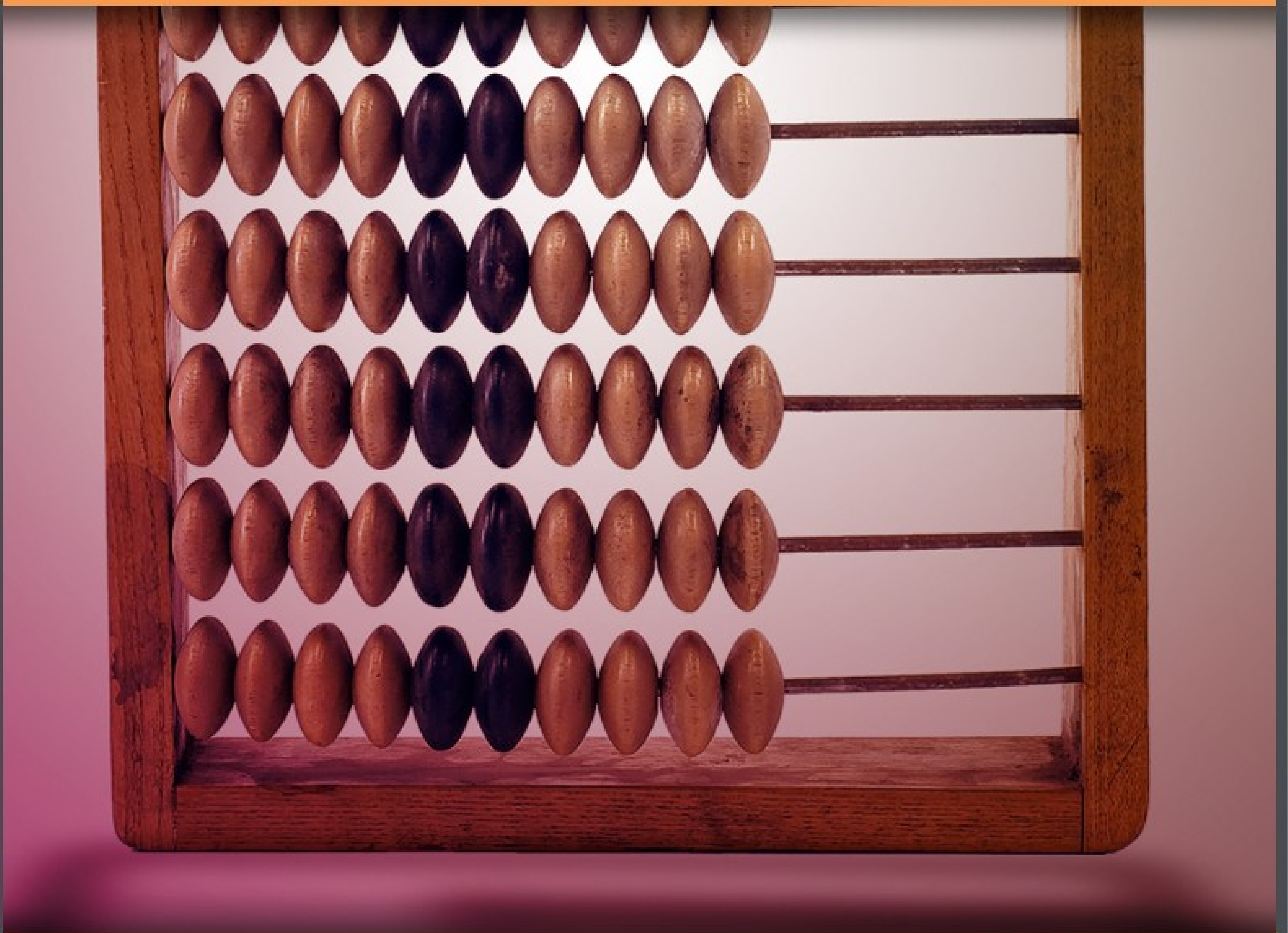
Real Functions of Several Variables - Basic Con...

Leif Mejlbro



Leif Mejlbro

Real Functions of Several Variables

Examples of Basic Concepts, Examination of Functions, Level Curves and Level Surfaces, Description of Curves

Calculus 2c-1

Real Functions of Several Variables - Examples of Basic Concepts, Examination of Functions, Level Curves and Level Surfaces, Description of Curves – Calculus 2c-1

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Contents

Preface

In this volume I present some examples of *Basic Concepts, Examination of Functions, Level Curves and Level Surfaces* and *Description of Curves*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
17th October 2007

1 Point Sets

Example 1.1 *Sketch the point set A, the interior A°, the boundary ∂A and the closure $\overline{(}A)$ in each of the following cases.*
Furthermore, examine whether A is open, closed or nothing of that kind.
Finally, check whether A is bounded or unbounded.

1) $\{(x,y) \mid xy \neq 0\}$.

2) $\{(x,y) \mid 0 < x < 1,\ 1 \leq y \leq 3\}$.

3) $\{(x,y) \mid y \geq x^2,\ |x| < 2\}$.

4) $\{(x,y) \mid x^2 + y^2 - 2x + 6y \leq 15\}$.

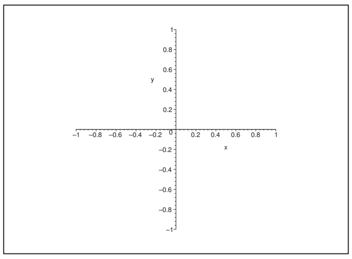
A Examination of point sets in the plane.

D Each set is analyzed on a figure.

I 1) The set $A = \{(x, y) \mid xy \neq 0\}$ is the whole plane with the exception of the X and the Y axes. It is obvious that it is *open*,

$$A = A^{\circ}.$$

The boundary ∂A is the union of the X and the Y axes.
The closure is $\overline{A} = A^{\circ} \cup \partial A = \mathbb{R}^2$, i.e. the whole plane.
Finally, A is clearly *not bounded*.



2) It is easy to sketch the rectangle $A =]0, 1[\times [1, 3]$. We see that

$$A^{\circ} =]0, 1[\times]1, 3[.$$

The boundary of the rectangle is rather complicated to describe formally:

$$\begin{aligned} \partial A \quad &= \quad \{(x, y) \mid 0 \leq x \leq 1,\ y = 1\} \cup \{(x, y) \mid 0 \leq x \leq 1,\ y = 3\} \\ &\quad \cup \{(x, y) \mid x = 0,\ 1 \leq y \leq 3\} \cup \{(x, y) \mid x = 1,\ 1 \leq y \leq 3\}. \end{aligned}$$

This example shows why one shall often prefer a figure instead of a formally correct mathematical description.

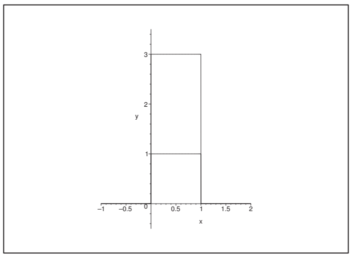
The closure is

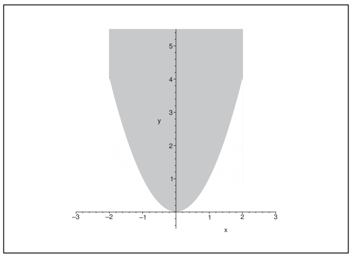
$$\overline{A} = [0, 1] \times [1, 3].$$

The set A is neither open nor closed.
Obviously, the set is bounded (it is e.g. contained in the disc of centrum $(0, 0)$ and radius 4).

3) The set

$$A = \{(x, y) \mid y > x^2,\ |x| < 2\},$$





is also easily sketched. Here

$$A^\circ = \{(x, y) \mid y > x^2,\ |x| < 2\}$$

and

$$\partial A = \{(x, y) \mid x = -2,\ y \geq 4\} \cup \{(x, y) \mid |x| \leq 2,\ y = x^2\} \cup \{(x, y) \mid x = 2,\ y \geq 4\},$$

and

$$\overline{A} = \{(x, y) \mid y \geq x^2,\ |x| \leq 2\}.$$

The set A is neither open nor closed.
The set is clearly not bounded.

4) Since

$$x^2 + y^2 - 2x + 6y \leq 15$$

can be rewritten as

$$x^2 - 2x + 1 + y^2 + 6y + 9 \leq 9 \leq 15 + 1 + 9 = 25 = 5^2,$$

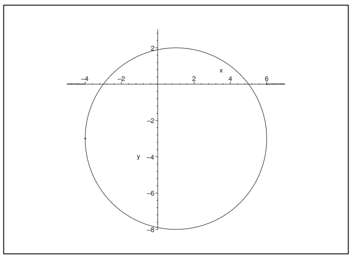
i.e. put into the form

$$(x-1)^2 + (y+3)^2 \leq 25 = 5^2,$$

it follows that

$$A = \{(x, y) \mid (x-1)^2 + (y+3)^2 \leq 5^2\} = \overline{K}((1, 3); 5).$$

This describes a closed disc of centrum $(1, -3)$ and radius 5, thus $A = \overline{A}$.



Then

$$A^\circ = K((1,-3);5) = \{(x,y) \mid (x-1)^2 + (y+3)^2 < 5^2\}$$

and

$$\partial A = \{(x,y) \mid (x-1)^2 + (y+3)^2 = 5^2\},$$

and $A = \overline{A}$ is closed and bounded.

REMARK. Notice that whenever a set like the one under consideration is described by an inequality between simple algebraic expressions, one will *usually* obtain the open set A° by only using the inequality signs $<$ or $>$ without equality sign, obtain the closed set by using $\leq$ or $\geq$ everywhere, and finally get the boundary by only using equality sign $=$. This is unfortunately only a rule of thumb, and one must be aware of that there are exceptions from this rule. $\Diamond$

Example 1.2 *Sketch in each of the following cases the point set A. Examine whether A is open or closed or none of the kind.*

1) $\{(x,y) \mid 3x^2 + 2y^2 < 6\}$.

2) $\{(x,y) \mid x^2 + y^2 \le 1,\ y > 0\}$.

3) $\{(x,y) \mid x^2(1 - x^2 - y^2) > 0\}$.

4) $\{(x,y) \mid 0 < x - y \le 1,\ y > 4\}$.

5) $\left\{(x,y) \mid x^2 + y^2 \ge \sqrt{x^2+y^2}\right\}$.

6) $\{(x,y) \mid \max\{|x|,|y|\} \le 1\}$.

7) $\{(x,y) \mid |x| + |y| < 1\}$.

8) $\{(x,y) \mid x \le y \le 4 - x^2\}$.

9) $\{(x,y) \mid (x-1)(x^2+y^2) \ge 0\}$.

10) $\{(x,y) \mid (y^2-1)(y-3) > 0\}$.

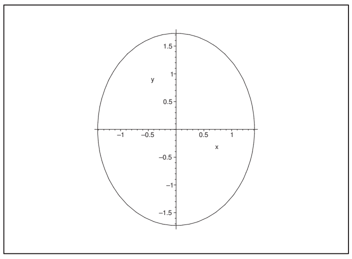
A Examination of point sets in the plane.

D Analyze each set on a figure, e.g. by first examining the function. (Neither LaTeX nor MAPLE er may be well fit for the sketches in every one of the cases).

I 1) It follows from the rearrangement

$$A = \{(x,y) \mid 3x^2 + 2y^2 < 6\} = \left\{(x,y) \;\middle|\; \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 < 1\right\}$$

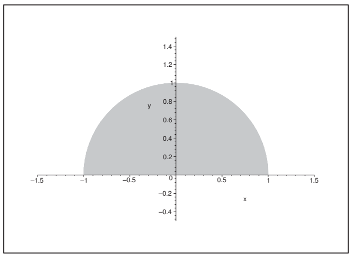
that the set is an open ellipsoidal disc of centrum $(0,0)$ and length of the half axes $\sqrt{2}$ and $\sqrt{3}$. The set is open.

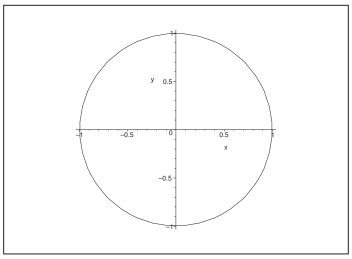


2) The set

$$A = \{(x,y) \mid x^2 + y^2 \le 1,\ y > 0\}$$

is the intersection of the closed unit disc and the open upper half plan. The set is neither open nor closed.





3) The set

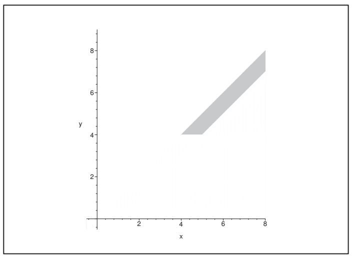
$$A = \{(x,y) \mid x^2(1-x^2-y^2) > 0\} = \{(x,y) \mid x \neq 0,\, x^2+y^2 < 1\}$$

is the open unit disc wich the exception of the points on the Y axis (where $x=0$).
The set is open.

4) The set $A = \{(x,y) \mid 0 < x-y \leq 1,\, y > 4\}$ is the intersection of the three half planes

$$\{(x,y) \mid x > y\}, \qquad \{(x,y) \mid y \geq x-1\}, \qquad \{(x,y) \mid y > 4\}.$$

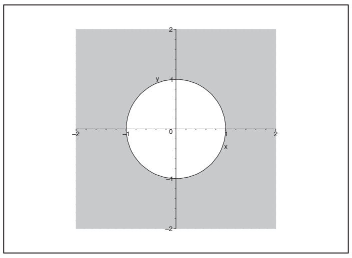
This set is neither open nor closed.



5) The set

$$\begin{aligned} A &= \{(x,y) \mid x^2+y^2 \geq \sqrt{x^2+y^2} \geq 1\} \\ &= \{(0,0)\} \cup \{(x,y) \mid \sqrt{x^2+y^2} \geq 1\} \\ &= \{(0,0)\} \cup \{(x,y) \mid x^2+y^2 \geq 1\} \end{aligned}$$

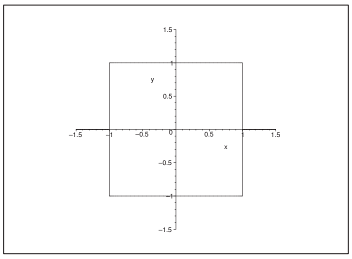
is the complementary set of a disc (centrum $(0,0)$ and radius 1), supplied with the point $(0,0)$.
The set is closed.



6) The set

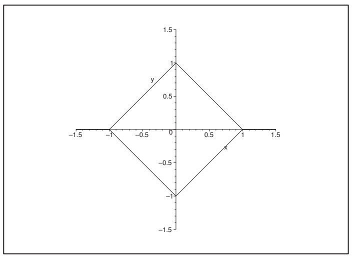
$$A = \{(x,y) \mid \max\{|x|,|y|\} \leq 1\} = [-1,1] \times [-1,1]$$

is a closed square.



7) The set $A = \{(x,y) \mid |x| + |y| < 1\}$ is the open square bounded by the lines

$$x + y = 1, \quad -x + y = 1, \quad x - y = 1, \quad -x - y = 1.$$



8) The set $A = \{(x,y) \mid x \leq y \leq 4 - x^2\}$ lies above the line $y = x$ and below the parabola $y = 4 - x^2$. These curves cut each other when $x^2 + x = 4$, i.e. when $x = -\dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{17}$.

9) Since we always have $x^2 + y^2 \geq 0$ and $x^2 + y^2 = 0$ only for $(x,y) = (0,0)$, we get that

$$A = \{(x,y) \mid (x-1)(x^2+y^2) \geq 0\} = \{(0,0)\} \cup \{(x,y) \mid x \geq 1\}$$

is the union of a point $(0,0)$ and a closed half plane $x \geq 1$. It follows that A is closed.

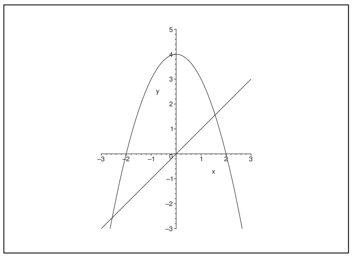


Figure 1: The figure of **Example 1.2.8.**

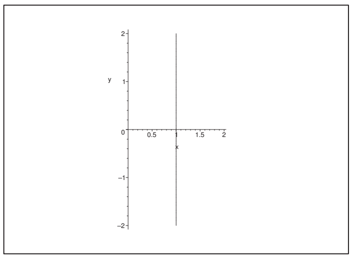
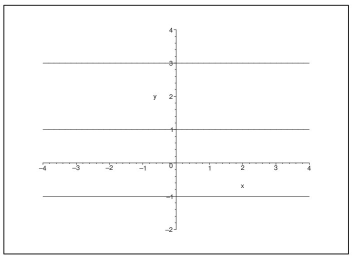


Figure 2: The figure of **Example 1.2.9.**

10) The set

$$\begin{aligned} A &= \{(x,y) \mid (y^2-1)(y-3)>0\} = \{(x,y) \mid (y+1)(y-1)(y-3)>0\} \\ &= \{(x,y) \mid -1<y<1\} \cup \{(x,y) \mid 3<y\} \end{aligned}$$

is open.



Example 1.3 *Examine in each of the following cases, possibly by means of a sketch, the given point set. Do these sets have names?*

1) $A = \{(x, y, z) \mid \max\{|x|, |y|, |z| \leq 1\}$.

2) $A = \{(x, y, z) \mid |x| + |y| + |z| \leq 1\}$.

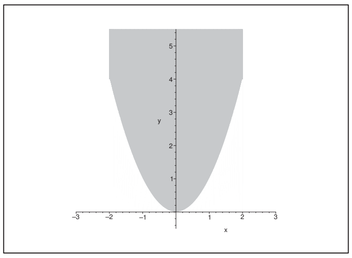
3) $A = \{(x, y, z) \mid x > 0,\; y > 0,\; z > 0\}$.

4) $A = \{(x, y, z) \mid 0 < x < y\}$.

5) $A = \{(x, y, z) \mid 0 < y\}$.

6) $A = \{(x, y, z) \mid x^2 + 2y^2 \leq 8\}$.

REMARK. It is difficult in all cases to let LaTeX or MAPLE sketch the three dimensional figures. The readers are kindly asked to sketch them themselves. $\Diamond$



A Point sets in the three dimensional space $\mathbb{R}^3$.

D Analyze each set, possibly on a figure.

I 1) The set

$$A = \{(x,y,z) \mid \max\{|x|,|y|,|z|\} \leq 1\} = [-1,1]^3$$

is a closed cube of centrum $(0,0,0)$ and edge length 2.

2) The set

$$A = \{(x,y,z) \mid |x|+|y|+|z| \leq 1\}$$

is a closed dodecahedron. It is obtained by taking the intersection of the eight half spaces

$$\begin{array}{ll} x+y+z \leq 1, & x+y+z \geq -1, \\ x+y-z \leq 1, & x+y-z \geq -1, \\ x-y+z \leq 1, & x-y+z \geq -1, \\ x-y-z \leq 1, & x-y-z \geq -1. \end{array}$$

3) The set

$$A = \{(x,y,z) \mid x>0,\ y>0,\ z>0\}$$

is the open first octant.

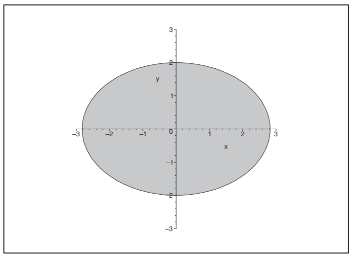
4) The set $A = \{(x,y,z) \mid 0 < x < y\}$ is the intersection of two open half spaces, hence itself open. The axis of the set is the Z axis, and the projection onto the XY plane in the direction of the Z axis is the angular set which lies between the line $y = x$ and the Y axis in the first quadrant.

5) The set $A = \{(x,y,z) \mid 0 < y\}$ is the open half space which is given by the inequality $y > 0$, i.e. bounded by the XZ plane where $y = 0$.

6) The set

$$A = \{(x,y,z) \mid x^2 + 2y^2 \leq 8\} = \left\{(x,y,z) \;\middle|\; \left(\frac{x}{2\sqrt{2}}\right)^2 + \left(\frac{y}{2}\right)^2 \leq 1\right\}$$

is the closed cylinder over the ellipse in the XY plane with centrum $(0,0)$ and half axes $2\sqrt{2}$ and 2. The figure shows the projection of the set onto the XY plane in the direction of the Z axis, hence a cross section.



Example 1.4 *In each of the following cases a plane point set A is given in polar coordinates. Sketch the point set and find a name of it.*

1) $0 \le \varphi \le \dfrac{\pi}{2}, \qquad 0 \le \varrho \le a\cos\varphi.$

2) $0 \le \varphi \le \dfrac{\pi}{4}, \qquad 0 \le \varrho \le a\cos\varphi + a\sin\varphi.$

3) $-\pi < \varphi \le \pi, \qquad (\varrho - a)^2 \ge |\varrho - a|a.$

4) $\begin{cases} 0 \le \varphi \le \ \text{Arctan}\,\dfrac{b}{a}, & 0 \le \varrho \le \dfrac{a}{\cos\varphi}, \\ \text{Arctan}\,\dfrac{b}{a} \le \varphi \le \dfrac{\pi}{2}, & 0 \le \varrho \le \dfrac{b}{\sin\varphi}. \end{cases}$

A Point sets in the plane given in polar coordinates.

D Analyze the point sets and sketch them.

I 1) When $0 \le \varrho \le a\cos\varphi$ we get by multiplying by $\varrho \ge 0$,

$$0 \le \varrho^2 \le a\varrho\cos\varphi,$$

i.e.

$$x^2 + y^2 \le ax,$$

and then by a rearrangement

$$\left(x - \frac{a}{2}\right)^2 + y^2 \le \left(\frac{a}{2}\right)^2.$$

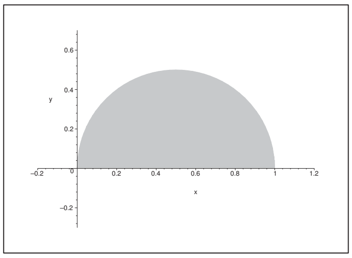
Since $0 \le \varphi \le \dfrac{\pi}{2}$, we get a closed half disc in the first quadrant of centrum $\left(\dfrac{a}{2}, 0\right)$ and radius $\dfrac{a}{2}$.

2) By a multiplication by ϱ we get

$$\varrho^2 \le a\varrho\cos\varphi + a\varrho\sin\varphi,$$

thus in rectangular coordinates

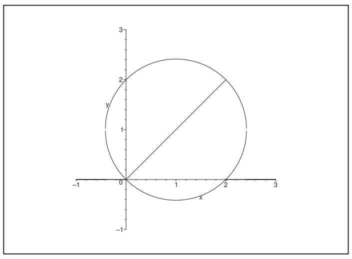
$$x^2 + y^2 \le ax + ay,$$



which is reduced to

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{a}{2}\right)^2\le\frac{a^2}{2}=\left(\frac{a}{\sqrt{2}}\right)^2.$$

This expression describes a closed disc of centrum $\left(\frac{a}{2},\frac{a}{2}\right)$ and radius $\frac{a}{\sqrt{2}}$. From the condition $0\le\varphi\le\frac{\pi}{4}$ follows that the set A is that part of the disc, which lies in in this angular set (a circumference angle).



3) It follows from $(\varrho-a)^2\ge|\varrho-a|a$ that either $\varrho=a$ or $|\varrho-a|\ge a$, hence

$$\varrho-a\ge a \qquad \text{or} \qquad \varrho-a\le -a.$$

Summarizing we get

$$\varrho=a \quad \text{or} \quad \varrho\ge 2a \quad \text{or} \quad \varrho=0,$$

since $\varrho<0$ is not possible.

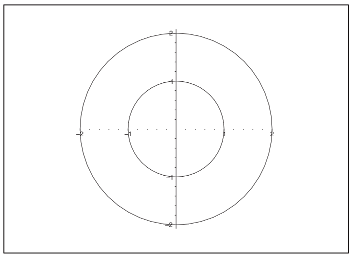
The point set is the union of a point $\{(0,0)\}$, a circumference $\varrho=a$ and the closed complementary set of a disc $\varrho\ge 2a$, since we have no restrictions on the angle $-\pi<\varrho\le\pi$.

4) Since $\cos\varphi>0$ for $0\le\varphi\le\ \text{Arctan}\ \frac{b}{a}$, the condition $0\le\varrho\le\frac{a}{\cos\varphi}$ is equivalent to

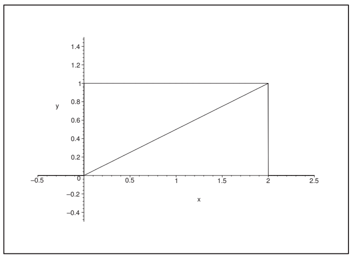
$$0\le\varrho\cos\varphi=x\le a, \qquad 0\le\varphi\le\ \text{Arctan}\ \frac{b}{a}.$$

Analogously, $\sin\varphi>0$ for $\text{Arctan}\ \frac{b}{a}\le\varphi\le\frac{\pi}{2}$, thus $0\le\varrho\le\frac{b}{\sin\varphi}$ is equivalent to

$$0\le\varrho\sin\varphi=y\le b, \qquad \text{Arctan}\ \frac{b}{a}\le\varphi\le\frac{\pi}{2}.$$



The two cases are described by each their triangle, and the conclusion is that the set in rectangular coordinates is just the rectangle $A = [0, a] \times [0, b]$.



Example 1.5 *Sketch and describe in polar coordinates the set A, where A is given below in rectangular coordinates.*

1) $A = \left\{(x,y) \mid x \geq 0,\ \left(x^2+y^2\right)^2 \geq x^2+y^2\right\}$.

2) $A = \left\{(x,y) \;\middle|\; x > 0,\ \dfrac{1}{2}+y^2 \leq x^2 \leq 1-y^2\right\}$.

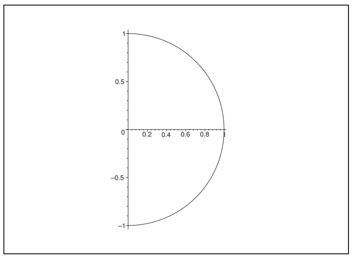
A Point sets in the plan, given in rectangular coordinates should be described in polar coordinates instead.

D Sketch the sets and use that $x = \varrho\cos\varphi$ and $y = \varrho\sin\varphi$.

I 1) The point set A is the intersection of a closed complementary set of a disc and the closed right half plane supplied by the point $(0,0)$.

In polar coordinates this is described by

$$-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \qquad \text{og} \qquad \varrho^2 \geq \varrho.$$



2) Since $x > 0$, the point set lies in the open right half plane. It follows from $x^2 \leq 1-y^2$ that $x^2+y^2 \leq 1$, so the point set lies in the unit disc.

Finally, $\dfrac{1}{2}+y^2 \leq x^2$ describes the interior of a branch of a hyperbola. The two limiting curves

$$x^2+y^2 = 1 \qquad \text{and} \qquad x^2-y^2 = \frac{1}{2}$$

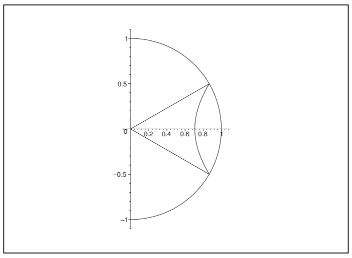
cut each other at the points $\left(\dfrac{\sqrt{3}}{2}, \pm\dfrac{1}{2}\right)$, so A lies in the angular set $-\dfrac{\pi}{6} \leq \varphi \leq \dfrac{\pi}{6}$.

In polar coordinates the upper is described by $\varrho \leq 1$, and the lower bound is given by

$$\frac{1}{2}+\varrho^2\sin^2\varrho \leq \varrho^2\cos^2\varphi,$$

hence by a rearrangement,

$$\frac{1}{2} \leq \varrho^2\left\{\cos^2\varphi - \sin^2\varphi\right\} = \varrho^2\cos 2\varphi.$$



Summarizing we get the following polar description

$$-\frac{\pi}{6} \le \varphi \le \frac{\pi}{6} \qquad \text{og} \qquad \frac{1}{\sqrt{2\cos 2\varphi}} \le \varrho \le 1.$$

Example 1.6 *Sketch the following subsets of* $\mathbb{R}^2$, *and if any of them is star shaped.*

1) $\{(x,y) \mid y > 3x^2\}$.

2) $\{(x,y) \mid x^2 + y^2 > 1\}$.

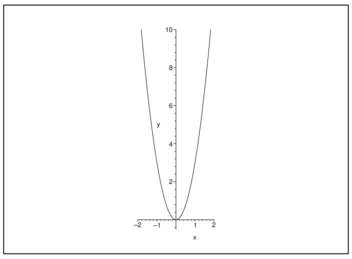
3) $\{(x,y) \mid y > -x^2\}$.

4) $\{(x,y) \mid x > 0,\, y > -x^2\}$.

A Analysis of sets concerning if they are star shaped.

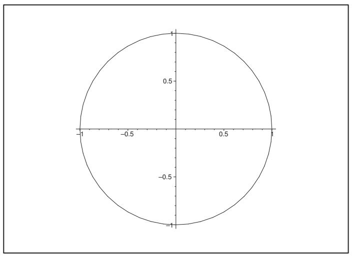
D Start by sketching the sets. In this case I have had problems with the sketching programs, so the sets are only given by their boundaries and not by the more desirable hatching.

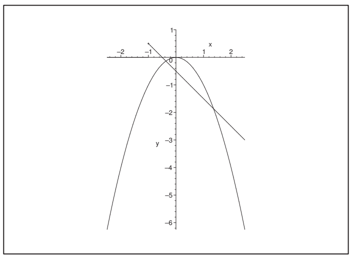
I 1) Here, A in the interior of a parabola. Obviously, this set is star shaped (and even convex).



2) This set is the complementary of a disc, so it cannot be star shaped. For any point from the set the unit disc shades for some other points.

3) The set is the exterior of a parabola. If $(x_0, y_0) \in A$ is any point we can always find a straight line through (x_0, y_0), which cuts the parabola in two different points. The points on the line beyond the most distant intersection point cannot be connected with (x_0, y_0) by a straight line inside A, so A is not star shaped seen from any point.





4) This set A is a part of the set in **Example 1.6.3**, hence it lies in the right half plane.
First notice that

$$y + \lambda^2 = -2\lambda(x - \lambda)$$

is a tangent of the parabola for every $\lambda > 0$. This can also be written

$$y + 2\lambda x = \lambda^2, \qquad \lambda > 0.$$

Indirect proof. Assume that A indeed was star shaped from a point (x, y). Then

$$y + 2\lambda x \geq \lambda^2 \qquad \text{for all } \lambda > 0,$$

which can also be written

$$y \geq \lambda(\lambda - 2x) \qquad \text{for alle } \lambda > 0.$$

Of course this is not possible for any $(x, y) \in A$. In fact, the right hand side of this inequality tends to $+\infty$ for $\lambda \to +\infty$, while y remains constant, and the inequality is violated.

We conclude from this contradiction that A is not star shaped.

Example 1.7 . *Sketch the point sets given below, and indicate which ones are convex.*

1) $\{(x,y) \mid -5 < y < -3x^2\}$.

2) $\{(x,y) \mid x^2 + 3y^2 > 2\}$.

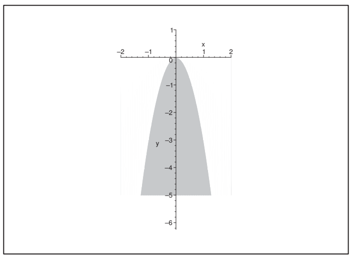
3) $\{(x,y) \mid y > -x^2\}$.

4) $\{/x,y) \mid x \geq 0,\, y \leq 0\}$.

A Examination of convexity.

D Skets the sets and analyze.

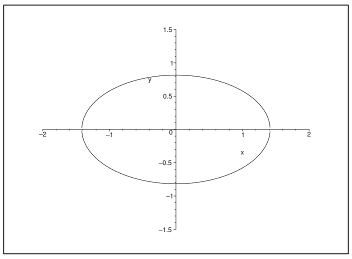
I 1) The set is the interior of a parabola where we furthermore have the restriction $-5 < y < 0$. Obviously, this set is convex.



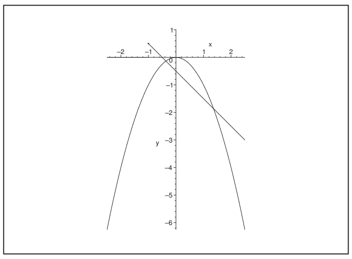
2) The set

$$\left\{(x,y) \mid x^2+3y^2>2\right\} = \left\{(x,y) \;\middle|\; \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{\sqrt{\frac{2}{3}}}\right)^2 > 1\right\}$$

is the complementary of an ellipse of centrum $(0,0)$ and half axis $\sqrt{2}$ and $\sqrt{\dfrac{2}{3}}$. It is clearly not convex.



3) This set is the complementary of a parabola (actually the same set as in **Example 1.6.2**). It is not star shaped, and therefore not convex either.



4) This set is the closed fourth quadrant. It is clearly convex. There is no need to sketch it.

Example 1.8 *Find and sketch in each of the following cases the domain of the given function.*

1) $f(x,y) = \ln|1-x^2-y^2|$.

2) $f(x,y) = \sqrt{-x^2-y^2}$.

3) $f(x,y) = \ln(1-x^2-y^2) + \sqrt{\left(x-\frac{1}{2}\right)(x^2+y^2)}$.

4) $f(x,y) = \ln[y(x^2+y^2+2y)]$.

5) $f(x,y) = y\sqrt{2-x^2} + \text{ Arctan } \dfrac{y}{x}$.

6) $f(x,y) = \sqrt{3-x^2-y^2} + 2\,\text{Arcsin}(x^2-y^2)$.

7) $f(x,y) = \text{ Arcsin}(2-x^2-y)$.

8) $f(x,y) = \sqrt{xy-1}$.

9) $f(x,y) = \sqrt{y+\sin x} + \sqrt{-y+\sin x}$.

10) $f(x,y) = x^y$.

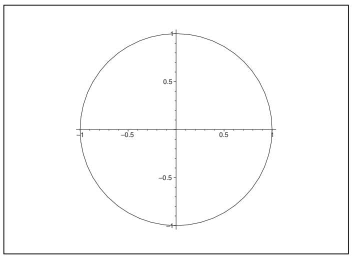
11) $f(x,y) = \ln y + \ln(x^2+y^2+2y)$.

A Domain of a function.

D Analyze the domain and the sketch the set.

I 1) The function $\ln|1-x^2-y^2|$ is defined for $|1-x^2-y^2| > 0$, i.e. for $x^2+y^2 \neq 1$. The domain is $\mathbb{R}^2$ with the exception of the unit circle:

$$\mathbb{R}^2 \setminus \{(x,y) \mid x^2+y^2 = 1\}.$$



2) The requirement of the function $\sqrt{-x^2-y^2}$ is that $-x^2-y^2 \geq 0$, i.e. the domain is only the point $\{(0,0)\}$.

3) The function $\ln(1-x^2-y^2)+\sqrt{(x-\frac{1}{2})(x^2+y^2)}$ is defined for

$$1-x^2-y^2>0 \qquad \text{and} \qquad \left(x-\frac{1}{2})(x^2+y^2\right)\geq 0.$$

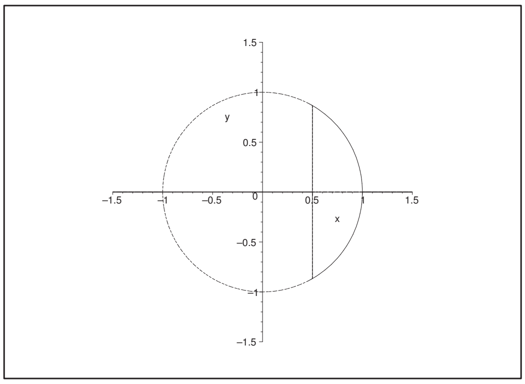
We first conclude that $x^2+y^2<1$, so the domain must be contained in the open unit disc.

Then notice that both requirements are fulfilled for $(x,y)=(0,0)$, thus $(0,0)$ belongs to the domain.

Finally, when $0<x^2+y^2<1$ we also have the requirement $x \geq \frac{1}{2}$.

Summarizing the domain is

$$\{(0,0)\}\cup\{(x,y)\mid x\geq\frac{1}{2},\, x^2+y^2<1\}.$$



4) The function $\ln(y(x^2+y^2+2y))$ is defined for

$$y(x^2+y^2+2y)=y\{x^2+(y+1)^1-1\}>0.$$

Here we get two possibilities:

a) When both $y>0$ and $x^2+(y+1)^2>1$, we see that we can reduce to $y>0$, because then also $(y+1)^2>1$.

b) The second possibility is that $y<0$ and $x^2+(y+1)^2<1$. In this case we reduce to $x^2+(y+1)^2<1$, because this inequality determines an open disc in the lower half plane of centrum $(0,-1)$ and radius 1, and $y<0$ is automatically satisfied.

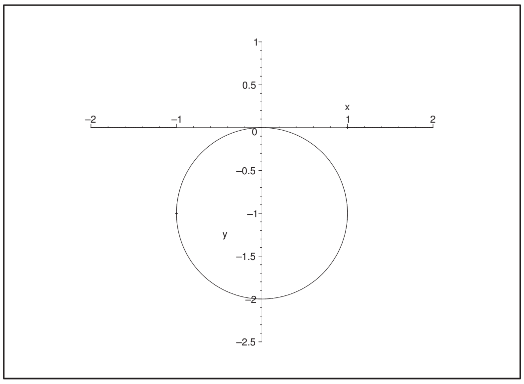
Summarizing we obtain the domain

$$\{(x,y)\mid y>0\}\cup\{(x,y)\mid x^2+(y+1)^1<1\},$$

i.e. the union of the upper half plane and the afore mentioned circle in the lower half plane.

5) The function $y\sqrt{2-x^2}+\text{ Arctan }\dfrac{y}{x}$ is defined for

$$2-x^2\geq 0 \qquad \text{and} \qquad x\neq 0,$$

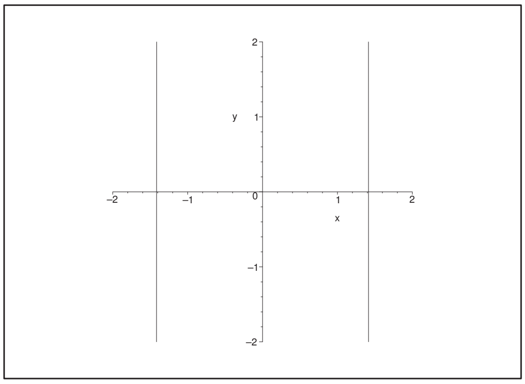


i.e. the domain is the union of two vertical strips, which are neither open nor closed,

$$\{(x,y) \mid -\sqrt{2} \le x < 0\} \cup \{(x,y) \mid 0 < x \le \sqrt{2}\}.$$

This can also be written

$$[-\sqrt{2},\sqrt{2}] \times \mathbb{R} \setminus \{(0)\} \times \mathbb{R}.$$



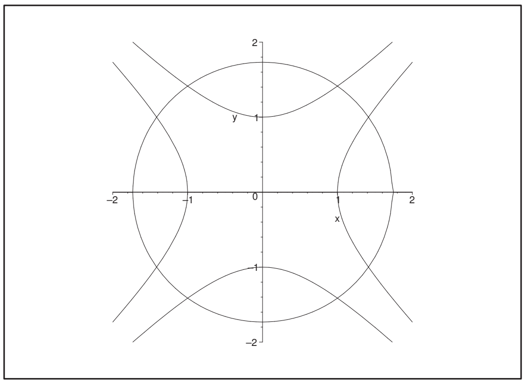
6) The function $\sqrt{3-x^2-y^2} + 2\,\mathrm{Arcsin}(x^2-y^2)$ is defined for

$$x^2+y^2 \le 3 \qquad \text{and} \qquad -1 \le x^2-y^2 \le 1,$$

i.e. for

$$\sqrt{x^2+y^2} \le \sqrt{3}, \qquad x^2-y^2 \le 1, \qquad y^2-x^2 \le 1.$$

The domain is that component of the intersection with the disc which also contains the point $(0,0)$.



7) The function $\mathrm{Arcsin}(2-x^2-y)$ is defined for

$$-1 \le 2-x^2-y \le 1,$$

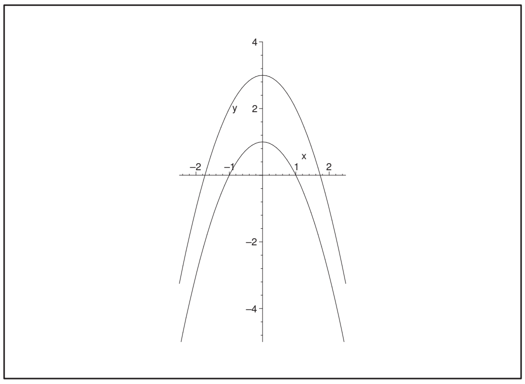
i.e. when the following two conditions are fulfilled:

$$y \le 3-x^2 \qquad \text{and} \qquad y \ge 1-x^2.$$

Summarizing the domain becomes

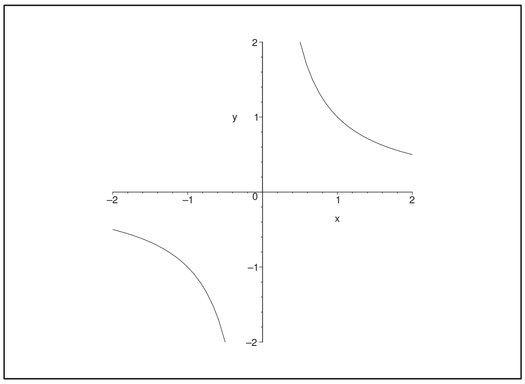
$$\{(x,y) \mid 1-x^2 \le y \le 3-x^2\},$$

which is the closed set which lies between the two arcs of parabolas.



8) The function $\sqrt{xy-1}$ is defined for $xy \geq 1$ i.e. the sets in the first and third quadrant, which are bounded by the hyperbola $y = \dfrac{1}{x}$ and which is not close to any of the axes:

$$\{(x,y) \mid x > 0,\, y > 0,\, xy \geq 1\} \cup \{(x,y) \mid x < 0,\, y < 0,\, xy \geq 1\}.$$



9) The function $\sqrt{y+\sin x} + \sqrt{-y+\sin x}$ is defined when both

$$y + \sin x \geq 0 \qquad \text{and} \qquad -y + \sin x \geq 0,$$

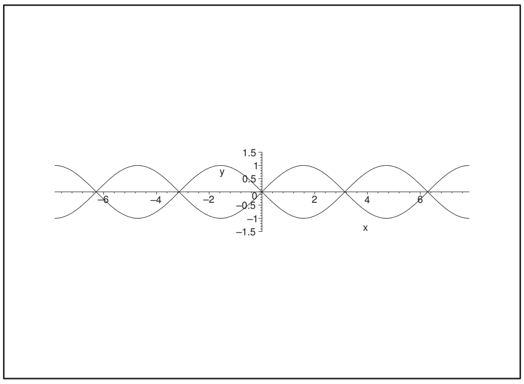
i.e. when

$$-\sin x \leq y \leq \sin x.$$

Hence the condition $\sin x \geq 0$, i.e. $x \in [2p\pi, \pi + 2p\pi]$, $p \in \mathbb{Z}$, and the domain is

$$\bigcup_{p\in\mathbb{Z}} \{(x,y) \mid 2p\pi \leq x \leq 2p\pi + \pi,\ |y| \leq \sin x\}.$$

On the figure the domain is the union of every second of the connected subsets.



10) This is a very difficult example. First notice that the function $f(x,y) = x^y$ is at least defined when $x,\, y > 0$.

When $x = 0$ the function is defined for every $y > 0$.

When $x < 0$, the function is defined for every $y = \dfrac{p}{2q+1}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}_0$.

When $y < 0$ is not a rational number of odd denominator, we must necessarily require that $x > 0$.

When $y = -\dfrac{p}{2q+1}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, then x^y is also defined for $x < 0$, though not for $x = 0$.

Remark. It is a matter of definition whether one can put $x^0 = 1$ for $x < 0$. This may be practical in some cases, though not in everyone. $\Diamond$.

This domain is fairly complicated:

$$\{(x,y) \mid x > 0\} \cup \{(0,y) \mid y > 0\} \cup \bigcup_{p,\, q \in \mathbb{N}_0} \{(x,y) \mid x < 0,\ y = -p/(2q+1)\},$$

where one may discuss whether the point $(0,0)$ should be included or not.

11) When the function $f(x,y) = \ln y + \ln(x^2 + y^2 + 2y)$ is defined, we must at least require that $y > 0$, because $\ln y$ in particular should be defined.

If on the other hand $y > 0$, then clearly also $x^2 + y^2 + 2y > 0$, no matter the choices of x and $y > 0$, thus $f(x,y)$ is defined for $y > 0$, i.e. in the upper half plane.

Example 1.9 *Describe in each of the following cases the domain of the given function.*

1) $f(x,y,z) = \sqrt{1-|x|-|y|-|z|}$.

2) $f(x,y,z) = \ln(\sqrt{1-|x|-|y|-|z|})$.

3) $f(x,y,z) = \text{ Arcsin}(x^2+y^2-4)$.

4) $f(x,y,z) = \sqrt[4]{x^2+4y^2+9z^2-1}$.

5) $f(x,y,z) = \text{ Arctan } \dfrac{x+z}{y}$.

6) $f(x,y,z) = \exp(3x+2y+5z)$.

A Domain of functions in three variables.

D Analyze in each case the function. There will here be given no sketches.

I 1) The function $\sqrt{1-|x|-|y|-|z|}$ is defined for $|x|+|y|+|z| \le 1$,

$$\{(x,y,z) \mid |x|+|y|+|z| \le 1\}.$$

This set is a closed tetrahedron in the space.

2) The function $\ln(\sqrt{1-|x|-|y|-|z|})$ is defined in the corresponding *open* tetrahedron in space,

$$\{(x,y,z) \mid |x|+|y|+|z| < 1\}.$$

3) The function $\operatorname{Arcsin}(x^2+y^2+z^2-4)$ is defined when

$$-1 \le x^2+y^2+z^2-4 \le 1,$$

i.e. in the shell

$$\left\{(x,y,z) \mid (\sqrt{3})^2 \le x^2+y^2+z^2 \le (\sqrt{5})^2\right\},$$

of centrum $(0,0,0)$, inner radius $\sqrt{3}$ and outer radius $\sqrt{5}$.

4) The function $\sqrt[4]{x^2+4y^2+9z^2-1}$ is defined outside an ellipsoid,

$$\left\{(x,y,z) \;\middle|\; x^2+\left(\frac{y}{\frac{1}{2}}\right)^2+\left(\frac{z}{\frac{1}{3}}\right)^2 \ge 1\right\},$$

where the half axes are 1, $\dfrac{1}{2}$ and $\dfrac{1}{3}$.

5) The function $\operatorname{Arctan}\dfrac{x+z}{y}$ is defined for $y \ne 0$.

6) The function $\exp(3x+2y+5z)$ is of course defined in the whole space $\mathbb{R}^2$.

Example 1.10 *Let*

$$B = \{(x,y) \in [0,1]\times[0,1] \mid x \textit{ is rational and } y \textit{ is rational}\}.$$

Find the interior B°, the boundary ∂B and the closure $\overline{B}$.

A Interior, exterior, boundary and closure of a point set. This is the classical "strange" example, which should shock the reader.

D First prove that $B^\circ = \emptyset$, and then $\overline{B]} = [0,1]\times[0,1]$.

I If $(x_0,y_0) \in B$, then $K((x_0,y_0);r)$, $r>0$, i.e. the ball of centrum (x_0,y_0) and any positive radius r, will always contain points (x,y), of which at least one of the coordinates is irrational, thus

$$K((x_0,y_0);r) \setminus B \ne \emptyset \qquad \text{for every } r>0.$$

We conclude from this that $B^\circ = \emptyset$.

Let $(x_0,y_0) \in [0,1]\times[0,1]$ be any point in the bigger set. Then the ball $K((x_0,y_0);r)$ of centrum (x_0,y_0) and any radius $r>0$ will always contain points from B. This means that $(x_0,y_0) \in \overline{B}$, i.e.

$$\overline{B} \supseteq [0,1]\times[0,1].$$

It is on the other hand trivial that $\overline{B} \subseteq [0,1] \times [0,1]$, hence we must have equality,

$$\overline{B} = [0,1] \times [0,1].$$

Finally, the boundary is found by means of the definition,

$$\partial B = \overline{B} \setminus B^{\circ} = [0,1] \times [0,1] \setminus \emptyset = [0,1] \times [0,1] = \overline{B}.$$

Example 1.11 *In each of the following cases there is given a solid tetrahedron by its four corners. Sketch the tetrahedron T – invisible edges are dotted – and set up equations of the four planes, which bound T. Then derive the inequalities which the points of T must fulfil, and finally set up expressions of the form*

$$T = \{(x,y,z) \mid (x,y) \in B,\ Z_1(x,y) \leq z \leq Z_2(x,y)\}$$

and

$$T = \{(x,y,z) \mid \alpha \leq z \leq \beta,\ (x,y) \in B(z)\};$$

sketch the sets B and $B(z)$.

1) $(0,0,0)$, $(2,0,0)$, $(0,1,0)$, $(0,0,2)$.

2) $(0,0,0)$, $(2,0,2)$, $(0,1,2)$, $(0,0,2)$.

3) $(1,0,0)$, $(0,0,4)$, $(0,2,2)$, $(-1,0,0)$.

4) $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(1,0,4)$.

5) $(1,0,0)$, $(0,0,4)$, $(0,2,0)$, $(-1,0,0)$.

A Analysis of tetrahedra.

D The text describes very carefully what should be done. Here we shall deviate a little because figures in space take a very long time to construct in the given programs. There are therefore left to the reader.

I 1) It follows immediately from the missing figure (which the reader should add himself), that three of the planes are described by

$$x = 0, \qquad y = 0 \quad \text{and} \quad z = 0.$$

In fact, the plane $x = 0$ contains the points

$$(0,0,0), \qquad (0,1,0), \qquad (0,0,2),$$

the plane $y = 0$ contains the points

$$(0,0,0), \qquad (2,0,0), \qquad (0,0,2),$$

and the plane $z = 0$ contains the points

$$(0,0,0), \qquad (2,0,0), \qquad (0,1,0).$$

A parametric description of the fourth plane is e.g.

$$\begin{aligned}(x,y,z) &= (2,0,0)+u\{(0,1,0)-(2,0,0)\}+v\{(0,0,2)-(2,0,0)\}\\ &= (2,0,0)+u(-2,1,0)+v(-2,0,2)\\ &= (2-2u-2v,u,2v),\end{aligned}$$

from which $y = u$ and $z = 2v$.
When we eliminate u and v, we get

$$x = 2-2u-2v = 2-2y-z,$$

and the equation of the fourth plane is

$$z = 2-x-2y.$$

The points of T must satisfy the inequalities

$$0 \le x\,(\le 2), \quad 0 \le y\left(\le 1-\frac{x}{2}\right), \quad 0 \le z \le 2-x-2y.$$

We immediately get

$$T = \{(x,y,z) \mid (x,y) \in B,\, 0 \le z \le 2-x-2y\},$$

where

$$B = \{(x,y) \mid 0 \le x \le 2,\, 0 \le y \le 1-\frac{x}{2}\}.$$

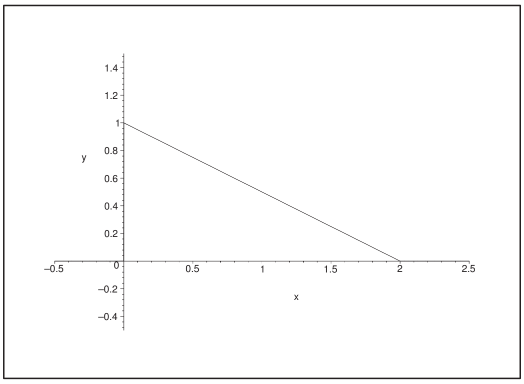


Figure 3: The domain B

If we instead keep $z \in [0,2]$ fixed, the tetrahedron is cut into a triangle $B(z)$, bounded by

$$0 \le \ (\le 2-z), \qquad 0 \le y \le 1-\frac{x}{2}-\frac{z}{2},$$

i.e.

$$B(z) = \left\{ (x,y) \;\middle|\; 0 \le x \le 2 - z,\, 0 \le y \le 1 - \frac{z}{2} - \frac{x}{2} \right\}, \quad 0 \le 0 \le z \le 2,$$

and

$$T = \{(x,y,z) \mid (x,y) \in B(z),\, 0 \le z \le 2\}.$$

It follows that $B(z)$ is similar to B above with the factor of similarity $1 - \dfrac{z}{2}$.

2) We see in the same way as in **Example 1.11.1** that three of the planes are described by

$$x = 0, \qquad y = 0 \quad \text{and} \quad z = 2.$$

A parametric description of the fourth plane is e.g.

$$(x,y,z) = (0,0,0) + u(2,0,2) + v(0,1,2) = (2u, v, 2u + 2v),$$

from which $x = 2u$ and $y = v$. When u and v are eliminated we get

$$z = 2u + 2v = x + 2y,$$

which is an equation of the fourth plane.

The points of T must satisfy the inequalities

$$0 \le x\,(\le 2), \quad 0 \le y\,\left(\le 1 - \frac{x}{2}\right), \quad x + 2y \le z \le 2.$$

Hence,

$$T = \{(x,y,z) \mid (x,y) \in B,\, x + 2y \le z \le 2\},$$

where

$$B = \left\{(x,y) \;\middle|\; 0 \le x \le 2,\, 0 \le y \le 1 - \frac{x}{2}\right\}.$$

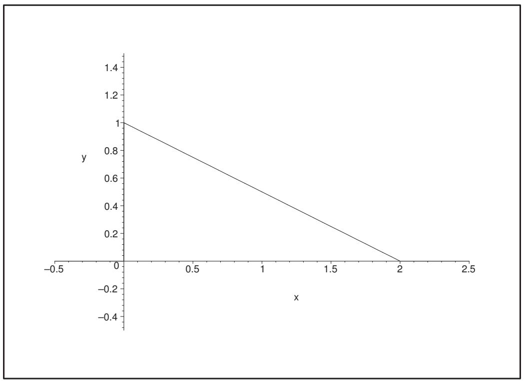


Figure 4: The domain B

If we instead keep $z \in [0,2]$ fixed, the tetrahedron is cut into a triangle $B(z)$, bounded by

$$0 \le x \le z, \qquad 0 \le y \le \frac{z}{2} - \frac{x}{2},$$

i.e.

$$B(z) = \left\{(x,y) \;\middle|\; 0 \le x \le z,\, 0 \le y \le \frac{z}{2} - \frac{x}{2}\right\}, \quad 0 \le z \le 2,$$

and

$$T = \{(x,y,z) \mid (x,y) \in (z),\, 0 \le z \le 2\}.$$

We see that $B(z)$ is similar to B with the constant of similarity $\dfrac{z}{2}$.

3) Here a trivial boundary plane is given by $y = 0$.

The points $(1,0,0)$, $(0,2,2)$, $(0,0,4)$ lie in the plane of the parametric description

$$(x,y,z) = (1,0,0) + u(-1,2,2) + v(-1,0,4) = (1-u-v, 2u, 2u+4v),$$

i.e.

$$x = 1-u-v, \qquad y = 2u, \qquad z = 2u+4v,$$

from which

$$u = \frac{y}{2}, \qquad v = 1-u-x = 1-\frac{y}{2}-x,$$

so

$$z = 2u+4v = y+4\left(1-\frac{y}{2}-x\right) = 4-4x-y,$$

which is the equation of this plane.

The points $(-1,0,0)$, $(0,2,2)$, $(0,0,4)$ lie in the plane of the parametric description

$$(x,y,z) = (-1,0,0) + u(1,2,2) + v(1,0,4) = (-1+u+v, 2u, 2u+4v),$$

i.e.

$$x = -1+u+v, \qquad y = 2u, \qquad z = 2u+4v,$$

from which

$$u = \frac{y}{2}, \qquad v = 1+x-u = 1+x-\frac{y}{2},$$

hence

$$z = 2u+4v = y+4+4x-2y = 4+4x-y,$$

which is the equation of this plane.

The points $(1,0,0)$, $(-1,0,0)$, $(0,2,2)$ lie in the plane of the parametric description

$$(x,y,z) = (-1,0,0) + u(2,0,0) + v(1,2,2) = (2u-1, 2v, 2v),$$

from which

$$x = 2u-1, \qquad y = 2v, \qquad z = 2v.$$

We see that the equation of the plane is $z = y$.

Summarizing we have obtained the four planes

$$y = 0, \quad z = 4-4x-y, \quad z = 4+4x-y, \quad z = y.$$

The projection of T onto the XY plane is the triangle B of the corners $(-1,0)$, $(1,0)$, $(0,2)$. This can be described by

$$0 \le y \le 2, \quad \frac{y}{2}-1 \le x \le 1-\frac{y}{2} \quad \left(|x| \le 1-\frac{y}{2}\right),$$

i.e.

$$B = \left\{(x,y) \;\middle|\; 0 \le y \le 2,\, |x| \le 1-\frac{y}{2}\right\}.$$

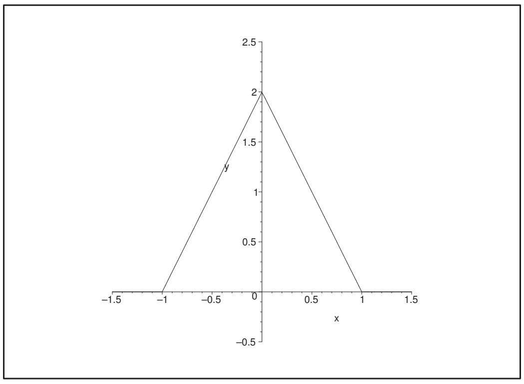


Figure 5: The domain B

When $(x, y) \in B$, it is seen from the figure that

$$\begin{cases} y \leq z \leq 4 - 4x - y & \text{for } x \geq 0, \\ y \leq z \leq 4 + 4x - y & \text{for } x \leq 0, \end{cases}$$

i.e.

$$T = \left\{(x,y,z) \;\middle|\; 0 \le y \le 2,\, |x| \le 1 - \frac{y}{2},\, y \le z \le 4 - 4|x| - y\right\}.$$

The plane $z = \text{constant} \in [2,4]$ cuts T in a triangle $B(z)$ given by

$$0 \le y \le 4 - z, \qquad |x| \le 2 - \frac{y}{2} - \frac{z}{2},$$

hence

$$B(z) = \left\{(x,y) \;\middle|\; 0 \le y \le 4 - z,\, |x| \le 2 - \frac{y}{2} - \frac{z}{2}\right\} \quad \text{for } z \in [2,4].$$

It follows that $B(z)$ is similar to B with the factor of similarity $2 - \dfrac{z}{2}$.

Then let $z \in]0,2[$ be fixed. This plane cuts T in a trapeze, which is obtained by cutting a triangle out of B at height z. Thus, for $z \in [0,2[$,

$$B(z) = \left\{(x,y) \;\middle|\; 0 \le y \le z,\, |x| \le 1 - \frac{y}{2}\right\} \quad \text{for } z \in [0,2[.$$

We now have the following description of the tetrahedron:

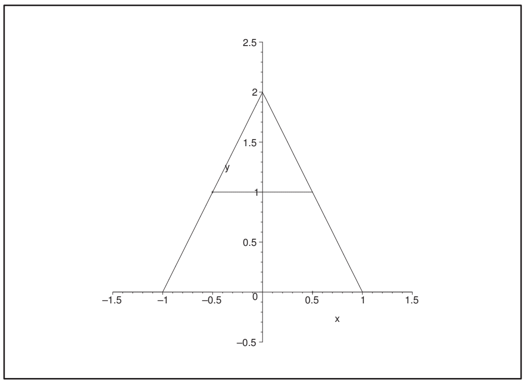


Figure 6: The domain $B(z)$ for $z = 1 \in [0,2[$

$$\begin{aligned} T &= \left\{(x,y,z) \;\middle|\; 0 \le z \le 2,\, 0 \le y \le z,\, |x| \le 1 - \frac{y}{2}\right\} \\ &\quad \cup \left\{(x,y,z) \;\middle|\; 2 \le z \le 4,\, 0 \le y \le 4 - z,\, |x| \le 2 - \frac{y}{2} - \frac{z}{2}\right\}. \end{aligned}$$

4) The obvious planes are here

$$\begin{aligned} y &= 0, \quad && [\text{points } (0,0,0),\, (1,0,0),\, (1,0,4)], \\ z &= 0, && [\text{points } (0,0,0),\, (1,0,0),\, (1,1,0)], \\ x &= 1, && [\text{points } (1,0,0),\, (1,1,0),\, (1,0,4)]. \end{aligned}$$

Finally, the points $(0,0,0)$, $(1,0,4)$, $(1,1,0)$ lie in the plane of the parametric description

$$(x,y,z) = u(1,0,4) + v(1,1,0) = (u+v, v, 4u),$$

from which

$$v = y, \qquad u = x - v = x - y \quad \text{og} \quad z = 4u = 4x - 4y.$$

The points in T must satisfy the inequalities

$$0 \le x \le 1, \qquad 0 \le y \le x, \qquad 0 \le z \le 4x - 4y.$$

In particular, the triangle B is

$$B = \{(x, y) \mid 0 \le x \le 1,\, 0 \le y \le x\},$$

and we get

$$T = \{(x, y, z) \mid 0 \le x \le 1,\, 0 \le y \le x,\, 0 \le z \le 4x - 4y\}.$$

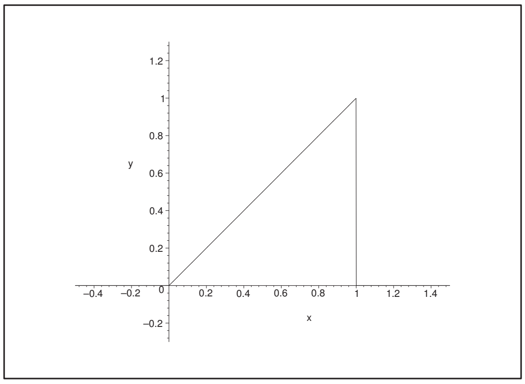


Figure 7: The domain B

The plane $z = \text{constant} \in [0, 4]$ cuts the tetrahedron in a triangle which is similar to B of the similarity factor $1 - \frac{z}{4}$ for $z \in [0, 4]$), thus

$$B(z) = \left\{(x, y) \;\middle|\; 0 \le x \le 1 - \frac{z}{4},\, 0 \le y \le x\right\},$$

and accordingly,

$$T = \left\{(x, y, z) \;\middle|\; 0 \le z \le 4,\, 0 \le x \le 1 - \frac{z}{4},\, 0 \le y \le x\right\}.$$

5) The obvious planes are

$$\begin{aligned} y &= 0, \qquad && [\text{points } (1,0,0),\ (-1,0,0),\ (0,0,4)], \\ z &= 0, \qquad && [\text{points } (1,0,0),\ (0,2,0),\ (-1,0,0)]. \end{aligned}$$

The points $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 4)$ lie in the plane of the parametric description

$$(x, y, z) = (1, 0, 0) + u(-1, 2, 0) + v(-1, 0, 4) = (1 - u - v, 2u, 4v).$$

Hence,

$$u = \frac{y}{2}, \quad v = \frac{z}{4}, \quad x = 1 - u - v = 1 - \frac{y}{2} - \frac{z}{4},$$

and we get the following equation of the plane,

$$z = 4 - 4x - 2y.$$

Due to the symmetry the points $(-1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 4)$ must lie in the plane of the equation

$$z = 4 + 4x - 2y.$$

The projection of T onto the XY plane is the triangle

$$B = \left\{ (x, y) \;\middle|\; 0 \le y \le 2,\, |x| \le 1 - \frac{y}{2} \right\}.$$

When $(x, y) \in B$, we get for $(x, y, z) \in T$ that

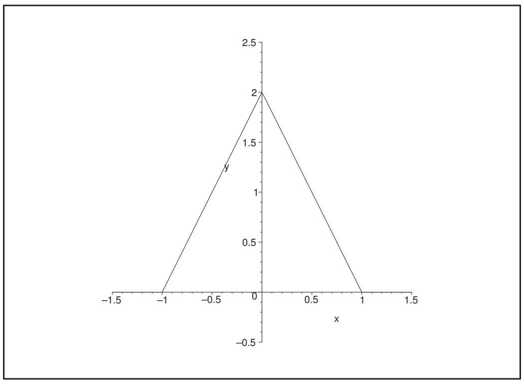


Figure 8: The domain B

$$\begin{cases} 0 \le z \le 4 - 4x - 2y, & \text{for } x \ge 0, \\ 0 \le z \le 4 + 4x - 2y, & \text{for } x \le 0, \end{cases}$$

i.e.

$$T = \left\{ (x, y, z) \;\middle|\; 0 \le y \le 2,\, |x| \le 1 - \frac{y}{2},\, 0 \le z \le 4 - 4|x| - 2y \right\}.$$

At the height $z \in [0, 4]$ the tetrahedron T is cut into a triangle

$$B(z) = \left\{ (x, y) \;\middle|\; 0 \le y \le 2 - \frac{z}{2},\, |x| \le 1 - \frac{y}{2} - \frac{z}{4} \right\},$$

where $B(z)$ is similar to B of the similarity factor $1 - \dfrac{z}{4}$, hence

$$T = \left\{ (x, y, z) \;\middle|\; 0 \le z \le 4,\, 0 \le y \le 2 - \frac{z}{2},\, |x| \le 1 - \frac{y}{2} - \frac{z}{4} \right\}.$$

Example 1.12 *Sketch of a figure the set A, where*

$$A = \{(x,y) \in \mathbb{R}^2 \mid x + 2y \le 2,\ |x-y| \le 2\}.$$

On the figure one should indicate the boundary ∂A. Finally, explain why A is not bounded.

A Sketch of a set in the plane.

D Start by analyzing the lines, which bound the set.

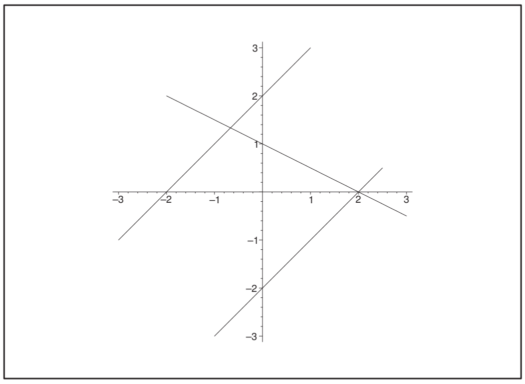


Figure 9: The domain A is that component of the plane, which contains the point $(0,0)$.

I It follows from the definition of A that we have the three restrictions

$$x + 2y \le 2, \qquad x - y \le 2, \qquad y - x \le 2.$$

We note that $(0,0)$ satisfies all three inequalities. Thus, the domain A is the closed component (the intersection of three closed half planes), which contains $(0,0)$. The boundary ∂A consists of pieces of the lines

$$x + 2y = 2, \quad x - y = 2, \qquad y - x = 2.$$

Now, the unbounded half line

$$\{(x,y) \mid y = x - 2, x \le 2\}$$

lies in A, so A must also be unbounded.

2 Examinations of functions

Example 2.1 *Draw a sketch for each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ indicating*

(i) *the curves where $f(x, y) = 0$,*

(ii) *the point sets where $f(x, y) > 0$,*

(iii) *the point sets where $f(x, y) < 0$.*

1) $f(x, y) = xy(x^2 - 2x + y^2)$.

2) $f(x, y) = xy^2(4x^2 + y^2 - 2y - 3)$.

3) $f(x, y) = (x + y - 1)(x - y - 1)(y + 4)$.

4) $f(x, y) = (y - x^2)(y - 3x^2)$.

A The functions are defined and continuous in $\mathbb{R}^2$. One shall find the zero sets, which due to the continuity divide $\mathbb{R}^2$ into open connected subsets. In each of these subsets $f(x, y)$ is either positive or negative.

D Factorize the function. Then the zero sets are found by successively putting each factor equal to 0. These curves are then sketched.

I 1) Here we write

$$f(x, y) = xy(x^2 - 2x + y^2) = xy\{(x - 1)^2 + y^2 - 1\},$$

thus $f(x, y) = 0$, in each of the following cases,

$$x = 0 \quad \text{or} \quad y = 0 \quad \text{or} (x - 1)^2 + y^2 = 1.$$

These curves are either the coordinate axes or the circle of centrum $(1, 0)$ and radius 1.

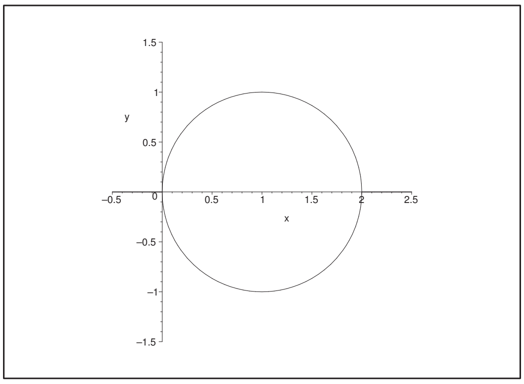


Figure 10: The function is positive in the unbounded part of the first quadrant, in the whole of the third quadrant, and in the half disc in the fourth quadrant

The plane $\mathbb{R}^2$ is divided by the zero curves into six connected subsets. Due to the continuity $f(x, y)$ has a fixed sign in each of these. The sign is e.g. found by insertion of a point from the subset. The function $f(x, y)$ is positive in "every second" subset, where we start with the unbounded part of the first quadrant.

2) Here

$$f(x,y) = xy^2(4x^2 + y^2 - 2y - 3) = xy^2\{(2x)^2 + (y-1)^2 - 2^2\}.$$

The zero set is the union of the axes and the ellipse of centrum $(0,1)$ and half axes 1 and 2. The rest of $\mathbb{R}^2$ is then divided into eight subsets, each of a fixed sign of $f(x,y)$. The subsets where $f(x,y) > 0$ lies either inside the ellipse in the second and the third quadrant, or outside the ellipse in the first and fourth quadrant.

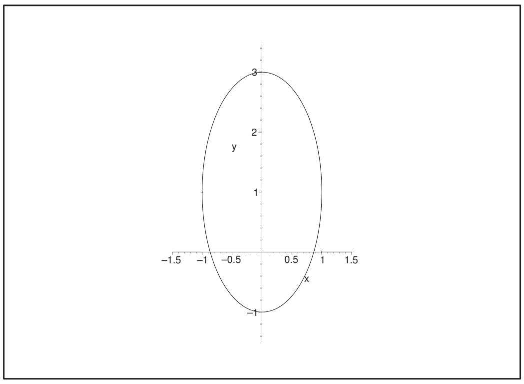


Figure 11: The function is positive inside the ellipse in the second and the third quadrant, and outside the ellipse in the first and the fourth quadrant

3) The zero set is the union of the lines

$$y = -x + 1, \qquad y = x - 1, \qquad y = -4.$$

The rest of $\mathbb{R}^2$ is hereby divided into seven subsets, where two neighbouring subsets always carry opposite signs.

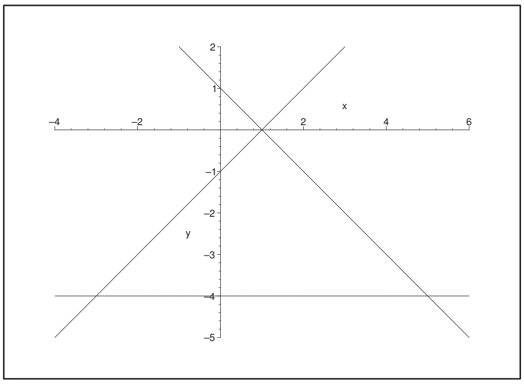


Figure 12: The sign is negative in the triangle and in the angular spaces away from the triangle. In the unbounded subsets neighbouring the edges of the triangle the sign of the function is positive.

4) The zero set is the union of the two parabolas

$$y = x^2 \qquad \text{and} \qquad y = 3x^2.$$

The plane is by these curves divided into four subsets where $f(x,y)$ is positive in the interior subset with respect to both parabolas and in the subset which is exterior to both the parabolas. It is negative in the two subsets which lie between the parabolas.

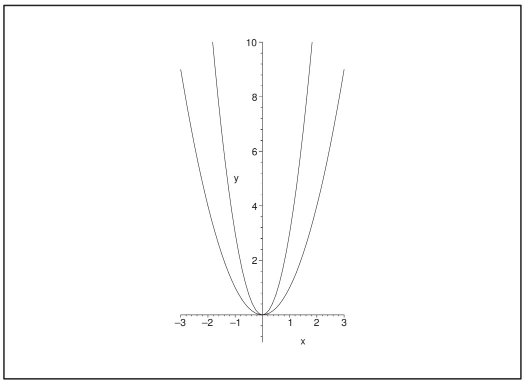


Figure 13: The function $f(x,y)$ is negative between the two parabolas, and positive otherwise.

Remark. The original example also contained the following question:

Example 2.1.5. $f(x,y) = \sin x + \sin y + \sin(x,y)$.
(Rewrite $f(x,y)$ as a product of three factors by using the trigonometric addition formulæ).

This is a really difficult problem. For completeness it is treated here, though it shall no longer be found in any normal curriculum.

One can no longer assume that the following trick is known. Let $x = u+v$ and $y = u-v$, i.e. $u = \dfrac{x+y}{2}$ and $v = \dfrac{x-y}{2}$. Then

$$\sin x = \sin(u+v) = \sin u \cdot \cos v + \cos u \cdot \sin v,$$

$$\sin y = \sin(u-v) = \sin u \cdot \cos v - \cos u \cdot \sin v,$$

hence by an addition,

$$\sin x + \sin y = 2\sin u \cdot \cos v = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

By insertion and transformation into the half of the angle we get

$$\begin{aligned}
f(x,y) &= \sin x + \sin y + \sin(x+y) \\
&= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right) \\
&= 2\sin\left(\frac{x+y}{2}\right)\left\{\cos\left(\frac{x-y}{2}\right) + \cos\left(\frac{x+y}{2}\right)\right\} \\
&= 2\sin\left(\frac{x+y}{2}\right)\left\{\cos\frac{x}{2}\cos\frac{y}{2} + \sin\frac{x}{2}\sin\frac{y}{2} + \cos\frac{x}{2}\cos\frac{y}{2} - \sin\frac{x}{2}\sin\frac{y}{2}\right\} \\
&= 4\sin\left(\frac{x+y}{2}\right)\cdot\cos\frac{x}{2}\cdot\cos\frac{y}{2}.
\end{aligned}$$

It follows that $f(x,y) = 0$, if and only if at least one of the the following possibilities holds,

$$\sin\left(\frac{x+y}{2}\right) = 0 \quad \text{or} \quad \cos\frac{x}{2} = 0 \quad \text{or} \quad \cos\frac{y}{2} = 0,$$

i.e. if and only if

$$\frac{x+y}{2} = p\pi \text{ or } \frac{x}{2} = \frac{\pi}{2} + p\pi \text{ or } \frac{y}{2} = \frac{\pi}{2} + p\pi, \qquad p \in \mathbb{Z}.$$

Hence the zero set is the union of the lines

$$x + y = 2p\pi \text{ eller } x = (2p+1)\pi \text{ eller } y = (2p+1)\pi, \qquad p \in \mathbb{Z}.$$

Due to the periodicity it suffices to investigate the sign in $[-\pi,\pi]\times[-\pi,\pi]$. This square is divided by the line $x+y=0$ from the upper left corner to the lower right corner into two triangles, which then are repeated periodically. The sign of the function is positive in the upper right triangle.

The reader who has come so far should now be capable of sketching the figure, which is left out here. ◊

Example 2.2 *Prove that the function*

$$g(u) = \frac{u^2}{u-1}, \qquad u \in \,]1,+\infty[,$$

satisfies

$$g(u) > g(2) \qquad \textit{for} \qquad u \neq 2.$$

Then find the minimum of the function

$$f(x,y) = \frac{x^2y^2}{(x-1)(y-1)}, \qquad x > 1, \quad y > 1.$$

Finally one shall prove that f does not have any maximum.

A A simple examination of a function.

D Differentiate $g(u) > 0$. Then use that $f(x,y) = g(x)g(y)$.

I We get by a differentiation,

$$g'(u) = \frac{2u(u-1)-u^2}{(u-1)^2} = \frac{u^2-2u}{(u-1)^2}.$$

When $u \in \,]1,+\infty[$, then $g'(u)$ can only be 0 for $u = 2$. Since $g(u) \to +\infty$ for both $u \to 1+$ and $u \to +\infty$, we see that $u = 2$ necessarily corresponds to a global minimum, and we have proved that

$$g(u) > g(2) = \frac{2^2}{2-1} = 4 \qquad \text{for every } u \neq 2 \text{ and } u > 1.$$

Since $g(u) > 0$ for $u > 1$, the product

$$f(x,y) = g(x)g(y)$$

is according to the first item smallest for $(x,y) = (2,2)$.

The minimum is

$$f(2,2) = g(2) \cdot g(2) = 4 \cdot 4 = 16.$$

As mentioned above, $g(u) \to +\infty$ for $u \to 1+$ or for $u \to +\infty$. Accordingly, $f(x,y)$ does not have a maximum.

Example 2.3 *Find the minimum of the function*

$$f(x,y) = \exp|x-y| + \exp|x+y|, \qquad (x,y) \in \mathbb{R}^2.$$

Check whether f has a maximum.

A Minimum and maximum.

D Examine the function exp and also $|x-y|$ and $|x+y|$.

I The exponential is increasing, and $|x-y|$ has its minimum for $x=y$, while $|x+y|$ has its minimum for $x=-y$. Both $|x+y|$ and $|x-y|$ are smallest when $(x,y)=(0,0)$, which corresponds to the fact that $f(x,y)$ has the minimum

$$f(0,0) = e^0 + e^0 = 2.$$

The function does not have a maximum. In fact, there are no stationary points, and if $x^2+y^2 \to +\infty$, then at least one of the two expressions $|x-y|$ or $|x+y|$ will tend towards $+\infty$.

Example 2.4 *Prove that the function*

$$g(u) = u^2 \exp\left(-u^2\right), \qquad u \in [0,+\infty[,$$

has a maximum for $u=1$, such that $g(u) < g(1)$ for $u \neq 1$. Then find the maximum and minimum of the function

$$f(x,y) = x^2 \exp\left(-x^2-y^2\right), \qquad (x,y) \in \mathbb{R}^2.$$

A Maximum and minimum.

D Differentiate $g(u)$. If possible, apply a substitution.
Either write $f(x,y) = g(x)\exp\left(-y^2\right)$, or apply polar coordinates.

I We differentiation we get

$$g'(u) = 2u\exp\left(-u^2\right) - 2u^3\exp\left(-u^2\right) = 2u\left(1-u^2\right)\exp\left(-u^2\right).$$

When $u>0$, this is only zero for $u=1$.

Since $g(u)>0$ for $u>0$, and $g(0)=0$ and $g(u)\to 0$ for $u\to+\infty$, we see that $u=1$ corresponds to the maximum:

$$g(u) = u^2\exp\left(-u^2\right) < g(1) = \frac{1}{e} \qquad \text{for } u \neq 1, \quad u \geq 0.$$

ALTERNATIVELY one applies the substitution $t=u^2$, $t \in [0,+\infty[$. Then

$$g_1(t) = g(u) = u^2\exp\left(-u^2\right) = te^{-t}, \qquad t \in [0,+\infty[,$$

and

$$g_1'(t) = 1\cdot e^{-t} - te^{-t} = (1-t)e^{-t}$$

is zero for $t = 1$, corresponding to $u = +\sqrt{1} = 1$, and then one continues as above. $\Diamond$

It follows from the rearrangement

$$g(x,y) = x^2 \exp\left(-x^2 - y^2\right) = g(x) \cdot \exp\left(-y^2\right),$$

where the variables are separated that $f(x,y)$ has a maximum for $x = 1$ [this follows from the first item] and for $y = 0$, i.e. the maximum is

$$f(1,0) = f(-1,0) = \frac{1}{e} \cdot 1 = \frac{1}{e}.$$

Furthermore, $f(x,y) > 0$ for $x \neq 0$, hence the minimum is

$$f(0,y) = 0 \qquad \text{for every } y \in \mathbb{R}.$$

ALTERNATIVELY we get by polar coordinates,

$$f(x,y) = x^2 \exp\left(-x^2-y^2\right) = \varphi^2 \cos^2\varphi \cdot \exp\left(-\varrho^2\right) = g\left(\varrho^2\right)\cos^2\varphi,$$

where the variables are separated. It follows that $f(x,y) \geq 0$, and that $f(x,y) = 0$ for either $\varrho = 0$ or for $\varphi = \pm\frac{\pi}{2}$, corresponding to a minimum for

$$f(0,y) = 0, \qquad \text{for every } y \in \mathbb{R}.$$

By maximizing each factor it follows from the first item that the maximum is obtained for $\varrho = 1$ and $\varphi = 0$, π, corresponding to

$$f(1,0) = f(-1,0) = \frac{1}{e}.$$

Example 2.5 *Find the maximum of the function*

$$f(x,y,z) = \sin(x+y-z) - (x+y+z)^2, \qquad (x,y,z) \in \mathbb{R}^3,$$

and find a set in which we get the maximum. Does f have a minimum?

A Maximum of a function in three variables.

D Estimate each of the two terms on the right hand side.

I Obviously, $-(x+y+z)^2$ is largest (and $= 0$), when

$$x+y+z = 0.$$

Furthermore, $\sin(x+y-z)$ is largest (and $= 1$), when

$$x+y-z = \frac{\pi}{2} - 2p\pi, \qquad p \in \mathbb{Z}.$$

It follows by subtraction of these two equations that

$$2z = -\frac{\pi}{2} + 2p\pi, \qquad p \in \mathbb{Z},$$

i.e.

$$z = -\frac{\pi}{4} + p\pi, \qquad p \in \mathbb{Z}.$$

This corresponds to

$$x+y = -z = \frac{\pi}{4} - p\pi, \qquad p \in \mathbb{Z},$$

i.e.

$$y = \frac{\pi}{4} - p\pi - x, \qquad p \in \mathbb{Z}.$$

Summarizing we get that the maximum

$$f(x,y,z) = 1+0 = 1$$

is obtained everywhere in the set

$$\left\{(x,y,z) \;\middle|\; x=t,\, y=\frac{\pi}{4}-p\pi-t,\, z=-\frac{\pi}{4}+p\pi,\, t\in\mathbb{R},\, p\in\mathbb{Z}\right\}.$$

Since $-(x+y+z)^2 \to -\infty$ for $|x+y+z| \to +\infty$, we conclude that $f(x,y,z)$ does not have a minimum.

Example 2.6 *Find in each of the following cases the maximum and the minimum of the function by first estimating the maximum and the minimum with respect to x for any fixed y.*

1) $f(x,y)=\sqrt{x^2+2y^2}-2y^2, \qquad x^2+4y^2\le 3.$

2) $f(x,y)=\sqrt{x^2+16y^2}-y^4, \qquad x^2+36y^2\le 81.$

3) $f(x,y)=2x^2+y^2-2y, \qquad x^2+y^2\le 2.$

4) $f(x,y)=\exp\left(-y^2\right)\cdot\cos x, \qquad (x,y=\in\mathbb{R}.$

A Maximum and minimum.

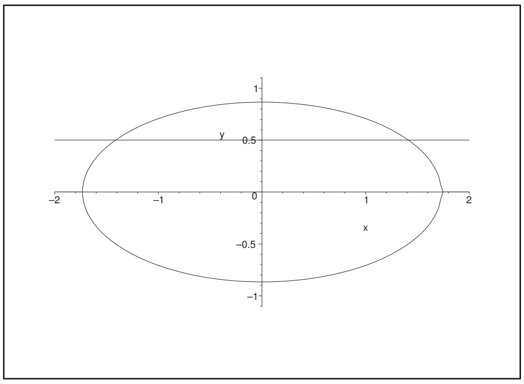
D Sketch the domain of the function. Keep y fixed and estimate with respect to x

Finally, estimate with respect to the variable y.

I 1) Here $x^2+4y^2\le 3$ describes the ellipsoidal disc in its canonic form

$$\left(\frac{x}{\sqrt{3}}\right)^2+\left(\frac{y}{\frac{\sqrt{3}}{2}}\right)^2\le 1$$

of centrum $(0,0)$ and half axes $\sqrt{3}$ and $\frac{\sqrt{3}}{2}$. When $y\in\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$ is kept fixed, then x goes



through the interval $\left[-\sqrt{3-^4 y^2},\sqrt{3-4y^2}\right]$. Obviously,

$$f^\star(y)=\max_x f(x,y)=\sqrt{3-4y^2+2y^2}-2y^2=\sqrt{3-2y^2}-2y^2$$

and

$$f_\star(y) = \min_x f(x,y) = \sqrt{2y^2} - 2y^2 = \sqrt{2}\cdot|y| - 2y^2.$$

Hence

$$\max_{x,y} f(x,y) = \max_y f^\star(y) = \left[\sqrt{2-2y^2} - 2y^2\right]_{y=0} = \sqrt{3},$$

which is obtained for $(x,y) = (\pm\sqrt{3}, 0)$, and for

$$\begin{aligned}\min_{x,y} f(x,y) &= \min_y f_\star(y) = \min_{|y|\le\sqrt{3/2}}\left\{\sqrt{2}\cdot|y| - 2y^2\right\}\\ &= \min_{0\le|y|\sqrt{3}/2}\left\{(\sqrt{2}\cdot|y|) - (\sqrt{2}\cdot|y|)^2\right\} = \min_{0\le t\le\sqrt{6}/2}\left\{t-t^2\right\}\\ &= \min\left\{0-0^2, \frac{1}{2}-\frac{1}{4}, \frac{\sqrt{6}}{2}-\frac{6}{4}\right\} = \min\left\{0, \frac{1}{4}, \frac{\sqrt{6}-3}{2}\right\}\\ &= \frac{\sqrt{6}-3}{2} < 0,\end{aligned}$$

and accordingly, $(x,y) = \left(0, \pm\sqrt{\dfrac{3}{2}}\right)$.

2) We shall here use the same method and notation as above. The domain is the ellipse described by

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{\frac{3}{2}}\right)^2 \le 1,$$

of centrum $(0,0)$ and the half axes 9 and $\dfrac{3}{2}$.

When $|y| \le \dfrac{3}{2}$ is kept fixed, then x satisfies

$$0 \le x^2 \le 81 - 36y^2.$$

Then

$$f^\star(y) = \max_x f(x,y) = \sqrt{81-36y^2+16y^2} - y^4 = \sqrt{81-20y^2} - y^4,$$

and therefore,

$$\max_{x,y} f(x,y) = \max f^\star(y) = \left[\sqrt{81-20y^2} - y^4\right]_{y=0} = \sqrt{81} = 9,$$

which is obtained for $(x,y) = (\pm 9, 0)$.

Analogously,

$$f_\star(y) = \sqrt{16y^2} - y^4 = 4y^2 = 4 - (y^2-2)^2, \quad 0 \le y^2 \le \frac{9}{4}.$$

It follows that $(t-2)^2$ for $t \in \left[0, \dfrac{9}{4}\right]$ is largest when $t = y^2 = 0$, hence

$$\min_{x,y} f(x,y) = \min_y f_\star(y) = \left[4-(y^2-2)^2\right]_{y=0} = 0,$$

which is obtained for $(x,y) = (0,0)$.

3) The set is the disc of centrum $(0,0)$ and radius $\sqrt{2}$. We get the maximum for

$$f(x,y) = 2x^2 + y^2 - 2y,$$

in x for fixed $y \in [-\sqrt{2}, \sqrt{2}]$, when $x = \pm\sqrt{2-y^2}$, corresponding to

$$f^\star(y) = \max_x f(x,y) = 2(2-y^2) + y^2 - 2y = 4 - y^2 - 2y = 5 - (y+1)^2.$$

The maximum of $f^\star(y)$ for $y \in [-\sqrt{2}, \sqrt{2}]$ is obtained for $y = -1$, corresponding to $x = \pm 1$, and

$$\max_{(x,y)} f(x,y) = \max_y f^\star(y) = 5$$

at the points $(\pm 1, -1)$.

When $y \in [-\sqrt{2}, \sqrt{2}]$ is fixed we get the minimum for $x = 0$, which corresponds to

$$f_\star(y) = \min_x f(x,y) = y^2 - 2y = (y-1)^2 - 1.$$

The minimum for $f_\star(y)$ is obtained for $y=1$, which corresponds to

$$\min_{(x,y)} f(x,y) = -1,$$

which is attained at the point $(0,1)$.

4) For fixed $y \in \mathbb{R}$ we get

$$f^\star(y) = \max_x \exp\left(-y^2\right)\cos x = \exp\left(-y^2\right) \quad \text{for } x = 2p\pi,\ p \in \mathbb{Z},$$

corresponding to

$$\max_{(x,y)} f(x,y) = \max_y f^\star(y) = 1,$$

and it is attained at the points $(2p\pi, 0)$, $p \in \mathbb{Z}$.

Furthermore,

$$f_\star(y) = \min_x \exp\left(-y^2\right)\cos x = -\exp\left(-y^2\right), \quad \text{for } x = \pi + 2p\pi,\ p \in \mathbb{Z},$$

corresponding to

$$\min_{(x,y)} f(x,y) = \min_y f_\star(y) = -1,$$

which is attained at the points $(\pi + 2p\pi, 0)$, $p \in \mathbb{Z}$.

Example 2.7 *Use the trigonometric additional formulæ to rewrite the function*

$$f(x,y) = \sin x + \sin y + \sin(x+y)$$

as a product of three factors, one of which is $\sin \dfrac{x+y}{2}$. *Then draw a sketch like in Example 2.1.*

Remark. Here we get the explanation why the previous **Example 2.1.5** has been removed from later editions of the textbook, in which I found this example in a short form. $\Diamond$

A The function is defined and continuous in $\mathbb{R}^2$. Transform into the half angles.

D Factorize the function by using the mentioned trick. Find the zero set and then analyze.

I The first step in this procedure is a technique, which no longer is well known, so it shall be painstakingly described. If we put $x = u+v$ and $y = u-v$, then $u = \dfrac{x+y}{2}$ and $v = \dfrac{x-y}{2}$, so

$$\sin x = \sin(u+v) = \sin u \cdot \cos v + \cos u \cdot \sin v,$$

$$\sin y = \sin(u-v) = \sin u \cdot \cos v - \cos u \cdot \sin v,$$

and hence by an addition

$$\sin x + \sin y = 2\sin u \cdot \cos v = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

Then we get by an insertion,

$$\begin{aligned}
f(x,y) &= \sin x+\sin y+\sin(x+y)\\
&= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)\\
&= 2\sin\left(\frac{x+y}{2}\right)\left\{\cos\left(\frac{x-y}{2}\right)+\cos\left(\frac{x+y}{2}\right)\right\}\\
&= 2\sin\left(\frac{x+y}{2}\right)\left\{\cos\frac{x}{2}\cos\frac{y}{2}+\sin\frac{x}{2}\sin\frac{y}{2}+\cos\frac{x}{2}\cos\frac{y}{2}-\sin\frac{x}{2}\sin\frac{y}{2}\right\}\\
&= 4\sin\left(\frac{x+y}{2}\right)\cos\frac{x}{2}\cos\frac{y}{2}.
\end{aligned}$$

It follows that $f(x,y)=0$, if (and only if)

$$\sin\left(\frac{x+y}{2}\right)=0 \quad\text{or}\quad \cos\frac{x}{2}=0 \quad\text{or}\quad \cos\frac{y}{2}=0,$$

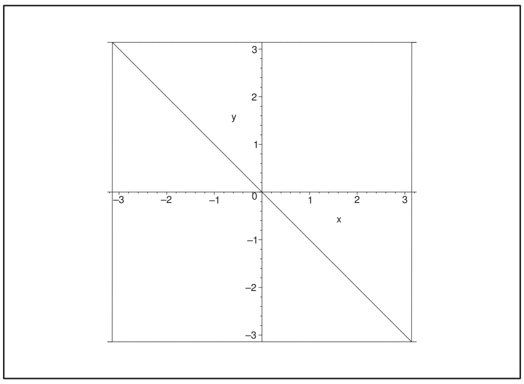
i.e. if and only if

$$\frac{x+y}{2}=p\pi \quad\text{or}\quad \frac{x}{2}=\frac{\pi}{2}+p\pi \quad\text{or}\frac{y}{2}=\frac{\pi}{2}+p\pi, \qquad p\in\mathbb{Z}.$$

The zero set is characterized by the possibilities

$$x+y=2p\pi \quad\text{or}\quad x=(2p+1)\pi \quad\text{or}\quad y=(2p+1)\pi, \qquad p\in\mathbb{Z}.$$

It suffices due to the periodicity to examine the sign in $[-\pi,\pi]\times[-\pi,\pi]$. The function is positive



in the upper triangle in this square. Then this figure is continued periodically of period 2π in both the direction of the X axis and the Y axis.

Example 2.8 *1) Sketch the domain A of the function f, given by*

$$f(x,y)=\ln\left(y-\sqrt{1+x^2}\right)-\frac{y}{2}-\sqrt{1+x^2},$$

and find possible stationary points of f.

2) Find for each fixed $a\in\,]1,+\infty[$ the range of the function

$$G_a(x)=f(x,a),\qquad x\in I_a,$$

where $I_a=\{x\in\mathbb{R}\mid (x,a)\in A\}$.

3) Finally, find the range $f(A)$ of the function f, and its possible extrema.

A Domain, stationary points, range.

D The text gives some good guidelines which should be followed.

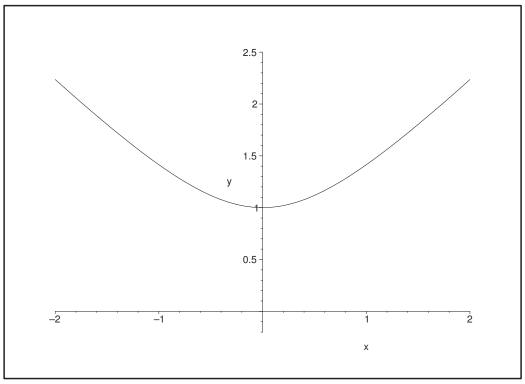


Figure 14: The domain D lies above the branch of the hyperbola.

I 1) The function is defined for $t>\sqrt{1+x^2}>1$, i.e. for

$$y^2-x^2>1\qquad\text{and}\qquad y>1.$$

Since

$$(1)\quad \frac{\partial f}{\partial x}=\frac{1}{y-\sqrt{1+x^2}}\cdot\left(-\frac{x}{\sqrt{1+x^2}}\right)-\frac{x}{\sqrt{1+x^2}}=-\frac{x}{\sqrt{1+x^2}}\left\{1+\frac{1}{y-\sqrt{1+x^2}}\right\},$$

where the latter factor of (1) is >1, we conclude that possible stationary points must satisfy $x=0$.

Since

$$(2)\quad \frac{\partial f}{\partial y}(0,y)=\frac{1}{y-1}-\frac{1}{2},$$

is zero for $y-1=2$, i.e. for $y=3$, we conclude that $(0,3)$ is the only stationary point. The value of the function is here

$$f(0,3)=\ln(3-1)-\frac{3}{2}-1=\ln 2-\frac{5}{2}.$$

2) Let $a \in]1,+\infty[$. Then I_a is described by $x^2 < a^2 - 1$, i.e.

$$I_a = \left] -\sqrt{a^2-1}\,,\ \sqrt{a^2-1}\right[.$$

Putting $y = a$ into (1), we get

$$G_a'(x) = -\frac{x}{\sqrt{1+x^2}}\left\{1+\frac{1}{a-\sqrt{1+x^2}}\right\},$$

thus G_a is increasing in $]-\sqrt{a^2-1},0[$ and decreasing in $]0,\sqrt{a^2-1}[$. Since $G_a(x) \to -\infty$ for $x \to \pm\sqrt{a^2-1}$, the range of G_a is given by

$$]-\infty, G_a(0)] = \left]-\infty, \ln(a-1) - \frac{a}{2} - 1\right].$$

3) It follows from (2) when $y = a$ that

$$\frac{dG_a(0)}{da} = \frac{1}{a-1} - \frac{1}{2},$$

hence $G_a(0)$ is increasing for $a \in]1,3[$ and decreasing for $a \in]3,+\infty[$. Since $G_a(0) \to -\infty$ for $a \to 1+$ and for $a \to +\infty$, we see that

$$G_3(0) = f(0,3) = \ln 2 - \frac{5}{2}$$

is a global maximum.

The range is

$$f(A) = \left]-\infty, \ln 2 - \frac{5}{2}\right].$$

Example 2.9 *Let*

$$w(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}, \qquad (x,t) \in A.$$

1) Sketch the domain A

2) Prove that w fulfils the differential equation

$$(x-t)\,\frac{\partial w}{\partial x} = t\,\frac{\partial w}{\partial t}.$$

A Domain, and a check of a differential equation.

D Analyze the condition $1 - 2xt + t^2 > 0$. Then insert w into the differential equation.

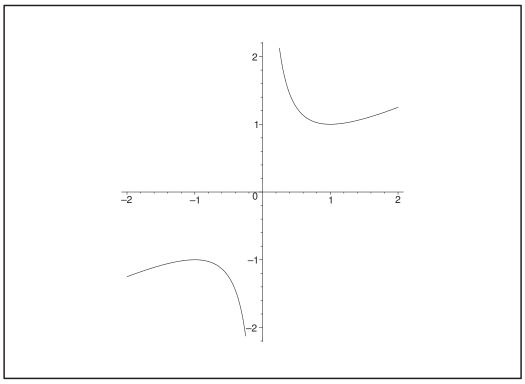


Figure 15: The domain A lies between the two branches of hyperbola $x = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right)$.

I The domain is described by $1 - 2xt + t^2 > 0$. Since $(0,0)$ in particular satisfies this condition, it follows from the continuity that A is that component of the plane, defined by the branches of the hyperbola, which also contains $(0,0)$, i.e. A lies between the two branches. Obviously, $w \in C^\infty(D)$.

When $(x,y) \in A$, then $w(x,y) \neq 0$. A calculation gives

$$\frac{\partial w}{\partial x} = -\frac{1}{2} \cdot \frac{1}{w^3} \cdot (-2t) = \frac{t}{w^3}$$

and

$$\frac{\partial w}{\partial t} = -\frac{1}{2} \cdot \frac{1}{w^3} \cdot (-2x+2t) = \frac{x-t}{w^3},$$

hence

$$(x-t)\,\frac{\partial w}{\partial x} = \frac{(x-t)t}{w^3} = t\,\frac{\partial w}{\partial t},$$

and we have proved that the differential equation is fulfilled.

Example 2.10 *Let*

$$w(x,t) = \frac{1}{1-t}\,\exp\left(-\frac{xt}{1-t}\right), \qquad t \neq 1.$$

Prove that w satisfies the differential equation

$$t(1-t)\,\frac{\partial w}{\partial t} = (x+t-1)\,\frac{\partial w}{\partial x}.$$

A Check of a differential equation.

D When we differentiate we exploit the structure of w.

I Clearly, $w(x,t)$ is defined and C^∞ for $t \neq 1$.

Then notice that if $t \neq 1$, then

$$w(x,t) = \frac{1}{1-t}\,\exp\left(-\frac{xt}{1-t}\right) = \frac{1}{1-t}\,\exp\left(x - \frac{x}{1-t}\right),$$

where we shall apply the latter expression when we differentiate with respect to t.

We get by differentiation

$$\frac{\partial w}{\partial x} = w(x,t) \cdot \left(-\frac{t}{1-t}\right) = -\frac{t}{1-t}\,w(x,t),$$

and

$$\frac{\partial w}{\partial t} = \frac{1}{1-t}\,w(x,t) + w(x,t) \cdot \left(-\frac{x}{(1-t)^2}\right) = \frac{1-t-x}{(1-t)^2}\,w(x,t).$$

Hence

$$t(1-t)\,\frac{\partial w}{\partial t} = \frac{t}{1-t}\,(1-t-x)\,w(x,t)$$

and

$$(x+t-1)\,\frac{\partial w}{\partial x} = -\frac{t}{1-t}\,(x+t-1)\,w(x,t) = t(1-t)\,\frac{\partial w}{\partial t},$$

and we have proved that the differential equation is fulfilled for $t \neq 1$.

Example 2.11 *Sketch the domain A of the function*

$$f(x,y) = \frac{\ln(y-x^2)}{y-4},$$

and sketch, or describe the interior A°, the closure $\overline{A}$ and the boundary ∂A of A.

A Domain; interior, closure and boundary of this set.

D Analyze the function $f(x,y)$.

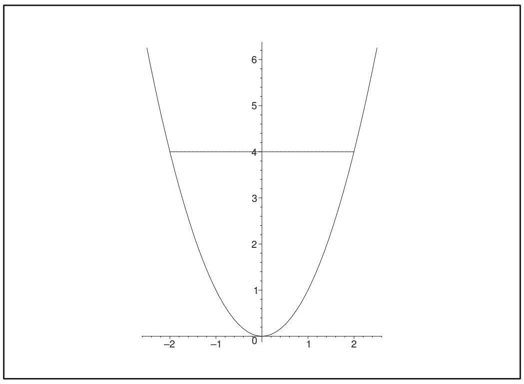


Figure 16: Sketch of the open set A; the boundary is the the union of the parabola and a piece of the straight line at height $y = 4$. Notice that A is the union of two components.

I The function f is defined for $y > x^2$ and $y \neq 4$, i.e. $A = A^\circ$ is given by

$$\begin{aligned} A = A^\circ &= \{(x,y) \mid x^2 < y < 4,\ -2 < x < 1\} \cup \{(x,y) \mid y > \max\{4, x^2\},\ x \in \mathbb{R}\} \\ &= \{(x,y) \mid y \in \mathbb{R}_+ \setminus \{4\},\ x \in \mathbb{R},\ y > x^2\}. \end{aligned}$$

The closure is

$$\overline{A} = \left\{(x,y) \mid y \geq x^2\right\},$$

and the boundary is

$$\partial A = \left\{(x,y) \mid y = x^2\right\} \cup \left\{(x,4) \mid -2 < x < 2\right\}.$$

Example 2.12 *Sketch the domain A of the function*

$$f(x,y)=\sqrt{\frac{4-x^2-y^2}{x+y-2}},$$

and sketch or describe the interior A° and the boundary ∂A.

A Domain.

D Find the zero sets of the numerator and the denominator in the fraction and then examine the signs of these.

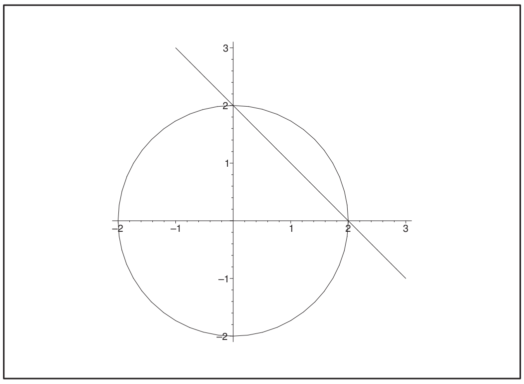


Figure 17: The zero sets of the numerator and the denominator. The domain A is the union of the circular section in the first quadrant and the complementary sets of the disc in the second, third and fourth quadrant, supplied with the circle itself except for the points on the line $x + y = 2$.

I Clearly, $4 - x^2 - y^2$ is zero on the boundary of the circle of centrum $(0, 0)$ and radius 2. The numerator is positive on the open disc and negative outside the closed disc.

The denominator is zero on the line $y = 2 - x$, where the fraction is not defined. Furthermore, the denominator is negative in the half plane which contains the point $(0, 0)$, and positive in the other open half plane.

The domain A is the point set, in which the fraction is defined and ≥ 0, i.e. the union of the circular section in the first quadrant, and the sets outside the disc in the second, the third and the fourth quadrant, included the points on the circle $x^2 + y^2 = 2^2$, and excluded in all cases mentioned above on the line $y = 2 - x$.

We obtain the interior A° by removing the circle $x^2 + y^2 = 2^2$ from A.

The boundary ∂A is the union of the circle $\{(x, y) \mid x^2 + y^2 0 = 2^2\}$ and the line $\{(x, y) \mid x + y = 2\}$.

Example 2.13 *Sketch the domain A of the function*

$$f(x, y) = \sqrt{y} \cdot \ln(1 + x^2 - y).$$

Describe the interior, the exterior and the boundary of A. Explain why A is connected.

A Domain.

D Examine each factor.

I The function $f(x, y)$ is defined, when both $\sqrt{y}$ and $\ln(1 + x^2 - y)$ are defined, i.e. when $y \geq 0$ and $1 + x^2 - y > 0$, or put in another way,

$$0 \leq y > 1 + x^2.$$

It follows that A is the set which lies between the X axis and the parabola, inclusive the X axis and exclusive the parabola.

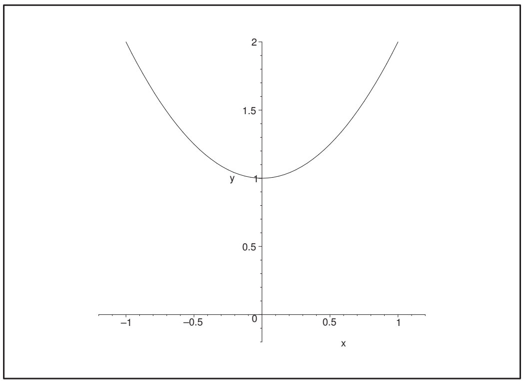


Figure 18: The domain is the point set between the X axis and the parabola, included the X axis and excluded the parabola.

The boundary is the union of the X axis and the parabola $y = 1 + x^2$.

The interior of A is the set which lies between the X axis and the parabola, exclusive both curves.

The exterior is the union of the open lower half plane and the open set inside the parabola.

The set A arc wise connected, because any two points (x_1, y_1), $(x_2, y_2) \in A$ can be joined by a step curve ℓ, which lies entirely in A, where ℓ is composed of the straight lines between (x_1, y_1), $(x_1, 0)$, $(x_2, 0)$ and (x_2, y_2).

Example 2.14 *Sketch the domain A of the function*

$$f(x, y) = \ln\left(x^2 + 2y^2 - 1\right) + \sqrt{4 - x^2 - y^2}.$$

Check whether the point set A is star shaped.

A Domain; star shaped set.

D Analyze the figure.

I The function is defined for

$$x^2 + 2y^2 > 1 \qquad \text{and} \qquad x^2 + y^2 \le 2^2,$$

i.e. outside the ellipse of centrum $(0, 0)$ and half axes 1, $\dfrac{\sqrt{2}}{2}$, and inside or on the circle of centrum $(0, 0)$ and radius 2.

No matter which point we choose in A, we see that the inner ellipse "shadows" for some points of A seen from this point, and A is not star shaped.

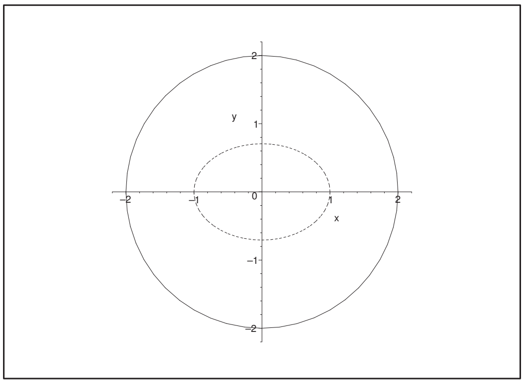


Figure 19: the domain A lies between the ellipse and the circle; the circle is included in A, while the ellipse does not lie in A.

Example 2.15 *Let* $B = \{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1,\, -x < y \leq 2x\}$.

1) Sketch B *and its closure* $\overline{B}$.

2) Find the range of the function

$$f(x,y) = (x+y)e^{-xy}, \qquad (x,y) \in \overline{B}.$$

3) Find the range of the restriction of the function f *to the point set* B.

A Ranges.

D Just follow the text and apply the standard methods.

I 1)

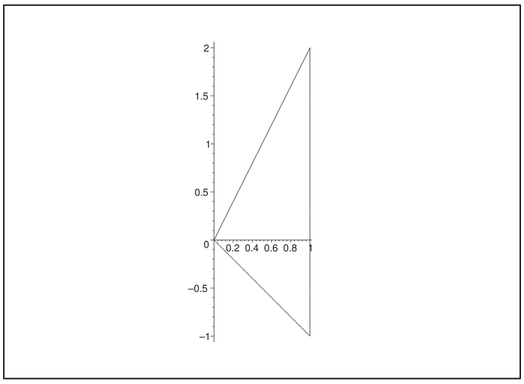


Figure 20: The set B.

2) The function $f(x,y) = (x+y)e^{-xy}$ is continuous on the closed and bounded set $\overline{B}$. By the second main theorem of continuous functions f has a maximum and a minimum on $\overline{B}$. Since $f \in C^\infty\left(\mathbb{R}^2\right)$, these values are only obtained at stationary points in the interior B° or at points on the boundary ∂B.

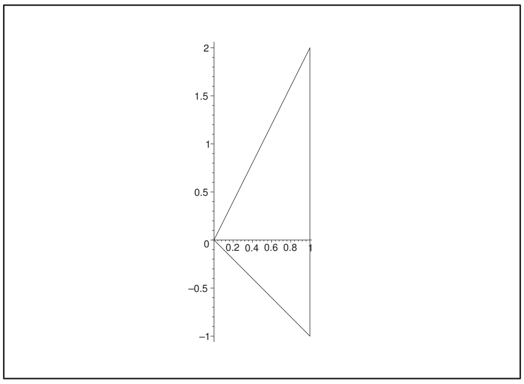


Figure 21: The closed set $\overline{B}$.

Stationary points.
These are given by the solutions of

$$\frac{\partial f}{\partial x} = e^{-xy} - y(x+y)e^{-xy} = \{1 - y(x+y)\}e^{-xy} = 0,$$

$$\frac{\partial f}{\partial y} = e^{-xy} - x(x+y)e^{-xy} = \{1 - x(x+y)\}e^{-xy} = 0.$$

Since $e^{-xy} \neq 0$, these equations are reduced to

$$x(x+y) = y(x+y) = 1.$$

Here, $x + y \neq 0$, so $y = x$, and the equations are further reduced to

$$x \cdot 2x = 2x^2 = 1 \quad \text{and} \quad y = x.$$

The stationary points are

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \qquad \text{og} \qquad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Of these, only $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ lies in the interior B°. The value of the function is here

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \cdot e^{-\frac{1}{2}} = \sqrt{\frac{2}{e}}.$$

Examination of the boundary.

a) For $y = 2x$, $x \in [0, 1]$, we get the restriction

$$\varphi_1(x) = f(x, 2x) = 3x\,\exp(-2x^2)$$

where

$$\varphi_1'(x) = 3\left\{1 - 4x^2\right\}\exp(-2x^2).$$

In the interval $[0,1]$ we get $\varphi_1'(x) = 0$ for $x = \dfrac{1}{2}$, corresponding to

$$\varphi_1\left(\frac{1}{2}\right) = f\left(\frac{1}{2},1\right) = \frac{3}{2}\exp\left(-\frac{1}{2}\right) = \frac{3}{2\sqrt{e}}.$$

At the end points of the interval we get

$$\varphi_1(0))f(0,0) = 0 \quad \text{and} \quad \varphi_1(1) = f(1,2) = 3\exp(-2).$$

b) For $y = -x$ we find the restriction $\varphi_2(x) = f(x,-x) = 0$ for every $x \in [0,1]$.

c) For $x = 1$ and $y \in [-1,2]$ the restriction is

$$\varphi_3(y) = f(1,y) = (1+y)e^{-y}$$

where

$$\varphi_3'(y) = \{1-(1+y)\}e^{-y} = -y\,e^{-y},$$

which is 0 for $y = 0$. This corresponds to

$$\varphi_3(0) = f(1,0) = 1 \cdot e^{-0} = 1.$$

For completeness, the function values at the end points are

$$\varphi_3(-1) = f(1,-1) = 0 \quad \text{and} \quad \varphi_3(2) = f(1,2) = 3 \cdot e^{-2}.$$

Notice, however, that we have already found these values above.

MAXIMUM AND MINIMUM IN $\overline{B}$.

We find these by a simple numerical comparison between the values

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{\frac{2}{e}},$$

$$f\left(\frac{1}{2}, 1\right) = \frac{3}{2\sqrt{e}}, \quad f(0,0) = 0, \quad f(1,2) = \frac{3}{e^2},$$

$$f(x,-x) = 0 \qquad \text{for every } x \in [0,1],$$

$$f(1,0) = 1, \quad f(1,-1) = 0, \quad f(1,2) = \frac{3}{e^2}.$$

Clearly, the minimum in $\overline{B}$ is 0, and we get this value on the line $y = -x$, $x \in [0,1]$.

Since

$$\sqrt{\frac{2}{e}} < 1, \quad \frac{3}{2\sqrt{e}} = \frac{3}{\sqrt{4e}} < \frac{3}{\sqrt{10}} < 1, \quad \text{and} \quad \frac{3}{e^2} < 1,$$

the maximum in $\overline{B}$ is given by

$$f(1,0) = 1.$$

Then apply the first main theorem of continuous function, i.e. connected sets are mapped into connected sets by continuous maps, (and both B and $\overline{B}$ are connected). It then follows that the range on $\overline{B}$ is

$$f(\overline{B}) = [M,S] = [0,1].$$

3) *The range of f on B.*
Since B is obtained from $\overline{B}$ by removing the boundary $y = -x$, on which f takes on the value 0, it follows by the continuity of f that

$$]0,1] \subseteq f(B) \subseteq [0,1].$$

Since f is not 0 at any point of B, we finally conclude that

$$f(B) =]0,1].$$

3 Level curves and level surfaces

Example 3.1 *Let*

$$f(x,y) = \ln(2-2x^2-3y^2)+2-4x^2-6y^2, \qquad (x,y)\in A.$$

1) Sketch the domain A.

2) Describe the level curves of the function. It is convenient to introduce a new variable u, such that $f(x,y)=F(u(x,y))$.

3) Sketch the level curve corresponding to $f(x,y)=0$.

4) Find the range $f(A)$.

A Domain and level curves.

D Describe the set given by $2-2x^2-3y^2>0$, where $f(x,y)$ is defined. Then change the parameter to u.

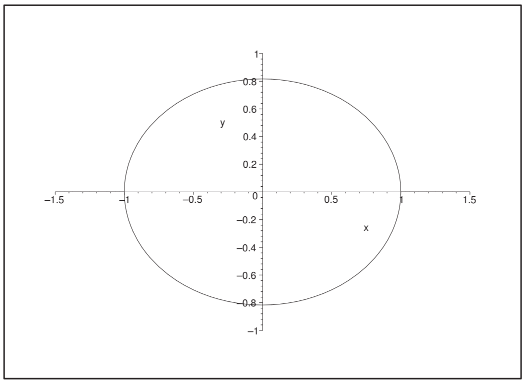
I 1) The function is defined, if and only if

$$u = u(x,y) = 2-2x^2-3y^2 > 0,$$

i.e. for

$$\left(\frac{x}{1}\right)^2 + \left(\frac{y}{\sqrt{\frac{2}{3}}}\right)^2 < 1,$$

which describes an open ellipsoidal disc of centrum $(0,0)$ and half axes 1 and $\sqrt{\dfrac{2}{3}}$.



2) If we define

$$u = u(x,y) = 2-2x^2-3y^2 > 0,$$

i.e. $u \in]0,2]$, then

$$\begin{aligned} f(x,y) &= \ln(2-2x^2-3y^2)+2-4x^2-6y^2 \\ &= \ln(2-2x^2-3y^2)+2(2-2x^2-3y^2)-2 \\ &= \ln u + 2u - u. \end{aligned}$$

This is clearly an increasing function in $u \in]0,2]$. Every level curve

$$f(x,y) = \ln u + 2u - 2 = c$$

corresponds to

$$u = 2 - 2x^2 - 3y^2 = k \in]0,2],$$

where k is unique according to the above.

Then by a rearrangement,

$$2x^2 + 3y^2 = 2 - k, \qquad k \in]0,2].$$

If $k = 2$, then the level "curve" degenerates to the point $(0,0)$.

If $0 < k < 2$, then the level curve is an ellipse

$$\left(\frac{x}{\sqrt{\frac{2-k}{2}}}\right)^2 + \left(\frac{y}{\sqrt{\frac{2-k}{3}}}\right)^2 = 1$$

with the half axes $\sqrt{\dfrac{2-k}{2}}$ and $\sqrt{\dfrac{2-k}{3}}$.

3) When

$$f(x,y) = \ln u + 2u - 2 = 0,$$

it follows that $u = 1$ is a solution. Since the function of u is strictly increasing, it follows that $u = 1$ is the only solution, so $k = 1$.

According to 2) the level curve $f(x,y) = 0$ is the ellipse

$$\left(\frac{x}{\sqrt{\frac{1}{2}}}\right)^2 + \left(\frac{y}{\sqrt{\frac{1}{3}}}\right)^2 = 1$$

af centrum $(0,0)$ and half axes $\sqrt{\dfrac{1}{2}}$ and $\sqrt{\dfrac{1}{3}}$.

4) We obtain the range by changing the variable to u,

$$f(x,y) = F(u) = \ln u + 2u - 2, \qquad u \in]0,2],$$

because the value u is attained precisely on one level curve.

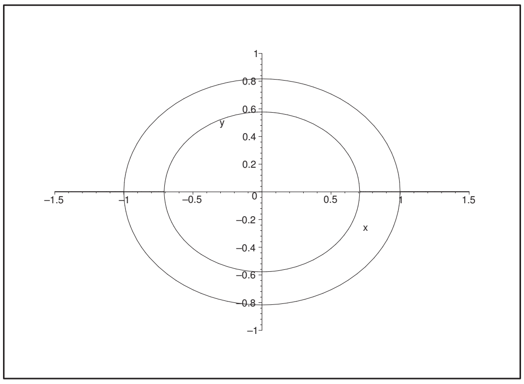
Since $F'(u) = \dfrac{1}{u} + 2$, we see that $F(u)$ is increasing.

When $n \to 0+$, we get $F(u) \to -\infty$. When $u = 2$, we get

$$F(u) = \ln 2 + 4 - 2 = 2 + \ln 2.$$

Since $F(u)$ is continuous, the connected interval $]0,2]$ is mapped into the connected interval $]-\infty, 2 + \ln 2]$. Here we apply the third main theorem of continuous functions.

The range is $f(A) =]-\infty, 2 + \ln 2]$.



Example 3.2 *Sketch for each for the functions* $f : \mathbb{R}^2 \to \mathbb{R}$ *below the level curves given by* $f(x,y) = C$ *for the given values of the constant* C.

1) $f(x,y) = x^2 + y^2, \qquad C \in \{1,2,3,4,5\},$

2) $f(x,y) = x^2 - 4x + y^2, \qquad C \in \{-3,-2,-1,0,1\},$

3) $f(x,y) = x^2 - 2y, \qquad C \in \{-2,-1,0,1,2\},$

4) $f(x,y) = \max\{|x|,|y|\}, \qquad C \in \{1,2,3\},$

5) $f(x,y) = |x| + |y|, \qquad C \in \{1,2,3\},$

6) $f(x,y) = (x^2 + y^2 + 1)^2 - 4x^2, \qquad C \in \{\frac{1}{2},1,3\},$

7) $f(x,y) = x^2 + y^2(1+x)^3, \qquad C \in \{-4,0,\frac{1}{4},1,4\}.$

A Level curves.

D Whenever it is necessary, start by analyzing the given function.

I 1) The level curves are circles of centrum $(0,0)$ and radii $\sqrt{C}$, i.e. 1, $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$.

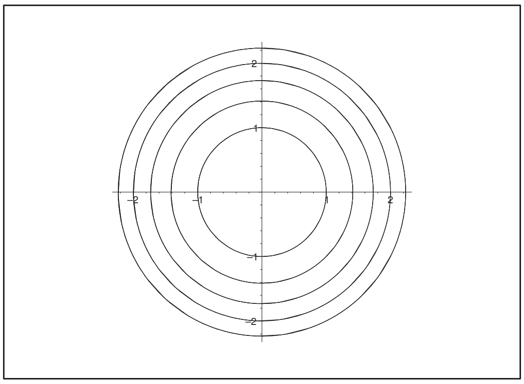


Figure 22: The level curves $x^2 + y^2 = C$, $C = 1$, $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$.

2) Since

$$f(x,y) = x^2 - 4x + y^2 = (x-2)^2 + y^2 - y,$$

we can also write the equation $f(x,y) = C$ of the level curves in the form

$$(x-2)^2 + y^2 = 4 + C.$$

The level curves are circles of centrum $(2,0)$ and radius $\sqrt{4+C}$, i.e. 1, $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$.

It follows that we obtain the same system as in 1), only translated to the centre $(2, 0)$.

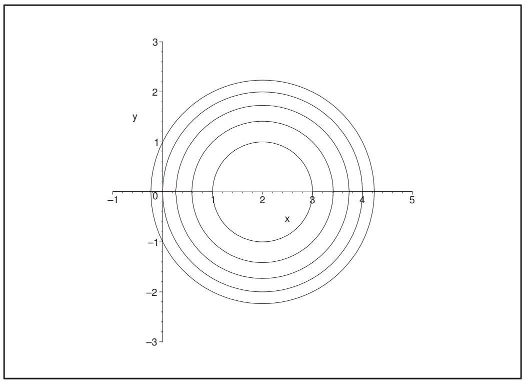


Figure 23: The level curves $x^2 - 4x + y^2 = C$, $C = -3,\ -2,\ -1,\ 0,\ 1$.

3) The equation of the level curves $f(x, y) = C$ can also be written

$$y = \frac{1}{2}\,x^2 - \frac{C}{2}, \qquad C \in \{-2, -1, 0, 1, 2\}.$$

These are parabolas of top points at $\left(0, -\dfrac{C}{2}\right)$.

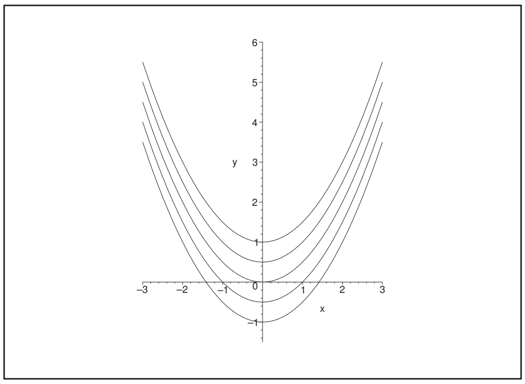


Figure 24: The level curves $x^2 - 2y = C$, $C = -2,\ -1,\ 0,\ 1,\ 2$.

4) The level curves are the boundary of the the squares of centrum $(0, 0)$ and edge length $2C$.
5) The level curves are the boundaries of the squares of centrum $(0, 0)$ and the corners $(\pm C, 0)$ and $(0, \pm C)$.
6) First note that

$$\begin{aligned} f(x, y) &= (x^2 + y^2 + 1)^2 - 4x^2 \\ &= (x^2 + y^2 + 1 - 2x)(x^2 + y^2 + 1 + 2x) \\ &= \{(x-1)^2 + y^2\}\{(x+1)^2 + y^2\}. \end{aligned}$$

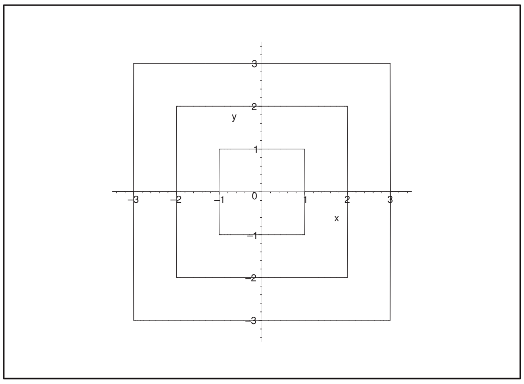


Figure 25: The level curves $\max\{|x|,|y|\} = C$, $C = 1, 2, 3$.

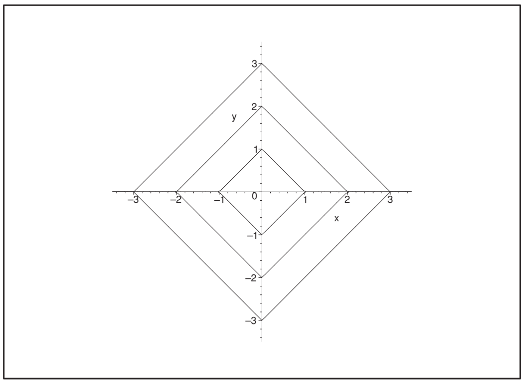


Figure 26: The level curves $|x| + |y| = C$, $C = 1, 2, 3$.

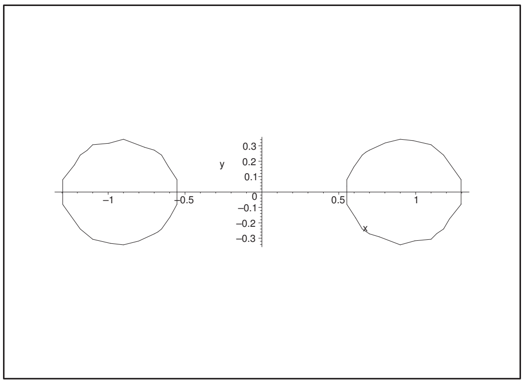


Figure 27: The level curve $(x^2 + y^2 + 1)^2 - 4x^2 = \dfrac{1}{2}$.

The level curves $f(x, y) = C$ can then be interpreted as the curves composed of the points (x, y), for which the *product* of the distances to $(1, 0)$ and $(-1, 0)$ is equal to $\sqrt{C}$.

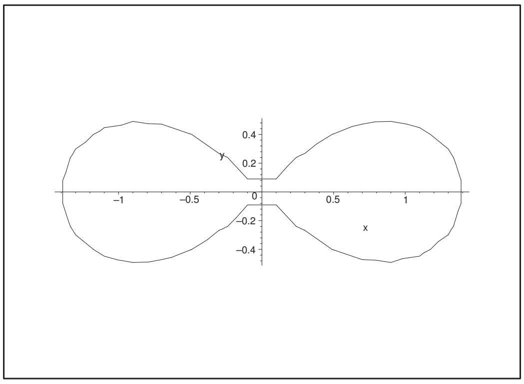


Figure 28: The level curve $(x^2 + y^2 + 1)^2 - 4x^2 = 1$. Though it cannot be seen (due to some error in the programme of sketching) the curves continue through $(0, 0)$.

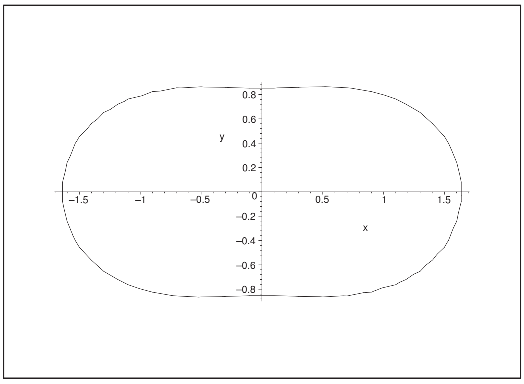


Figure 29: The level curve $(x^2 + y^2 + 1)^2 - 4x^2 = 3$.

7) First note that when $x = -1$, then $f(-1, 0) = 1$. This means that we shall be particular careful in the case of $C = 1$.

Here we get five cases which are treated successively.

a) When $C = -4$, it follows from our first remark that $x \neq -1$. Clearly, $y \neq 0$, because $x^2 = -4$ does not have any real solution. The level curves are given by

$$y^2 = -\frac{4 + x^2}{(1 + x)^3} > 0.$$

Accordingly, $x < -1$, and

$$y^2 = -\frac{1}{(1 + x)^2}\left(x - 1 + \frac{5}{1 - x}\right),$$

i.e.

$$y = \pm\frac{1}{|1 + x|}\sqrt{1 - x - \frac{5}{1 + x}} = \pm\frac{1}{|1 + x|}\sqrt{2 + |1 + x| + \frac{5}{|1 + x|}},$$

for $x < -1$.

We get two level curves, which lie symmetrically with respect to the X axis where the line $x = -1$ and the X axis are the asymptotes.

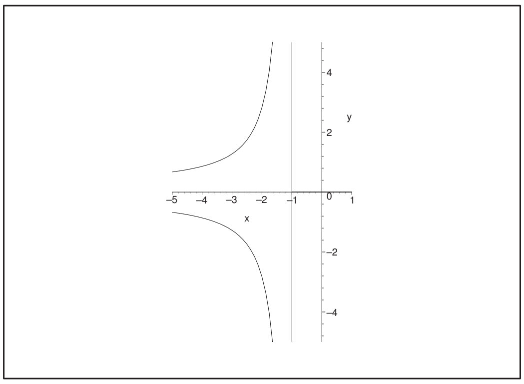


Figure 30: The level curves $x^2 + y^2(1+x)^3 = -4$.

b) When $C = 0$, we again find that $x \neq -1$. Notice that if $y = 0$, then $x = 0$ is a solution, hence the point $(0,0)$ belongs to the solutions. When $y \neq 0$, we get

$$y = \pm \left| \frac{x}{1+x} \right| \cdot \frac{1}{\sqrt{|1+x|}}, \qquad x < -1.$$

The level "curves" are the point $(0,0)$ and two symmetric curves with respect to the X axis. These are closer the asymptotes than the level curves for $C = -4$.

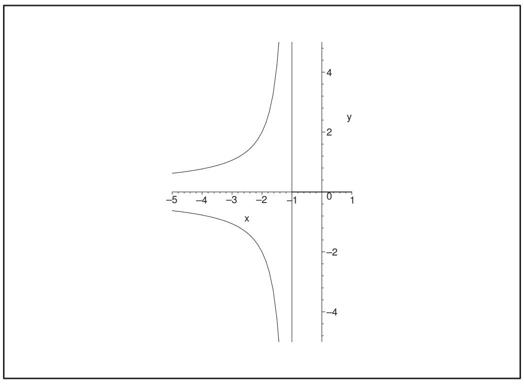


Figure 31: The level curves $x^2 + y^2(1+x)^3 = 0$, where the point $(0,0)$ should be added.

c) If $C = \dfrac{1}{4}$, then $x \neq -1$, and

$$y^2 = \frac{\frac{1}{4} - x^2}{(1+x)^3} = -\frac{(x-\frac{1}{2})(x+\frac{1}{2})}{(x+1)^2} \geq 0.$$

We note that $y = 0$, if and only if $x = \pm\dfrac{1}{2}$.

Then the right hand side is positive, when either $|x| < \dfrac{1}{2}$ or $x < -1$.

The level curves are two symmetric curves for $x < -1$ with respect to the X axis, where the X axis and the line $x = -1$ are the asymptotes, supplied with a closed curve for $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

d) For $C = 1$ we are in the exceptional case mentioned above where $x = -1$ is a level curve. When $x \neq -1$, we get

$$y^2 = \frac{1-x^2}{(1+x)^3} = \frac{1-x}{(1+x)^2} \geq 0,$$

thus $x \leq 1$. When $x = 1$, we only get the solution $y = 0$, i.e. we get the point $(1,0)$.

The level curves are the line $x = -1$, two symmetric curves with respect to the X axis for $x < -1$, and a curve with the X axis as an axis of symmetry for $x \in]-1,1]$ and the line $x = -1$ as an asymptote.

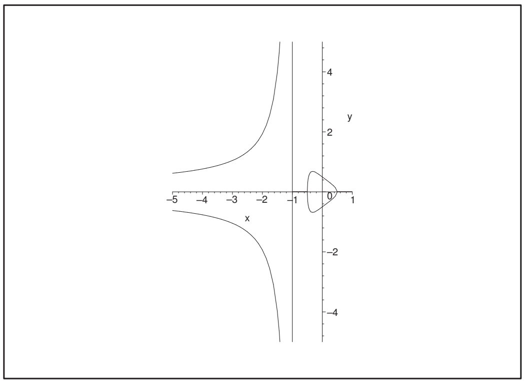


Figure 32: The level curves $x^2 + y^2(1 + x)^3 = \dfrac{1}{4}$.

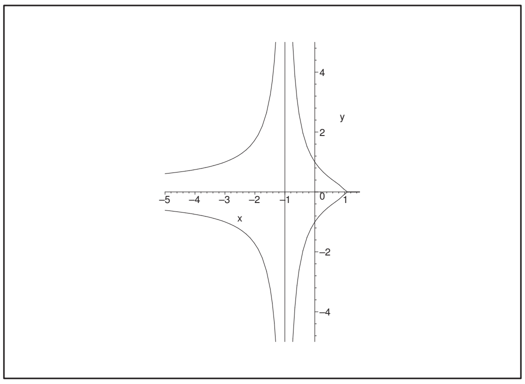


Figure 33: The level curves $x^2 + y^2(1 + x)^3 = 1$.

e) When $C = 4$, we get

$$y^2 = \frac{4 - x^2}{(1 + x)^3} \geq 0.$$

It follows that $(\pm 2, 0)$ are solutions and that we only get solutions for either $x \leq -2$ or $-1 < x \leq 2$.

We obtain two curves, each symmetric with respect to the X axis. Furthermore, one of these curves has $x = -1$ as an asymptote.

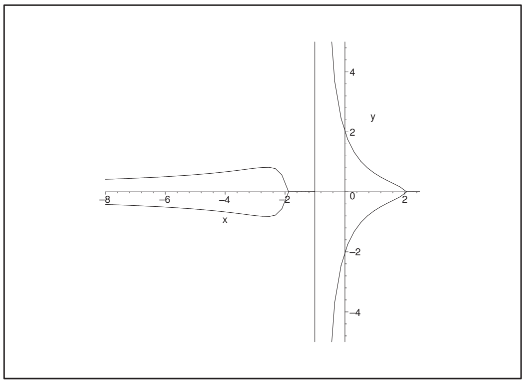


Figure 34: The level curves $x^2 + y^2(1+x)^3 = 4$.

Example 3.3 *Describe the level surfaces for the following functions:*

1) $f(x,y,z) = x$ *for* $(x,y,z) \in \mathbb{R}^3$,

2) $f(x,y,z) = \max\{|x|,|y|,|z|\}$ *for* $(x,y,z) \in \mathbb{R}^3$,

3) $f(x,y,z) = \sqrt{\max\{|x|,|y|,|z|\}}$ *for* $(x,y,z) \in \mathbb{R}^3$,

4) $f(x,y,z) = z - x^2 - y^2$ *for* $(x,y,z) \in \mathbb{R}^3$,

5) $f(x,y,z) = \dfrac{x^2+y^2+z^2-a^2}{z}$ *for* $z \neq 0$.

A Level surfaces in space.

D Analyze the function. The sketches are left to the reader, because there are difficulties here with the MAPLE programs. (I am not clever enough to get the right drawings.)

I 1) Obviously, the level surfaces

$$f(x,y,z) = x = c$$

are all planes parallel to the YZ plane, where $c \in \mathbb{R}$.

2) The level surfaces are the boundaries of all cubes of centrum $(0,0,0)$ and edge length $2c$ for $c > 0$, supplied with the point $(0,0,0)$ when $c = 0$.

Only $c \geq 0$ is possible.

3) The level surfaces are the same as in 2), only the edge length is here $2c^2$ for $c > 0$. When $c = 0$ we obtain as before the point $(0,0,0)$.

4) Since $f(x,y,z) = z - x^2 - y^2 = c$ can also be written

$$z - c = x^2 + y^2,$$

we obtain all paraboloids of revolution with top point at $(0,0,c)$, through the unit circle in the plane $z = 1 + c$ and with the Z axis as the axis of revolution.

5) First we rewrite

$$f(x,y,z) = \frac{x^2+y^2+z^2-a^2}{z} = c, \qquad z \neq 0,$$

to

$$x^2+y^2+z^2-a^2 = cz, \qquad z \neq 0,$$

i.e.

$$x^2+y^2+\left(z-\frac{C}{2}\right)^2 = a^2+\frac{c^2}{4}, \qquad z \neq 0+.$$

The level surfaces are spheres of centrum $\left(0,0,\frac{c}{2}\right)$ and radius $\sqrt{a^2+\frac{c^2}{4}}$, with the exception of the points in the XY plane, i.e. with the exception of the circle

$$x^2+y^2=a^2, \qquad z=0.$$

Example 3.4 *Consider the function $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{e}$, $\mathbf{x} \in \mathbb{R}^k$, where $\mathbf{e}$ is a constant unit vector.*

1) Sketch the level curves of the function in the case of $k = 2$.

2) Describe the level surfaces of the function in the case of $k = 3$.

A Level curves and level surfaces.

D Sketch if possible a figure and analyze.

I 1) The level curves are all the straight lines ℓ, which are perpendicular to the line generated by the vector $\mathbf{e}$.

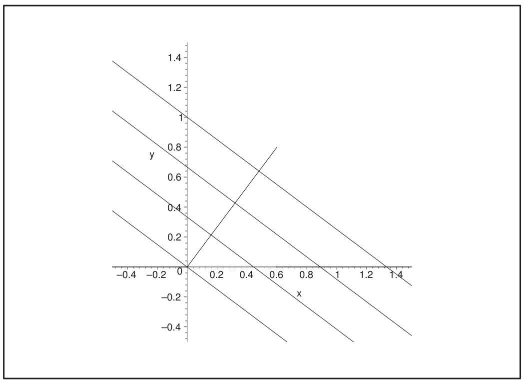


Figure 35: Some level curves when $\mathbf{e} = \left(\frac{3}{5}, \frac{4}{5}\right)$.

2) Analogously the level surfaces for $k = 3$ are all planes π, which are perpendicular to the line generated by the vector $\mathbf{e}$.

Example 3.5 *Let a be a positive constant. Find the domain of the function*

$$f(x, y, z) = \ln\left(a^2 - 3x^2 - y^2 - 2z^2\right).$$

The describe the level surfaces for f, and find the range of the function.

A Domain, level surfaces, range.

D Just follow the text.

I The function is defined for

$$3x^2 + y^2 + 2z^2 < a^2,$$

which describes the open ellipsoid with the half axes

$$\frac{a}{\sqrt{3}}, \quad a, \quad \frac{a}{\sqrt{2}}.$$

The level surfaces are all the ellipsoidal surfaces

$$3x^2 + y^2 + 2 <^2= b^2, \qquad 0 < b < a,$$

with the half axes

$$\frac{b}{\sqrt{3}}, \quad b, \quad \frac{b}{\sqrt{2}}.$$

The value of the function on such a level surface is $\ln\left(a^2 - b^2\right)$.

The range of f is the same as the range of the function

$$g(t) = \ln\left(a^2 - t^2\right), \qquad t \in [0, a[,$$

so the range is $]-\infty, 2\ln a]$.

Example 3.6 *Sketch the domain A of the function*

$$f(x,y) = \ln\left(225 - 25x^2 - 9y^2\right).$$

Indicate the boundary ∂A of A, and sketch the level curve of f, which contains the point

$$(x,y) = \left(\frac{3}{2}, \frac{5}{2}\right).$$

A Domain and level curve.

D Since ln is only defined on $\mathbb{R}_+$, the domain is given by the requirement that the expression inside the ln is positive.

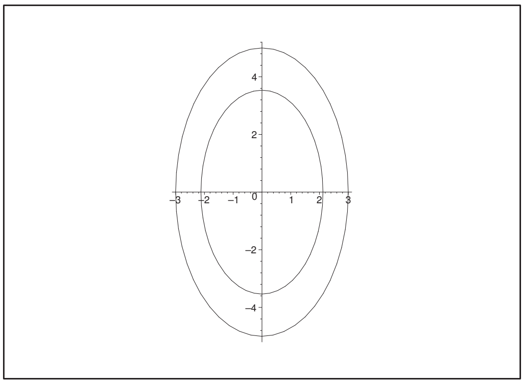


Figure 36: The domain A and the level curve through $\left(\frac{3}{2}, \frac{5}{2}\right)$.

I The function is defined for

$$225 - 25x^2 - 9y^2 > 0, \quad \text{i.e. for} \quad (5x)^2 + (3y)^2 < 15^2,$$

hence

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 < 1.$$

The domain is an open ellipsoidal disc of centrum $(0,0)$ and half axes 3 and 5.

the level curve is given by

$$\ln\left(225 - 25x^2 - 9y^2\right) = f\left(\frac{3}{2}, \frac{5}{2}\right) = \ln\left\{225 - \left(5 \cdot \frac{3}{2}\right)^2 - \left(3 \cdot \frac{5}{2}\right)^2\right\},$$

i.e. by

$$225 - 25x^2 - 9y^2 = 225\left(1 - \frac{1}{4} - \frac{1}{4}\right) = \frac{225}{2},$$

hence by a rearrangement,

$$(5x)^2 + (3y)^2 = \left(\frac{15}{\sqrt{2}}\right)^2.$$

This can also be written

$$\left(\frac{x}{\frac{3}{2}\sqrt{2}}\right)^2 + \left(\frac{y}{\frac{5}{2}\sqrt{2}}\right)^2 = 1.$$

Thus the level curve is an ellipse of centrum $(0,0)$ and half axes $\frac{3}{2}\sqrt{2} = \frac{3}{\sqrt{2}}$ and $\frac{5}{2}\sqrt{2} = \frac{5}{\sqrt{2}}$.

4 Conics

Example 4.1 *A conic $\mathcal{F}$ is given by the equation*

$$2x^2 - 2y^2 + \alpha z^2 = 1,$$

where α is a real constant.

1) *Find the values of α, for which $\mathcal{F}$ is a surface of revolution. Indicate in each of these cases the type of the surface and its axis of symmetry.*

2) *Prove that there is one value of α, for which the surface $\mathcal{F}$ is a cylindric surface. Indicate for this value of α the type of the surface and its axis of symmetry.*

A Conic sections.

D Analyze each of the three cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.

I 1) a) When $\alpha < 0$, the conic is an hyperboloid of two nets:

$$1 = \left(\frac{x}{1\sqrt{2}}\right)^2 - \left\{\left(\frac{y}{1/\sqrt{2}}\right)^2 + \left(\frac{z}{\sqrt{1/|\alpha|}}\right)^2\right\}.$$

This is an hyperboloid of revolution for $\alpha = -2$, where the X axis is the axis of revolution.

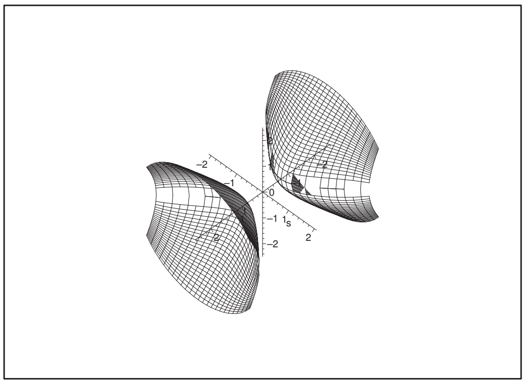


Figure 37: The surface of revolution for $\alpha = -2$.

b) When $\alpha > 0$, the conic is an hyperboloid of one net:

$$1 = \left\{\left(\frac{x}{\frac{1}{\sqrt{2}}}\right)^2 + \left(\frac{z}{\sqrt{\frac{1}{\alpha}}}\right)^2\right\} - \left(\frac{y}{\frac{1}{\sqrt{2}}}\right)^2.$$

This becomes an hyperboloid of revolution when $\alpha = 2$, with the Y axis as its axis of revolution.

2) When $\alpha = 0$, we get an hyperbolic cylindric surface with the Z axis as its axis of generation,

$$1 = \left(\frac{x}{\frac{1}{\sqrt{2}}}\right)^2 - \left(\frac{y}{\frac{1}{\sqrt{2}}}\right)^2.$$

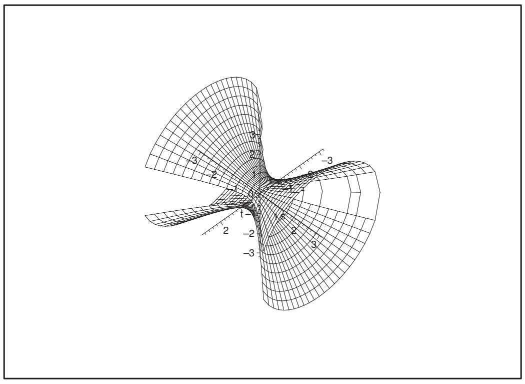


Figure 38: The surface of revolution for $\alpha = 2$.

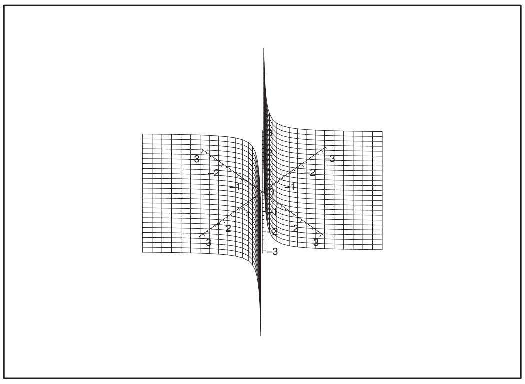


Figure 39: The surface for $\alpha = 0$.

Example 4.2 *Find the type and position of the conic of the equation*

$$x^2 + 2y^2 - x + 6y + \frac{3}{4} = 0.$$

A Conic section.

D Translate the coordinates.

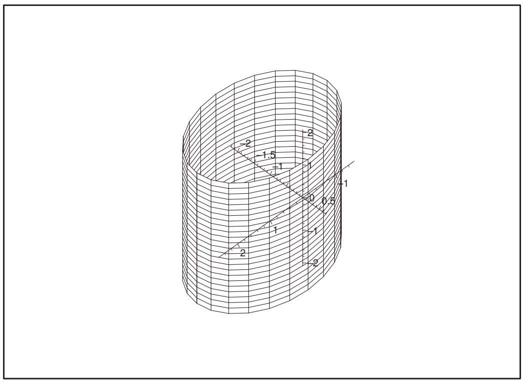
I By a rearrangement we get

$$\begin{aligned} 0 &= x^2 + 2y^2 - x + 6y + \frac{3}{4} \\ &= \left(x^2 - 2\cdot\frac{1}{2}\,x + \frac{1}{4}\right) + 2\left(y^2 + 2\cdot\frac{3}{2}\,y + \frac{9}{4}\right) - 2\cdot\frac{9}{4} + \frac{3}{4} \\ &= \left(x - \frac{1}{2}\right)^2 + 2\left(y + \frac{3}{2}\right)^2 - 4, \end{aligned}$$

i.e. in the canonical form

$$\left(\frac{x - \frac{1}{2}}{2}\right)^2 + \left(\frac{y + \frac{3}{2}}{\sqrt{2}}\right)^2 + 0\cdot z^2 = 1,$$

because z does not appear in the equation.



The surface is an *elliptic cylindric surface* with the Z axis as its axis of generation, and with the ellipse of centrum $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and the half axes 2 and $\sqrt{2}$ as generating curve.

Example 4.3 *Let a, b, c be constant different from zero satisfying the equation*

$$a + b + c = 0.$$

Prove that the plane of the equation

$$x + y + z = 0$$

cuts the conic given by

$$\frac{yz}{a} + \frac{zx}{b} + \frac{xy}{c} = 0$$

in two straight lines (generators), which form an angle of $\frac{2\pi}{3}$.

A Intersection of two surfaces.

D Start by e.g. eliminating $z = -x - y$.

I Clearly, $(0,0,0)$ lies in the intersection of the two surfaces. Furthermore, if two of the variables are 0, e.g. $x = y = 0$, then we have a point on the conic, no matter the value of the third variable (here z). We conclude that the X, the Y and the Z axes all lie on the conic section. Of course, none of then are contained in the oblique plane $x + y + z = 0$.

If we keep off the coordinate planes, i.e. we assume in the following that $xyz \neq 0$, then the equation of the conic can also be written

$$0 = \frac{yz}{a} + \frac{zx}{b} + \frac{xy}{x} = xyz\left(\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}\right),$$

i.e.

$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=0 \qquad \text{for } xyz \neq 0.$$

Since $z=-(x+y)$ on the plane, we get by insertion into the reduced equation of the conic that

$$0=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{ax}+\frac{1}{by}-\frac{1}{c(x+y)}.$$

When we put everything here in the same fraction and reduce we get

$$(3)\;\; 0=\frac{1}{a}\,(x+y)y+\frac{1}{b}\,(x+y)x-\frac{1}{c}\,xy,$$

which is an homogeneous polynomial of second degree in (x,y).

Now $x = 0$, if and only if $y = 0$, so the solutions must have the structure

(4) $y = \alpha\, x, \qquad \alpha \neq 0.$

It follows that the intersection of the two surfaces must have the structure

$$\mathbf{r}(t) = (t, \alpha t, -(1+\alpha)t) = t(1, \alpha, -(1+\alpha)), \qquad t \in \mathbb{R},$$

because $z = -x - y$, and because we can trivially continue to $(0,0,0)$.

When (4) is put into (3), we get that α is a solution of a polynomial of second degree with the roots α_1 and α_2, corresponding to two straight lines. (According to the geometry the solutions exist, so we must necessarily have the the roots α_1 and α_2 are real numbers).

By insertion of $(x,y,z) = (1, \alpha, -(1+\alpha))$ we get for $\alpha \neq -1$ that

$$0 = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{a} + \frac{1}{b\alpha} - \frac{1}{c(1+\alpha)} = \frac{bc\alpha(1+\alpha) + ac(1+\alpha) - ab\alpha}{abc\alpha(1+\alpha)},$$

which is reduced to

$$0 = \alpha(1+\alpha) + \frac{a}{b}\,(1+\alpha) - \frac{a}{c}\,\alpha = \alpha^2 + \left(1 + \frac{a}{b} - \frac{a}{c}\right)\alpha + \frac{a}{b} = \alpha^2 + a\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\alpha + \frac{a}{b},$$

hence

$$\alpha_1 + \alpha_2 = a\left(\frac{1}{c} - \frac{1}{a} - \frac{1}{b}\right) = a\left(\frac{1}{c} - \frac{a+b}{ab}\right) = \frac{a}{c} + \frac{c}{b}$$

and

$$\alpha_1\alpha_2 = \frac{a}{b}.$$

Since $(1, \alpha, -(1+\alpha))$ is of length

$$\sqrt{1 + \alpha^2 + (1+\alpha)^2} = \sqrt{2(1+\alpha+\alpha^2)},$$

The angle φ between the two lines (which both pass through $(0,0,0)$) is given by

$$\cos\varphi = \frac{(1, \alpha_1, -(1+\alpha_1))}{\sqrt{2(1+\alpha_1+\alpha_1^2)}} \cdot \frac{(1, \alpha_2, -(1+\alpha_2))}{\sqrt{2(1+\alpha_2+\alpha_2^2)}} = \frac{1}{2} \cdot \frac{1 + \alpha_1\alpha_2 + (1+\alpha_1)(1+\alpha_2)}{\sqrt{(1+\alpha_1+\alpha_1^2)(1+\alpha_2+\alpha_2^2)}}.$$

Here the numerator is

$$\begin{aligned}1 + \alpha_1\alpha_2 + (1+\alpha_1)(1+\alpha_2) &= 2 + (\alpha_1+\alpha_2) + 2\alpha_1\alpha_2 = 2 + \frac{ab+c^2}{bc} + 2\,\frac{a}{b} \\ &= \frac{1}{bc}\,\{2bc - (b+c)b + c^2 - 2(b+c)c\} = -\frac{1}{bc}\,(b^2 + bc + c^2),\end{aligned}$$

and the radicand is

$$
\begin{aligned}
&(1+\alpha_1+\alpha_1^2)(1+\alpha_2+\alpha_2^2)\\
&\quad= 1+\alpha_1+\alpha_2+\alpha_1^2+\alpha_2^2+\alpha_1\alpha_2+\alpha_1+\alpha_2^2+\alpha_1^2\alpha_2+\alpha_1^2\alpha_2^2\\
&\quad= 1+(\alpha_1+\alpha_2)+(\alpha_1+\alpha_2)^2-\alpha_1\alpha_2+\alpha_1\alpha_2(\alpha_1+\alpha_2)+(\alpha_1\alpha_2)^2\\
&\quad= 1+\frac{ab+c^2}{bc}+\left(\frac{ab+c^2}{bc}\right)^2-\frac{a}{b}+\frac{a}{b}\cdot\frac{ab+c^2}{bc}+\frac{a^2}{b^2}\\
&\quad= \frac{1}{b^2c^2}\{b^2c^2+ab^2c+bc^3+a^2b^2+2abc^2+c^4-abc^2+a^2bc+ac^3+a^2c^2\}\\
&\quad= \frac{1}{b^2c^2}\{b^2c^2+bc^3+c^4+a(b^2c+2bc^2-bc^2+c^3)+a^2(b^2+bc+c^2)\}\\
&\quad= \frac{1}{b^2c^2}\{c^2(b^2+bc+c^2)+ac(b^2+bc+c^2)+a^2(b^2+bc+c^2)\}\\
&\quad= \frac{1}{b^2c^2}\,(b^2+bc+c^2)(c^2+ac+a^2)=\frac{1}{b^2c^2}\,(b^2+bc+c^2)(c^2+(-b-c)(-b))\\
&\quad= \frac{1}{b^2c^2}\,(b^2+bc+c^2)^2.
\end{aligned}
$$

Then by insertion

$$\cos\varphi=\frac{1}{2}\cdot\frac{1+\alpha_1\alpha_2+(1+\alpha_1)(1+\alpha_2)}{\sqrt{(1+\alpha_1+\alpha_1^2)(1+\alpha_2+\alpha_2^2)}}=\frac{1}{2}\cdot\frac{-\dfrac{1}{bc}\,(b^2+bc+c^2)}{\left|\dfrac{1}{bc}\,(b^2+bc+c^2)\right|}.$$

Since $b^2+bc+c^2=\left(b+\dfrac{1}{2}\,c\right)^2+\dfrac{3}{4}\,c^2>0$, we have

$$\cos\varphi=-\frac{1}{2}\,\frac{|bc|}{bc}=-\frac{1}{2}\,\frac{bc}{|bc|}=\begin{cases}\ \ \frac{1}{2}, & \text{hvis } bc<0,\\ -\frac{1}{2}, & \text{hvis } bc>0.\end{cases}$$

Hence $\varphi=\dfrac{\pi}{3}$, if $bc<0$, and $\varphi=\dfrac{2\pi}{3}$ $\left(\text{or }\ -\dfrac{\pi}{3}\right)$, if $bc>0$.

If we do not include the sign of the angle we get $\varphi=\dfrac{\pi}{3}$.

Example 4.4 *Indicate for each value of the constant k the type of the conic $\mathcal{F}$, which is given by the equation*

$$x^2+(4-k^2)y^2+k(2-k)z^2=2k,$$

and find in particular those values of k, for which $\mathcal{F}$ is a surface of revolution.
Finally, think about if it makes sense to put k equal to $+\infty$ or $-\infty$.

A Conics.

D Discuss the sign of the coefficients and then consider the various cases.

I By considering the signs we get the scheme

	$k<-2$	$k=-2$	$-2<k<0$	$k=0$	$0<k<2$	$k=2$	$k>2$
$4-k^2$	$-$	0	$+$	$+$	$+$	0	$-$
$k(2-k)$	$-$	$-$	$-$	0	$+$	0	$-$
$2k$	$-$	$-$	$-$	0	$+$	$+$	$+$
	1	2	3	4	5	6	7

1) When $k<-2$, we get the canonical form (notice the absolute values)

$$\begin{aligned}1 &= -\frac{1}{2|k|}\,x^2+\left|\frac{4-k^2}{2k}\right|\,y^2+\left|\frac{2-k}{2}\right|\,z^2\\ &= -\frac{1}{2|k|}\,x^2+\frac{4-k^2}{2k}\,y^2+\frac{2-k}{2}\,z^2.\end{aligned}$$

Since we have 2 plusses and 1 minus we conclude that we have an *hyperboloid of one net.*

2) When $k=-2$, the equation is written

$$x^2-8z^2=-4, \quad \text{dvs.} \quad -\left(\frac{x}{2}\right)^2+\left(\frac{z}{1/\sqrt{2}}\right)^2=1,$$

which describes an *hyperbolic cylindric surface.*

3) When $-2<k<0$, the canonical form becomes

$$-\frac{1}{2|k|}\,x^2-\left|\frac{4-k^2}{2k}\right|\,y^2+\left|\frac{2-k}{2}\right|\,z^2=1.$$

With 1 plus and 2 minusses we conclude that we get an *hyperboloid with two nets.*

4) When $k=0$, the equation is written

$$x^2+4y^2=0,$$

which is satisfied for the Z *axis.* (Degenerated "surface of revolution").

5) When $0<k<2$, we rewrite to the canonical form

$$\left|\frac{1}{2k}\right|\,x^2+\left|\frac{4-k^2}{2k}\right|\,y^2+\left|\frac{2-k}{2}\right|\,z^2=1.$$

With 3 plusses we get an *ellipsoid.*

6) When $k=2$, the equation is written

$$x^2=4,$$

which describes two planes $x=\pm 2$, parallel to the YZ plane.

7) When $k>2$, we get

$$\left|\frac{1}{2k}\right|\,x^2-\left|\frac{4-k^2}{2k}\right|\,y^2-\left|\frac{2-k}{k}\right|\,z^2=1.$$

With 1 plus and 2 minusses we see that we get an *hyperboloid with two nets.*

We obtain surfaces of revolution when

1) $x^2 + (4 - k^2)y^2 = x^2 + y^2$, i.e. when $k = \pm\sqrt{3}$.
2) $x^2 + k(2 - k)z^2 = x^2 + z^2$, i.e. when $k = 1$.
3) $4 - k^2 = k(2 - k)$, i.e. $k = 2$, which however produces degenerated surfaces of revolution.
4) $k = 0$ gives Z axis as the degenerated "surface of revolution".

1) When $k = -\sqrt{3}$ we are in case 3., so we have an *hyperboloid of revolution with two nets* where the Z axis is the axis of revolution.
2) When $k = 0$ we are in case 4., which is the degenerated case of the Z axis. The Z axis is clearly the axis of revolution.
3) When $k = 1$ we are in case 5., and we get an *ellipsoid of revolution* with the Y axis as the axis of revolution.
4) When $k = \sqrt{3}$ we are again in case 5., so we get an *ellipsoid of revolution* with the Z axis as the axis of revolution.
5) When $k = 2$ we are in the degenerated case 6. The two planes have clearly the X axis as the axis of revolution-

When $k \neq 0$, we get by dividing by $-k^2$ that

$$-\frac{1}{k^2}\,x^2+\left(1-\frac{4}{k^2}\right)y^2+\left(1-\frac{2}{k}\right)z^2=-\frac{2}{k}.$$

Then by taking the limits $k \to +\infty$ or $k \to -\infty$ it follows immediately

$$y^2+z^2=0,$$

so $y=z=0$, while x is free. Therefore, by taking the limits we get the X axis

Example 4.5 *The surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by the equations*

$$x^2+2y^2=z+1, \qquad x^2+2y^2=-1+3z^2.$$

1) Indicate the type and the top point(s) of both $\mathcal{F}_1$ and $\mathcal{F}_2$.

2) Prove that the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ consists of two ellipses, lying in planes, which are parallel to the (X,Y) plane.

A Conics and conic sections.

D In 1) we just reformulate the equations to the canonical form. In 2) we first eliminate x^2+2y^2 in order to get an equation in z. Then insert the solutions in z into one of the original expressions.

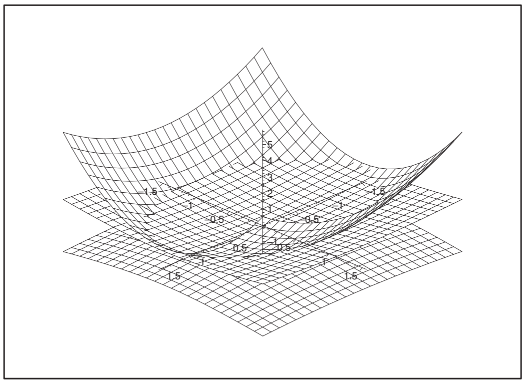


Figure 40: The surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$.

I 1) If we put $z_1=z+1$, we see that the equation of the surface $\mathcal{F}_1$ can be written in its canonical form

$$\frac{x^2}{1^2}+\frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2}=z_1,$$

which shows that $\mathcal{F}_1$ is an elliptic paraboloid with top point $(0,0,-1)$.

Then the equation of $\mathcal{F}_2$ is written in the following way:

$$-\frac{x^2}{1^2}-\frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{z^2}{\left(\frac{1}{\sqrt{3}}\right)^2}=1.$$

This equation describes an hyperboloid with two nets. The top points are

$$\left(0,0,\pm\frac{1}{\sqrt{3}}\right).$$

2) The equation of the intersection is obtained by eliminating the common expression x^2+2y^2 i (x,y). This gives

$$z+1=-1+3z^2, \qquad \text{i.e.} \qquad 3z^2-2-2=3(z-1)\left(z+\frac{2}{3}\right)=0.$$

The solutions are $z=1$ and $z=-\frac{2}{3}$, so the intersection curves lie in these two planes which are parallel to the (X,Y) plane.

a) When we put $z=1$, we get $x^2+2y^2=2$, which in its canonical form becomes

$$\frac{x^2}{(\sqrt{2})^2}+\frac{y^2}{1^2}=1.$$

This is an equation of an ellipse in the plane $z=1$ of centrum $(0,0)$ and half axes $\sqrt{2}$ and 1.

b) If we put $z=-\frac{2}{3}$, we get $x^2+2y^2=\frac{1}{3}$, which is written in its canonical form in the following way

$$\frac{x^2}{\left(\frac{1}{\sqrt{3}}\right)^2}+\frac{y^2}{\left(\frac{1}{\sqrt{6}}\right)^2}=1.$$

This is an equation of an ellipse in the plane $z=-\frac{2}{3}$ of centrum $(0,0)$ and half axes $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{6}}$.

5 Continuous functions

Example 5.1 *The range of each of the following functions in two variables is not the whole plane but $\mathbb{R}^2 \setminus M$, where $M \neq \emptyset$. Find the point set M in each case and explain why $f : \mathbb{R}^2 \setminus \to \mathbb{R}$ is continuous. Finally, check whether the function has a continuous extension to either $\mathbb{R}^2$ or to $\mathbb{R}^2 \setminus L$, where $L \subset M$.*

1) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$,

2) $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$,

3) $f(x,y) = \dfrac{x^2y}{\sqrt{x^2+y^2}}$,

4) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$,

5) $f(x,y) = \dfrac{3x-2y}{2x-3y}$,

6) $f(x,y) = \dfrac{x^2-y^2}{\operatorname{Arctan}(x-y)}$,

7) $f(x,y) = \dfrac{x^3-y^3}{x-y}$,

8) $f(x,y) = \dfrac{1-e^{xy}}{xy}$.

A Examination of functions, continuous extension.

D Find the set of exceptional points. Since the numerator and the denominator are continuous in $\mathbb{R}^2$ in all cases, it is only a matter of determining the zero set of the denominator. A possible continuous extension can only take place at points in which both the numerator and the denominator are zero, so this set should be examined too.

I 1) The denominator is clearly only zero at $(0,0)$, so $M = \{(0,0)\}$.

If we use polar coordinates, we get for $\varrho > 0$,

$$f(x,y) = \frac{\varrho^2\cos^2\varphi - \varphi^2\sin^2\varphi}{\varphi^2} = \cos^2\varphi - \sin^2\varphi = \cos 2\varphi,$$

and it is obvious that we cannot have a continuous extension to $(0,0)$, because there is no restriction on φ.

2) Here also $M = \{(0,0)\}$. By using polar coordinates we get

$$f(x,y) = \frac{\varrho^3\cos^3\varphi + \varphi^3\sin^3\varphi}{\varrho^2} = \varrho\{\cos^3\varphi + \sin^3\varphi),$$

which tends to 0 for $\varrho \to 0$. Hence, the function has a continuous extension to $(0,0)$ given by $f(0,0) = 0$.

3) Again $M = \{(0,0)\}$. By using polar coordinates we get

$$f(x,y) = \frac{\varrho^3 \cos^2\varphi \sin\varphi}{\varrho} = \varrho^2 \cos^2\varphi \sin\varphi,$$

which tends to 0 for $\varrho \to 0$. Hence the function has a continuous extension given by $f(0,0) = 0$.

4) Also here $M = \{(0,0)\}$. Again by polar coordinates,

$$f(x,y) = \frac{\varrho^2 \sin\varphi \cos\varphi}{\varrho} = \varrho \sin\varphi \cos\varphi \to 0 \qquad \text{for } \varrho \to 0.$$

By continuous extension we get $f(0,0) = 0$.

5) Here

$$M = \{(x,y) \mid 2x = 3y\} = \left\{(x,y) \;\middle|\; y = \frac{2}{3}\,x\right\}.$$

The only possibility of a continuous extension must take place on that subset where the numerator is also zero, i.e. on $\{(0,0)\}$. Using polar coordinates we get

$$f(x,y) = \frac{3\cos\varphi - 2\sin\varphi}{2\cos\varphi - 3\sin\varphi},$$

which clearly does not have a limit, when $\varrho \to 0$, and $\varphi \in [0, 2\pi[$. In this case we do not have a continuous extension.

6) Here $M = \{(x,y) \mid y = x\}$. Since

$$f(x,y) = \frac{x+y}{\frac{\text{Arctan}(x-y)}{x-y}}, \qquad (x,y) \notin M,$$

where

$$\frac{\text{Arctan}\, t}{t} \to 1 \qquad \text{for } t \to 0,$$

it is possible to extend the function to all of M by

$$f(x,x) = 2x, \qquad (x,x) \in M.$$

7) Here we also have $M = \{(x,y) \mid y = x\}$. We get by a division

$$f(x,y) = \frac{x^3 - y^3}{x-y} = x^2 + xy + y^2, \qquad (x,y) \notin M.$$

Clearly, the latter expression can be continuously extended to all of $\mathbb{R}^2$. On M we get

$$f(x,x) = 3x^2, \qquad (x,x) \in M.$$

8) Here $M = \{(x,y) \mid x = 0 \text{ or } y = 0\}$, i.e. the union of the coordinate axes.

Since

$$\frac{1 - e^t}{t} = -\frac{e^t - e^0}{t - 0} \to -1 \qquad \text{for } t \to 0,$$

it follows from an application of the substitution $t = xy$ that f can be extended to the axes by

$$f(0,y) = f(x,0) = -1.$$

Example 5.2 *In each of the following cases one shall find the domain D of the given function f, and explain why f is continuous. Then show that f has a continuous extension to a point set B, where $B \supset D$.*

1) $f(x,y) = \dfrac{x+y-1}{\sqrt{x}-\sqrt{1-y}}$,

2) $f(x,y) = (x+y)\ln\sinh(x+y)$,

3) $f(x,y) = \dfrac{\text{Arcsin}(xy-2)}{\text{Arctan}(3xy-6)}$,

4) $f(x,y) = \exp\left(-\dfrac{1}{(x-y)^2}\right)$.

A Examination of functions and continuous extensions.

D Find the point set where the numerator and the denominator are defined and continuous.

Then check a possible extension to the set where both the numerator and the denominator are zero.

I 1) The numerator is defined in $\mathbb{R}^2$. The numerator is defined and continuous when $x \geq 0$ and $1-y \geq 0$, i.e. for $y \leq 1$.

The denominator is zero, when $\sqrt{x} = \sqrt{1-y}$ for $x \geq 0$ and $y \leq 1$. A squaring shows that the denominator is zero when

$$x+y=1, \qquad x \geq 0, \qquad y \leq 1,$$

and we see that the numerator is zero on the same set. We see that the domain is

$$D = \{(x,y) \mid x \geq 0,\, y \leq 1,\, x+y \neq 1\} = D_1 \cup D_2.$$

In the two subdomains D_1 (the "lower triangular domain") and D_2 (the "upper triangular domain") both the numerator and the denominator are continuous, and the denominator is not zero in these two sets, so the function us continuous on D.

It has already above been given a hint that there is a possible continuous extension to the line $x+y=1$ for $x \geq 0$ and $y \leq 1$, because both the numerator and the denominator are here 0. We get by a simple rearrangement for $(x,y) \in D$, i.e. in particular for $x+y \neq 1$, that

$$f(x,y) = \frac{x-(1-y)}{\sqrt{x}-\sqrt{1-y}} = \frac{(\sqrt{x})^2-(\sqrt{1-y})^2}{\sqrt{x}-\sqrt{1-y}} = \sqrt{x}+\sqrt{1-y}.$$

This expression is continuous on the set

$$\{(x,y) \mid x \geq 0,\, y \leq 1\},$$

and we have found our continuous extension of the original function.

2) Here $f(x,y)$ is defined and continuous for $\sinh(x+y) > 0$, i.e. when $x+y > 0$, and the domain is

$$D = \{(x,y) \mid x+y > 0\}.$$

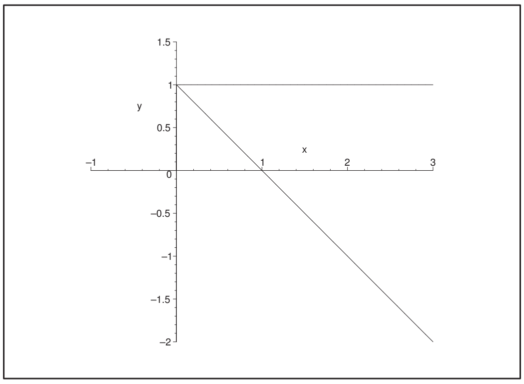


Figure 41: The domain of $f(x,y) = \dfrac{x+y-1}{\sqrt{x}-\sqrt{1-y}}$.

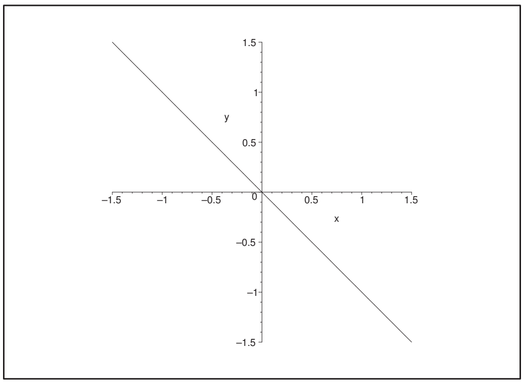


Figure 42: The domain of $f(x,y) = (x+y)\ln\sinh(x+y)$ lies above the oblique line.

By putting $t = x+y > 0$ we exploit that $f(x,y)$ actually is a function in $x+y$. Then

$$f(x,y) = g(t) = t\ln\sinh t = \frac{t}{\sinh t}\{\sinh t \cdot \ln\sinh t\}.$$

Here, $\dfrac{t}{\sinh t} \to 1$ for $t \to 0+$, and $\sinh t \cdot \ln\sinh t \to 0$ for $\sinh t \to 0+$, i.e. for $t \to 0+$. We therefore conclude for $z = \sinh t$ that

$$\lim_{t\to 0+} t\,\ln\sinh t = 0.$$

Then by the substitution $t = x+y$,

$$(x+y)\ln\sinh(x+y) \to 0 \qquad \text{for } x+y \to 0+\,.$$

Hence, the function can be extended continuously to the set

$$\overline{D} = \{(x,y) \mid x+y \geq 0\},$$

where we for $x + y = 0$ put

$$\overline{f}(x, -x) = 0, \qquad x \in \mathbb{R}.$$

3) The numerator $\text{Arcsin}(xy - 2)$ is defined and continuous, when $-1 \leq xy - 2 \leq 1$, i.e. when $1 \leq xy \leq 3$.

The denominator $\text{Arctan}(3xy - 6)$ is defined and continuous for every $(x, y) \in \mathbb{R}^2$.

The denominator is zero for $xy = 2$, and we see that the numerator is zero on the same set.

Thus the domain is

$$D = \{(x, y) \mid 1 \leq xy < 2 \text{ or } 2 < xy \leq 3\}.$$

We see that the domain has four connected components.

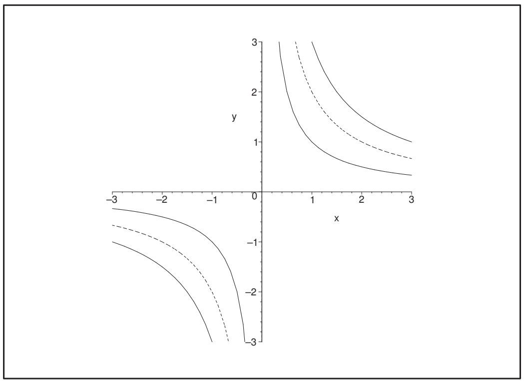


Figure 43: The domain of $f(x,y) = \dfrac{\text{Arcsin}(xy-2)}{\text{Arctan}(3xy-6)}$ is the union of the sets which lie between the hyperbolas in the first and third quadrant, with the exception of the dotted hyperbola in the "middle" of each set.

Since both the numerator and the denominator are zero on the exceptional hyperbola of the equation $xy = 2$, there is a possibility of a continuous extension to this hyperbola. We shall now examine this possibility.

First notice that

$$\frac{\text{Arcsin}\, t}{\text{Arctan}\, 3t} = \frac{1}{3} \cdot \frac{\text{Arcsin}\, t}{t} \cdot \frac{3t}{\text{Arctan}\, 3t} \to \frac{1}{3} \qquad \text{for } t \to 0.$$

Then by the substitution $t = xy - 2$,

$$f(x,y) = \frac{\text{Arcsin}(xy-2)}{\text{Arctan}(3xy-6)} \to \frac{1}{3} \qquad \text{for } xy \to 2.$$

Hence, we can extend f continuously to the set

$$B = \{(x,y) \mid \leq xy \leq 3\}$$

by putting

$$\overline{f}(x,y) = \begin{cases} \dfrac{\text{Arcsin}(xy-2)}{\text{Arctan}(3xy-6)} & \text{for } xy \in [1,3] \setminus \{2\}, \\ \dfrac{1}{3} & \text{for } xy = 2. \end{cases}$$

4) The function is defined and continuous for $y \neq x$, so the domain is given by

$$D = \{(x,y) \mid y \neq x\}.$$

Since

$$\lim_{t \to 0} \exp\left(-\frac{1}{t^2}\right) = 0,$$

it follows by the substitution $t = x - y$ that $f(x,y)$ can be extended to all of $\mathbb{R}^2$ by the continuous extension

$$\overline{f}(x,y) = \begin{cases} \exp\left(-\dfrac{1}{(x-y)^2}\right) & \text{for } y \neq x, \\ 0 & \text{for } y = x. \end{cases}$$

Example 5.3 *Sketch in each of the cases below the domain of the given function or vector function. Then examine whether the (vector) function has a limit for* $(x, y) \to (0, 0)$, *and indicate this when it exists.*

1) $f(x,y) = \dfrac{\sin(xy)}{x}$,

2) $f(x,y) = \dfrac{1}{x}\,\sin y$,

3) $f(x,y) = x\,\sin \dfrac{1}{y}$,

4) $\mathbf{f}(x,y) = \left(\dfrac{\ln(1+x^2+y^2)}{\sqrt{x^2+y^2}}, \dfrac{\ln x + \ln y}{\ln(xy)}\right)$,

5) $\mathbf{f}(x,y) = \left(\dfrac{x\sin y}{\sqrt{x^2+y^2}}, \dfrac{x^2y^2+x^2+y^2}{x^2+3y^2}\right)$,

6) $f(x,y) = \left(\dfrac{x}{x+y}, \sqrt{x+y}\right)$.

A Domains; limits.

D Analyze the function; take the limit.

I 1) The function is defined for $x \neq 0$, i.e. everywhere except for the Y axis,

$$D = \{(x,y) \mid x \neq 0\}.$$

There is of course no need to sketch the domain in this case.

By using polar coordinates we get from $x = \varrho\cos\varphi \neq 0$ in D that $\varrho > 0$ and $\cos\varphi \neq 0$. This shows that in D,

$$|f(x,y)| = \left|\frac{\sin(\varrho^2\cos\varphi\sin\varphi)}{\varrho\cos\varphi}\right| \leq \frac{\varrho^2|\cos\varphi|\,|\sin\varphi|}{\varrho|\cos\varphi|} = \varrho|\sin\varrho|,$$

which tends to 0 for $\varrho \to 0+$, hence

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Alternatively one can use directly that

$$|f(x,y) - 0| = \left|\frac{\sin(xy)}{x}\right| \leq \frac{|xy|}{|x|} = |y| \to 0$$

for $|y| \leq \sqrt{x^2+y^2} \to 0$.

2) The domain is the same as in 1).
The limit does not exist, because e.g.

$$f(x,x) = \frac{\sin x}{x} \to 1 \qquad \text{for } x \to 0,$$

$$f(x,-x) = -\frac{\sin x}{x} \to -1 \qquad \text{for } x \to 0.$$

3) The function is defined for $y \neq 0$, i.e. at the points outside the X axis. There is no need either to sketch this set.

   The limit is 0, because

$$|f(x,y) - 0| = |x| \cdot \left|\sin \frac{1}{y}\right| \leq |x| \to 0 \qquad \text{for } (x,y) \to (0,0).$$

4) The vector function is defined (and continuous), when
   a) $1 + x^2 + y^2 > 0$ (always fulfilled),
   b) $x^2 + y^2 > 0$ (i.e. $(x,y) \neq (0,0)$),
   c) $x > 0$,
   d) $y > 0$,
   e) $xy > 0$,
   f) $xy \neq 1$.

   Summarizing we see that the domain is the open first quadrant, with the exception of a branch of a hyperbola,

$$D = \{(x,y) \mid x > 0,\ y > 0\} \setminus \{(x,y) \mid xy = 1\}.$$

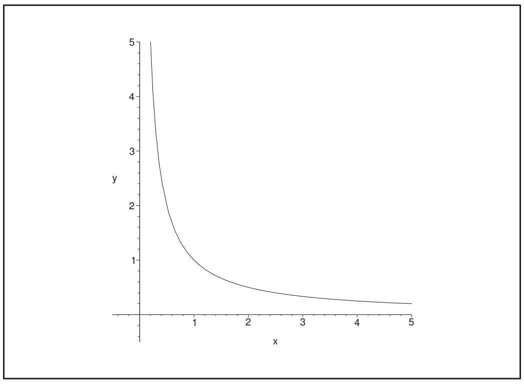


Figure 44: The vector function is defined in the first quadrant with the exception of the branch of the hyperbola.

Since

$$\begin{aligned} \frac{\ln(1+x^2+y^2)}{\sqrt{x^2+y^2}} &= \frac{1}{\sqrt{x^2+y^2}}\left\{(x^2+y^2) + (x^2+y^2)\varepsilon(x^2+y^2)\right\} \\ &= \sqrt{x^2+y^2}\{1 + \varepsilon(x^2+y^2)\} \to 0 \end{aligned}$$

for $(x,y) \to (0,0)$, and

$$\frac{\ln x + \ln y}{\ln(xy)} = \frac{\ln(xy)}{\ln(xy)} = 1 \qquad \text{for } (x,y) \in D,$$

we conclude that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D}} f(x,y) = (0,1).$$

5) The vector function is defined for $(x,y) \neq (0,0)$.

Let us estimate the first coordinate function,

$$\left|\frac{x\sin t}{\sqrt{x^2+y^2}}\right| = \frac{|x|}{\sqrt{x^2+y^2}}\,|\sin y| \leq 1\cdot|\sin y| \to 0$$

for $(x,y)\to(0,0)$. We see that the first coordinate function converges towards 0 by the limit.

In the examination of the second coordinate function we use polar coordinates $0<\varphi<\dfrac{\pi}{2}$, $\varrho>0$. We get by insertion

$$\frac{x^2y^2+x^2+y^2}{x^2+3y^2} = \frac{\varrho^4\cos^2\varphi\cdot\sin^2\varphi+\varrho^2}{\varrho^2(1+2\sin^2\varphi)} = \frac{1}{1+2\sin^2\varphi}+\varrho^2\cdot\frac{\sin^2\varphi\cos^2\varphi}{1+2\sin^2\varphi}.$$

The latter term converges towards 0 for $\varrho \to 0$; but the first term depends on φ and not on ϱ.

Since the second coordinate function cannot be extended continuously to $(0,0)$, neither can the vector function itself be extended continuously to $(0,0)$.

6) The vector function

$$\mathbf{f}(x,y) = \left(\frac{x}{x+y}, \sqrt{x+y} \right)$$

is defined for $x+y \neq 0$ and $x+y \geq 0$, so the domain is

$$\{(x,y) \mid x+y > 0\}.$$

The first coordinate function does not have a limit for $(x,y) \to (0,0)$ in the domain. In fact

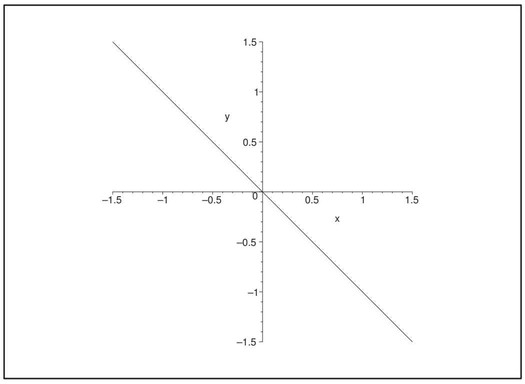


Figure 45: **Example 5.3.5**. The domain is the half plane which lies above the line.

if we in particular restrict ourselves to the positive X axis where $y = 0$, then

$$\lim_{x \to 0+} f_1(x,0) = \lim_{x \to 0+} \frac{x}{x+0} = 1.$$

If we instead restrict ourselves to the positive Y axis we get

$$\lim_{y \to 0+} f_1(0,y) = \lim_{y \to 0+} \frac{0}{0+y} = 0.$$

Since $1 \neq 0$, the limit does not exist.

Example 5.4 *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *be given by*

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}.$$

Show that

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0,$$

and that f *nevertheless* does not *have a limit for* $(x,y) \to (0,0)$.

A Limits.

D Calculate the successive limits and finally the limit along the line $y = x$.

I If $x \neq 0$, then

$$x^2y^2 \to 0 \text{ and } x^2y^2 + (x-y)^2 \to x^2 \neq 0 \qquad \text{for } y \to 0,$$

hence

$$\lim_{y\to 0} f(x,y) = 0 \qquad \text{for } x \neq 0.$$

Note also that

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

Since $f(x,y) = f(y,x)$, it follows immediately that

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0.$$

Then consider the limit $(x,y) \to (0,0)$ along the line $y = x$. This is given by

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^4}{x^4 + 0^2} = 1 \neq 0.$$

We conclude that f does not have a limit for $(x,y) \to (0,0)$.

Example 5.5 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x,y) = \begin{cases} \sin \dfrac{1}{x} \sin y, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that $f(x,y) \to 0$ *for* $(x,y) \to (0,0)$*; and that we nevertheless do not have*

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right).$$

A Limits.

D Use the definition of a limit in 1), and the rules of calculations in 2).

I If $x \neq 0$, then

$$|f(x,y) - f(0,0)| = \left|\sin \frac{1}{x}\right| \cdot |\sin y| \leq |\sin y| \to 0 \quad \text{for } (x,y) \to (0,0),$$

and it follows trivially for $x = 0$ that

$$|f(0,y) - f(0,0)| = 0 \to 0 \qquad \text{for } (x,y) \to (0,0).$$

We conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Then it follows immediately that

$$\lim_{y\to 0} f(x,y) = \begin{cases} \lim_{y\to 0} \sin \dfrac{1}{x} \cdot \sin y = 0, & \text{for } x \neq 0, \\ \lim_{y\to 0} 0 = 0, & \text{for } x = 0, \end{cases}$$

thus

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 0.$$

On the other hand, $\sin \dfrac{1}{x} \cdot \sin y$ for $y \neq p\pi$, $p \in \mathbb{Z}$, does not have a limit for $x \to 0$, so

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$$

is not defined.

Example 5.6 *Find the domain A of*

$$f(x,y)=\frac{xy}{x+y}.$$

Show that f cannot be continuously extended to a point set $B \supset A$.
Then let

$$D=\{(x,y)\mid 0\le x,\,0\le y,\,x^2+y^2>0\},$$

and consider the function $g : D \to \mathbb{R}$ given by

$$g(x,y)=\frac{xy}{x+y}.$$

Sketch D, and prove that g has a continuous extension to the point set $D\cup\{(0,0)\}$. Compare with the formula of the resulting resistance of a connection in parallel of two resistances.

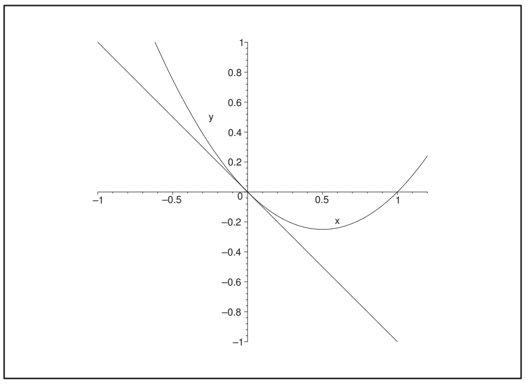
A Domain; continuous extension; limit.

D Find the point set, in which the denominator is 0, and then indicate A. Examine the limit in D.

I Clearly,

$$A = \{(x,y) \mid y \neq -x\}.$$

Furthermore, $(0,0)$ is the only point in which both the numerator and the denominator are zero, so there is only a possibility of a continuous extension to the set $A \cup \{(0,0)\}$.



When we restrict ourselves to the curve $y = -x + x^2$, $x \neq 0$, we get

$$\lim_{x \to 0} f(x, -x+x^2) = \lim_{x \to 0} \frac{-x^2+x^3}{x^2} = -1.$$

On the other hand, it is obvious that $f(x,0) = 0 \to 0$ for $x \to 0$, so we get two different limits by approaching $(0,0)$ along two different curves. Hence, the limit does not exist, and f cannot be extended continuously.

The set D is the closed first quadrant with the exception of the point $(0,0)$. Since $x \geq 0$ and $y \geq 0$ in D, we have the estimate

$$0 < \max\{x,y\} \leq x + y \qquad \text{for every } (x,y) \in D,$$

and thus

$$|g(x,y) - 0| = \left| \frac{x}{x+y} \right| \cdot |y| \leq |y| \to 0 \qquad \text{for } (x,y) \to (0,0) \text{ i } D.$$

This shows that g can be extended continuously to $(0,0)$ by defining $g(0,0) = 0$.

By the rearrangement

$$\frac{1}{g(x,y)} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}, \qquad x > 0, \quad y > 0,$$

we get the connection to the formula of the resulting resistance for a connection in parallel. From the above follows that

$$g(x,y) = \frac{xy}{x+y}$$

in D° can be extended to $D^\circ \cup \{(0,0)\}$.

6 Description of curves

Example 6.1 *In the following there are given some curves. In each case one shall find an equation of the curve by eliminating the parameter t. Indicate the name of the curve.*

1) $\mathbf{r}(t) = \left(a\,\dfrac{1-t^2}{1+t^2}, b\,\dfrac{2t}{1+t^2}\right)$*, for* $t \in \mathbb{R}$*.*

2) $\mathbf{r}(t) = \left(\dfrac{a}{\cos t}, b\,\tan t\right)$*, for* $t \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[\cup \left]\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right[$.

3) $\mathbf{r}(t) = \left(at^2, 2at\right)$*, for* $t \in \mathbb{R}$*.*

4) $\mathbf{r}(t) = (a\sin t, a\cos 2t)$*, for* $t \in [-\pi, \pi]$*.*

A Description of curves.

D Eliminate the parameter.

I 1) It follows from $x = a\,\dfrac{1-t^2}{1+t^2}$ and $y = b\,\dfrac{2t}{1+t^2}$ that

$$\frac{x}{a} = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad \frac{y}{b} = \frac{2t}{1+t^2},$$

where the idea is that the two right hand sides are independent of the arbitrary constants a and b.
We get by squaring and adding

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \frac{(1-2t^2+t^4)+4t^2}{(1+t^2)^2} = \frac{1+2t^2+t^t}{1+2t^2+t^4} = 1.$$

Thus the curve is a subset of an ellipse of centrum $(0,0)$ and the half axes a and b.

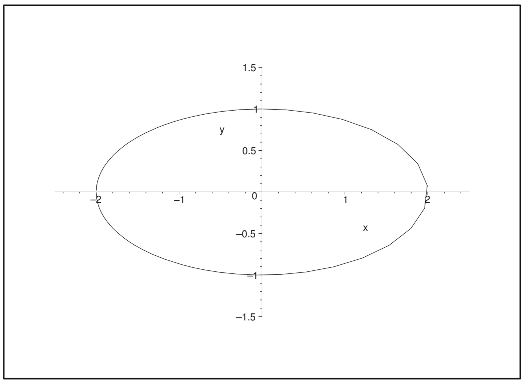


Figure 46: the curve for $a = 2$ and $b = 1$.

Now,

$$\frac{1-t^2}{1+t^2} = -1 + \frac{2}{1+t^2} \leq 1$$

with equality for $t=0$, so $\dfrac{1-t^2}{1+t^2}$ goes through the interval $]-1,1]$ (twice), when t goes through $\mathbb{R}$. Since $\dfrac{2t}{1+t^2}$ changes its sign for $t=0$, we conclude that the arc of the curve is the ellipse with the exception of the point $(-a,0)$.

2) It follows from $x = \dfrac{a}{\cos t}$ and $y = b\tan t$ that

$$\frac{x}{a} = \frac{1}{\cos t} \qquad \text{and} \qquad \frac{y}{b} = \frac{\sin t}{\cos t},$$

so the parameter t is eliminated by

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \frac{1-\sin^2 t}{\cos^2 t} = 1.$$

This describes an hyperbola of the half axes a and b and of centrum $(0,0)$. The two intervals

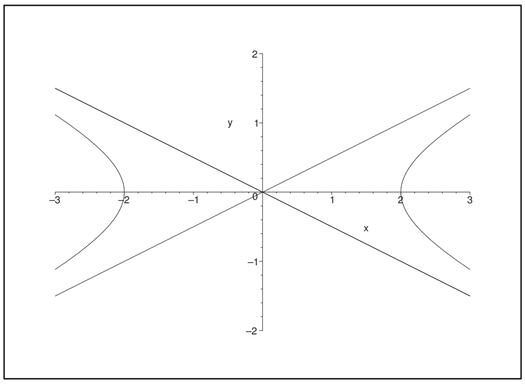


Figure 47: The curves for $a=2$ and $b=1$.

$\left]-\dfrac{\pi}{2},\dfrac{\pi}{2}\right[$ and $\left]\dfrac{\pi}{2},\dfrac{3\pi}{2}\right[$ corresponds to the two branches.

3) Here, $x = at^2$ and $y = 2at$, so $t = \dfrac{y}{2a}$. Then by insertion,

$$x = at^2 = \frac{1}{4a}\,y^2,$$

which is the equation of a parabola with top point $(0,0)$ and the X axis as its axis.

4) When

$$(x,y) = \mathbf{r}(t) = (a\sin t, a\cos 2t), \qquad t \in [-\pi,\pi],$$

and $a>0$, it follows that

$$y = a\cos 2t = a\left(1-2\sin^2 t\right) = 1 - \frac{2}{a}(a\sin t)^2 = a - \frac{2}{a}\,x^2,$$

i.e.

$$y = a - \frac{2}{a}\,x^2, \qquad x \in [-\pi,\pi],$$

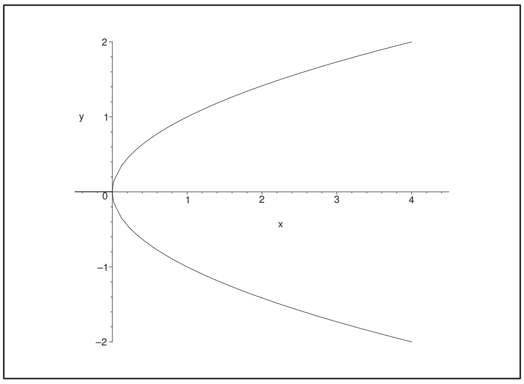


Figure 48: The curve for $a = \dfrac{1}{4}$.

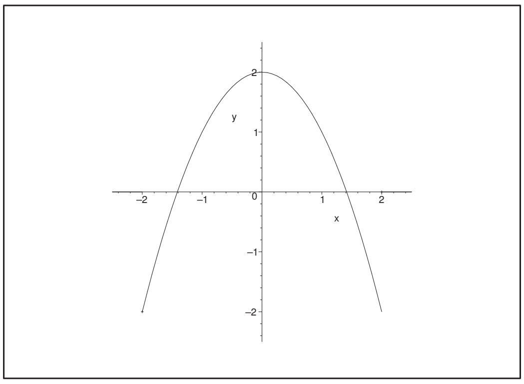


Figure 49: The curve for $a = 2$.

which is a part of a parabolic arc. Note that we use that

$$|x| = |a \sin t| \leq a,$$

when we find the domain $[-a, a]$, where we can have both $x = -a$ and $x = a$.

Example 6.2 *Prove that the curve given by*

$$\mathbf{r}(t) = \left(3r + t^2, t - t^2, 3 - 5t + t^2\right), \qquad t \in \mathbb{R},$$

lies in a plane, and find an equation of this plane.

A Space curve lying in a plane.

D Put the coordinate functions of the curve into the general equation of a plane and find the coefficients.

I In general the equation of a plane is given by

$$ax + by + cz = k.$$

Then by insertion of

$$(x, y, z) = (3t + t^2, t - t^2, 3 - 5t + t^2),$$

we get

$$\begin{aligned} k &= a(3t + t^2) + b(t - t^2) + c(3 - 5t + t^2) \\ &= t^2(a - b + c) + t(3a + b - 5c) + 3c, \qquad t \in \mathbb{R}. \end{aligned}$$

This should hold for every t, so we must necessarily have

$$\begin{cases} a - b + c &= 0, \\ 3a + b - 5c &= 0, \\ 3c &= k. \end{cases}$$

It follows that if $k = 0$, then we only get $(a, b, c) = (0, 0, 0)$ as a solution.

By choosing $k \neq 0$, e.g. $k = 3$, we get $c = 1$, and then by insertion

$$\begin{cases} a - b &= -c = -1, \\ 3a + b &= 5c = 5. \end{cases}$$

An addition shows that $4a = 4$, i.e. $a = 1$, and it follows that $b = 2$.

Thus an equation of the plane is

$$x + 2y + z = 3,$$

and we have at the same time proved that the curve lies in this plane.

Example 6.3 *Prove that the curve given by*

$$\mathbf{r}(t) = \left(2t\sqrt{1-t}, 2(1-t)\sqrt{t}, 1-2t\right), \qquad t \in [0, 1],$$

lies on a sphere of centrum $(0, 0, 0)$*.*

A A space curve lying on a sphere.

D Put the coordinate functions into the equation of the sphere and find its radius r.

I The general equation of a sphere of centrum $(0, 0, 0)$ is

$$x^2 + y^2 + z^2 = r^2.$$

By putting

$$x = 2r\sqrt{1-t}, \qquad y = 2(1-t)\sqrt{t}, \qquad z = 1 - 2t,$$

we get

$$\begin{aligned} x^2 + y^2 + z^2 &= 4t^2(1-t) + 4(1-t)^2 t + (1-2t)^2 \\ &= 4t(1-t)\{t + (1-t)\} + (1-2t)^2 \\ &= (4t - 4t^2) + (1 - 4t + 4t^2) = 1, \end{aligned}$$

and we conclude that the curve lies on the unit sphere.

Example 6.4 *Prove that the curve given by*

$$\mathbf{r}(t) = \left(a(1-\sin t)\cos t, b(\sin t+\cos^2 t), c\cos t\right), \qquad t \in [-\pi, \pi],$$

lies on an hyperboloid.

A A space curve lying on an hyperboloid.

D Calculate $\left(\dfrac{x}{a}\right)^2$, $\left(\dfrac{y}{b}\right)^2$ and $\left(\dfrac{z}{c}\right)^2$, which are three expressions which are independent of the constants a, b and c. Then compare.

I We calculate

$$\begin{aligned}
\left(\frac{x}{a}\right)^2 &= (1-\sin t)^2\cos^2 t &&= (1-2\sin t+\sin^2 t)\cos^2 t\\
& &&= \cos^2 t - 2\sin t\cos^2 t + \sin^2 t\cos^2 t,\\
\left(\frac{y}{b}\right)^2 &= (\sin t+\cos^2 t)^2 &&= \sin^2 t + 2\sin t\cos^2 t + \cos^4 t\\
\left(\frac{z}{c}\right)^2 &= &&= \cos^2 t.
\end{aligned}$$

Thus

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1+\cos^2 t = 1+\left(\frac{z}{c}\right)^2,$$

and by a rearrangement

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^1 - \left(\frac{z}{c}\right)^2,$$

and we conclude that the curve lies on an hyperboloid with one net.

Example 6.5 *Sketch the so-called* cycloid *given by*

$$\mathbf{r}(t) = (a(t-\sin t), a(1-cost)), \qquad t \in \mathbb{R}.$$

A Sketch of a curve.

D If one does not have MAPLE at hand, start by finding some points of the curve. One may exploit the geometrical meaning of

$$\mathbf{r}(t) = a(t,1) - a(\sin t, \cos t), \qquad t \in \mathbb{R},$$

where the former term on the right hand side is a rectilinear and even movement, while the latter term is a circular movement. Thus the curve describes the movement of a point on a wheel, which is rolling along the X axis.

I Clearly, $\mathbf{r}(t)$ is periodical of period 2π, so it suffices to sketch one period and a little bit of the neighbouring periods.

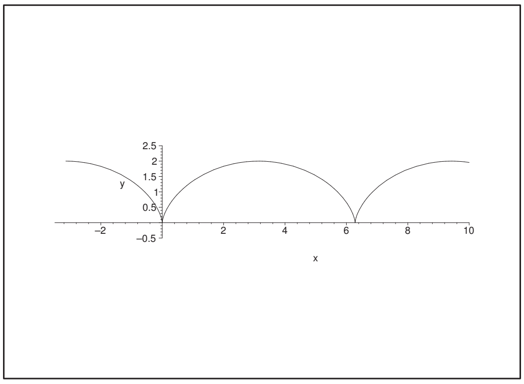


Figure 50: The cycloid for $a = 1$.

Example 6.6 *Find in each of the cases below an equation of the given curve by eliminating the parameter t, and then sketch the curve.*

1) $\mathbf{r}(t) = \left(\dfrac{3t}{1+t^3}, \dfrac{3t^2}{1+t^3}\right)$, *for* $t \in \mathbb{R} \setminus \{-1\}$.

2) $\mathbf{r}(t) = (\cos t, \sin t \cos t)$, *for* $t \in \mathbb{R}$.

3) $\mathbf{r}(t) = \left(a\,\cos^3 t, a\,\sin^3 t\right)$, *for* $t \in [-\pi, \pi]$.

4) $\mathbf{r}(t) = \left(a(1-3t^2), at(3-t^2)\right)$, *for* $t \in \mathbb{R}$.

A Description of curves.

D Eliminate the parameter.

I 1) When $t \neq -1$, we get

$$x = \frac{3t}{1+t^3} \qquad \text{and} \qquad y = \frac{3t^2}{1+t^3}.$$

For $t = 0$ we get the point $(x, y) = (0, 0)$.
For $t \neq 0$ and $t \neq -1$ we get $t = \dfrac{y}{x}$, where $x \neq 0$ and $y \neq 0$, so by insertion

$$x = \frac{3t}{1+t^3} = \frac{3y/x}{1+(y/x)^3} = \frac{3x^2y}{x^3+y^3}.$$

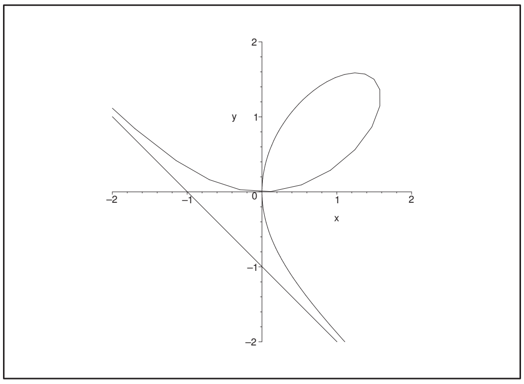
When $x \neq 0$ this is reduced to

$$x^3 + y^3 = 3xy.$$

Finally, we see that $(x, y) = (0, 0)$, which corresponds to $t = 0$, also satisfies this equation, so we can remove the restriction.
Note that the line $y = x$ is an axis of symmetry.

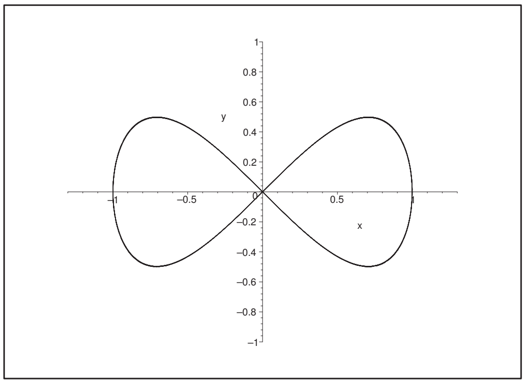
2) Here, $x = \cos t$ and $y = \sin t \cos t$, hence

$$y^2 = \sin^2 t \cos^2 = (1 - \cos^2 t)\cos^2 = (1-x)x^2,$$



or written more conveniently,

$$y^2 = (1 - x^2)x^2, \qquad \text{hence } y = \pm|x|\sqrt{1 - x^2}, \quad x \in [-1, 1].$$



3) From

$$x = a\,\cos^3 t, \qquad y = a\,\sin^3 t,$$

we get by elimination

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\left\{\cos^2 t + \sin^2 t\right\} = a^{\frac{2}{3}}.$$

Note that

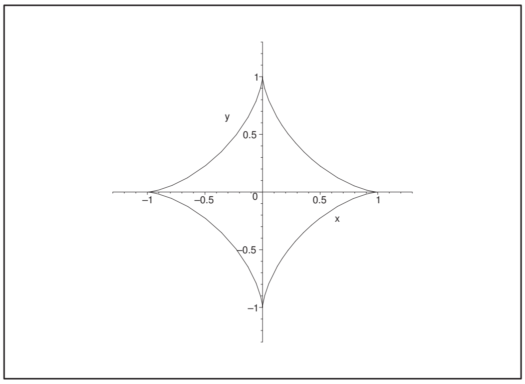


Figure 51: The curve for $a = 1$.

$$\mathbf{r}'(t) = 3a\,\sin t \cdot \cos(-\cos t, \sin t)$$

is $\mathbf{0}$ for $t = p \cdot \dfrac{-\pi}{2}$, $p = -2,\,-1,\,0,\,1,\,2$, corresponding to the cusps on the curve.

4) First notice that $\mathbf{r}'(t) = a(-6t, 3-3t^2)$, so $y'(t) = 0$ for $t = \pm 1$. It follows from

$$x(t) = a(1-3t^2) \qquad \text{and} \qquad y(t) = at(3-t^2)$$

that $x(t)$ is largest for $t = 0$, corresponding to $x(t) \le x(0) = a$. For this value the point on the curve is $\mathbf{r}(0) = (a, 0)$.

Furthermore, we see that the X axis is an axis of symmetry.

Notice

a) that $x(t) = 0$ for $t = \pm\frac{1}{\sqrt{3}}$, corresponding to

$$(x, y) = \left(0, \pm\frac{8a}{3\sqrt{3}}\right),$$

b) that the curve has a horizontal tangent for $y'(t) = 0$, i.e. for $t = \pm 1$, corresponding to

$$(x, y) = (-2a, \pm 2a),$$

c) and that $y(t) = 0$ for $t = 0$ and $t = \pm\sqrt{3}$, corresponding to

$$(0, 0) \qquad \text{and} \qquad (-8a, 0).$$

d) that y and t have the same sign for $0 < |t| < \sqrt{3}$, and opposite sign for $|t| > \sqrt{3}$. The latter means that we are allowed to square by the elimination of t.

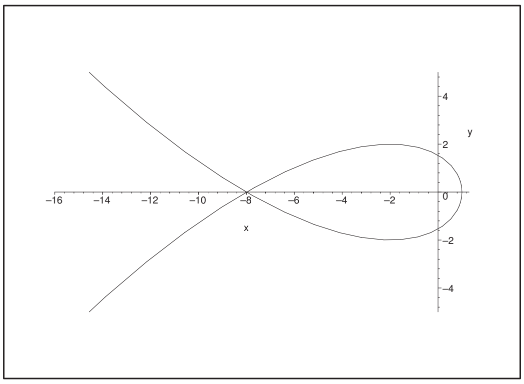


Figure 52: The curve for $a = 1$.

It follows from

$$\frac{x}{a} = 1 - 3t^2 \qquad \text{and} \qquad \frac{y}{a} = t(3 - t^2)$$

that

$$t^2 = \frac{1}{3}\left\{1 - \frac{x}{a}\right\},$$

so we finally get by a squaring,

$$\frac{y^2}{a^2} = t^2\left(3 - t^2\right)^2 = \frac{1}{3}\left\{1 - \frac{x}{a}\right\}\left(3 - \frac{1}{3}\left\{1 - \frac{x}{a}\right\}\right)^2 = \frac{1}{27}\left(1 - \frac{x}{a}\right)\left(8 + \frac{x}{a}\right)^2,$$

thus

$$y^2 = \frac{1}{27a}(a - x)(8a + x)^2.$$

Notice that $|y|$ tends faster towards $+\infty$ than $|x|$ for $|t| \to +\infty$.

Example 6.7 *Sketch the point set A in the first quadrant of the plane, which is bounded by the three curves given by*

$$\mathbf{r}(t) = (\cos t, 1 + \sin t), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathbf{r}(t) = (1 + \cos t, sint), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathbf{r}(t) = (2\cos t, 2\sin t), \quad t \in \left[0, \frac{\pi}{2}\right].$$

A A set bounded by given curves.

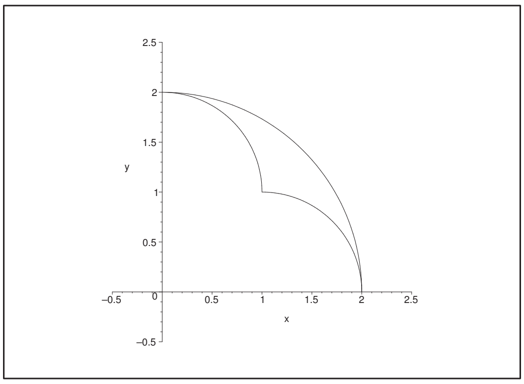
D Identify the curves and sketch the set.

I All three curves are quarter circles, which follows from

$$\mathbf{r}_1(t) = (0,1) + (\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathbf{r}_2(t) = (1,0) + (\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathbf{r}_3(t) = (0,0) + 2(\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2}\right].$$



Example 6.8 *Let α be a non-negative constant, and let the curve $\mathcal{K}$ be given by the equation*

$$\varrho = \frac{c}{1 + a\,\cos\varphi}, \qquad \varphi \in I,$$

where I is a symmetric interval around the point 0, which is as big as possible. Prove that $\mathcal{K}$ is (a part of) a conic section.

A Conic section in polar coordinates.

D Multiply by the denominator and reduce to rectangular coordinates, where c as usual denotes some positive constant.

I Since $\varrho \geq 0$ and $c > 0$, we must have $1 + \alpha\cos\varphi > 0$. Therefore, in order to find I we must find possible zeros of the denominator, i.e. we shall examine the equation

$$1 + \alpha\cos\varphi = 0, \qquad \text{i.e. } \cos\varphi = -\frac{1}{\alpha}.$$

Since $\alpha \geq 0$, we have to distinguish between the cases

$$\alpha = 0, \qquad 0 < \alpha < 1, \qquad \alpha = 1, \qquad \alpha > 1.$$

1) If $\alpha = 0$, then $\varrho = c$, which is the polar equation of a circle of radius $c > 0$.
The circle is clearly a conic section, and $I = \mathbb{R}$.

2) If $0 < \alpha < 1$, then $1 + \alpha \cos\varphi \geq 1 - \alpha > 0$ for every φ, and the denominator is always positive, and we get $I = \mathbb{R}$. When we multiply by the denominator we get by using rectangular coordinates,

$$c = \varrho + \alpha\varrho\cos\varphi = \alpha x,$$

hence by a rearrangement,

$$\sqrt{x^2+y^2} = c - \alpha x \geq 0.$$

We get in particular the condition $x < \dfrac{c}{\alpha}$, which should be checked at the very end of this example.
When this restriction is satisfied we can square, thus obtaining

$$x^2 + y^2 = c^2 - 2\alpha c x + \alpha^2 x^2.$$

Then by a rearrangement,

$$(1-\alpha^2)x^2 + 2\alpha c x + y^2 = c^2, \qquad 0 < \alpha < 1,$$

i.e.

$$(1-\alpha^2)\left\{x^2 + \frac{2\alpha c}{1-\alpha^2}\,x + \left(\frac{\alpha c}{1-\alpha^2}\right)^2\right\} + y^2 = c^2 + \frac{\alpha^2 c^2}{1-\alpha^2} = \frac{c^2}{1-\alpha^2}.$$

This can be written in the canonical way

$$\left\{\frac{x + \dfrac{\alpha c}{1-\alpha^2}}{\dfrac{c}{1-\alpha^2}}\right\}^2 + \left\{\frac{y}{\dfrac{c}{\sqrt{1-\alpha^2}}}\right\}^2 = 1.$$

This is the equation of an ellipse, hence a conic section of

$$\text{centrum: } \left(-\frac{\alpha c}{1-\alpha^2}, 0\right) \qquad \text{and half axes: } \frac{c}{1-\alpha^2} \quad \text{and} \quad \frac{c}{\sqrt{1-\alpha^2}}.$$

Note that

$$-\frac{\alpha c}{1-\alpha^2} + \frac{c}{1-\alpha^2} = \frac{c}{1+\alpha} < \frac{c}{\alpha} \quad \text{for } 0 < \alpha < 1,$$

and we conclude that the earlier restriction for the squaring is automatically fulfilled.

3) If $\alpha = 1$, the denominator is $1 + \cos\varphi = 0$ for $\varphi =$ an odd multiple of π, and > 0 otherwise. The wanted interval is $I =]-\pi, \pi[$.

By a multiplication by the denominator we get

$$c = \varrho + \varrho\cos\varphi = \sqrt{x^2+y^2} + x,$$

hence

$$\sqrt{x^2+y^2} = c - x \geq 0, \qquad \text{dvs. } x \leq c.$$

Under this condition we get by squaring,

$$x^2+y^2=c^2-2cx+x^2,$$

so after some reduction we obtain the equation of the parabola

$$x=-\frac{1}{2c}\,y^2+\frac{c}{2}.$$

Clearly, this expression is $\leq c$, so $\mathcal{K}$ is the whole of the parabola, and a parabola is also a conic section.

4) If $\alpha>1$, then $1+\alpha\cos\varphi=0$ for

$$\cos\varphi=-\frac{1}{\alpha}\in\,]-1,0[,$$

i.e. the largest possible symmetric domain interval I is

$$I=\left]-\text{Arccos}\left(-\frac{1}{\alpha}\right),\text{Arccos}\left(-\frac{1}{\alpha}\right)\right[.$$

In this interval we get as in 2) that

$$\sqrt{x^2+y^2}=c-\alpha x\geq 0,\qquad \text{i.e. } x\leq\frac{c}{\alpha},$$

and the calculations are then continued in the usual way under this assumption by a squaring,

$$x^2+y^2=c^2+\alpha^2x^2-2\alpha cx\qquad \text{for } x\leq\frac{c}{\alpha}.$$

Then by a rearrangement,

$$(1-\alpha^2)\left\{x^2+\frac{2\alpha c}{1-\alpha^2}\,x+\left(\frac{\alpha c}{1-\alpha^2}\right)^2\right\}+y^2=\frac{c^2}{1-\alpha^2}<0,$$

thus by norming

$$\left\{\frac{x-\dfrac{\alpha c}{\alpha^2-1}}{\dfrac{c}{\alpha^2-1}}\right\}^2-\left\{\frac{y}{\dfrac{c}{\sqrt{\alpha^2-1}}}\right\}^2=1.$$

Thus, for $\varphi\in I$ we get an arc of an hyperbola, which again is a conic section.

7 Connected sets

Example 7.1 *Examine if the point set*

$$A = \{(x,y) \mid (x^2+y^2+2x)(y^2-x) < 0\}$$

is connected.

A Connected set.

D First find the boundary curves of A. Sketch a figure.

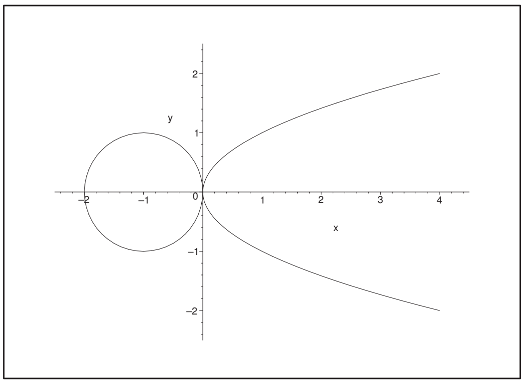


Figure 53: The set A consists of the points which either lies inside the circle or inside the parabola.

I Since $(x^2+y^2+2x)(y^2-x)$ is continuous in $\mathbb{R}^2$, the boundary ∂A is given by

$$0 = (x^2+y^2+2x)(y^2-x) = \{(x+1)^2+y^2-1\}(y^2-x),$$

i.e. the boundary is composed of the circle of equation

$$(x+1)^2+y^2 = 1$$

of centrum $(-1,0)$ and radius 1, and the parabola of equation $x = y^2$. The plane $\mathbb{R}^2$ is in this way divided into three subregions in which $f(x,y)$ due to the continuity must have a fixed sign in each of these.

The set A is characterized by the condition $f(x,y) < 0$.

Inserting the centrum $(-1,0)$ of the circle we get

$$f(-1,0) = -1 \cdot 1 = -1 < 0,$$

so by the continuity it follows that the open disc is contained in A.

The point $(1,0)$ lies inside the parabola, and the value is

$$f(1,0) = 3 \cdot (-1) = -3 < 0,$$

so the interior of the parabola is also a subset of A.

This is now sufficient to declare that the set is not connected, because it is impossible to connect $(-1,0) \in A$ with $(1,0) \in A$ by any continuous curve without intersecting at least one of the zero curves, which do *not* lie in A. We therefore conclude that A is not connected.

Remark. Since $(0,1)$ is a point in the latter component, and

$$f(0,1) = 1 \cdot 1 = 1 > 0,$$

the third component of $\mathbb{R}^2$ does not contain any point from A, and A consists of precisely the union of the open disc and the open interior of the parabola. However, one was not asked this question. $\lozenge$

Example 7.2 *Give an example of a point set which fulfils the following condition: A is not connected, but its closure $\overline{A}$ is connected.*

A Connected sets.

D Analyze the concept of connected sets and give examples.

I According to **Example 7.5** below an extreme example is $A = \mathbb{Q}$, which is not connected in $\mathbb{R}$, while $\overline{A} = \mathbb{R}$ is connected.

A simpler example is $A = \mathbb{R} \setminus \{0\}$ where $\overline{A} = \mathbb{R}$.

Another example is given by **Example 7.1**, because one by the closure also include the point $(0, 0)$, which can be reached by a continuous curve from both components.

Example 7.3 *Show by an example that two connected point sets A and B do not necessarily have a connected intersection.*

A Connected sets.

D Sketch "amoebe" in the plane.

I Sketch two "half moons" which only intersect in their tips, we see that the intersection has got two components, and the intersection is not connected. Clearly, each "half moon" is connected.

The sketches are left to the reader.

Example 7.4 *Examine if the domain of the function*

$$f(x, y) = \operatorname{Arcsin}(x^2 + y^2 - 3)$$

is imply connected.

A Simply connected sets.

D Find the domain and analyze.

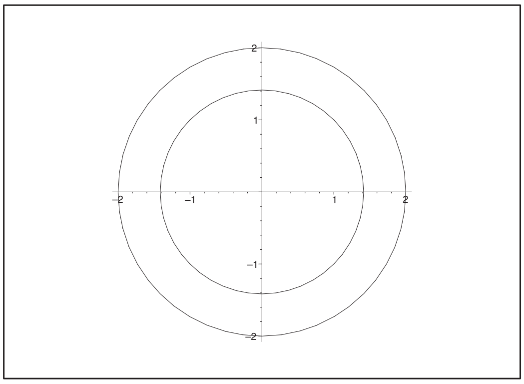
I The function $f(x, y)$ is defined for

$$-1 \le x^2 + y^2 - 3 \le 1,$$

i.e. for

$$2 \le x^2 + y^2 \le 4.$$

This set is an annular set (a set containing a "hole") of inner radius $\sqrt{2}$ and outer radius 2. This set is clearly not simply connected.



Example 7.5 *Prove that the set of rational numbers is not connected. Formulate a similar result for a set in the plane.*

A Connected sets.

D Analyze the definition of connected sets.

I Let x, $y \in \mathbb{Q}$, where e.g. $x < y$. Then every continuous curve in $\mathbb{R}$, which connects x and y, will then contain the interval $[x, y]$, which also contains irrational numbers, i.e. points outside $\mathbb{Q}$. We conclude that $\mathbb{Q}$ is not connected.

The set $\{(x, y) \mid x \in \mathbb{Q},\, y \in \mathbb{Q}\}$ is not connected in the plane.

Example 7.6 *Check in each of the cases below if the domain of the given function is connected.*

1) $f(x, y, z) = \ln|1 - x^2 - y^2 - z^2|$.

2) $f(x, y, z) = \ln(1 - x^2 - y^2 - z^2)$.

3) $f(x, y, z) = \sqrt{y^2 - x^2} + \sqrt{z^2 - 1}$.

4) $f(x, y, z) = \sqrt{y - x} + \sqrt{z - 1}$.

5) $f(x, y, z) = \ln(1 - y^2) + \sqrt{x^2 - 4} + \sqrt{9 - x^2}$.

A Connected domains.

D First find the domain. Then analyze.

1) The function is defined for $x^2 + y^2 + z^2 \neq 1$, i.e. everywhere with the exception of the unit sphere. The set is obviously falling into two connected components, so it is not connected.

2) In this case the domain is the open unit ball, which is connected.

3) It suffices to realize that the domain has one part lying in the half space $z \geq 1$ and another part in the half space $z \leq -1$ and no point in between. Hence the set is not connected.

4) The domain is given by $y \geq x$ and $z \geq 1$, i.e. the union of two *half spaces* (convex sets) and thus connected.

5) The function is independent of z, and defined for

$$1-y^2>0, \qquad x^2-4\geq 0, \qquad 9-x^2\geq 0,$$

so the domain is

$$[-3,-2]\times\,]-1,1[\,\times\mathbb{R}\cup[2,3]\times\,]-1,1[\,\times\mathbb{R}.$$

This set contains two connected components, hence it is not connected itself.