, 1].$$

Example 2.11 *Let*

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 - 1 \leq y \leq 3\},$$

and consider the function $f : A \rightarrow \mathbb{R}$ given by

$$f(x, y) = 1 - x^2 - y + 2x^2y, \quad (x, y) \in A.$$

- 1) *Sketch A , and explain why the function f has both a maximum value S and a minimum value M .*
- 2) *Find the stationary points of the function f .*
- 3) *Find S and M .*
- 4) *Find the range $f(A)$ of the function.*

A Maximum and minimum and range of a function.

D Standard example.

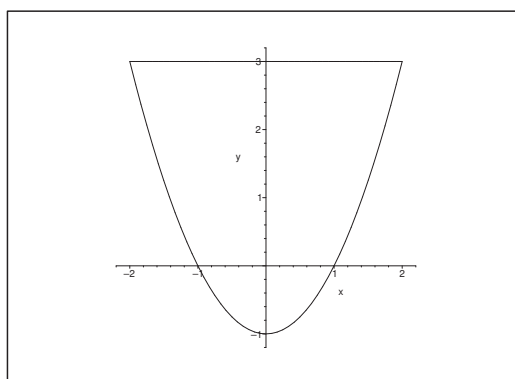


Figure 43: The domain A .

- I 1) The set A is closed and bounded, and the polynomial f is continuous on A . It follows from the second main theorem for continuous functions that f has both a maximum value S and a minimum value M on A . Since f is of class C^∞ in the interior of A , the values S and M are either attained at a stationary point or at a boundary point.
- 2) The equations of the stationary points are

$$\begin{cases} \frac{\partial f}{\partial x} = -2x + 4xy = 0, & \text{i.e.} \quad 4x \left(y - \frac{1}{2} \right) = 0, \\ \frac{\partial f}{\partial y} = -1 + 2x^2 = 0, & \text{i.e.} \quad x^2 = \frac{1}{2}. \end{cases}$$

We get from the latter equation that $x = \pm \frac{1}{\sqrt{2}}$, which put into the first equation gives $y = \frac{1}{2}$.

Hence, the stationary points are

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{2} \right) \quad \text{and} \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{2} \right),$$

and we note that they both lie in the interior of A :

$$x^2 - 1 = -\frac{1}{2} < y = \frac{1}{2} < 3.$$

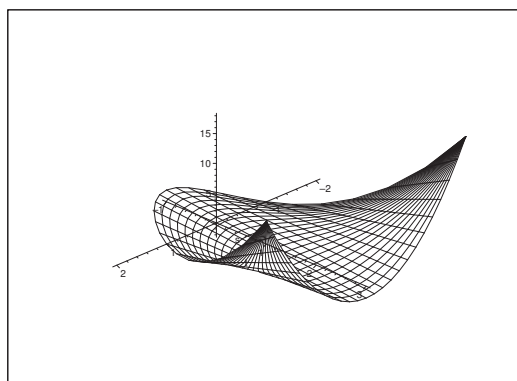


Figure 44: The graph of f over A .

- 3) The values at the stationary points are

$$f\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right) = 1 - \frac{1}{2} - \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

THE BOUNDARY.

- a) By restriction to the line segment $y = 3$, $x \in [-2, 2]$,

$$\varphi(x) = f(x, 3) = 1 - x^2 - 3 + 6x^2 = 5x^2 - 2,$$

which clearly is smallest when $x = 0$, and largest when $x = \pm 2$. We get the candidates

$$f(0, 3) = -2 \quad \text{and} \quad f(\pm 2, 3) = 18.$$

b) Considering the restriction to the parabolic arc we have two alternatives:

i) By the restriction to $y = x^2 - 1$, $x \in [-2, 2]$, we get

$$\psi(x) = f(x, x^2 - 1) = 2(1 - x^2) + 2x^2(x^2 - 1) = 2(x^2 - 1)^2,$$

which is smallest when $x^2 = 1$, and largest when $x^2 = (\pm 2)^2$. The candidates are

$$f(\pm 1, 0) = 1 - 1 = 0 \quad \text{and} \quad f(\pm 2, 3) = 18.$$

ii) ALTERNATIVELY, $x^2 = y + 1$, $y \in [-1, 3]$, so

$$\eta(y) = -2y + 2(y + 1)y = 2y^2, \quad y \in [-1, 3],$$

which is smallest for $y = 0$, and largest for $y = 3$, corresponding to

$$f(\pm 1, 0) = 0 \quad \text{and} \quad f(\pm 2, 3) = 18.$$

Then by a numerical comparison of the candidates,

$$\text{SP: } f\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \frac{1}{2},$$

$$\text{a) } f(0, 3) = -2 \quad \text{and} \quad f(\pm 2, 3) = 18,$$

$$\text{b) } f(\pm 1, 0) = 0 \quad \text{and} \quad f(\pm 2, 3) = 18,$$

we finally get

$$S = f(\pm 2, 3) = 18 \quad \text{and} \quad M = f(0, 3) = -2.$$

c) Since A is connected, and f is continuous, it follows from the first main theorem for continuous functions, that $f(A) \subseteq \mathbb{R}$ is connected, hence an interval.

We have in 3) shown that $M = -2$ and $S = 18$, so the range is

$$f(A) = [-2, 18].$$

Example 2.12 Find the interval of range of the function

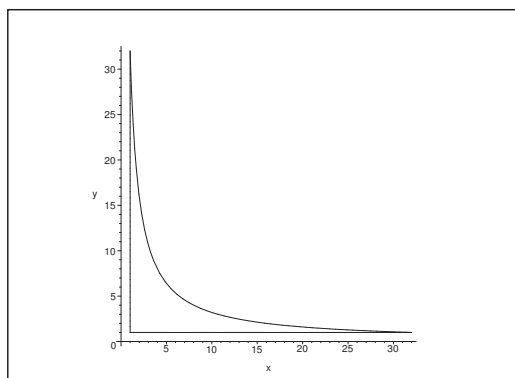
$$f(x, y) = xy + 64 \left(\frac{1}{x} + \frac{1}{y} \right), \quad (x, y) \in A,$$

where

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 1, y \geq 1, xy \leq 32\}.$$

A Maximum, minimum, interval of range.

D Sketch A and conclude that A is closed and bounded. Apply the first and second main theorems for continuous functions. Find the possible stationary points. Check the boundary. Conclude by a numerical comparison.

Figure 45: The closed and bounded domain A .

I It follows from the figure and the definition of A that A is connected and closed and bounded. Since $f(x, y)$ is *continuous* on the *closed and bounded* set A , it follows from the second main theorem for continuous functions that $f(x, y)$ has a maximum and a minimum on A . Since f is of class C^∞ , the maximum and minimum values are to be found among the values at the possible inner stationary points and at the boundary points.

Since A is also *connected*, and f is *continuous*, it follows from the first main theorem for continuous functions that the range is connected, hence a closed interval, which must necessarily be

$$f(A) = [f_{\min}, f_{\max}].$$

The equations of the possible stationary points are

$$\frac{\partial f}{\partial x} = y - \frac{64}{x^2} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = x - \frac{64}{y^2} = 0,$$

thus

$$x^2 y = 64 \quad \text{and} \quad x y^2 = 64.$$

Now $x, y > 0$ in A , so it follows by a division that $\frac{x}{y} = 1$, i.e. $y = x$. Then by insertion $x^3 = 64$, so $x = y = 4$. Since $xy = 16 < 32$ and $4 > 1$ we conclude that $(4, 4)$ lies in the interior of A . Hence $(4, 4)$ is a stationary point, and the value of the function is here

$$f(4, 4) = 16 + 64 \left(\frac{1}{4} + \frac{1}{4} \right) = 16 + 32 = 48.$$

THE BOUNDARY.

1) For $y = 1$, $1 \leq x \leq 32$, we get the restriction

$$\varphi(x) = f(x, 1) = x + \frac{64}{x} + 64$$

where

$$\varphi'(x) = 1 - \frac{64}{x^2} = 0 \quad \text{for } x = 8.$$

We notice the values

$$f(1, 1) = \varphi(1) = 1 + 64 + 64 = 129,$$

$$f(32, 1) = \varphi(32) = 32 + 2 + 64 = 98,$$

$$f(8, 1) = \varphi(8) = 8 + 8 + 64 = 80.$$

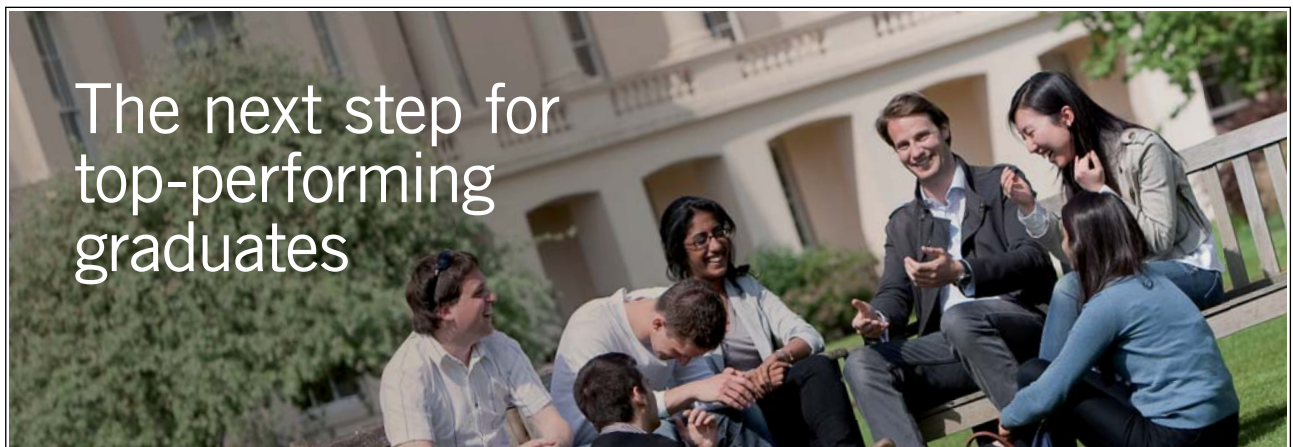
2) For $x = 1$, $1 \leq y \leq 32$, it follows by the symmetry,

$$f(1, 1) = 129, \quad f(1, 32) = 98, \quad f(1, 8) = 80.$$

3) If $xy = 32$, i.e. $y = \frac{32}{x}$, $1 \leq x \leq 32$, we get the restriction

$$\psi(x) = xy + 64 \cdot \frac{x+y}{xy} = 32 + 2 \left(x + \frac{32}{x} \right) = 2x + \frac{64}{x} + 32$$

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* Figures taken from London Business School's Masters in Management 2010 employment report

where

$$\psi'(x) = 2 - \frac{64}{x^2} = 0 \quad \text{for } x = \sqrt{32} = 4\sqrt{2} = y.$$

This is the only additional value, because we have already checked the end points of the interval above,

$$f(\sqrt{32}, \sqrt{32}) = \psi(\sqrt{32}) = 32 + (\sqrt{32} + \sqrt{32}) = 32 + 8\sqrt{2}.$$

Then compare the values of the candidates,

$$f(4, 4) = 48, \quad f(1, 1) = 129, \quad f(8, 1) = f(1, 8) = 80,$$

$$f(32, 1) = f(1, 32) = 98, \quad f(\sqrt{32}, \sqrt{32}) = 32 + 8\sqrt{2} < 48.$$

It follows that the minimum value is

$$f(\sqrt{32}, \sqrt{32}) = 32 + 8\sqrt{2},$$

and the maximum value is

$$f(1, 1) = 129.$$

The range is connected, so the interval of the range is

$$f(A) = [32 + 8\sqrt{2}, 129].$$

Example 2.13 *Given the function*

$$f(x, y) = 2 \ln(1 + x^2 + y^2) + x\sqrt{2} + y, \quad x^2 + y^2 \leq 4.$$

1) *Explain why the function has a maximum value S and a minimum value M .*

2) *Show that the stationary points of the function are $(-\sqrt{2}, -1)$ and $(-\frac{1}{3}\sqrt{2}, -\frac{1}{3})$.*

3) *Find S and M .*

4) *Find the range of the function.*

A Maximum and minimum, range.

D Sketch a figure. Follow the guidelines.

I 1) The domain A is a closed disc of centrum $(0, 0)$ and radius 2, thus A is closed and bounded and connected.

Clearly, f is continuous on A and of class C^∞ in the interior of A . According to the second main theorem for continuous functions, f has a maximum value S and a minimum value M on A . These values are either attained at a stationary point or on the boundary, because there are no exceptional points.

Notice also that A is connected, so the range $f(A) = [M, S]$ is connected according to the the first main theorem for continuous functions. This will be used in 4).

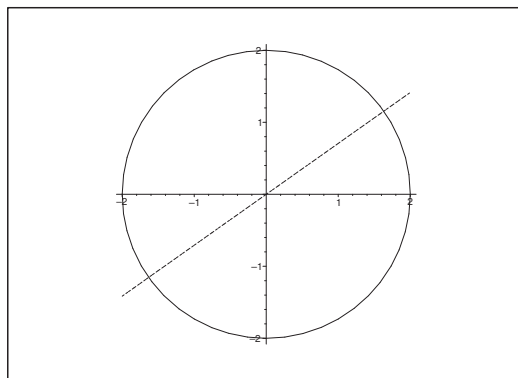


Figure 46: The domain A and the line $y = \frac{1}{\sqrt{2}}x$.

2) The equations of the stationary points are

$$\frac{\partial f}{\partial x} = \frac{4x}{1+x^2+y^2} + \sqrt{2} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{4y}{1+x^2+y^2} + 1 = 0.$$

It follows that at a stationary point we must have $x \neq 0$ and $y \neq 0$, so the equations are equivalent to

$$\frac{4xy}{1+x^2+y^2} = -\sqrt{2}y \quad \text{and} \quad \frac{4xy}{1+x^2+y^2} = -x, \quad x \neq 0, \quad y \neq 0.$$

Accordingly, the possible stationary points must lie on the line $x = \sqrt{2}y$.

When we eliminate x in e.g. the latter equation of the stationary points, then

$$-1 = \frac{4y}{1+x^2+y^2} = \frac{4y}{1+3y^2},$$

hence

$$3y^2 + 4y + 1 = 0.$$

The solutions are $y = -1$ and $y = -\frac{1}{3}$. From $x = \sqrt{2}y$ follows that the only *possible* stationary points are

$$(-\sqrt{2}, -1) \quad \text{and} \quad \left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right).$$

It remains to be proved that they are both stationary points.

a) They satisfy the equations:

i) For $(-\sqrt{2}, -1)$ we get

$$\frac{-4\sqrt{2}}{1+2+1} + \sqrt{2} = -\sqrt{2} + \sqrt{2} = 0 \quad \text{and} \quad \frac{-4}{1+2+1} + 1 = -1 + 1 = 0.$$

ii) For $\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right)$ we get

$$\frac{-\frac{4}{3}\sqrt{2}}{1+\frac{2}{9}+\frac{1}{9}} + \sqrt{2} = \frac{-\frac{4}{3}\sqrt{2}}{\frac{4}{3}} + \sqrt{2} = 0 \text{ and } \frac{-\frac{4}{3}}{1+\frac{2}{9}+\frac{1}{9}} + 1 = \frac{-\frac{4}{3}}{\frac{4}{3}} + 1 = 0.$$

b) They both lie in A :

i) For $(-\sqrt{2}, -1)$ we get

$$x^2 + y^2 = 2 + 1 = 3 < 4.$$

ii) For $\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right) = \frac{1}{3}(-\sqrt{2}, 1)$ we get

$$x^2 + y^2 = \frac{1}{9} \cdot 3 = \frac{1}{3} < 4.$$

We have now proved that the stationary points are

$$(-\sqrt{2}, -1) \quad \text{and} \quad \left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right).$$

3) The values of the function at the stationary points are

$$f(-\sqrt{2}, -1) = 2 \ln(1 + 2 + 1) - \sqrt{2} \cdot \sqrt{2} - 1 = 2 \ln 4 - 3,$$

$$f\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right) = 2 \ln\left(1 + \frac{2}{9} + \frac{1}{9}\right) - \frac{1}{3}\sqrt{2} \cdot \sqrt{2} - \frac{1}{3} = 2 \ln \frac{4}{3} - 1.$$

THE BOUNDARY.

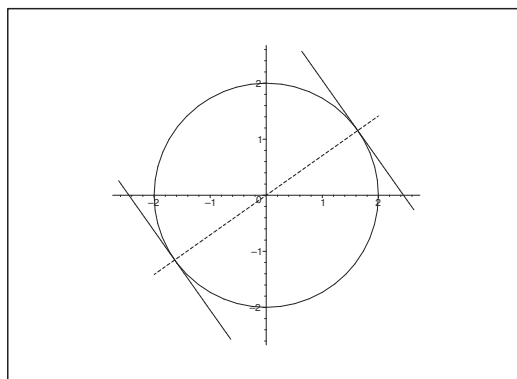


Figure 47: The geometrical analysis of the maximum and the minimum on the boundary.

On the boundary $2 \ln(1 + x^2 + y^2) = 2 \ln 5$ is constant. The maximum and minimum values on the boundary are attained at those points in which $x\sqrt{2} + y$ attains its maximum and minimum value on the circle. Geometrically these are given by the two tangents of the circle, which are parallel to the line $x\sqrt{2} + y = 0$, i.e. at the intersection points of the circle $x^2 + y^2 = 4$ and

perpendicular to the line $x = \sqrt{2}y$, on which the stationary points lie. This gives the new candidates

$$\left(2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) \quad \text{og} \quad \left(-2\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right).$$

Since all candidates of S and M lie on the line $x = \sqrt{2}y$, we get the following one-dimensional variant of the identification of S and M :

The restriction to the line $x = \sqrt{2}y$ is

$$g(y) = 2(1 + 3y^2) + 3y, \quad y \in \left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right],$$

where

$$g'(y) = \frac{12y}{1 + 3y^2} + 3, \quad y \in \left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right].$$

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As before we get $g'(y) = 0$ for $y = -1$ and $y = -\frac{1}{3}$. The variation of $g'(y)$ is:

y	$-\frac{2}{\sqrt{3}} \leq y < -1$	-1	$-1 < y < -\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3} < y \leq \frac{2}{\sqrt{3}}$
$g'(y)$	+	0	-	0	+

Thus we have a local maximum for $y = -1$ and a local minimum for $y = -\frac{1}{3}$.

The candidates of S are

$$g(-1) = 2 \ln 4 - 3 \quad \text{and} \quad g\left(-\frac{1}{3}\right) = 2 \ln 5 + 2\sqrt{3}.$$

It follows that

$$S = g\left(-\frac{1}{3}\right) = f\left(2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = 2 < \ln 5 + 2\sqrt{3} \quad [\approx 6,6830].$$

Analogously, the candidates of M are

$$\left(-\frac{2}{\sqrt{3}}\right) = 2 \ln 5 - 2\sqrt{3} \quad \text{og} \quad g\left(-\frac{1}{3}\right) = 2 \ln \frac{4}{3} - 1.$$

By a numerical comparison on a pocket calculator we get

$$M = g\left(-\frac{1}{3}\right) = f\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right) = 2 \ln \frac{4}{3} - 1 \quad [\approx -0,4246].$$

ALTERNATIVELY a parametric description of the boundary is

$$(x, y) = (2 \cos \varphi, 2 \sin \varphi), \quad \varphi \in [0, 2\pi].$$

Hence the restriction to the boundary becomes

$$h(\varphi) = f(2 \cos \varphi, 2 \sin \varphi) = 2 \ln 5 + 2\sqrt{2} \cos \varphi + 2 \sin \varphi$$

where

$$h'(\varphi) = -2\sqrt{2} \sin \varphi + 2 \cos \varphi = 0$$

for $\tan \varphi = \frac{1}{\sqrt{2}}$, corresponding to

$$\cos \varphi = \pm \frac{1}{\sqrt{1 + \tan^2 \varphi}} = \pm \frac{1}{\sqrt{1 + \frac{1}{2}}} = \pm \sqrt{\frac{2}{3}}$$

and

$$\sin \varphi = \cos \varphi \tan \varphi = \pm \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{1}{2}} = \pm \frac{1}{\sqrt{3}},$$

where the signs are belonging together.

We get the candidates

$$f\left(2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) \quad \text{and} \quad \left(-2\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right)$$

with the values of the function

$$\left(2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = 2 \ln \left(1 + \frac{8}{3} + \frac{4}{3}\right) + 2\sqrt{\frac{2}{3}} \cdot \sqrt{2} + \frac{2}{\sqrt{3}} = 2 \ln 5 + 2\sqrt{3},$$

and

$$f\left(-2\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right) = 2 \ln 5 - 2\sqrt{3}.$$

NUMERICAL COMPARISON.

We shall find the maximum and the minimum value among

$$f(-\sqrt{2}, -1) = 2 \ln 4 - 3, \quad f\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right) = 2 \ln \frac{4}{3} - 1,$$

$$f\left(2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = 2 \ln 5 + 2\sqrt{3}, \quad f\left(-2\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right) = 2 < \ln 5 - \sqrt{3}.$$

Clearly,

$$S = f\left(2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = 2 \ln 5 + 2\sqrt{3} \approx 6,6830.$$

Then by a numerical comparison on a pocket calculator,

$$M = f\left(-\frac{1}{3}\sqrt{2}, -\frac{1}{3}\right) = 2 \ln \frac{4}{3} - 1 \approx -0.4246.$$

- 4) Since A is connected, and f is continuous on A , the range $f(A)$ is connected according to the first main theorem for continuous functions. We have already found S and M in 3), so the range is the interval

$$f(A) = [M, S] = \left[2 \ln \frac{4}{3} - 1, 2 \ln 5 + 2\sqrt{3}\right].$$

Example 2.14 Given the function

$$f(x, y) = 2y \sin x - x, \quad (x, y) \in A,$$

where the domain A is given by the inequalities

$$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad 0 \leq y \leq \cos x.$$

- 1) Explain why the function has a maximum value S and a minimum value M .
- 2) Find S and M .
- 3) Find the range of the function.

A Maximum and minimum values; range.

D Sketch A , and apply the second and the first main theorem for continuous functions.

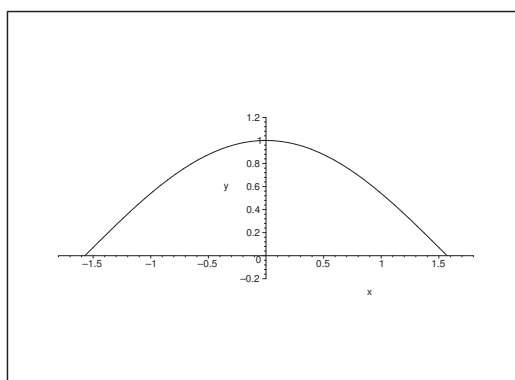


Figure 48: The set A .

- I** 1) The set A is closed and bounded, and f is continuous. According to the second main theorem for continuous functions, f has both a maximum value S and a minimum value M on A .
- 2) Since f is of class C^∞ , the values S and M are either attained at an inner stationary point or at a boundary point.

STATIONARY POINTS.

The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 2y \cos x - 1 = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2 \sin x = 0.$$

It follows from the latter equation that $x = p\pi$, $p \in \mathbb{Z}$, i.e. $x = 0$, if the point shall also lie in A . When this is put into the former equation we get $2y - 1 = 0$, so $y = \frac{1}{2}$. Thus the only stationary point in A is $\left(0, \frac{1}{2}\right)$. The value of the function at this point is

$$f\left(0, \frac{1}{2}\right) = 2 \cdot \frac{1}{2} \cdot \sin 0 - 0 = 0.$$

THE BOUNDARY.

a) On the boundary curve $y = 0$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have the restriction

$$\varphi(x) = f(x, 0) = -x,$$

with the maximum value

$$\varphi\left(-\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}, 0\right) = \frac{\pi}{2}$$

and the minimum value

$$\varphi\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}, 0\right) = -\frac{\pi}{2}.$$

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b) On the boundary curve $y = \cos x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we get the restriction

$$\psi(x) = f(x, \cos x) = 2 \cos x \cdot \sin x - x = \sin 2x - x,$$

where

$$\psi'(x) = 2 \cos 2x - 1 = 0, \quad \cos 2x = \frac{1}{2},$$

so

$$x = \left(\pm \frac{\pi}{3} + 2p\pi\right) \cdot \frac{1}{2} = \pm \frac{\pi}{6} + p\pi, \quad p \in \mathbb{Z}.$$

Here $p = 0$, because we shall stay inside A , hence

$$\psi\left(\frac{\pi}{6}\right) = f\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right) = \sin \frac{\pi}{3} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} = \frac{3\sqrt{3} - \pi}{6},$$

and

$$\psi\left(-\frac{\pi}{6}\right) = f\left(-\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right) = -\sin \frac{\pi}{3} + \frac{\pi}{6} = -\frac{3\sqrt{3} - \pi}{6}.$$

Then by a numerical comparison,

$$S = f\left(-\frac{\pi}{2}, 0\right) = \frac{\pi}{2} \quad \text{and} \quad M = f\left(\frac{\pi}{2}, 0\right) = -\frac{\pi}{2}.$$

c) Since A is connected, and f is continuous, it follows from the first main theorem for continuous functions that $f(A)$ is connected, i.e. an interval. When we apply the results from 2), we get

$$f(A) = [M, S] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$