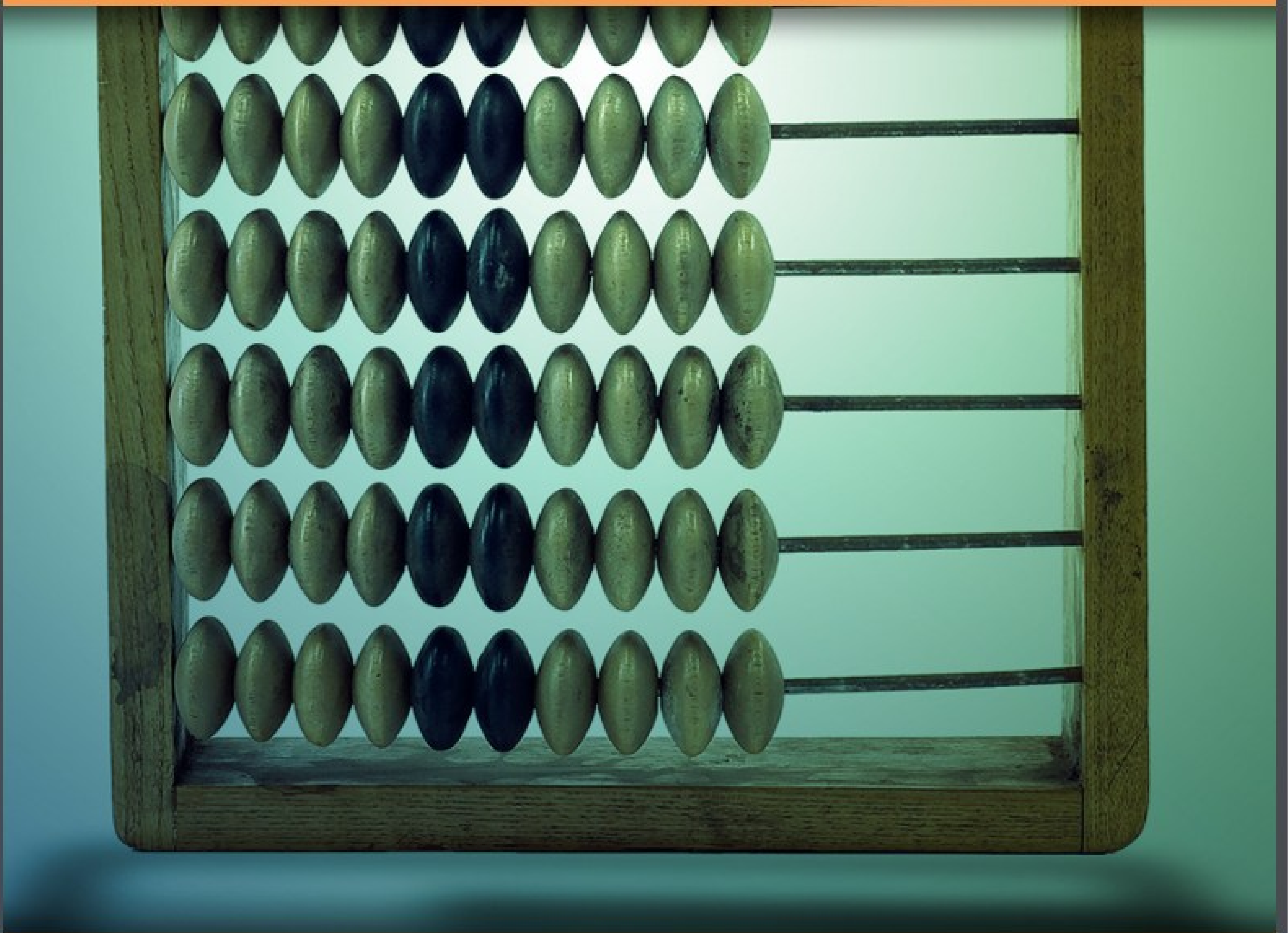
Real Functions of Several Variables - Tangents...

Leif Mejlbro



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Leif Mejlbro

Real Functions of Several Variables

Examples of Tangents to Curves, Tangent Planes to Surfaces Elementary Integrals Extrema

Calculus 2c-3

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Preface

In this volume I present some examples of *tangents to curves, tangent planes to surfaces, elementary integrals* and *Extrema*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
17th October 2007

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1 Tangents to curves

Eksempel 1.1 *Find in each of the following cases an equation or a parametric description of the tangent to the given curve at the given point.*

1) *The curve is given by* $x^3 - y^3 + 2x - 3y + 1 = 0$ *and the point is* $(1, 1)$.

2) *The curve is given by* $x^y - y^x = 0$ *and the point is* $(2, 4)$.

3) *The curve is given by* $\mathbf{r}(t) = (\cos t, \sin t, e^t)$ *and the point is* $(1, 0, 1)$.

4) *The curve is given by* $\mathbf{r}(t) = (t - \sin t, 1 - cost)$, *and the point is* $\left(\frac{\pi}{2} - 1, 1\right)$, [cf. **Example 1.3.3**]

5) *The curve is given by* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \frac{5}{4}$ *and the point is* $\left(\frac{1}{8}, 1\right)$.

6) *The curve is given by* $\mathbf{r}(t) = \left(\ln t, \cos(t-1), 2t^4 - t^2\right)$ *and the point is* $(0, 1, 1)$.

7) *The curve is given by* $\mathbf{r}(t) = (\text{Arcsin}\, t, \text{Arctan}(2t), \text{Arccot}(2t))$ *and the point is* $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6}\right)$.

8) *The curve is given by* $\mathbf{r}(t) = (2\sin t, -\cos t, 3t)$ *and the point is* $(0, 1, 3\pi)$.

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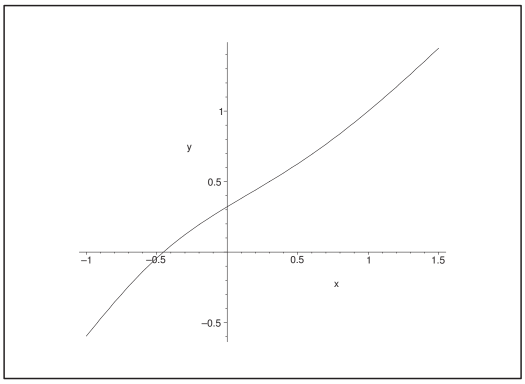
A Find the tangent to a curve at a point.

D First check if the point lies on the curve. Find the slope of the curve at the point. Write down the equation of the tangent. Notice that the case where the curve is given by an equation is treated differently from a curve given a parametric description.

I 1) By putting $(x, y) = (1, 1)$ into the equation we get $1^3 - 1^3 + 2 - 3 + 1 = 0$, proving that $(1, 1)$ lies on the curve.



Figur 1: The curve in 1).

When we differentiate the equation of the curve with respect to x we get

$$0 = -(3y^2 + 3)\frac{dy}{dx} + 3x^2 + 2.$$

When we here put $(x, y) = (1, 1)$, we get $\dfrac{dy}{dx} = \dfrac{5}{6}$, so the equation of the tangent becomes

$$y - 1 = \frac{5}{6}(x - 1).$$

2) By putting $(x, y) = (2, 4)$ into the equation of the curve we get $2^4 - 4^2 = 0$, proving that $(2, 4)$ lies on the curve. When we differentiate the equation of the curve with respect to x, we get

$$\begin{aligned} 0 &= \frac{d}{dx}(x^y - y^x) = \frac{d}{dx}\left(e^{y \ln x}\right) - \frac{d}{dx}\left(e^{x \ln y}\right) = x^y \frac{d}{dx}(y \ln x) - y^x \frac{d}{dx}(x \ln y) \\ &= x^y \left\{ \ln x \cdot \frac{dy}{dx} + \frac{y}{x} \right\} - y^x \left\{ \ln y + \frac{x}{y} \cdot \frac{dy}{dx} \right\}. \end{aligned}$$

When we put $(x, y) = (2, 4)$, we get

$$0 = 16\left\{ \ln 2 \cdot \frac{dy}{dx} + 2 \right\} - 16\left\{ 2 \ln 2 + \frac{1}{2}\frac{dy}{dx} \right\} = 16\left\{ \left(\ln 2 - \frac{1}{2} \right) \frac{dy}{dx} - (2 \ln 2 - 2) \right\},$$

hence

$$\frac{dy}{dx} = \frac{2 \ln 2 - 2}{\ln 2 - \frac{1}{2}} = 4 \cdot \frac{\ln 2 - 1}{2 \ln 2 - 1}.$$

The equation of the tangent becomes

$$y - 4 = -4 \cdot \frac{1 - \ln 2}{2 \ln 2 - 1}\,(x - 2).$$

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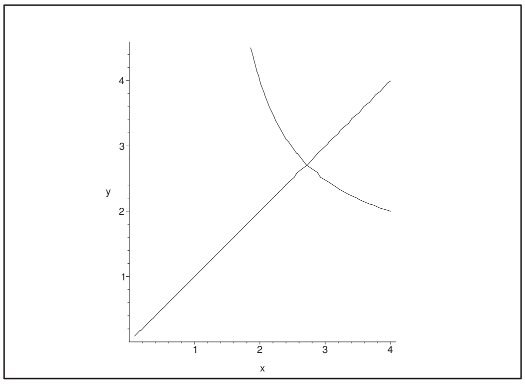
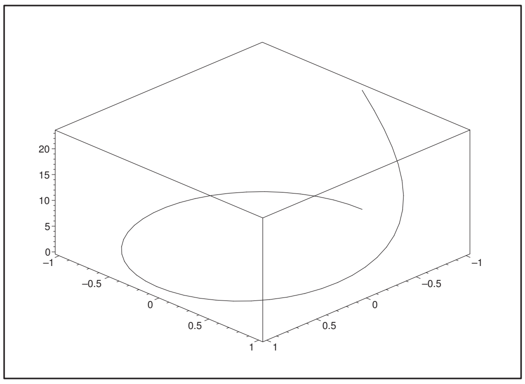


Figur 2: The curve in 2); note that the curve contains a self intersection.

3) It is immediately seen that $\mathbf{r}(0) = (1,0,1)$, so the point lies on the curve. Furthermore,



Figur 3: The curve in 3.

$$\mathbf{r}'(t) = (-\sin t, \cos t, e^t), \qquad \mathbf{r}'(0) = (0,1,1),$$

so a parametric description of the tangent is

$$(x(u), y(u), z(u)) = \mathbf{r}(0) + u \cdot \mathbf{r}'(0) = (1,0,1) + u(0,1,1), \qquad u \in \mathbb{R}.$$

4) It follows immediately that $\mathbf{r}\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2} - 1, 1\right)$, proving that the point lies on the curve corresponding to the value of the parameter $t = \frac{\pi}{2}$. Furthermore,

$$\mathbf{r}'(t) = (1 - \cos t, \sin t), \qquad \mathbf{r}'\left(\frac{\pi}{2}\right) = (1,1).$$

A parametric description of the tangent is

$$(x(u), y(u)) = \left(\frac{\pi}{2}, 1\right) + u \cdot (1,1), \qquad u \in \mathbb{R}.$$

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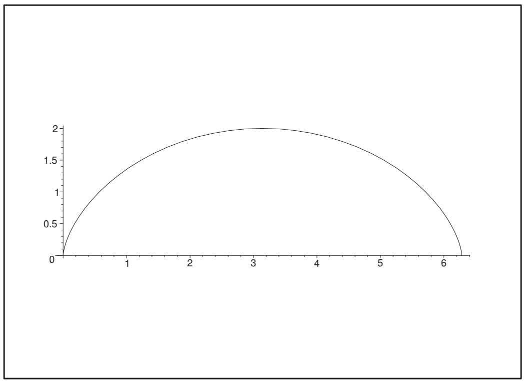
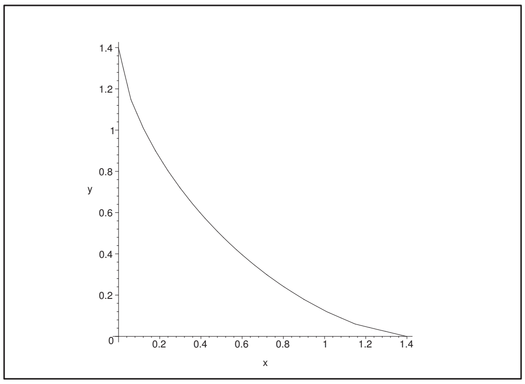


Figur 4: The curve in 4.

5) It follows from

$$\left(\frac{1}{8}\right)^{\frac{2}{3}} + 1^{\frac{2}{3}} = \left(\frac{1}{2}\right)^{3\cdot\frac{2}{3}} + 1 = \frac{1}{4} + 1 = \frac{5}{4},$$

that the point $\left(\frac{1}{8}, 1\right)$ lies on the curve.



Figur 5: The curve in 5.

If we put $f(x,y) = x^{2/3} + y^{2/3}$, then

$$\nabla f(x,y) = \frac{2}{3}\left(\frac{1}{\sqrt[3]{x}}, \frac{1}{\sqrt[3]{y}}\right) \qquad \text{for } x \neq 0 \text{ og } y \neq 0,$$

and hence

$$\nabla f\left(\frac{1}{8}, 1\right) = \frac{2}{3}(2,1),$$

which indicates the direction of the *normal* of the curve at the point. The direction of the tangent is then perpendicular to the normal ∇f, e.g. $(1,-2)$.

A parametric description of the tangent is

$$(x(t), y(t)) = \left(\frac{1}{8}, 1\right) + t\,(1, -2) = \left(t + \frac{1}{8}, -2t + 1\right), \qquad t \in \mathbb{R}.$$

This implies that $t = x - \frac{1}{8}$, so $y = 1 - 2t = 1 - 2x + \frac{1}{4} = \frac{5}{4} - 2x$. Finally, we can write the equation of the tangent

$$y + 2x = \frac{5}{4}.$$

ALTERNATIVELY the equation $f(x, y) = \frac{5}{4}$ is differentiated with respect to x. Then

$$\frac{2}{3}\,x^{-\frac{1}{3}} + \frac{2}{3}\,y^{-\frac{1}{3}}\,\frac{dy}{dx} = 0 \qquad \text{for } x \neq 0 \text{ og } y \neq 0.$$

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At the point $\left(\dfrac{1}{8}, 1\right)$ we find the slope

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}} = -2,$$

so the equation of the tangent becomes

$$y - 1 = -2\left(x - \frac{1}{8}\right) = -2x + \frac{1}{4},$$

i.e.

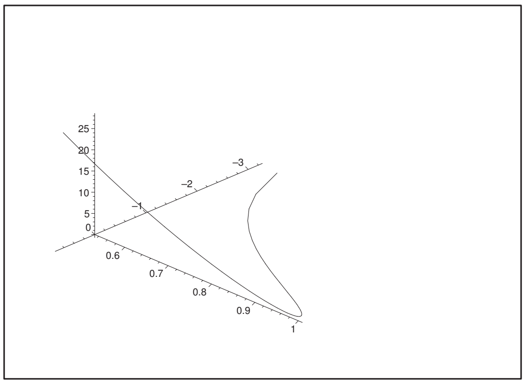
$$y + 2x = \frac{5}{4}.$$

Remark. The methods fail when either $x = 0$ or $y = 0$. This is in accordance with the fact that we have cusps in the corresponding points of the curve. $\Diamond$

6) Putting $t = 1$ we get

$$\mathbf{r}(1) = (\ln 1, \cos(1-1), 2 \cdot 1^4 - 1^2) = (0, 1, 1),$$

so $(0, 1, 1)$ lies on the curve. Furthermore,



Figur 6: The curve in 6).

$$\mathbf{r}'(t) = \left(\frac{1}{t}, -\sin(t-1), 8t^3 - 2t\right), \qquad \mathbf{r}'(1) = (1, 0, 6),$$

so a parametric description of the tangent is

$$(x(u), y(u), z(u)) = \mathbf{r}(1) + u\,\mathbf{r}'(1) = (0, 1, 1) + u\,(1, 0, 6).$$

7) If we choose $t = \dfrac{\sqrt{3}}{2}$, we get by insertion,

$$\mathbf{r}\left(\frac{\sqrt{3}}{2}\right) = \left(\operatorname{Arcsin}\left(\frac{\sqrt{3}}{2}\right), \operatorname{Arctan}(\sqrt{3}), \operatorname{Arccot}(\sqrt{3})\right) = \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6}\right),$$

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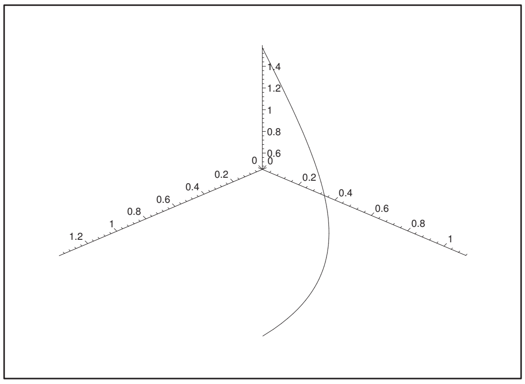


Figur 7: The curve in 7).

so $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6}\right)$ lies on the curve. Furthermore,

$$\mathbf{r}'(t) = \left(\frac{1}{\sqrt{1-t^2}}, \frac{2}{1+4t^2}, -\frac{2}{1+4t^2}\right),$$

so

$$\mathbf{r}'\left(\frac{\sqrt{3}}{2}\right) = \left(\frac{1}{\sqrt{1-\frac{3}{4}}}, \frac{2}{1+4\cdot\frac{3}{4}}, -\frac{2}{1+4\cdot\frac{3}{4}}\right) = \left(2, \frac{1}{2}, -\frac{1}{2}\right),$$

and a parametric description of the tangent is

$$(x(u), y(u), z(u)) = \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6}\right) + u\,(4, 1, -1).$$

8) If we choose $t = \pi$ we see that

$$\mathbf{r}(\pi) = (2\sin\pi, -\cos\pi, 3\pi) = (0, 1, 3\pi),$$

so $(0, 1, 3\pi)$ lies on the curve. Furthermore,

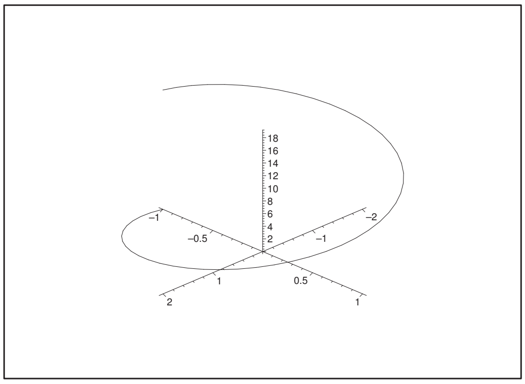
$$\mathbf{r}'(t) = (2\cos t, \sin t, 3)$$

where

$$\mathbf{r}'(\pi) = (2\cos\pi, \sin\pi, 3) = (-2, 0, 3).$$

A parametric description of the tangent is

$$(x(u), y(u), z(u)) = (0, 1, 3\pi) + u\,(-2, 0, 3).$$



Figur 8: The curve in **Example 1.1.8**.

Eksempel 1.2 *A curve is given by the parametric description*

$$x = a\{\ln(1+\sin t) - \ln\cos t - \sin t\}, \qquad y = -a\cos t, \qquad t \in \left[0, \frac{\pi}{2}\right[.$$

1) Prove that

$$\frac{dx}{dt} = \frac{a\,\sin^2 t}{\cos t},$$

and then find the direction of the tangent of the curve in the point $P(t)$ corresponding to the value $t > 0$ of the parameter.

2) Find an equation of a parametric description of the tangent at $P(t)$.

3) Finally find the length of the straight line from $P(t)$ to the intersection of the tangent with the X axis,

A Tangent for a curve, which is given by a parametric description.

D Follow the guidelines of the text.

1) By a differentiation,

$$\begin{aligned}
\frac{dx}{dt} &= a\left(\frac{\cos t}{1+\sin t} + \frac{\sin t}{\cos t} - \cos t\right) = a\left(\frac{\cos t(1-\sin t)}{1-\sin^2 t} + \frac{\sin t}{\cos t} - \cos t\right) \\
&= a\left(\frac{\cos t(1-\sin t)}{\cos^2 t} + \frac{\sin t}{\cos t} - \cos t\right) = a\left(\frac{1-\sin t}{\cos t} + \frac{\sin t}{\cos t} - \cos t\right) \\
&= a\left(\frac{1}{\cos t} - \cos t\right) = a \cdot \frac{1-\cos^2 t}{\cos t} = a \cdot \frac{\sin^2 t}{\cos t}.
\end{aligned}$$

This gives us the direction of the tangent

$$\mathbf{r}'(t) = \left(\frac{dx}{dt}, \frac{dy}{dy}\right) = a\left(\frac{\sin^2 t}{\cos t}, \sin t\right) = a\tan t\,(\sin t, \cos t)$$

for $t \in \left]0, \frac{\pi}{2}\right[$.

Notice that the latter form is very convenient, because it immediately gives us the vector $(\sin t, \cos t)$ of the direction.

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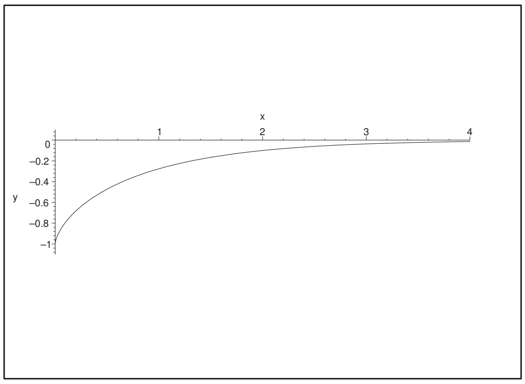

Figur 9: The curve in **Example 1.2**

2) From the result of 1) we get a parametric description of the tangent at the point $P(t)$,

$$(x(u), y(u)) = a(\ln(1 + \sin t) - \ln \cos t - \sin t, -\cos t) + au\,(\tan t)\,(\sin t, \cos t),$$

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where $t \in \left]0, \frac{\pi}{2}\right[$, and where one may put the factor $(\tan t)$ into the parameter u, hence obtaining an equivalent (and simpler) solution without the factor $(\tan t)$.

When we apply the variant above, we get

$$\begin{cases} au \cdot \sin t \cdot \tan t &= x - a\{\ln(1+\sin t) - \ln\cos t - \sin t\}, \\ au \cdot \sin t &= y + a\cos t, \end{cases}$$

thus in particular,

$$\begin{aligned} \tan t \cdot (y + a\cos t) &= \tan t \cdot y + a\sin t = au\,\sin t \cdot \tan t \\ &= x - a\ln(1+\sin t) + a\ln\cos t + a\sin t. \end{aligned}$$

The equation of the tangent is obtained by a reduction,

(1) $\tan t \cdot y = x - a\ln(1+\sin t) + a\ln\cos t,$

which can also be written in one of the two forms

$$\begin{cases} y = \cot t \cdot \{x - a\ln(1+\sin t) + a\ln\cos t\}, \\ \sin t \cdot y = \cos t \cdot \{x - a\ln(1+\sin t) + a\ln\cos t\}. \end{cases}$$

3) The intersection point of the tangent with the X axis is found by putting $y = 0$ into the equation of the tangent and the solve the equation. Here we have two variants.

a) If we use the parametric description we get

$$y(u) = -a \cdot \cos t + au \cdot \sin t = 0,$$

which is fulfilled for $u = \cot t$. Using this u we get

$$x(u) = a\{\ln(1+\sin t) - \ln\cos t - \sin t + \sin t\} = a\{\ln(1+\sin t) - \ln\cos t\}.$$

Since $P(t)$ has the coordinates $(x(0), y(0))$ from the equation of the tangent, the length of the straight line on the tangent between $P(t)$ and the intersection point with the X axis is

$$\begin{aligned} L &= \sqrt{\{x(0) - x(u)\}^2 - \{y(0)\}^2} \\ &= \sqrt{\{\ln(1+\sin t) - \ln\cos t - \sin t - \ln(1+\sin t) + \ln\cos t\}^2 + \{-\cos t\}^2} \\ &= a\sqrt{\sin^2 t + \cos^2 t} = a. \end{aligned}$$

b) If we instead use the equation of the tangent, it follows from (1) that

$$0 = x - a\ln(1+\sin t) + a\ln\cos t,$$

so the abscissa of the intersection point is given by

$$x = a\{\ln(1+\sin t) - \ln\cos t\},$$

thus $x - x_0 = -\sin t$, and

$$L = a\sqrt{\{x - x_0\}^2 + y_0^2} = a\sqrt{\sin^2 t + \cos^2 t} = a.$$

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Eksempel 1.3 *There are below given some plane curves by a parametric description*

$$\mathbf{x} = \mathbf{r}(t), \qquad t \in I.$$

Explain in each case why the curve is a C^1-curve with $\mathbf{r}'(0) = \mathbf{0}$. Then check whether the curve has a cusp at the point $\mathbf{r}(0)$. If the curve does not *have a cusp, then find another parametric description of the curve,*

$$\mathbf{x} = \mathbf{R}(u), \qquad u \in J,$$

such that $u = 0$ corresponds to $t = 0$, and such that $\mathbf{R}'(0) \neq \mathbf{0}$.

1) $\mathbf{r}(t) = (t^2, t^3)$ *for* $t \in \mathbb{R}$.

2) $\mathbf{r}(t) = (t^3, t^6)$ *for* $t \in \mathbb{R}$.

3) $\mathbf{r}(t) = (t - \sin t, 1 - \cos t)$ *for* $t \in [-\pi, \pi]$. [Cf. **Example 1.1.4**].

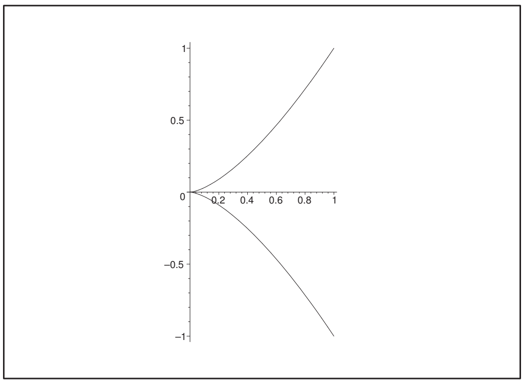
4) $\mathbf{r}(t) = \left(\cos^3 t, \sin^3 t\right)$ *for* $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

5) $\mathbf{r}(t) = \left(t^3, \sin^2 t\right)$ *for* $t \in \left[-\frac{\pi}{2}, \frac{pi}{2}\right]$.

A C^1-curves with or without cusps.

D Follow the guidelines. One may also sketch the curve.

I 1) The coordinate functions of $\mathbf{r}(t) = (t^2, t^3)$, $t \in \mathbb{R}$ are clearly C^∞-functions in t.



Figur 10: The curve in 1).

Then by a differentiation,

$$\mathbf{r}'(t) = (2t, 3t^2) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

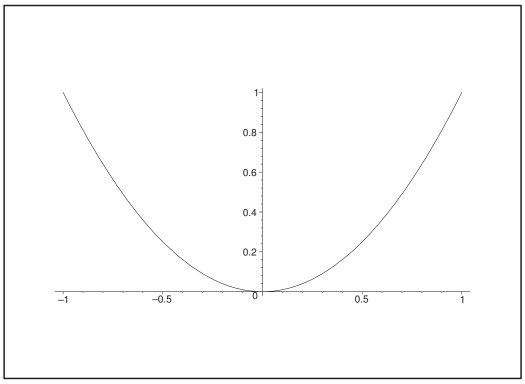
The figure indicates that we have a cusp. However, one must never in situations like this trust the figure 100 %. An argument is needed! Now, $\dfrac{dx}{dt}$ changes its sign, when t goes through 0, while $\dfrac{dy}{dt}$ does not change its sign, and it goes also faster towards 0 than $\dfrac{dx}{dt}$. We therefore conclude that we indeed have a cusp for $t = 0$

It is here possible to eliminate t, and one gets

$$x = y^{2/3}, \qquad y \in \mathbb{R}.$$

Remark. If instead one tries to express y by x, then we get the more confused expression $|y| = x^{3/2}$, due to the fact that the square root occurs latently. Always be careful, whenever the *square root* enters a problem. “If one can handle the square root, then one can handle anything inside mathematics.”

2) The coordinate functions of $\mathbf{r}(t) = (t^3, t^6)$, $t \in \mathbb{R}$, are clearly C^∞-functions in t.



Figur 11: The curve in 2).

We get by a differentiation

$$\mathbf{r}'(t) = (3t^2, 6t^5) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

It follows immediately that $y = t^6 = (t^3)^2 = x^2$, so the curve is a parabola, which does not have a cusp.

An obvious alternative parametric description is $u = t^3$, by which $u = 0$ for $t = 0$, and

$$\mathbf{R}'(u) = (u, u^2), \qquad u \in \mathbb{R},$$

where

$$\mathbf{R}'(u) = (1, 2u) \quad \text{and} \quad \mathbf{R}'(0) = (1, 0) \neq \mathbf{0}.$$

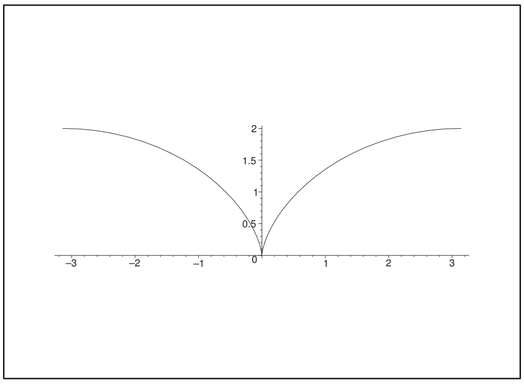
3) Clearly, the coordinate functions of

$$\mathbf{r}(t) = (t - \sin t, 1 - \cos t), \qquad t \in [-\pi, \pi],$$

are C^∞-functions in $t \in]-\pi, \pi[$.
The curve is a part of the cycloid with a cusp at $\mathbf{0}$. We get by a differentiation,

$$\mathbf{r}'(t) = (1 - \cos t, \sin t) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

Since $\dfrac{dx}{dt}$ does not change its sign, while $\dfrac{dy}{dt}$ changes its sign when we pass through $t = 0$, and since



Figur 12: The curve in 3).

$$1 - \cos t = \frac{1}{2}\,t^2 + t^2\varepsilon(t)$$

tends faster towards zero than $\sin t = t + t\varepsilon(t)$ for $t \to 0$, we conclude that the curve has a cusp.

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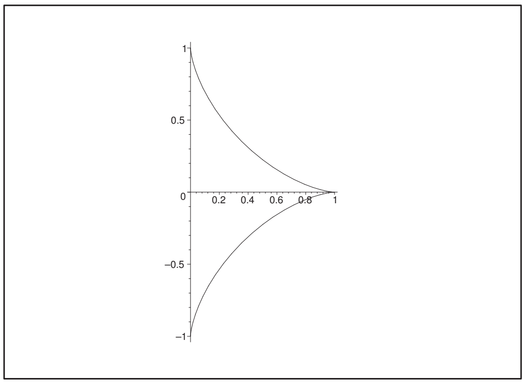
4) The coordinate functions of $\mathbf{r}(t) = (\cos^3 t, \sin^3 t)$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, are clearly C^∞-functions in the open interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.



Figur 13: The curve in 4).

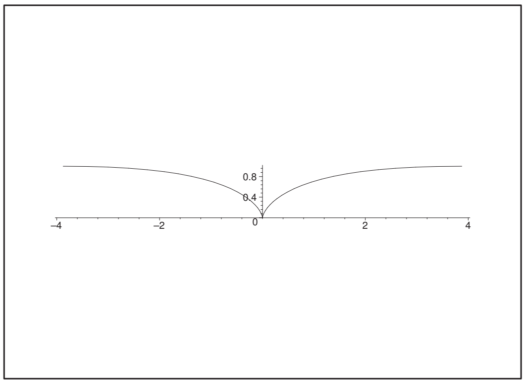
The figure indicate that we may have a cusp for $t = 0$, i.e. at the point $(1, 0)$.

Then by a differentiation,

$$\mathbf{r}'(t) = 3(-\cos^2 t \sin t, \sin^2 t \cos t) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

Since $\frac{dx}{dt}$ changes its sign, while $\frac{dy}{dt}$ does not for $t \to 0$, and since $\frac{dy}{dt}$ tends faster towards 0 than $\frac{dx}{dt}$ for $t \to 0$, we conclude again that the curve has a cusp.

5) The coordinate functions of $\mathbf{r}(t) = (t^3, \sin^2 t)$ are clearly C^∞-functions in the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.



Figur 14: The curve in 5).

Then by a differentiation,

$$\mathbf{r}'(t) = (3t^2, 2\sin t \cos t) = (3t^2, \sin 2t) \qquad \text{where } \mathbf{r}'(0) = \mathbf{0}.$$

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The curve has a cusp for $t = 0$, because $\dfrac{dy}{dt}$ changes its sign, while $\dfrac{dx}{dt}$ does not, when we pass through $t = 0$, and because $\dfrac{dx}{dt} = 3t^2$ goes faster towards zero for $t \to 0$ than $\dfrac{dy}{dt} \approx 2t$.

Remark. Since $\sin t \approx t$ for small t, the curve lies in the neighbourhood of $t = 0$ close to

$$\tilde{\mathbf{r}}(t) = (t^3, t^2) \qquad \text{for } |t| \text{ small.}$$

Cf. 1). $\Diamond$

Eksempel 1.4 *A space curve $\mathcal{K}$ is given by the parametric description*

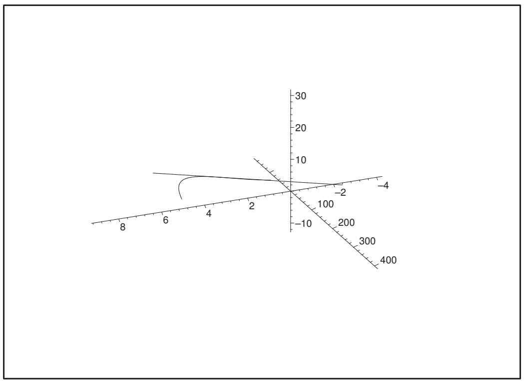
$$\mathbf{r}(r) = \left(t^2, e^{2t}, 4 + t^3\right), \qquad t \in \mathbb{R}.$$

1) Find a parametric description of the tangent to $\mathcal{K}$ at the point $\mathbf{r}(2)$.

2) Prove that this tangent intersects the Y axis at some point $(0, \beta, 0)$, and find β.

A Tangent to a space curve.

D Standard methods.



Figur 15: The curve $\mathcal{K}$ for $t \in [1, 3]$ and its tangent at the point $\mathbf{r}(2)$. Notice the different scales on the axes.

I 1) We get by a differentiation,

$$\mathbf{r}'(t) = \left(2t, 2e^{2t}, 3t^2\right), \quad \text{så} \quad \mathbf{r}'(2) = \left(4, 2e^4, 12\right).$$

Since $\mathbf{r}(2) = \left(4, e^4, 12\right)$, the parametric description of the tangent to $\mathcal{K}$ at the point $\mathbf{r}(2)$ is

$$\begin{aligned}
(x(t), y(t), z(t)) &= \mathbf{r}(2) + t \cdot \mathbf{r}'(2) \\
&= \left(4, e^4, 12\right) + t\left(4, 2e^4, 12\right) \\
&= \left(4(t+1), e^4(2t+1), 12(t+1)\right).
\end{aligned}$$

2) It follows immediately that $x(t) = z(t) = 0$ for $t = -1$, hence

$$(x(-1), y(-1), z(-1)) = \left(0, -e^4, 0\right) = (0, \beta, 0),$$

and we conclude that

$$\beta = -e^4.$$

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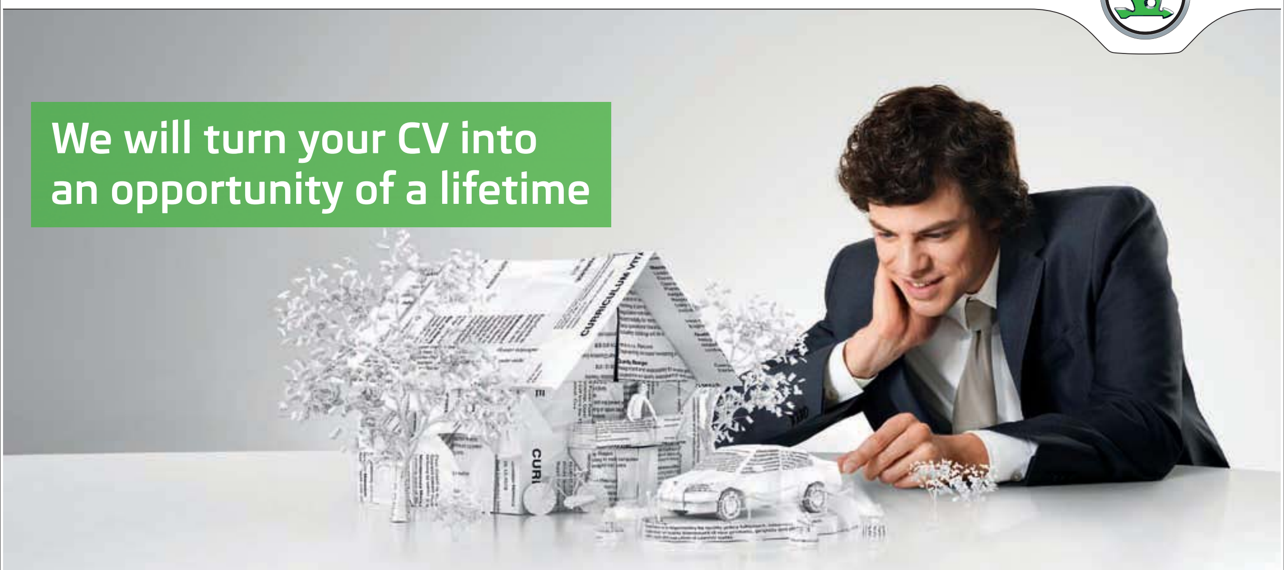

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2 Tangent plane to a surface

Eksempel 2.1 *Find in each of the following cases an equation of the tangent plane to the given surface at the given point.*

1) The surface is given by $\mathbf{r}(u,v) = (2u, u^2+v, v^2)$*, and the point is* $(0,1,1)$.

2) The surface is given by $\mathbf{r}(u,v) = u^2 - v^2, u+v, u^2+4v)$*, and the point is* $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$.

3) The surface is given by $xy - 2xz + xyz = 6$*, and the point is* $(-3,2,1)$.

4) The surface is given by $\mathbf{r}(u,v) = (u+v, u^2+v^2, u^3+v^3)$ *for* $u > v$*, and the point is* $(0,2,0)$

5) The surface is given by $\cos x - \cos y + \sin z = 1$*, and the point is* $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right)$.

6) The surface is given by $z^2 + 2xz + 2yz + 4y = 0$*, and the point is* $\left(1, \frac{1}{2}, -1\right)$.

7) The surface is given by $z =$ *Arctan*(xy)*, and the point is* $\left(1, 1, \frac{\pi}{4}\right)$.

A Tangent plane to a given surface.

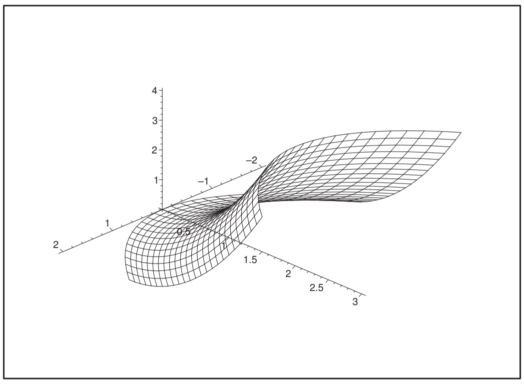
D First check that the given point lies on the surface.

If the surface is given by a parametric description, then calculate the normal vector

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}.$$

If the surface instead is given implicitly by $f(x,y,z) = c$, then it can be considered as a level surface, so its normal vector is $\bigtriangledown f(x,y,z)$.

I 1) Clearly, the point corresponds to the values of the parameters $u = 0$ and $v = 1$.



Figur 16: The surface in 1).

Then by partial differentiations

$$\frac{\partial \mathbf{r}}{\partial u} = (2, 2u, 0) \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial v} = (0, 1, 2v),$$

so the normal vector is

$$\frac{\partial \mathbf{r}}{\partial u}(0,1) \times \frac{\partial \mathbf{r}}{\partial v}(0,1) = (2,0,0)\times(0,1,2) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (0,-4,2).$$

The tangent plane is given by

$$0\cdot(x-0) - 4(y-1) + 2(z-1) = 0,$$

hence by a rearrangement,

$$2y - z = 1.$$

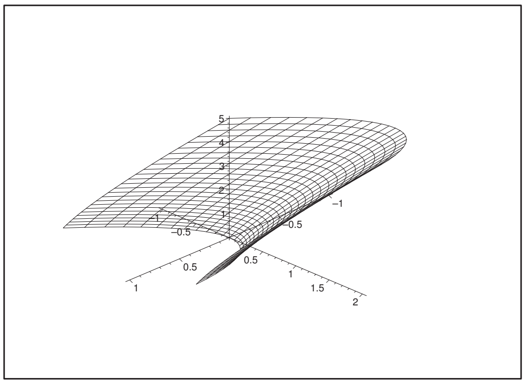
2) We first check that $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$ lies on the surface. We that solve the equations

$$\begin{cases} u^2 - v^2 &= -\dfrac{1}{4}, \\ u + v &= \dfrac{1}{2}, \\ u^2 + 4v &= 2. \end{cases}$$

When we divide the second equation into the first one we get

$$u - v = -\frac{1}{2} \qquad \text{and} \qquad u + v = \frac{1}{2},$$

thus $u = 0$ and $v = \frac{1}{2}$. These values solve the first two equations, and then we see by insertion that the third one also holds. Thus we have proved that the point $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$ lies on the surface corresponding to $(u,v) = \left(0, \frac{1}{2}\right)$.



Figur 17: The surface in 2).

Then we get by partial differentiation,

$$\frac{\partial \mathbf{r}}{\partial u} = (2u, 1, 2u), \qquad \frac{\partial \mathbf{r}}{\partial u}\left(0, \frac{1}{2}\right) = (0, 1, 0),$$

$$\frac{\partial \mathbf{r}}{\partial v} = (-2v, 1, 4), \qquad \frac{\partial \mathbf{r}}{\partial v}\left(0, \frac{1}{2}\right) = (-1, 1, 4).$$

The normal vector is

$$\frac{\partial \mathbf{r}}{\partial u}\left(0, \frac{1}{2}\right) \times \frac{\partial \mathbf{r}}{\partial v}\left(0, \frac{1}{2}\right) = (0,1,0) \times (-1,1,4) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 1 & 0 \\ -1 & 1 & 4 \end{vmatrix} = (4, 0, 1).$$

The tangent plane is given by

$$4\left(x + \frac{1}{4}\right) + 0 \cdot \left(y - \frac{1}{2}\right) + 1 \cdot (z - 2) = 0,$$

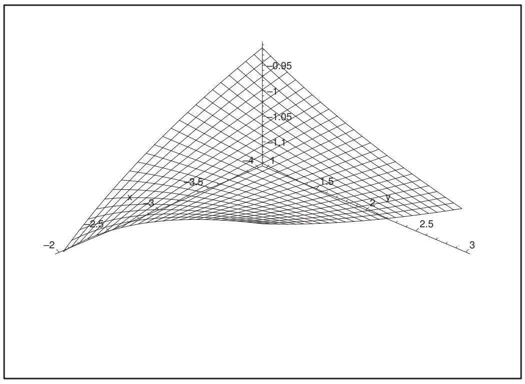
which is reduced to

$$4x + z = 1.$$

3) This is an implicitly given surface. Putting $(x, y, z) = (-3, 2, 1)$ into the left hand side of the equation we get

$$-3 \cdot 2 - 2(-3) \cdot 1 + 3 \cdot 2 \cdot 1 = -6 + 6 + 6 = 6,$$

so the point lies on the surface.



Figur 18: The surface in 3).

The gradient is given by

$$\triangledown f(x, y, z) = (y - 2z, x + 3z, -2x + 3y),$$

hence

$$\triangledown f(-3, 2, 1) = (2 - 2, -3 + 3, 606) = (0, 0, 12).$$

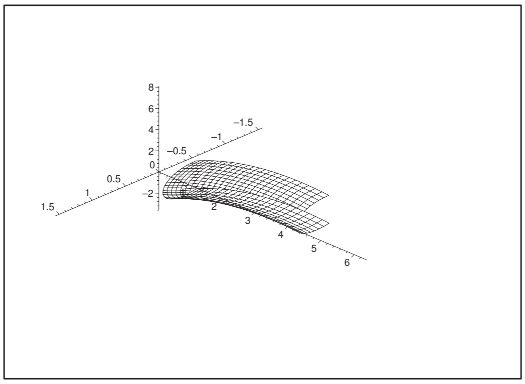
Thus a normal vector is $(0, 0, 1)$, and the tangent plane is given by

$$z = 1.$$

4) If $(0,2,0)$ lies on the surface we must have

$$\begin{cases} u+v=0, \\ u^2+v^2=2, \\ u^3+v^3=0, \end{cases}$$

where the solution should also satisfy $u>v$. It is easily seen that $(u,v)=(1,-1)$ is a solution.



Figur 19: The surface of 4).

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Remark. In the present formulation of the example one does not have to check whether there are other possible values of the parameters, which can be used instead. For completeness we prove that there actually are no other values. This follows from

$$0 = (u+v)^2 = (u^2+v^2) + 2uv = 2 + 2uv,$$

thus $uv = -1$. This implies that u and v are solutions of the equation of second degree

$$\lambda^2 - (u+v)\lambda + uv = \lambda^2 - 0 \cdot \lambda - 1 = \lambda^2 - 1 = 0,$$

i.e. $\lambda = \pm 1$. Since $u > v$, we see that $(u,v) = (1,-1)$ is the only solution. $\Diamond$.

Now,

$$\frac{\partial \mathbf{r}}{\partial u} = (1, 2u, 3u^2), \qquad \frac{\partial \mathbf{r}}{\partial u}(1,-1) = (1,2,3),$$

$$\frac{\partial \mathbf{r}}{\partial v} = (1, 2v, 3v^2), \qquad \frac{\partial \mathbf{r}}{\partial v}(1,-1) = (1,-2,3).$$

A normal vector is

$$\frac{\partial \mathbf{r}}{\partial u}(1,-1) \times \frac{\partial \mathbf{r}}{\partial v} = (1,2,3) \times (1,-2,3) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = 4(3,0,-1).$$

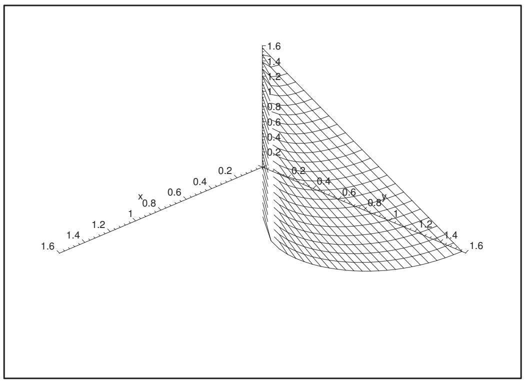
The equation of the tangent plane is

$$z = 3x.$$

5) When $(x,y,z) = \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ is put into the left hand side of the equation, we get

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}\right) = \cos \frac{\pi}{3} - \cdot \frac{\pi}{3} + \sin \frac{\pi}{2} = 1,$$

proving that the point lies on the surface.



Figur 20: The surface in 5).

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The gradient is

$$\bigtriangledown f(x,y,z) = (-\sin x, \sin y, \cos z)$$

and

$$\bigtriangledown f\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{2}\right) = \left(-\sin\frac{\pi}{3},\sin\frac{\pi}{3},\cos\frac{\pi}{2}\right) = -\sin\frac{\pi}{3}\cdot(1,-1,0).$$

A convenient normal of the surface at the point is therefore $(1,-1,0)$, and a tangent plane is

$$0 = (1,-1,0)\cdot\left(x-\frac{\pi}{3}, y-\frac{\pi}{3}, z-\frac{\pi}{2}\right) = x-\frac{\pi}{3}-\left(y-\frac{\pi}{3}\right) = x-y,$$

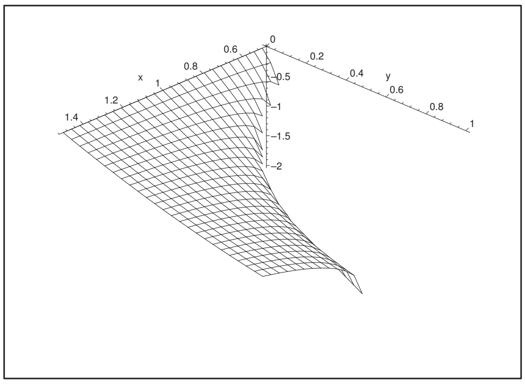
i.e.

$$y = x.$$

6) When $(x,y,z) = \left(1,\frac{1}{2},-1\right)$ is put into the left hand side of the equation we get

$$(-1)^2+2\cdot 1\cdot(-1)+2\cdot\frac{1}{2}\cdot(-1)+4\cdot\frac{1}{2} = 1-2-1+2 = 0,$$

proving that the point $\left(1,\frac{1}{2},-1\right)$ lies on the surface.



Figur 21: The surface in 6).

It follows from

$$\bigtriangledown F = (2z, 2z+4, 2z+2x+2y),$$

that

$$\bigtriangledown F\left(1,\frac{1}{2},-1\right) = (-2,-2+4,-2+2+1) = (-2,2,1)$$

is perpendicular to the surface at the point $\left(1,\frac{1}{2},-1\right)$.

The tangent equation now becomes

$$\begin{aligned} 0 &= \bigtriangledown F\left(1, \frac{1}{2}, -1\right) \cdot \left(x-1, y-\frac{1}{2}, z+1\right) \\ &= -2(x-1) + 2\left(y - \frac{1}{2}\right) + (z+1) \\ &= -2x + 2y + z + 2. \end{aligned}$$

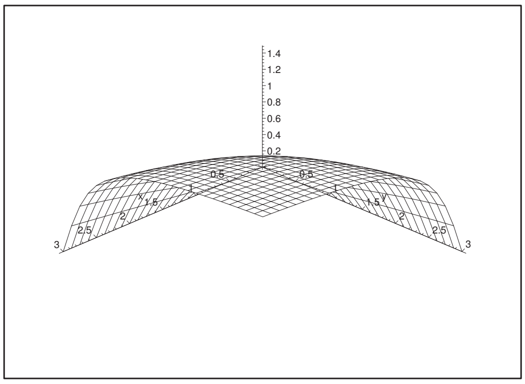
7) The equation is equivalent to

$$F(x,y,z) = \text{ Arctan}(x,y) - z = 0.$$

If we put $(x,y,z) = \left(1, 1, \frac{\pi}{4}\right)$, we get

$$F\left(1,1,\frac{\pi}{4}\right) = \text{ Arctan } 1 - \frac{\pi}{4} = 0,$$

and $\left(1,1,\frac{\pi}{4}\right)$ lies on the surface.



Figur 22: The surface of 7).

Now

$$\bigtriangledown F = \left(\frac{y}{1+(xy)^2}, \frac{x}{1+(xy)^2}, -1\right),$$

and

$$\bigtriangledown F\left(1,1,\frac{\pi}{4}\right) = \frac{1}{2}(1,1,-2),$$

so the equation of the tangent plane becomes

$$\begin{aligned} 0 &= 2 \bigtriangledown F\left(1,1,\frac{\pi}{4}\right) \cdot \left(x-1, y-1, z-\frac{\pi}{4}\right) \\ &= (1,1,-2) \cdot \left(x-1, y-1, z-\frac{\pi}{4}\right) \\ &= x - 1 + y - 1 - 2z + \frac{\pi}{2}, \end{aligned}$$

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hence by a rearrangement,

$$x + y - 2z = 2 - \frac{\pi}{2}.$$

Eksempel 2.2 *Let $\mathcal{F}$ be an hyperboloid with two nets, given by*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Consider $\mathcal{F}$ as a level surface and find an equation of the tangent plane of $\mathcal{F}$ through the point (ξ, η, ζ).

A Tangent plan for a level surface.

D Find the gradient and set up the equation of the tangent plane. We shall need that (ξ, η, ζ) lies on the surface.

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I If we put

$$F(x,y,z)=\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2},$$

we see that the hyperboloid $\mathcal{F}$ can be considered as the level surface $F(x,y,z)=1$.

Now let $(\xi,\eta,\zeta)\in\mathcal{F}$, i.e.

$$\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}=1.$$

The gradient in (ξ,η,ζ) is

$$\bigtriangledown F(\xi,\eta,\zeta)=2\left(\frac{\xi}{a^2},-\frac{\eta}{b^2},-\frac{\zeta}{c^2}\right),$$

so the equation of the tangent plane is

$$0=\frac{1}{2}\bigtriangledown F(\xi,\eta,\zeta)\cdot(x-\xi,y-\eta,z-\zeta)=\frac{\xi x-\xi^2}{a^2}-\frac{\eta y-\eta^2}{b^2}-\frac{\zeta z-\zeta^2}{c^2},$$

hence by a rearrangement

$$\frac{\xi x}{a^2}-\frac{\eta y}{b^2}-\frac{\zeta z}{c^2}=\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}=1.$$

Thus the equation of the tangent plane at $(\xi,\eta,\zeta)\in\mathcal{F}$ is

$$\frac{\xi x}{a^2}-\frac{\eta y}{b^2}-\frac{\zeta z}{c^2}=1.$$

Eksempel 2.3 *Let $\mathcal{F}$ be that part of the surface of equation $xyz=1$ which lies in the first octant.*

1) Find an equation of the tangent plane $\mathcal{T}$ of $\mathcal{F}$ at the point (ξ,η,ζ) on the surface.

2) Find the intersection points between $\mathcal{T}$ and the coordinate axes.

3) Find the volume of the tetrahedron, which is bounded by $\mathcal{T}$ and the three coordinate planes.

A Tangent plane of a level surface.

D Find the gradient and the the tangent plane.

I 1) When $F(x,y,z)=xyz$, the $\mathcal{F}$ is the level surface $F(x,y,z)=1$ in the first octant.

If $(\xi,\eta,\zeta)\in\mathcal{F}$, then

$$\eta\zeta=\frac{1}{\xi},\qquad \zeta\xi=\frac{1}{\eta},\qquad \xi\eta=\frac{1}{\zeta},$$

and the gradient is here

$$\bigtriangledown F(\xi,\eta,\zeta)=(\eta\zeta,\zeta\xi,\xi\eta)=\left(\frac{1}{\xi},\frac{1}{\eta},\frac{1}{\zeta}\right).$$

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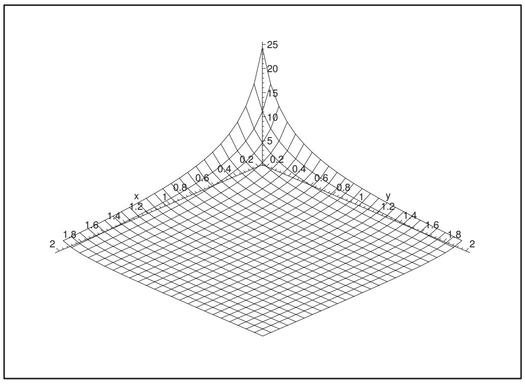

The equation of the tangent plane becomes

$$0 = \bigtriangledown F(\xi,\eta,\zeta)\cdot(x-\xi,y-\eta,z-\zeta) = \frac{x}{\xi}-1+\frac{y}{\eta}-1+\frac{z}{\zeta}-1,$$

i.e.

$$\frac{x}{\xi}+\frac{y}{\eta}+\frac{z}{\zeta}=3,$$

or alternatively,

$$x\eta\zeta+y\xi\zeta+z\xi\eta=3.$$

2) The X axis is characterized by $y=0$ and $z=0$, so the intersection point is $(3\xi,0,0)$.

We get analogously $(0,3\eta,0)$ on the Y axis and $(0,0,3\zeta)$ on the Z axis.

3) The volume is by the method of sections given by $\int_0^{3\zeta} A(z)\,dz$, where $A(z)$ denotes the area of a triangle which is cut out of the tetrahedron by a plane, parallel to the XY plane at the height z.

At the height $z=0$ we have

$$A(0)=\frac{9}{2}\,\xi\eta.$$

By the similarity we then get in general

$$A(z)=\frac{9}{2}\,\xi\eta\left(1-\frac{z}{3\zeta}\right)^2,\qquad z\in[0,3\zeta].$$

Finally, by insertion and calculation of the integral we get the volume

$$\begin{aligned} V &= \int_0^{3\zeta} A(z)\,dz = \frac{9}{2}\,\xi\eta\int_0^{3\zeta}\left(1-\frac{z}{3\zeta}\right)^2 dz \qquad \left[t=\frac{z}{3\zeta}\right]\\ &= \frac{9}{2}\,\xi\eta\cdot 3\zeta\int_0^1(1-t)^2dt = \frac{27}{2}\,\xi\eta\zeta\int_0^1 u^2du\\ &= \frac{27}{2}\cdot 1\cdot 13 = \frac{9}{2}. \end{aligned}$$

We note that the volume is constant $=\dfrac{9}{2}$, no matter which point $(\xi,\eta,\zeta)\in\mathcal{F}$ we have chosen.

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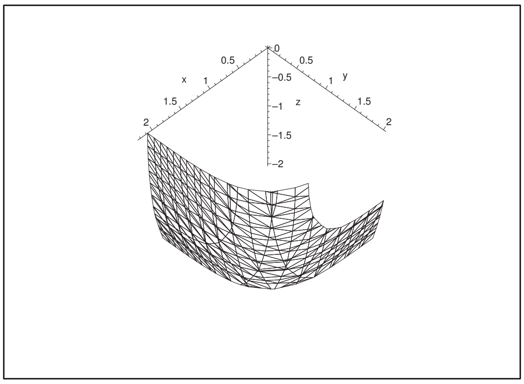
Eksempel 2.4 *The surface $\mathcal{F}$ is given by the equation $x^4 + y^4 + 2z^2 = 19$. Find an equation of the tangent plane of $\mathcal{F}$ through the point $(2, 1, -1)$.*

A Tangent plane.

D First check that $(2, 1, -1)$ lies on the surface. Then find a normal.



Figur 23: Part of the surface in the neighbourhood of the point $(2, 1, -1)$.

I It follows from

$$2^4 + 1^4 + 2(-1)^2 = 16 + 1 + 2 = 19,$$

that $(2, 1, -1)$ lies on $\mathcal{F}$. Then

$$\mathbf{N}(x, y, z) = (4x^3, 4y^3, 4z) = 4(x^3, y^3, z),$$

and

$$\mathbf{N}(2, 1, -1) = 4(8, 1, -1),$$

so the equation of the tangent plane is

$$0 = \frac{1}{4}\mathbf{N}(2, 1, -1) \cdot (x - 2, y - 1, z + 1) = 8(x - 2) + (y - 1) - (z + 1),$$

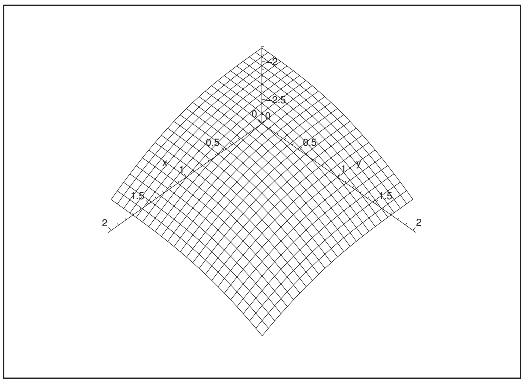
thus by a reduction

$$8x + y - z = 18.$$

Eksempel 2.5 *The surface $\mathcal{F}$ is given by the equation*

$$x^3 + y^3 + z^3 + 6 = 0.$$

Prove that the straight line given by $x = 1$ and $y = 1$ intersects $\mathcal{F}$ in a single point P, and then find an equation of the tangent plane of $\mathcal{F}$ at P.



Figur 24: The surface $\mathcal{F}$ in the neighbourhood of P. Note that the coordinate system is translated along the Z axis, so the origo does not occur on the figure.

A Tangent plane.

D Either exploit the geometric meaning of $\bigtriangledown F$, or find two tangents the cross products of which gives a normal.

I By putting $x = y = 1$ we get

$$0 = 1 + 1 + z^3 + 6 = z^3 + 8 = (z + 2)(z^2 - 2z + 4) = (z + 2)\{(z - 1)^2 + 3\}.$$

It follows that $z = -2$ is the only *real* solution, so the line of the parametric description $\mathbf{r}(z) = (1, 1, z)$ does only intersect the surface in the point P: $(1, 1, -2) \in \mathcal{F}$.

We shall find the tangent plane in two ways.

First variant. If we put $F(x, y, z) = x^3 + y^3 + z^3 + 6$, then $\bigtriangledown F$ indicates the direction in which the increase of F is largest, i.e. $\bigtriangledown F$ is a normal to the surface $F(x, y, z) = C$ at the surface point.

We find

$$\bigtriangledown F = 3(x^2, y^2, z^2), \qquad \bigtriangledown F(1, 1, -2) = 3(1, 1, 4),$$

so an equation of the tangent plane is e.g. given by

$$\begin{aligned} 0 &= \frac{1}{3} \bigtriangledown F(1, 1, -2) \cdot (x - 1, y - 1, z + 2) \\ &= (1, 1, 4) \cdot (x - 1, y - 1, z + 2) \\ &= x - 1 + y - 1 + 4z + 8 = x + y + 4z + 6, \end{aligned}$$

thus by a rearrangement

$$x + y + 4z = -6.$$

Second variant. A parametric description of the surface $\mathcal{F}$ is in a neighbourhood of the point $(1, 1, -2)$ given by

$$\mathbf{r}(u, v) = \left(u, v, -\sqrt[3]{6 + u^3 + v^3}\right), \qquad (u, v) \in K((1, 1, -2); r).$$

The parameter curves have the tangents

$$\frac{\partial \mathbf{r}}{\partial u} = \left(1, 0, -\left\{\frac{u}{\sqrt[3]{6+u^3+v^3}}\right\}^2\right), \qquad \frac{\partial \mathbf{r}}{\partial v} = \left(0, 1, -\left\{\frac{v}{\sqrt[3]{6+u^3+v^3}}\right\}^2\right).$$

At the point $(1, 1, -2)$ we get the tangents

$$\mathbf{r}'_u(1,1,-2) = \left(1, 0, -\frac{1}{4}\right) \quad \text{og} \quad \mathbf{r}'_v(1,1,-2) = \left(0, 1, -\frac{1}{4}\right).$$

Then a parametric description of the tangent plane is given by

$$\begin{aligned} \mathbf{r}(u,v) &= (x,y,z) = (1,1,-2) + u\left(1, 0, -\frac{1}{4}\right) + v\left(0, 1, -\frac{1}{4}\right) \\ &= \left(u+1, v+1, -\frac{1}{4}(u+v+8)\right), \qquad (u,v) \in \mathbb{R}^2. \end{aligned}$$

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When we eliminate u and v we get

$$z = -\frac{1}{4}(u+v+8) = -\frac{1}{4}\{(u+1)+(v+1)+6\} = -\frac{1}{4}(x+y+6),$$

hence by a rearrangement

$$x+y+4z=-6.$$

Remark. Here we might alternatively have found the normal instead,

$$\mathbf{N}(1,1,-2) = \mathbf{r}'_u(1,1,-2) \times \mathbf{r}'_v(1,1,-2).$$

The details are left to the reader. $\Diamond$

Eksempel 2.6 *Given the function*

$$f(x,y,z) = x^2 - 4y^2 + 4x + 8y - z, \qquad (x,y,z) \in \mathbb{R}^3,$$

and the level surface $\mathcal{F}$ of the equation

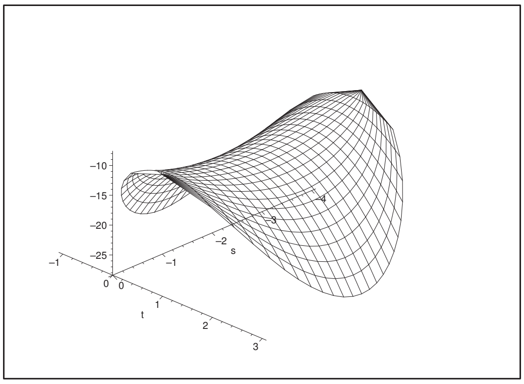
$$f(x,y,z) = 12.$$

1) Indicate the type of the surface and its top point.

2) Find $\nabla f(2,0,0)$, and set up an equation of the tangent plane of $\mathcal{F}$ at the point $(2,0,0)$.

A Level surface and tangent plane.

D Rewrite the equation of the level surface to one of the generic forms. The tangent plane is found by the standard method.



Figur 25: Part of the level surface $\mathcal{F}$.

I 1) The equation of the level surface

$$f(x,y,z) = x^2 - 4y^2 + 4x + 8y - z = (x+2)^2 - 4(y-1) - z = 12$$

is rewritten as

$$z + 12 = (x+2)^2 - 4(y-1)^2,$$

and we see that $\mathcal{F}$ is an hyperbolic paraboloid of top point $(-2,1,-12)$.

2) It follows from

$$\nabla f(x,y,z) = (2x+4, -8y+8, -1)$$

that the normal of the surface at the point $(2,0,0)$ is

$$\nabla f(2,0,0) = (8,8,-1).$$

Since

$$f(2,0,0) = 4 - 4\cdot 0^2 + 4\cdot 2 + 8\cdot 0 - 0 = 12,$$

we see that $(2,0,0)$ lies on the surface.

The equation of the tangent plane is

$$0 = \nabla f(2,0,0)\cdot(x-2, y-0, z-0) = (8,8,-1)\cdot(x-2,y,z) = 8x - 16 + 8y - z,$$

so by a rearrangement,

$$z = 8x + 8y - 16.$$

Eksempel 2.7 *Let α be a constant. Consider the surface $\mathcal{F}_\alpha$ given by the equation*

$$x^2 + 4y^2 - z^2 + \alpha\, z^4 = \alpha + \alpha^2.$$

Let $\mathcal{T}_\alpha$ denote the tangent plane of $\mathcal{F}_\alpha$ at the point $\left(\alpha, \frac{1}{2}, 1\right)$ on the surface.

1) Find an equation of $\mathcal{T}_\alpha$.

2) Check if α can be chosen such that the point $(1,1,0)$ belongs to the plane $\mathcal{T}_\alpha$.

3) Prove that there exists a value of α, for which $\mathcal{F}_\alpha$ is a conic section and indicate its type and centrum.

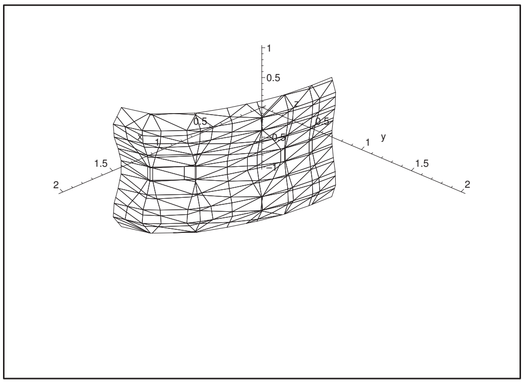
A Tangent plane; conic section.

D Check if $\left(\alpha, \frac{1}{2}, 1\right)$ lies on the surface. Find a normal and prove that this can always be chosen $\neq \mathbf{0}$. Set up an equation of the tangent plane. Insert $(1,1,0)$ and solve with respect to α. Finally, find α, such that every term is at most a square.

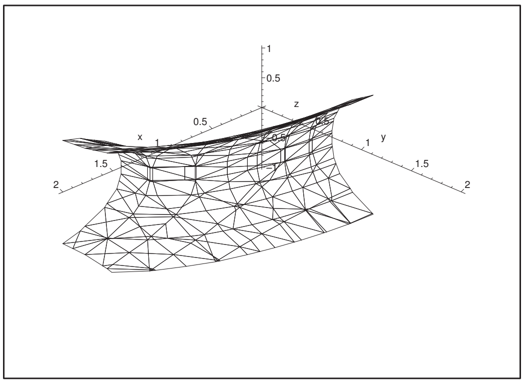
I 1) If we put $(x,y,z) = \left(\alpha, \frac{1}{2}, 1\right)$, then

$$\alpha^2 + 4\cdot\frac{1}{4} - 1 + \alpha = \alpha^2 + \alpha,$$

and the point $\left(\alpha, \frac{1}{2}, 1\right)$ lies on the surface $\mathcal{F}_\alpha$.



Figur 26: The surface $\mathcal{F}_\alpha$ for $\alpha = 1$ in the neighbourhood of the point $(1, 1, 0)$, cf. 2).



Figur 27: The surface $\mathcal{F}_\alpha$ for $\alpha = -2$ in the neighbourhood of the point $(1, 1, 0)$, cf. 2).

In general a normal is given by

$$\bigtriangledown F = (2x, 8y, -2 < +4\alpha z^3) = 2(x, 4y, -z + 2\alpha z^3).$$

By removing the factor 2 and inserting $\left(\alpha, \frac{1}{2}, 1\right)$ we get

$$\mathbf{n} = (\alpha, 2, -1 + 2\alpha) \neq \mathbf{0} \qquad \text{for every } \alpha.$$

Since $\mathbf{n}$ is perpendicular to the tangent plane, the equation of the tangent plane becomes

$$\begin{aligned} 0 &= \mathbf{n} \cdot \left(x - \alpha, y - \frac{1}{2}, z - 1\right) \\ &= \alpha(x - \alpha) + 2\left(y - \frac{1}{2}\right) + (-1 + 2\alpha)(z - 1) \\ &= \alpha x + 2y + (2\alpha - 1)z - \alpha^2 - 1 + 1 - 2\alpha \\ &= \alpha x + 2y + (2\alpha - 1)z - \alpha^2 - 2\alpha, \end{aligned}$$

which is rewritten as

$$\alpha x + 2y + (2\alpha - 1)z = \alpha^2 + 2\alpha.$$

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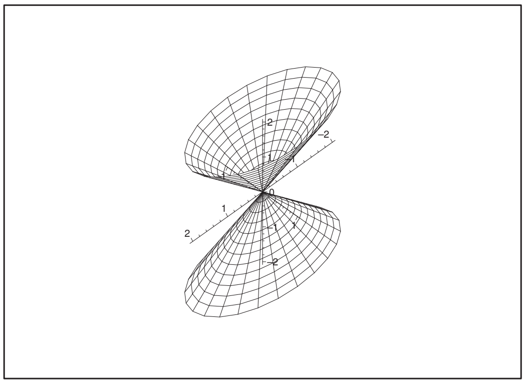

Figur 28: The conic section in 3), i.e. for $\alpha = 0$.

2) If we put $(x, y, z) = (1, 1, 0)$ into the equation of the tangent plane, we get the following equation of second order in α,

$$\alpha^2 + 2\alpha = \alpha + 2 + 0, \qquad \text{i.e.} \qquad \alpha^2 + \alpha - 2 = 0,$$

the roots of which are $\alpha = 1$ and $\alpha = -2$. For these values of α the point $(1, 1, 0)$ lies in $\mathcal{T}_\alpha$.

3) When we consider conic sections, the corresponding equation must at most contain terms of degree two. We are therefore forced to put $\alpha = 0$, corresponding to

$$x^2 + 4y^2 - z^2 = 0, \qquad \text{i.e.} \qquad z^2 = x^2 + 4y^2.$$

This is the equation of a cone of centrum $(0, 0, 0)$ and the Z axis as axis.

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Eksempel 2.8 *Given the function*

$$f(x,y,z) = \ln\left(16 - x^2 - 2y^2 - 4z^2\right), \qquad (x,y,z) \in A,$$

where A is given by the inequality

$$x^2 + 2y^2 + 4z^2 < 16.$$

1. *Indicate the boundary ∂A and explain why ∂A is a conic section; find the name of this and indicate its half axes.*

2. *Check if A is*

1) open,

2) closed,

3) star shaped.

3. *Find the gradient $\bigtriangledown f(x,y,z)$.*

Let $\mathcal{F}$ be that level surface of f, which contains the point $(2,1,1)$.

4. *Find the value of $f(x,y,z)$ for $(x,y,z) \in \mathcal{F}$.*

5. *Find an equation of the tangent plane of $\mathcal{F}$ at the point $(2,1,1)$.*

A Conic section; level surface; tangent plane.

D Follow the guidelines of the text.

I 1) It follows from the continuity of the function that the equation of the boundary ∂A is

$$x^2 + 2y^2 + 4z^2 = 16,$$

hence by a norming (division by 16),

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2\sqrt{2}}\right)^2 + \left(\frac{z}{2}\right)^1 = 1.$$

This normed form shows that ∂A is a conic section, in fact an ellipsoid of centrum $(0,0,0)$ and half axes 4, $2\sqrt{2}$ and 2 along the X axis, the Y axis and the Z axis respectively.

2) Since a polynomial is continuous and since we have “sharp” inequality signs, we conclude that A is open. Hence A is an open ellipsoid, therefore convex, and thus also star shaped. The set A is clearly not closed.

3) the gradient of f in A is

$$\bigtriangledown f(x,y,z) = \frac{1}{16 - x^2 - 2y^2 - 4z^2}\,(-2x, -4y, -8z).$$

4) By insertion we see that if $(x,y,z) \in \mathcal{F}$, then

$$f(x,y,z) = f(2,1,1)\ln(16-4-2-4) = \ln 6.$$

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5) We get at the point $(2,1,1)$ that

$$\nabla f(2,1,1) = \frac{1}{6}(-4,-4,-8) = -\frac{2}{3}(1,1,2).$$

The gradient is always perpendicular to the tangent plane, so an equation of the tangent plane of $\mathcal{F}$ at the point $(2,1,1)$ is given by

$$\begin{aligned} 0 &= -\frac{3}{2}\nabla f(2,1,1)\cdot(x-2,y-1,z-1) \\ &= (1,1,2)\cdot(x-2,y-1,z-1) \\ &= x-2+y-1+2z-2 \\ &= x+y+2z-5, \end{aligned}$$

thus after a rearrangement,

$$x+y+2z=5.$$

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3 Simple integrals in several variables

Eksempel 3.1 *Find for every of the given vector fields first the domain and then every indefinite integral, whenever such an integral exists.*

1) $\mathbf{V}(x,y) = (x,y)$.

2) $\mathbf{V}(x,y) = (y,x)$.

3) $\mathbf{V}(x,y) = \left(\dfrac{1}{x+y}, \dfrac{-x}{y(x+y)}\right)$.

4) $\mathbf{V}(x,y) = (3x^2+2y^2, 2xy)$.

5) $\mathbf{V}(x,y) = \left(3x^2+y^2+\dfrac{y}{1+x^2y^2}, 2xy-4+\dfrac{x}{1+x^2y^2}\right)$.

6) $\mathbf{V}(x,y) = \begin{pmatrix} \dfrac{-2x}{2-x^2-2y^2} + \dfrac{-x}{\sqrt{2-x^2-2y^2}} \\ \dfrac{-4y}{2-x^2-2y^2} + \dfrac{-2y}{\sqrt{2-x^2-2y^2}} \end{pmatrix}$.

7) $\mathbf{V}(x,y) = \left(\dfrac{x}{(x-y)^2}, \dfrac{-x^2}{y(x-y)^2}\right)$.

8) $\mathbf{V}(x,y) = \left(\dfrac{2x(1-e^y)}{(1+x^2)^2}, \dfrac{e^y}{1+x^2}\right)$.

9) $\mathbf{V}(x,y) = (\sin y + y\sin x + x, \cos x + x\cos y + y)$.

A Gradient fields; integrals.

D First find the domain. Then check if we are dealing with a differential, or use indefinite integration. Another alternative is to integrate along a step line within the domain.

I 1) The vector field $\mathbf{V}(x,y) = (x,y)$ is defined in the whole of $\mathbb{R}^2$.

a) First method. We get by only using the rules of calculation,

$$\mathbf{V}(x,y)\cdot(dx,dy) = x\,dx + y\,dy = d\left\{\frac{1}{2}(x^2+y^2)\right\},$$

which shows that $\mathbf{V}(x,y)$ has an indefinite integral,

$$F(x,y) = \frac{1}{2}(x^2+y^2).$$

b) Second method. We get by indefinite integration,

$$F_1(x,y) = \int x\,dx = \frac{1}{2}\,x^2,$$

hence

$$y - \frac{\partial}{\partial y}F_1(x,y) = y, \qquad \text{dvs. } F_2(x,y) = \frac{1}{2}\,y^2.$$

An integral is

$$F(x,y) = F_1(x,y) + F_2(x,y) = \frac{1}{2}(x^2+y^2).$$

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c) Third method. When we integrate along the step line

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

which lies in the domain, we get the candidate

$$\int_C \mathbf{V} \cdot (dx,dy) = \int_0^x t\,dt + \int_0^y t\,dt = \frac{1}{2}\,x^2 + \frac{1}{2}\,y^2.$$

d) Test. The test is *always* mandatory by the latter method; though it is not necessary in the two former ones, it is nevertheless highly recommended. Obviously,

$$\bigtriangledown F(x,y) = (x,y) = \mathbf{V}(x,y),$$

and we have checked our result.

2) The vector field $\mathbf{V}(x,y) = (y,x)$ is defined in $\mathbb{R}^2$.

a) First method. It follows by the rules of calculations that

$$\mathbf{V}(x,y) \cdot (dx,dy) = y dx + x dy = d(xy),$$

which shows that $\mathbf{V}(x,y)$ has an integral

$$F(x,y) = xy.$$

b) Second method. We get by indefinite integration

$$F_1(x,y) = \int y\,dx = xy,$$

thus

$$x - \frac{\partial}{\partial y} F_1(x,y) = x - x = 0, \qquad \text{i.e. } F_2(x,y) = 0.$$

An integral is

$$F(x,y) = xy.$$

c) Third method. When we integrate along the step line

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

which lies in the domain, then

$$\int_C \mathbf{V} \cdot (dx,dy) = \int_0^x 0\,dt + \int_0^y x\,dt = xy.$$

d) Test (which is mandatory by the third method). Clearly, $\bigtriangledown F = (y,x)$, so the calculations are all right.

3) The vector field $\mathbf{V}(x,y) = \left(\dfrac{1}{x+y}, \dfrac{-x}{y(x+y)} \right)$ is defined in the set

$$A = \{(x,y) \mid y \neq 0,\ y \neq -x\}.$$

This set is the union of four angular spaces, where one considers each of these separately when we solve the problem.

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a) First method. Here we get by some clever reductions,

$$\frac{1}{x+y}\,dx - \frac{x}{y(x+y)} = \frac{y^2}{y(x+y)}\left(\frac{1}{y}\,dx - \frac{x}{y^2}\,dy\right) = \frac{1}{1+\dfrac{x}{y}}\,d\left(\frac{x}{y}\right) = d\ln\left|1+\frac{x}{y}\right|,$$

so an integral in each of the four domains is

$$F(x,y) = \ln\left|1+\frac{x}{y}\right| = \ln|x+y| - \ln|y|.$$

b) Second method. We get by indefinite integration,

$$F_1(x,y) = \int \frac{1}{x+y}\,dx = \ln|x+y|,$$

thus

$$-\frac{x}{y(x+y)} - \frac{\partial F_1}{\partial y} = -\frac{x}{y(x+y)} - \frac{1}{x+y} = -\frac{1}{y}\,\frac{x+y}{x+y} = -\frac{1}{y}.$$

Hence by integration, $F_2(x,y) = -\ln|y|$, so an integral is

$$F(x,y) = F_1(x,y) + F_2(x,y) = \ln|x+y| - \ln|y| = \ln\left|1+\frac{x}{y}\right|.$$

c) Third method. In this case the integration along a step line is fairly complicated, because we shall choose a point and a step curve in each of the four angular spaces. It is possible to go through this method of solution, but since it is fairly long, we shall here leave it to the reader.

d) Test. Here

$$\triangledown F(x,y) = \left(\frac{1}{x+y}, \frac{1}{x+y} - \frac{1}{y}\right) = \left(\frac{1}{x+y}, -\frac{x}{y(x+y)}\right) = \mathbf{V}(x,y),$$

so our calculations are correct.

4) The vector field $\mathbf{V}(x,y) = (3x^2+2y^2, 2xy)$ is defined in $\mathbb{R}^2$.

a) First method Since

$$\begin{aligned}\mathbf{V}(x,y)\cdot(dx,dy) &= (3x^2+2y^2)dx + 2xy\,dy \\ &= d(x^3) + y^2dx + (y^2dx + xd(y^2)) \\ &= d(x^3+xy^2) + \underline{y^2dx},\end{aligned}$$

cannot be written as a differential, we conclude that $\mathbf{V}(x,y)$ is *not* a gradient field and no integral exists.

b) Second method. We get by indefinite integration,

$$F_1(x,y) = \int (3x^2+2y^2)dx = x^3 + 2xy^2,$$

and accordingly,

$$2xy - \frac{\partial F_1}{\partial y} = 2xy - 4xy = -2xy.$$

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This expression depends on x, which it should not if the field is a gradient field. Therefore, we conclude that the field is *not* a gradient field, and also that there does *not* exist any integral.

If one does not immediately see this, we get by the continuation,

$$F_2(x,y) = -\int 2xy\,dy = -xy^2,$$

so a *candidate* of the integral is

$$F(x,y) = F_1(x,y) + F_2(x,y) = x^3 + xy^2.$$

Then the test below will prove that this is not an integral.

c) Third method. Integration along the step curve

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y),$$

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in the domain gives

$$\int_C \mathbf{V} \cdot (dx, dy) = \int_0^x (3t^2 + 0)dt + \int_0^y 2xt\, dt = x^3 + xy^2,$$

in other words the same candidate as by the second method.

d) Test. We find

$$\bigtriangledown F(x, y) = (3x^2 + y^2, 2xy) \neq \mathbf{V}(x, y),$$

so the test is not successful. The field is *not* a gradient field.

5) The vector field

$$\mathbf{V}(x, y) = \left(3x^2 + y^2 + \frac{y}{1 + x^2y^2}, 2xy - 4 + \frac{x}{1 + x^2y^2}\right)$$

is defined in $\mathbb{R}^2$.

a) First method. Here

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= 3x^2dx + (y^2dx + 2xydy) - 4dy + \frac{1}{1 + x^2y^2}(ydx + xdy) \\ &= d(x^3 = +d(xy^2) - d(4t) + \frac{1}{1 + x^2y^2}d(xy) \\ &= d\{x^3 + xy^2 - 4y + \operatorname{Arctan}(xy)\}, \end{aligned}$$

so $\mathbf{V}(x, y)$ has an integral given by

$$F(x, y) = x^3 + xy^2 - 4y + \operatorname{Arctan}(xy).$$

b) Second method. We get by an indefinite integration,

$$F_1(x, y) = \int \left\{3x^2 + y^2 + \frac{y}{1 + x^2y^2}\right\} dx = x^3 + xy^2 + \operatorname{Arctan}(xy),$$

hence

$$\frac{\partial F_1}{\partial y} = 2xy + \frac{x}{1 + x^2y^2},$$

and whence

$$2xy - 4 + \frac{x}{1 + x^2y^2} - \frac{\partial F_1}{\partial y} = -4.$$

It follows immediately that $F_2(y) = -4y$. The vector field is a gradient field with an integral

$$F(x, y) = F_1(x, y) + F_2(y) = x^3 + xy^2 - 4y + \operatorname{Arctan}(xy).$$

c) Third method. If we integrate along the step curve

$$C : \quad (0, 0) \longrightarrow (x, 0) \longrightarrow (xy),$$

entirely in the domain, we get

$$\begin{aligned} \int_C \mathbf{V} \cdot d\mathbf{x} &= \int_0^x (3t^2 + 0 + 0)dt + \int_0^y \left\{2xt - 4 + \frac{x}{1 + x^2t^2}\right\} dt \\ &= x^3 + \{xy^2 - 4y + \operatorname{Arctan}(xy)\}. \end{aligned}$$

As mentioned above one *shall always* test the result by this method! The test is not necessary in the two former methods, but it is nevertheless highly recommended.

d) TEST. By some routine calculations,

$$\triangledown F(x,y) = \left(3x^2+y^2-0+\frac{y}{1+(xy)^2}, 0+2xy-4+\frac{x}{1+(xy)^2}\right) = \mathbf{V}(x,y).$$

We get the correct answer, so $\mathbf{V}(x,y)$ is a gradient field and an integral is

$$F(x,y) = x^3+xy^2-4y+\ \text{Arctan}(xy).$$

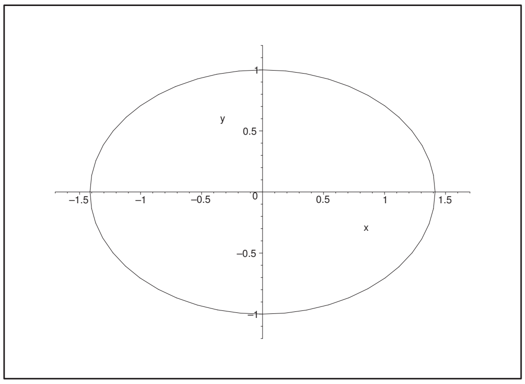
6) The vector field

$$\mathbf{V}(x,y) = \begin{pmatrix} \dfrac{-2x}{2-x^2-2y^2}+\dfrac{-x}{\sqrt{2-x^2-2y^2}} \\ \dfrac{-4y}{2-x^2-2y^2}+\dfrac{-2y}{\sqrt{2-x^2-2y^2}} \end{pmatrix}$$

is defined in the open ellipsoidal disc

$$A = \left\{(x,y)\ \middle|\ \left(\frac{x}{\sqrt{2}}\right)^2+y^2<1\right\}$$

of centrum $(0,0)$ and half axes $\sqrt{2}$ and 1.



Figur 29: The open domain of 6).

a) FIRST METHOD. By collecting terms which look more or less the same we get

$$\begin{aligned}
\mathbf{V}\cdot d\mathbf{x} &= \frac{1}{2-x^2-2y^2}(-2x\,dx-4y\,dy)+\frac{1}{\sqrt{2-x^2-2y^2}}(-x\,dx-2y\,dy) \\
&= \frac{1}{2-x^2-2y^2}d(2-x^2-2y^2)+\frac{1}{2}\,\frac{1}{\sqrt{2-x^2-2y^2}}d(2-x^2-2y^2) \\
&= d\left(\ln|2-x^2-2y^2|\right)+d\left(\sqrt{2-x^2-2y^2}\right) \\
&= d\left\{\ln(2-x^2-2y^2)+\sqrt{2-x^2-2y^2}\right\} \qquad \text{for } (x,y)\in A.
\end{aligned}$$

This vector field is a gradient field, an an integral in A is given by

$$F(x,y) = \ln(2-x^2-2y^2)+\sqrt{2-x^2-2y^2}.$$

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b) SECOND METHOD. We get by indefinite integration,

$$\begin{aligned} F_1(x,y) &= \int \left\{ \frac{-2x}{2-x^2-2y^2} + \frac{-x}{\sqrt{2-x^2-2y^2}} \right\} dx \\ &= \ln|2-x^2-2y^2| + \sqrt{2-x^2-2y^2} \\ &= \ln(2-x^2-2y^2) + \sqrt{2-x^2-2y^2}, \qquad\qquad (x,y)\in A, \end{aligned}$$

hence

$$\frac{\partial F_1}{\partial y} = \frac{-4y}{2-x^2-2y^2} + \frac{-2y}{\sqrt{2-x^2-2y^2}} = g(x,y).$$

Thus an integral in A is given by

$$F(x,y) = F_1(x,y) = \ln(2-x^2-2y^2) + \sqrt{2-x^2-2y^2},$$

and the vector field is a gradient field.

c) Since A is convex and symmetric about e.g. the X axis, the step curve

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y)$$

lies totally inside A for each $(x,y)\in A$. By an integration along this step curve we get

$$\begin{aligned} \int_C \mathbf{V}\cdot d\mathbf{x} &= \int_0^x \left\{ \frac{-2t}{2-t^2-0} + \frac{-t}{\sqrt{2-t^2-0}} \right\} dt \\ &\qquad + \int_0^y \left\{ \frac{-4t}{2-x^2-2t^2} + \frac{-2t}{\sqrt{2-x^2-2t^2}} \right\} dt \\ &= \{\ln(2-x^2) - \ln 2\} + \{\sqrt{2-x^2} - \sqrt{2}\} \\ &\qquad + \{\ln(2-x^2-2y^2) - \ln(2-x^2)\} \\ &\qquad + \{\sqrt{2-x^2-2y^2} - \sqrt{2-x^2}\} \\ &= \ln(2-x^2-2y^2) + \sqrt{2-x^2-2y^2} - \ln 2 - \sqrt{2}. \end{aligned}$$

Here we can of course neglect the constant $-\ln 2 - \sqrt{2}$.

d) TEST. We get by standard calculations

$$\triangledown F = \left(\frac{-2x}{2-x^2-2y^2} + \frac{-x}{\sqrt{2-x^2-2y^2}}, \frac{-4y}{2-x^2-2y^2} + \frac{-2y}{\sqrt{2-x^2-2y^2}} \right) = \mathbf{V}(x,y).$$

The test is OK, and $\mathbf{V}(x,y)$ is a gradient field with the obtained function $F(x,y)$ as an integral.

7) The vector field

$$\mathbf{V}(x,y) = \left(\frac{x}{(x-y)^2}, \frac{-x^2}{y(x-y)^2} \right)$$

is defined in the set

$$A = \{(x,y) \mid y \neq 0,\, y \neq x\},$$

with four angular spaces as its components.

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a) FIRST METHOD. Since $y \neq 0$ in A, it seems natural to put y^2 outside the denominator. Then

$$\begin{aligned}
\mathbf{V} \cdot d\mathbf{x} &= \frac{x}{(x-y)^2}dx - \frac{x^2}{y(x-y)^2}dy = \frac{\frac{x}{y}}{\left(\frac{x}{y}-1\right)^2} \cdot \frac{1}{y}\,dx + \frac{\frac{x}{y}}{\left(\frac{x}{y}-1\right)^2}\left(-\frac{x}{y^2}\right)dy \\
&= \frac{\frac{x}{y}}{\left(\frac{x}{y}-1\right)^2}\left\{\frac{1}{y}\,dx + x d\left(\frac{1}{y}\right)\right\} = \frac{\frac{x}{y}-1+1}{\left(\frac{x}{y}-1\right)^2}\,d\left(\frac{x}{y}\right) \\
&= \left\{\frac{1}{\frac{x}{y}-1} + \frac{1}{\left(\frac{x}{y}-1\right)^2}\right\} d\left(\frac{x}{y}\right) = d\left\{\ln\left|\frac{x}{y}-1\right| - \frac{1}{\frac{x}{y}-1}\right\}.
\end{aligned}$$

It follows that $\mathbf{V}(x,y)$ has an integral, e.g.

$$F(x,y) = \ln\left|\frac{x-y}{y}\right| + \frac{y}{y-x},$$

defined in each of the four connected components of A. Furthermore, $\mathbf{V}(x,y)$ is a gradient field.

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b) SECOND METHOD. We get by indefinite integration in A that

$$\begin{aligned} F_1(x,y) &= \int \frac{x}{(x-y)^2}\,dx = \int \frac{x-y+y}{(x-y)^2}\,dx = \int \frac{1}{x-y}\,dx + y\int \frac{1}{(x-y)^2}\,dx \\ &= \ln|x-y| - \frac{y}{x-y}, \end{aligned}$$

whence

$$\frac{\partial F_1}{\partial y} = \frac{-1}{x-y} - \frac{1}{x-y} - \frac{y}{(x-y)^2} = \frac{-2x+2y-y}{(x-y)^2} = \frac{-2x+y}{(x-y)^2},$$

so

$$-\frac{x^2}{y(x-y)^2} - \frac{\partial F_1}{\partial y} = -\frac{-x^2+2xy-y^2}{y(x-y)^2} = \frac{-(x-y)^2}{y(x-y)^2} = -\frac{1}{y},$$

and thus by an integration, $F_2(y) = -\ln|y|$, so an integral is

$$F(x,y) = \ln|x-y| - \frac{y}{x-y} - \ln|y| = \ln\left|\frac{x-y}{y}\right| - \frac{y}{x-y}.$$

c) THE THIRD METHOD can also be applied here but it is fairly difficult due to the structure of A, so here follows only a short description of the method. Choose a point in each of the four connected components. Then a geometric analysis shows that one should in the two angular spaces in which the angle is acute first integrate horizontally and then vertically, so the integration path has here the form

$$C: \quad (x_0,y_0) \longrightarrow (x,y_0) \longrightarrow (x,y),$$

because we have to stay inside the connected component.
In the other two angular spaces this path of integration may go beyond the connected component (sketch examples on a figure), so we introduce instead the following path of integration

$$C: \quad (x_0,y_0) \longrightarrow (x_0,y) \longrightarrow (x,y),$$

i.e. we first integrate vertically and then horizontally.
This is clearly a tedious procedure, and on the top of it one should also test the result before we can recognize the result as being correct.

d) TEST.

$$\begin{aligned} \nabla F(x,y) &= \left(\frac{\frac{1}{y}}{\frac{x-y}{y}} + \frac{y}{(x-y)^2}, \frac{1}{\frac{x-y}{y}} \cdot \left(-\frac{x}{y^2}\right) - \frac{x-y+y}{(x-y)^2} \right) \\ &= \left(\frac{x-y}{(x-y)^2} + \frac{y}{(x-y)^2}, -\frac{x}{y(x-y)} - \frac{x}{(x-y)^2} \right) \\ &= \left(\frac{x}{(x-y)^2}, \frac{-x}{y(x-y)^2}(x-y+y) \right) \\ &= \left(\frac{x}{(x-y)^2}, -\frac{x^2}{y(x-y)^2} \right) = \mathbf{V}(x,y), \end{aligned}$$

so we have checked that $\mathbf{V}(x,y)$ is indeed a gradient field with $F(x,y)$ as its integral.

8) The vector field

$$\mathbf{V}(x,y)=\left(\frac{2x(1-e^y)}{(1+x^2)^2},\frac{e^y}{1+x^2}\right)$$

is defined in $\mathbb{R}^2$.

a) First method. We get by some clever manipulation

$$\begin{aligned}\mathbf{V}\cdot d\mathbf{x} &= \frac{1-e^y}{(1+x^2)^2}\cdot 2x\,dx+\frac{1}{1+x^2}\,e^y dy=(1-e^y)\,\frac{1}{(1+x^2)^2}\,d(x^2)+\frac{1}{1+x^2}\,d(e^y)\\ &= (e^y-1)d\left(\frac{1}{1+x^2}\right)+\frac{1}{1+x^2}\,d(e^y-1)=d\left(\frac{e^y-1}{1+x^2}\right),\end{aligned}$$

so an integral is

$$F(x,y)=\frac{e^y-1}{1+x^2},$$

and $\mathbf{V}(x,y)$ is a gradient field.

b) Second method. We get by indefinite integration,

$$F_1(x,y)=\int\frac{2x(1-e^y)}{(1+x^2)^2}\,dx=(1-e^y)\int\frac{d(1+x^2)}{(1+x^2)^2}=\frac{e^y-1}{1+x^2},$$

where

$$\frac{\partial F_1}{\partial y}=\frac{e^y}{1+x^2}=g(x,y),$$

and $\mathbf{V}(x,y)$ is seen to be a gradient field.

c) Third method. Here there is plenty of space to integrate along the step curve

$$C:\quad (0,0)\longrightarrow(x,0)\longrightarrow(x,y),$$

no matter where $(x,y)\in\mathbb{R}^2$ lies. Then

$$\int_C\mathbf{V}\cdot d\mathbf{x}=\int_0^x 0\,dt+\int_0^y\frac{e^t}{1+x^2}\,dt=\frac{e^y-1}{1+x^2},$$

which is the *candidate*, which should be tested.

d) Test.

$$\triangledown F(x,y)=\left(-\frac{2x(e^y-1)}{(1+x^2)^2},\frac{e^y}{1+x^2}\right)=\mathbf{V}(x,y).$$

We conclude that $\mathbf{V}(x,y)$ is a gradient field and an integral is $F(x,y)$.

9) The vector field

$$\mathbf{V}(x,y)=(\sin y+y\sin x+x,\cos x+x\cos y+y)$$

is defined in $\mathbb{R}^2$.

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a) First method. We get by some simple manipulations

$$\begin{aligned}\mathbf{V}\cdot d\mathbf{x} &= \sin y\,dx + y\sin x\,dx + x\,dx + \cos x\,dy + x\cos y\,dy + y\,dy\\ &= \frac{1}{2}d\left(x^2+y^2\right) + (\sin y\,dx + x\,d\sin y) + (-y\,d\cos x + \cos x\,dy)\\ &= d\left\{\frac{1}{2}\,x^2+\frac{1}{2}\,y^2 + x\sin y - y\cos x\right\} + \underline{2\cos x\,dy},\end{aligned}$$

which clearly is *not* a differential, so the field is *not* a gradient field.

b) Second method. Indefinite integration gives

$$F_1(x,y) = \int (\sin y + y\sin x + x)dx = x\sin y - y\cos x + \frac{x^2}{2},$$

where

$$\frac{\partial F_1}{\partial y} = x\cos y - \cos x,$$

thus

$$\cos x + x\cos y + y - \frac{\partial F_1}{\partial y} = 2\cos x + y.$$

This expression depends on x, so we conclude that the vector field is *not* a gradient field.

c) Third method. Choose the step curve

$$C: \quad (0,0) \longrightarrow (x,0) \longrightarrow (x,y)$$

as the path of integration in $\mathbb{R}^2$. Then

$$\begin{aligned}\int_C \mathbf{V}\cdot d\mathbf{x} &= \int_0^x (0+0+t)dt + \int_0^y (\cos x + x\cos t + t)dt\\ &= \frac{1}{2}\,x^2 + y\cos x + x\sin y + \frac{1}{2}\,y^2\\ &= F(x,y),\end{aligned}$$

which is the *candidate*, and we *must* test it.

d) Test. It follows from

$$\begin{aligned}\bigtriangledown F(x,y) &= (x - y\sin x + \sin y, \cos x + x\cos y + y)\\ &= \mathbf{V}(x,y) - (2y\sin x, 0) \neq \mathbf{V}(x,y),\end{aligned}$$

that $\mathbf{V}(x,y)$ is not a gradient field.

Eksempel 3.2 *Sketch the domain A of the vector field*

$$\mathbf{V}(x,y)=\left(\frac{4x-y}{x(3x-y)},-\frac{1}{3}\left(\frac{1}{y}+\frac{1}{3x-y}\right)\right).$$

Prove that $\mathbf{V}$ *is a gradient field and find all its integrals.*
(Consider first a connected subset of A).

A Gradient field; integral.

D If there exists an integral, it can be found by one of the following three standard methods:

1) FIRST METHOD. Rules of calculation for differentials.
2) SECOND METHOD. Indefinite integration.
3) THIRD METHOD. Integration along some curve, typically a step curve. Notice that the *test is mandatory* by this method, because one may get "wrong solutions" by this method.

I The vector field is defined in the set

$$A=\{(x,y)\mid x\neq 0,\; y\neq 0,\; y\neq 3x\},$$

which is the union of six connected components. We shall in the following consider any one of these.

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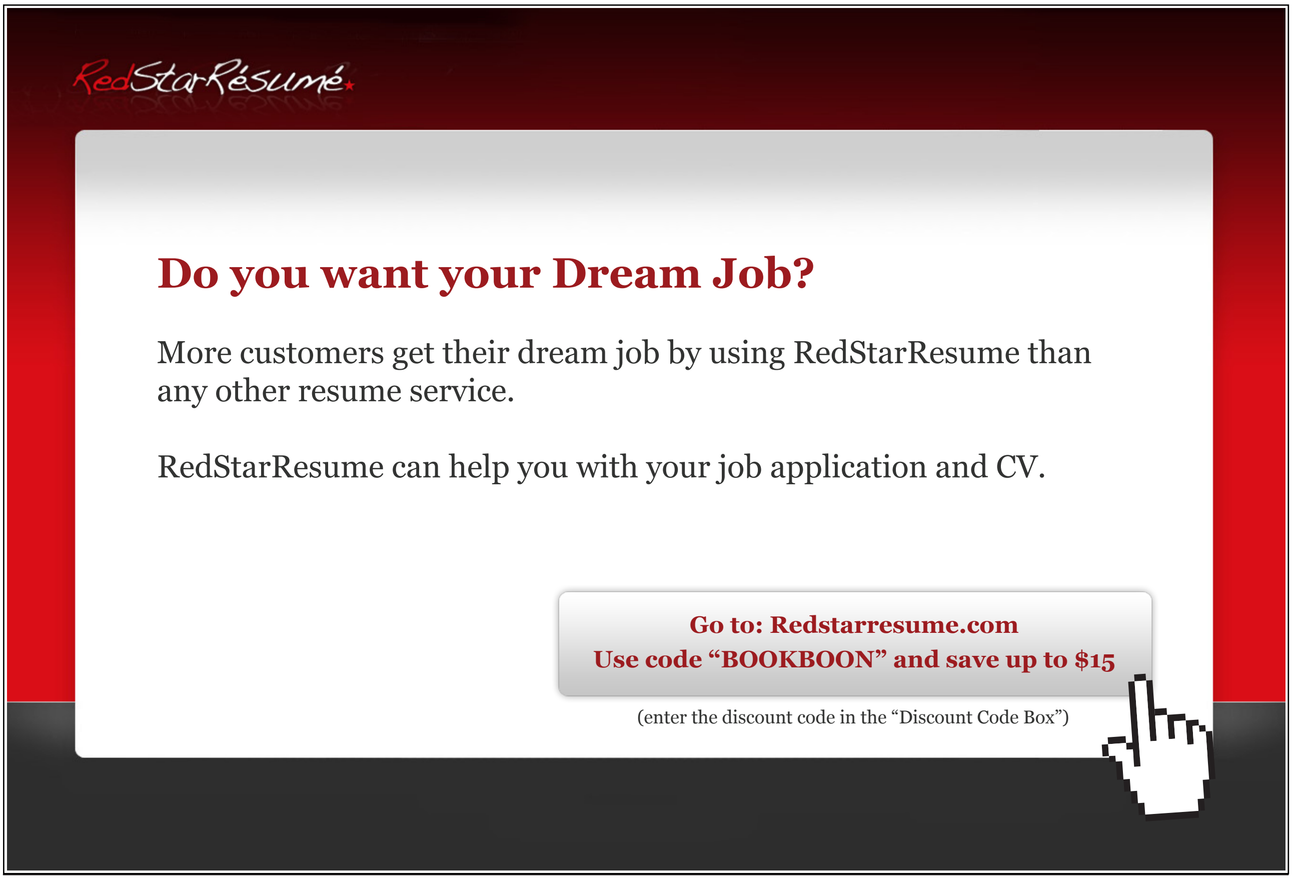

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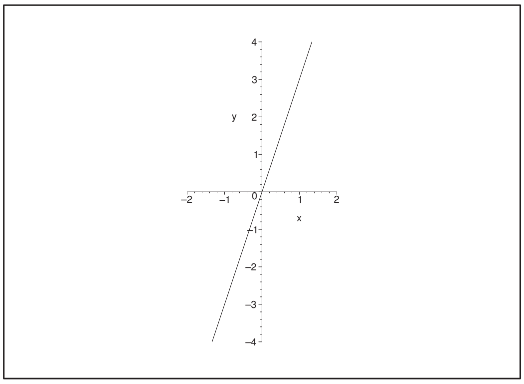

Figur 30: The six connected components of the domain of $\mathbf{V}(x,y)$.

ASIDE. Before we start on the calculations it will be quite convenient in advance to perform the following simple decomposition:

$$\frac{4x-y}{x(3x-y)}=\frac{x+(3x-y)}{x(3x-y)}=\frac{1}{x}+\frac{1}{3x-y},\qquad (x,y)\in A.$$

The vector field can then also be written

$$\mathbf{V}(x,y)=\left(\frac{1}{x}+\frac{1}{3x-y},-\frac{1}{3}\left(\frac{1}{y}+\frac{1}{3x-y}\right)\right),$$

which we shall exploit in the following. $\Diamond$

1) FIRST METHOD. The ides is that *if* $F(x,y)$ is an integral, then we can write

$$dF=\mathbf{V}(x,y)\cdot(dx,dy).$$

The task is therefore to prove that $\mathbf{V}(x,y)\cdot(dx,dy)$ can be written as a differential, dF, from which we directly get the integral $F(x,y)$.

We get in this case, where we always pair terms which are similar,

$$\begin{aligned}\mathbf{V}\cdot d\mathbf{x} &= \left(\frac{1}{x}+\frac{1}{3x-y}\right)dx-\frac{1}{3}\left(\frac{1}{y}+\frac{1}{3x-y}\right)dy\\ &= \frac{1}{x}\,dx-\frac{1}{3}\,\frac{1}{y}\,dy+\frac{1}{3}\cdot\frac{1}{3x-y}(3dx-dy)\\ &= d\ln|x|-\frac{1}{3}\,d\ln|y|+\frac{1}{3}\cdot\frac{1}{3x-y}\,d(3x-y)\\ &= \frac{1}{3}\,\ln\left|\frac{x^3}{y}\right|+\frac{1}{3}\,\ln|3x-y|=\frac{1}{3}\,\ln\left|\frac{x^3(3x-y)}{y}\right|.\end{aligned}$$

Since a possible test only consists of doing the same calculations in the reverse order, we conclude that all the integrals in A_i, $i=1,\dots,6$, are given by

$$F(x,y)=\frac{1}{3}\,\ln\left|\frac{x^3(3x-y)}{y}\right|+C_i,$$

where $C_i\in\mathbb{R}$ for $(x,y)\in A_i$, $i=1,\dots,6$.

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2) Second method. Indefinite integration. It follows from the form of the expression that it is most convenient to perform indefinite integration on the *latter* coordinate of the vector field. (It is not an error to choose the former coordinate instead; it is only that the calculations become somewhat more difficult in that case).

$$F_1(x,y) = -\frac{1}{3}\int\left(\frac{1}{y}+\frac{1}{3x-y}\right)dy = -\frac{1}{3}\,\ln|y|+\frac{1}{3}\,\ln|3x-y|.$$

Hence

$$\frac{\partial F_1}{\partial x} = \frac{1}{3}\cdot\frac{3}{3x-y} = \frac{1}{3x-y},$$

so

$$V_1(x,y)-\frac{\partial F_1}{\partial x} = \frac{4x-y}{x(3x-y)}-\frac{1}{3x-y} = \frac{4x-y-x}{x(3x-y)} = \frac{1}{x}.$$

As a weak control we note that this expression no longer depends on y, which is the variable which should have been removed by the integration above.

We get by another integration,

$$F_2(x) = \int\frac{1}{x}\,dx = \ln|x|.$$

We get in any connected component A_i, $i=1,\dots,6$, the integral

$$F(x,y) = F_1(x,y)+F_2(x) = -\frac{1}{3}\,\ln|y|+\frac{1}{3}\,\ln|3x-y|+\ln|x| = \frac{1}{3}\,\ln\left|\frac{x^3(3x-y)}{y}\right|,$$

and all integrals in a connected component A_i, $i=1,\dots,5$ is

$$F(x,y) = \frac{1}{3}\,\ln\left|\frac{x^3(3x-y)}{y}\right|+C_i,\quad C_i\in\mathbb{R},\quad (x,y)\in A_i,\quad i=1,\dots,6.$$

3) Third method. Because A is the union of six connected components, and since none of these have a natural starting point, the third method becomes somewhat complicated, so we shall leave this to the reader. In principal the calculations can be performed.

4) Test. It follows from

$$F(x,y) = -\frac{1}{3}\ln|y|+\frac{1}{3}\ln|3x-y|+\ln|x|+C_1 \quad \text{i } A_i$$

that

$$\begin{aligned}\triangledown F(x,y) &= \left(\frac{1}{3}\cdot\frac{3}{3x-y}+\frac{1}{x},-\frac{1}{3}\cdot\frac{1}{y}-\frac{1}{3}\cdot\frac{1}{3x-y}\right)\\ &= \left(\frac{4x-y}{x(3x-y)},-\frac{1}{3}\left(\frac{1}{y}+\frac{1}{3x-y}\right)\right) = \mathbf{V}(x,y).\end{aligned}$$

which shows that $\mathbf{V}$ is a gradient field.

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Eksempel 3.3 *Let A denote the point set which is obtained by removing the origo and the positive part of the Y axis from the plane $\mathbb{R}^2$,*

$$A = \{(x, y) \mid x \neq 0 \text{ or } (x = 0 \text{ and } y < 0)\}.$$

We define a function $f : A \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0, & y < 0, \\ y^2, & x > 0 \text{ og } y \geq 0, \\ -y^2, & x < 0 \text{ og } y \geq 0. \end{cases}$$

Prove that f is a C^1-function, and that its partial derivative $\dfrac{\partial f}{\partial x}$ is zero everywhere in A.

A A C^1-function, which is not identically 0, and where nevertheless $\dfrac{\partial f}{\partial x} = 0$ everywhere.

D Apply the definition of differentiability itself (and not one of the weaker rules of calculations) to prove that f is of class C^1. Then calculate $\dfrac{\partial f}{\partial x}$ by going to the limit in the difference quotient.

I Clearly, A is open, and $f(x, y)$ is continuous across the X axis (with the exception of $(0, 0)$, which is not included in the domain). Furthermore, $f(x, y)$ is of class C^∞, when $(x, y) \in A$ does not lie on the X axis.

Let us consider a point $(x_0, 0)$, $x_0 \neq 0$, on the X axis minus $(0, 0)$. If $x_0 < 0$, then

$$f(x,y) - f(x_0, 0) = \begin{cases} -y^2, & \text{for } y > 0, \\ \\ 0, & \text{for } y \leq 0, \end{cases} \qquad x < 0,$$

thus

$$|f(x,y) - f(x_0, 0)| \leq |y|^2 = 0 + \sqrt{(x-x_0)^2+y^2} \cdot \varepsilon(\sqrt{(x-x_0)^2+y^2}).$$

We get analogously for $x_0 > 0$ that

$$f(x,y) - f(x_0, 0) = \begin{cases} y^2, & \text{for } y > 0, \\ \\ 0, & \text{for } y \leq 0, \end{cases} \qquad x > 0,$$

i.e.

$$f(x,y) - f(x_0, y)| \leq |y|^2 = 0 + \sqrt{(x-x_0)^2+y^2} \cdot \varepsilon(\sqrt{(x-x_0)^2+y^2}).$$

It follows from these considerations that f is of class C^1 on the set of points on the X axis which also is included in A.

It is finally trivial that $\dfrac{\partial f}{\partial x} = 0$ everywhere in A.

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Eksempel 3.4 *Given the vector field*

$$\mathbf{V}(x,y)=\left(\frac{2x^3+2x+2xy}{1+x^2},\frac{x^2y^2+2y+x^2}{1+y^2}\right),\qquad (x,y)\in\mathbb{R}^2.$$

1) Prove that $\mathbf{V}$ *is a gradient field, and find all its integrals.*

2) Explain why any of the integrals has the range $\mathbb{R}$.

A Gradient field.

D Prove that $\omega=\mathbf{V}\cdot d\mathbf{x}$ is a differential. (Reduce!)

I 1) When we compute ω we get

$$\begin{aligned}\omega &= \mathbf{V}\cdot d\mathbf{x}=\frac{2x^3y+2x+2xy}{1+x^2}\,dx+\frac{x^2y^2+2y+x^2}{1+y^2}\,dy\\ &= \frac{2x}{1+x^2}\,dx+2xy\,dx+\frac{2y}{1+y^2}\,dy+x^2dy\\ &= d\,\ln(1+x^2)+d\,\ln(1+y^2)+\{2xy\,dx+x^2dy\}\\ &= d\left\{\ln\left(1+x^2\right)+\ln\left(1+y^2\right)+x^2y\right\}.\end{aligned}$$

This proves that $\mathbf{V}$ is a gradient field, and that all integrals are given by

$$F_C(x,y)=\ln\left(1+x^2\right)+\ln\left(1+y^2\right)+x^2y+C,\qquad C\in\mathbb{R}.$$

2) It follows from the rules of magnitude that

$$F_C(x,1)\to+\infty\qquad\text{for } x\to\pm\infty,$$

and that

$$F_C(x,-1)\to-\infty\qquad\text{for } x\to\pm\infty.$$

Since $F_C(x,y)$ is continuous in $(x,y)\in\mathbb{R}^2$ for every fixed $C\in\mathbb{R}$, we conclude that the range is all of $\mathbb{R}$.

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4 Extremum (two variables)

Eksempel 4.1 *Find in each of the following cases first the stationary points of the given function $f : \mathbb{R}^2 \to \mathbb{R}$. Then check if f in any of these points has an extremum; whenever this is the case, decide whether it is a maximum or a minimum.*

1) $f(x,y) = x^2 + 2y^2 - 2x - 2y$.

2) $f(x,y) = x^2 + y^2 + 2xy$.

3) $f(x,y) = xy^2$.

4) $f(x,y) = 3x^3 + 4y^3 + 6xy^2 - 9x^2$.

5) $f(x,y) = (x^2 + y^2 - 2y)(x^2 + y^2 - 6y)$

6) $f(x,y) = x^4 + y^4 - 2x^2y^2$.

7) $f(x,y) = 3x^4 + 4y^4 - 4x^2y^2$.

8) $f(x,y) = (\sin x)\cos y$.

9) $f(x,y) = x^2 + y^2 + e^{xy}$.

10) $f(x,y) = xy + 2\sinh(1 + x^2 + y^2)$.

11) $f(x,y) = x^3y - 2x^2y + xy^3$.

A Stationary points; extrema.

D Inspect the expression for a smart rearrangement. Find the stationary points. Check if these are extrema.

I 1) a) First variant. It is seen by inspection that

$$f(x,y) = x^2 + 2y^2 - 2x - 2y = (x-1)^2 + 2\left(y - \frac{1}{2}\right)^2 - \frac{3}{2}.$$

We conclude that $\left(1, \frac{1}{2}\right)$ is the only stationary point and that it is a minimum.

b) Second variant. Traditionally the equations of the stationary points are

$$\frac{\partial f}{\partial x} = 2x - 2 = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y - 2 = 0,$$

from which follows that $\left(1, \frac{1}{2}\right)$ is the only stationary point.

i) First subvariant. The approximating polynomial of at most second degree is found by translating the coordinate system to the point $\left(1, \frac{1}{2}\right)$, so we introduce the new variables

$$x = x_1 + 1, \qquad y = y_1 + \frac{1}{2}.$$

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Then by insertion,

$$P_2(x_1, y_1)\,[= f(x,y)] \;= x_1^2 + 2y_1^2 - \frac{3}{2}$$

[cf. the first variant], which clearly has a minimum for $(x_1, y_1) = (0,0)$, i.e. for $(x,y) = \left(1, \frac{1}{2}\right)$.

ii) SECOND SUBVARIANT. The (r,s,t)-method. It follows from

$$r = \frac{\partial^2 f}{\partial x^2} = 2, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0, \quad t = \frac{\partial^2 f}{\partial y^2} = 4,$$

that r, $t > 0$ and $s^2 < rt$, so we conclude that we have a minimum.

2) a) INSPECTION. It is immediately seen that

$$f(x,y) = x^2 + y^2 + 2xy = (x+y)^2,$$

which has a minimum ($= 0$) on the line $y = -x$. The points of this line are of course not proper minima.

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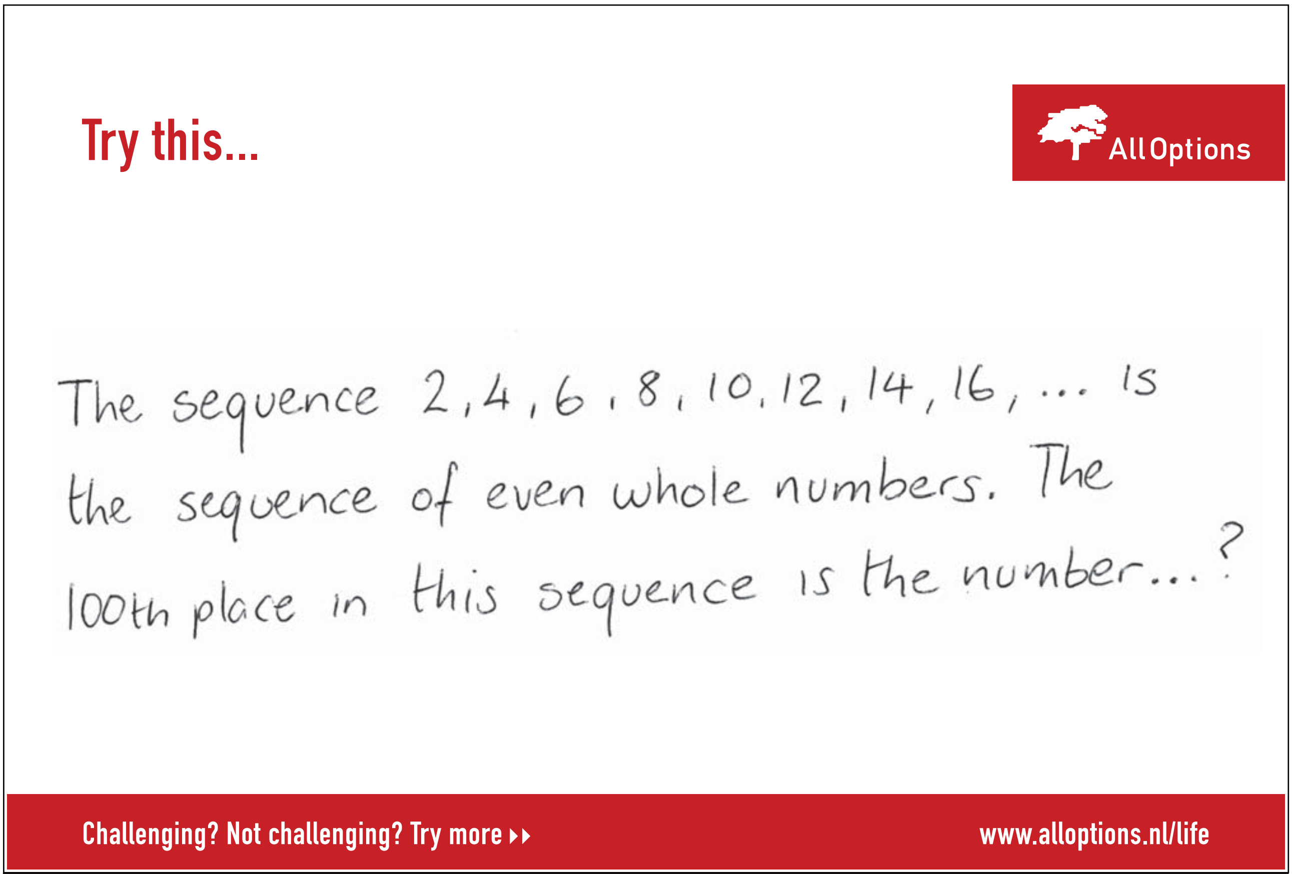

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b) The stationary points. These are the solutions of the equations

$$\frac{\partial f}{\partial x} = 2x + 2y = 2(x+y) = 0, \quad \frac{\partial f}{\partial y} = 2y + 2x = 2(x+y) = 0,$$

thus every point on the line $y = -x$ is a stationary point.

In this case we cannot conclude anything by the (r,s,t)-method. One should, however, be able to see that e.g.

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 2(x+y)(dx+dy) = d(x+y)^2,$$

so $f(x,y) = (x+y)^2$, and we are back to the first variant.

3) It follows from

$$\frac{\partial f}{\partial x} = y^2 = 0 \quad \text{og} \quad \frac{\partial f}{\partial y} = 2xy = 0$$

that the stationary points are the points on the X axis $y = 0$. In a neighbourhood of any point $(x_0, 0)$, $x_0 \neq 0$ we see that xy^2 has the same sign as x_0, and it is 0 on the X axis. This means that we have a minimum (though not a proper minimum) for every $(x_0, 0)$, $x_0 > 0$, and a maximum (though not a proper maximum) for every $(x_0, 0)$, $x_0 < 0$. In the stationary point $(0,0)$ we have neither a maximum nor a minimum, because the function in any neighbourhood of $(0,0)$ takes on both positive and negative values.

4) When $f(x,y) = 3x^3 + 4y^3 + 6xy^2 - 9x^2$, the equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = 9x^2 + 6y^2 - 18x = 0, \\[2ex] \dfrac{\partial f}{\partial y} = 12y^2 + 12xy = 12y(y+x) = 0. \end{cases}$$

We get two possibilities from the latter condition:

$$y = 0 \qquad \text{or} \qquad y = -x.$$

a) If we put $y = 0$ into the first equation we get

$$9x^2 - 18x = 9x(x-2) = 0,$$

so we conclude that $(0,0)$ and $(2,0)$ are stationary points.

b) If we put $y = -x$ into the first equation we get

$$15x^2 - 18x = 15x\left(x - \frac{6}{5}\right) = 0,$$

so we get the stationary points $(0,0)$ and $\left(\frac{6}{5}, -\frac{6}{5}\right)$.

Summarizing we get three different stationary points

$$(0,0), \qquad (2,0) \quad \text{og} \quad \left(\frac{6}{5}, -\frac{6}{5}\right).$$

These are now considered one by one.

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a) The point $(0,0)$ is not an extremum, because e.g. $f(0,y) = 4y^3$ takes on both positive and negative values in any neighbourhood of $(0,0)$.
In the other two cases we apply the (r,s,t)-method. We first calculate the general results

$$r = \frac{\partial^2 f}{\partial x^2} = 18x - 18, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 12y, \quad t = \frac{\partial^2 f}{\partial y^2} = 24y + 12x.$$

b) At $(2,0)$ we have $r = 18$, $s = 0$ and $t = 24$, so $rt > s^2$, $r > 0$, $t > 0$, corresponding to a proper minimum.

c) At $\left(\frac{6}{5}, -\frac{6}{5}\right)$ we have $r = \frac{18}{5}$, $s = -\frac{72}{5}$, $t = -\frac{72}{5}$, so $rt < s^2$ (e.g. s and t have different signs), and we have no extremum.

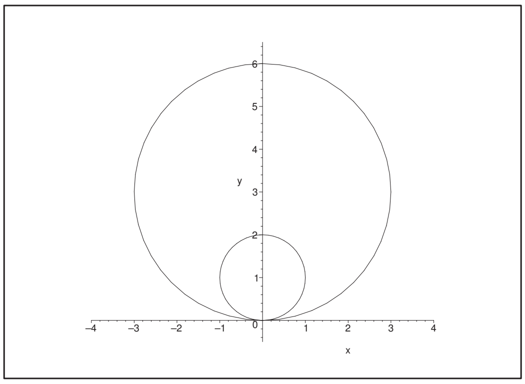
Summarizing we see that only $(2,0)$ is an extremum (a proper minimum).

5) It follows by the rearrangement

$$\begin{aligned} f(x,y) &= (x^2+y^2-2y)(x^2+y^2-6y) \\ &= \left(x^2 + \{y-1\}^2 - 1\right)\left(x^2 + \{y-3\}^2 - 3^2\right) \end{aligned}$$

that f is zero on the circles

$$x^2 + (y-1)^2 = 1 \quad \text{and} \quad x^2 + (y-3)^2 = 3^2.$$



Figur 31: Zero curves for $f(x,y)$.

The function is positive inside both circles (i.e. inside the smaller circle), and outside both circles. It is negative at every point inside the larger circle and outside the smaller circle. The function is continuous and 0 on both circles, so it follows from the main theorem that we must have a local maximum inside the smaller circle, and a local minimum inside the larger disc (and outside the smaller disc). Finally, f is both positive and negative in any neighbourhood of $(0,0)$, so this point cannot be an extremum.

Remark. It follows from the above that one can apply the main theorem in a qualitative way to decide where we must have extrema. Such analyses of figures are very useful. $\Diamond$

The stationary points. We shall now start on the tough calculations of the example. It follows from

$$f(x,y) = (x^2+y^2-2y)(x^2+y^2-6y)$$

that

$$\frac{\partial f}{\partial x} = 2x(x^2+y^2-6y) + 2x(x^2+y^2-2y) = 4x(x^2+y^2-4y),$$

and

$$\begin{aligned}
\frac{\partial f}{\partial y} &= (2y-2)(x^2+y^2-6y) + (2y-6)(x^2+y^2-2y) \\
&= (2y-4)(x^2+y^2-6y)+2(x^2+y^2-6y) + (2y-4)(x^2+y^2-2y)-2(x^2+y^2-2y) \\
&= 4(y-2)(x^2+y^2-4y) - 8y \\
&= 4\left\{(y-2)(x^2+y^2-4y) - 2y\right\}.
\end{aligned}$$

The two equations of the stationary points are therefore written more conveniently

$$(2) \quad \left\{ \begin{array}{rcl} x(x^2+y^2-4y) &=& 0, \\ \\ (y-2)(x^2+y^2-4y) &=& 2y. \end{array} \right.$$

It follows from the first equation that the stationary points (if any) either lies on the line $x=0$ or on the circle $x^2+(y-2)^2 = 2^2$.

a) If $x = 0$, then we get from the latter equation (2) that

$$\begin{aligned}
0 &= (y-2)(y^2-4y) - 2y = y\{(y-2)(y-4)-2\} \\
&= y\left\{y^2-6y+6\right\} = y\left\{(y-3)^2 - (\sqrt{3})^2\right\},
\end{aligned}$$

so either $y = 0$ or $y = 3 \pm \sqrt{3}$. Hence we get three stationary points,

$$(0,0), \qquad (0, 3+\sqrt{3}), \qquad (0, 3-\sqrt{3}).$$

b) If $x^2+y^2-4y = 0$, then it follows from the latter equation of (2) that $y = 0$, and thus $x = 0$, so we find again $(0,0)$.

Summarizing we get the three stationary points

$$(0,0), \qquad (0, 3+\sqrt{3}), \qquad (0, 3-\sqrt{3}).$$

Alternatively the expression gives the inspiration of using *polar coordinates*. We get

$$f(x,y) = (\varrho^2 - 2\varrho\sin\varphi)(\varrho^2 - 6\varrho\sin\varphi) = \varrho^4 - 8\varrho^3 \sin\varrho + 12\ varrho^2 \sin^2\varphi,$$

hence

$$\begin{aligned}
\frac{\partial f}{\partial \varrho} &= 4\varrho^3 - 24\varrho^2\sin\varphi + 24\varrho\sin^2\varphi = 4\varrho\left(\varrho^2 - 6\varrho\sin\varphi + 6\sin^2\varphi\right), \\
\frac{\partial f}{\partial \varphi} &= -8\varrho^3\cos\varphi + 24\varphi^2\sin\varphi\cos\varphi = 8\varrho^2\cos\varphi(-\varrho + 3\sin\varphi).
\end{aligned}$$

After a reduction the system of equations is written

$$(3) \quad \left\{ \begin{array}{l} \varrho\left(\varrho^2 - 6\varrho\sin\varphi + 6\sin^2\varphi\right) = 0, \\ \\ \varrho^2\cos\varphi(3\sin\varphi - \varrho) = 0. \end{array} \right.$$

From the latter equation we get the three possibilities $\varrho = 0$, $\cos\varphi = 0$ and $3\sin\varphi = \varrho$.

a) If $\varrho = 0$, then both equations are fulfilled so $(0,0)$ is a stationary point corresponding to $\varrho = 0$, hence to $(0,0)$ in rectangular coordinates.

b) If $\cos\varphi = 0$ (and $\varrho \geq 0$), then $\sin\varphi = \pm 1$.

i) When $\sin\varphi = -1$ it follows that

$$\varrho^2 - 6\varrho\sin\varphi + 6\sin^2\varphi = \varrho^2 + 6\varrho + 6 \geq 6 \quad \text{for } \varrho \geq 0,$$

so we are only left with the possibility $\varrho = 0$, which has already been treated above.

ii) When $\sin\varphi = 1$ (and $\varrho = y$, because $y = \varrho\sin\varphi$), then

$$\varrho^2 - 6\varrho\sin\varphi + 6\sin^2\varphi = \varrho^2 - 6\varrho + 6 = 0$$

has the two positive solutions $\varrho = 3 \pm \sqrt{3}$, corresponding to the stationary points

$$(0, 3+\sqrt{3}) \quad \text{and} \quad (0, 3-\sqrt{3})$$

in rectangular coordinates.

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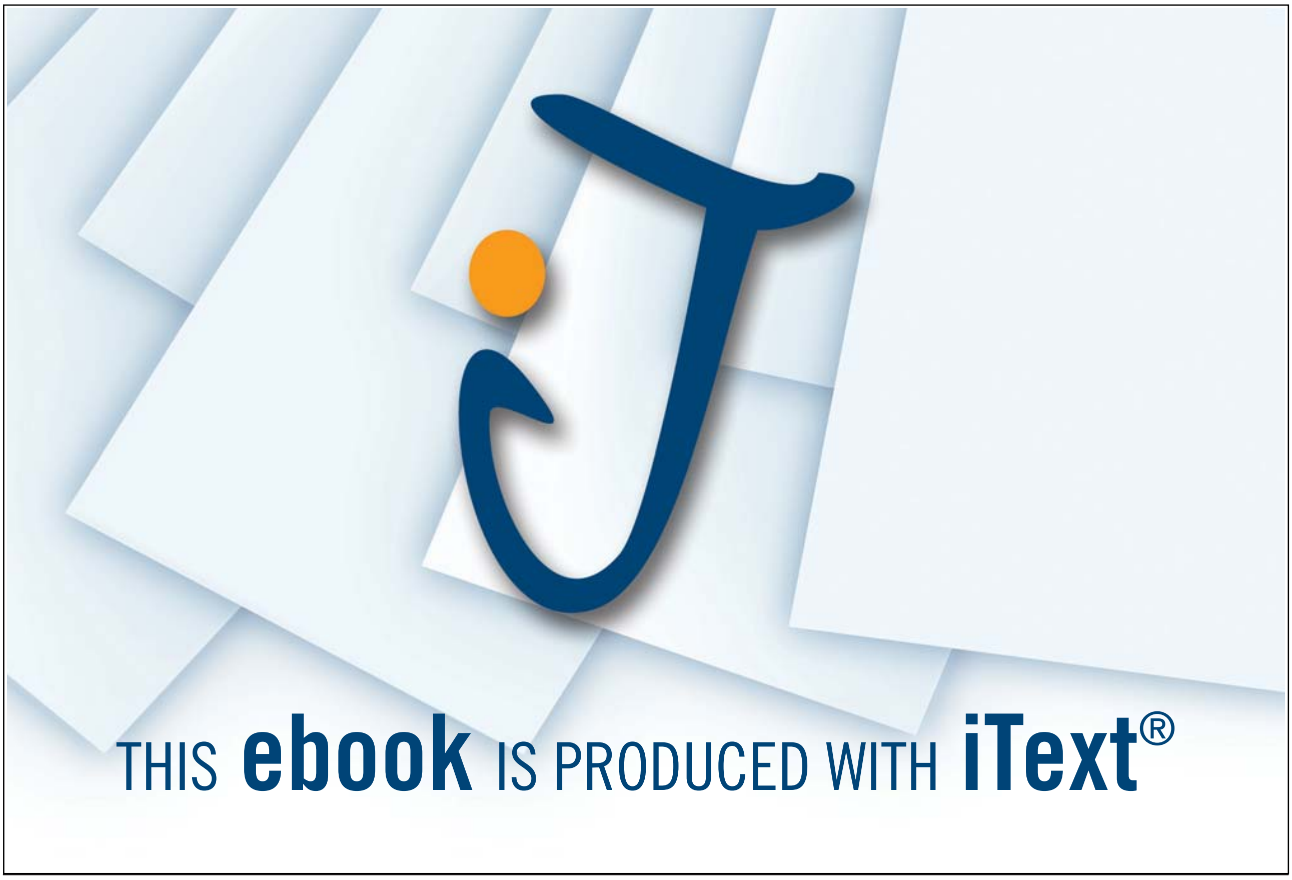

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c) If $3\sin\varphi = \varrho$, then

$$0 = 4\varrho\left(\varrho^2 - 2\varrho\cdot 3\sin\varphi + \frac{2}{3}(3\sin\varphi)^2\right) = 4\varrho\left(\varrho^2 - 2\varrho^2 + \frac{2}{3}\,\varrho^2\right) = -\frac{8}{3}\,\varrho^2,$$

with the only solution $\varrho = 0$, which we have already treated above.

Summarizing we get in rectangular coordinates the following three stationary points

$$(0,0), \qquad (0,3+\sqrt{3}), \qquad (0,3-\sqrt{3}).$$

Check of the type of the stationary points, inspection. It follows from the discussion of the sign in the beginning of the example that $(0,0)$ is *not* an extremum, because the function takes on both positive and negative values in any neighbourhood of $(0,0)$.

The point $(0,3-\sqrt{3})$ must be a *local maximum*. In fact, we have already by the main theorem concluded that there exists a local maximum in the smaller disc, and since $f \in C^\infty$, it can only be attained at a stationary point. Since $(0,3-\sqrt{3})$ is the only stationary point in the smaller disc the claim follows.

It follows analogously by the discussion of the sign that $(0,3+\sqrt{3})$ must be a *local minimum*.

Remark. Notice that by the application of the main theorem we obtain a much simpler analysis than by the traditional *standard* methods in the following. We have almost done everything! $\Diamond$

Investigation of the type of the stationary points, standard procedure.

a) We first check $(0,0)$.

i) First variant, the (r,s,t)-method. This breaks totally down because

$$r = s = t = 0,$$

and nothing can be concluded.

ii) Second variant. Approximating polynomials of at most second degree. This cannot be used either, because

$$P_2(x,y) \equiv 0,$$

and nothing can be concluded.

iii) Third variant. A dirty trick. If follows from

$$4ab = (a+b)^2 - (a-b)^2, \quad \text{dvs. } ab = \frac{1}{4}\left\{(a+b)^2 - (a-b)^2\right\},$$

that

$$a = x^2 + y^2 - 2y \quad \text{and} \quad b = x^2 + y^2 - 6y,$$

so

$$f(x,y) = \frac{1}{4}\left\{4(x^2+y^2-4y)^2 - 16y^2\right\} = (x^2+y^2-4y)^2 - 4y^2.$$

If we go towards $(0,0)$ along the line $y = 0$, then $f(x, 0) > 0$, and if we go towards $(0,0)$ along the circle $x^2 + y^2 - 4y = 0$, then $f(x, y) < 0$, so f takes on both positive and negative values in any neighbourhood of$(0,0)$. Therefore we cannot have an extremum at $(0,0)$.

REMARK. The trick above is one of the very oldest in mathematics. The Egyptians did not have tables of multiplication, though tables of squared numbers. They used the trick above to calculate products. $\diamondsuit$

ERRONEOUS VARIANT. One might try to investigate the limit along all lines through $(0,0)$.

i) If $y = 0$, then $f(x, 0) = x^4 > 0$ for $x \neq 0$.

ii) If $x = 0$, then

$$f(0, y) = (y^2 - 2y)(y^2 - 6^y) = y^2(2 - y)(6 - y) > 0 \quad \text{for } 0 < |y| < 2.$$

iii) If $y = ax$, $a \neq 0$, then

$$f(x, ax) = \cdots = x^2\{2a - (1 + a^2)x\}\{6a - (1 + a^2)x\},$$

where the product of the latter two factors tends towards $12a^2 > 0$ for $x \to 0$, and accordingly

$$f(x, ax) > 0 \quad \text{sufficiently close to } 0.$$

One might then **erroneously** conclude that $(0,0)$ is a minimum, what it is **not**.

b) Let us return to the points $(0, 3 \pm \sqrt{3})$. These are checked by the (r, s, t)-method. First calculate

$$\begin{aligned}
r &= \frac{\partial^2 f}{\partial x^2} = 4\{x^2 + y - 4y\} + 8x^2 = 12x^2 + 4y(y - 4), \\
s &= \frac{\partial^2 f}{\partial x \partial y} = 4x(2y - 4) = 8x(y - 2), \\
t &= \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2 - 4y) + 4(y - 2)(2y - 4) - 8 = 4x^2 + 12y(y - 4).
\end{aligned}$$

Since both stationary points satisfy $x = 0$ we can reduce in the following way

$$\begin{aligned}
r_{|x=0} &= 4y(y - 4), \\
s_{|x=0} &= 0, \\
t_{|x=0} &= 12y(y - 4) = 3r_{|x=0},
\end{aligned}$$

thus

$$r_{|x=0}t_{|x=0} = 3\left(r_{|x=0}\right)^2 \geq 0 = s_{|x=0}^2.$$

We therefore have extremum when $r_{|x=0} \neq 0$.

i) We have at the point $(0, 3 + \sqrt{3})$ that $3 + \sqrt{3} > 4$, so $r > 0$, and $t > 0$, so we have a minimum.

ii) We have at the point $(0, 3 - \sqrt{3})$ that $3 - \sqrt{3} < 4$, so $r < 0$, and $t < 0$, and we have a maximum.

6) Here

$$f(x, y) = x^4 + y^4 - 2x^2y^2 = \left(x^2 - y^2\right)^2 = (x - y)^2(x + y)^2,$$

corresponding to a minimum on the lines $y = x$ and $y = -x$.

ALTERNATIVELY,

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4x^3 - 4xy^2 = 4x(x^2 - y^2),\\ \frac{\partial f}{\partial y} &= 4y^3 - 4x^2y = 4y(y^2 - x^2).\end{aligned}$$

The stationary points are

$$(0,0), \qquad (x,x) \text{ and } (x,-x), \quad x \in \mathbb{R},$$

corresponding to the fact that the stationary points form the set consisting of the two lines $y = x$ and $y = -x$. On these we have a minimum, though not a proper minimum.

7) Here

$$f(x,y) = 3x^4 + 4y^4 - 4x^2y^2 = (4y^4 - 4x^2y^2 + x^4)02x^4 = \left(2y^2 - x^2\right)^2 + 2x^4,$$

so we have got a minimum for $y^2 = \frac{1}{2}\,x^2 = 0$, i.e. at $(0,0)$.

ALTERNATIVELY

$$\begin{aligned}\frac{\partial f}{\partial x} &= 12x^3 - 8xy^2 = 4x(3z^2 - 2y^2),\\ \frac{\partial f}{\partial y} &= 16y^3 - 8x^2y = 8y(2y^2 - x^2).\end{aligned}$$

It is almost obvious that $(0,0)$ is the only stationary point.

8) Here

$$\frac{\partial f}{\partial x} = \cos x \cdot \cos y \quad \text{and} \quad \frac{\partial f}{\partial y} = -\sin x \cdot siny.$$

These expressions are both zero, if and only if

$$(x,y) = \left(\frac{\pi}{2} + p\pi, q\pi\right), \qquad p,\, q \in \mathbb{Z},$$

hence

$$(x,y) = \left(p\pi, \frac{\pi}{2} + q\pi\right), \qquad p,\, q \in \mathbb{Z}.$$

These are the stationary points.

Since

$$f\left(\frac{\pi}{2} + p\pi, q\pi\right) = (-1)^p \cdot (-1)^q = (-1)^{p+q} \quad \text{og} \quad f\left(p\pi, \frac{\pi}{2} + q\pi\right) = 0,$$

and $|f(x,y)| \le 1$, it follows immediately that we have maxima at

$$\left(\frac{\pi}{2} + p\pi, q\pi\right) \qquad \text{for } p + q \text{ even},$$

and minima at

$$\left(\frac{\pi}{2} + p\pi, q\pi\right) \qquad \text{for } p+q \text{ odd.}$$

In the neighbourhood of any point of the form $\left(p\pi, \frac{\pi}{2} + q\pi\right)$ the function attains both positive and negative values, so these points are not extrema.

9) The equations of the stationary points for $f(x,y) = x^2 + y^2 + e^{xy}$ are

$$\frac{\partial f}{\partial x} = 2x + ye^{xy} = 0, \qquad \frac{\partial f}{\partial y} = 2y + xe^{xy} = 0.$$

These are clearly satisfied for $(x,y) = (0,0)$, so $(0,0)$ is a stationary point.

Assume that $x \neq 0$. Then also $y \neq 0$, and thus $xy \neq 0$. Accordingly,

$$2x^2 = -xye^{xy} = 2y^2,$$

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i.e. $x^2 = y^2$ and $xy \le 0$, so $y = -x$. If this restriction is put into the first equation, we get

$$0 = 2x - xe^{-x^2} = x\left(2 - e^{-x^2}\right),$$

which has only the solution $x = 0$.

Remark. This is actually a "false solution". On the other hand, we have already checked that $(0,0)$ is a stationary point. $\Diamond$

Hence, $(0,0)$ is the only stationary point.

The problem of a possible extremum at $(0,0)$ can be solved in various ways.

a) It follows from

$$\begin{aligned} f(x,y) &= x^2+y^2+e^{xy} \\ &= x^2+y^2+1+xy+xy\varepsilon(xy) \qquad \text{(Taylor)} \\ &= 1+\frac{1}{2}\left\{x^2+y^2+(x+y)^2\right\}+xy\cdot\varepsilon(x,y) \end{aligned}$$

that $(0,0)$ is a proper minimum.

b) We get from

$$\frac{\partial^2 f}{\partial x^2} = 2+y^2e^{xy}, \qquad r=2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{xy}+xye^{xy}, \qquad s=1,$$

$$\frac{\partial^2 f}{\partial y^2} = 2+x^2e^{xy}, \qquad t=2,$$

that $rt > s^2$, and since $r > 0$ and $t > 0$, the stationary point $(0,0)$ must be a proper minimum.

10) The equations for the stationary points for $f(x,y) = xy + 2\sinh(1+x^2+y^2)$ are

$$\begin{aligned} \frac{\partial f}{\partial x} &= y + 4x\cosh(1+x^2+y^2) = 0, \\ \frac{\partial f}{\partial y} &= x + 4y\cosh(1+x^2+y^2) = 0. \end{aligned}$$

By inspection, $(0,0)$ is clearly a stationary point. If there were other stationary points they should fulfil $xy \neq 0$. Assume that $(x,y) \neq (0,0)$ is such a stationary point. Then

$$y^2 = -4xy\cosh(1+x^2+y^2) = x^2,$$

so $y^2 = x^2$ and $xy < 0$, thus $y = -x$. By insertion the condition is reduced to

$$0 = -x + 4x\cosh(1+2x^2) = x(4\cosh(1+2x^2)-1).$$

Now $4\cosh(1+2x^2) - 1 > 0$, so $x = 0$ is the only solution (and strictly speaking we assumed that $x \neq 0$). On the other hand, he have already proved that $(0,0)$ is a stationary point. Accordingly, $(0,0)$ is the only stationary point.

Concerning extremum at $(0,0)$ we have again several possibilities.

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a) It follows from

$$\frac{\partial^2 f}{\partial x^2} = 4\cosh(1+x^2+y^2) + x\{\cdots\}, \qquad r = 4\cosh 1,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1 + x\{\cdots\}, \qquad s = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = 4\cosh(1+x^2+y^2) + y\{\cdots\}, \qquad t = 4\cosh 1,$$

that $rt > s^2$ and $r > 0$ and $t > 0$, corresponding to a minimum.

b) It is this time more tricky just to inspect the function, though it is still possible:

$$\begin{aligned} f(x,y) &= xy + 2\sinh(1+x^2+y^2) \\ &= xy + 2\sinh 1 \cdot \cosh(x^2+y^2) + 2\cosh 1 \cdot \sinh(x^2+y^2) \\ &= xy + 2\sinh 1 \cdot \left\{1 + (x^2+y^2)\varepsilon(x^2+y^2)\right\} \\ &\qquad +2\cosh 1 \cdot \left\{x^2+y^2+(x^2+y^2)\varepsilon(x^2+y^2)\right\} \\ &= 2\sinh 1 + \frac{1}{2}(x+y)^2 + \left\{2\cosh 1 - \frac{1}{2}\right\}(x^2+y^2) \\ &\qquad +(x^2+y^2)\varepsilon(x^2+y^2). \end{aligned}$$

Since $\cosh 1 - \dfrac{1}{2} > 0$, it follows by this rearrangement that we have a minimum at $(0,0)$.

11) It follows from

$$\begin{aligned} f(x,y) &= x^3y - 2x^2y + xy^3 = xy\left\{x^2 - 2x + y^2\right\} \\ &= xy\left\{(x-1)^2 + y^2 - 1\right\}, \end{aligned}$$

that the zero curves for $f(x,y)$ are the coordinate axes and the circle of centrum $(1,0)$ and radius 1.

The plane is then divided into six subregions, in which the sign of the function is fixed. Every one of these open subregions have $(0,0)$ as a boundary point, so if one circles around $(0,0)$, then the sign of the function will be positive in every second subregion and negative in the others, because the sign is negative in the second quadrant and positive in the third quadrant.

The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 3x^2y - 4xy + y^3 = y(3x^2 - 4x + y^2) = 0,$$

and

$$\frac{\partial f}{\partial y} = x^3 - 2x^2 + 3xy^2 = x(x^2 - 2x + 3y^2) = 0.$$

If we put $y = 0$, then the first equation is fulfilled, and the second one is reduced to

$$0 = x(x^2 - 2x) = x^2(x-2).$$

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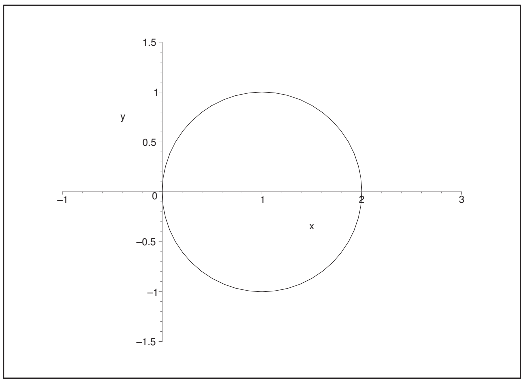

Figur 32: The zero curves for $f(x,y)$.

Therefore, in this case we get the stationary points $(0,0)$ and $(2,0)$.

If instead $y^2 = 4x - 3x^2 = 3x\left(\frac{4}{3} - x\right) \geq 0$, the first equation is again fulfilled. Then notice that this implies that $x \in \left[0, \frac{4}{3}\right]$. By insertion into the second equation we get

$$0 = x(x^2 - 2x + 12x - 9x^2) = x^2(10 - 8x) = 8x^2\left(\frac{5}{4} - x\right),$$

where the solutions are $x = 0$ and $x = \frac{5}{4} \in \left[0, \frac{4}{3}\right]$.

When $x = 0$ we get $y = 0$, so we find again the stationary point $(0,0)$.

When $x = \frac{5}{4}$ we get

$$y^2 = 4x - 3x^2 = 5 - \frac{73}{16} = \frac{5}{16}, \quad \text{i.e.} \quad y = \pm\frac{\sqrt{5}}{4},$$

corresponding to the stationary points

$$\left(\frac{5}{4}, \frac{\sqrt{5}}{4}\right) \quad \text{and} \quad \left(\frac{5}{4}, -\frac{\sqrt{5}}{4}\right).$$

Summarizing we have found the four stationary points

$$(0,0), \quad (2,0), \quad \left(\frac{5}{4}, \frac{\sqrt{5}}{4}\right), \quad \left(\frac{5}{4}, -\frac{\sqrt{5}}{4}\right).$$

It follows from the figure that $(0,0)$ and $(2,0)$ both lie on one of the zero curves, so

$$f(0,0) = f(2,0) = 0.$$

Then we conclude from the discussion of the sign that the function is both positive and negative in any neighbourhood of the points $(0,0)$ and $(2,0)$, so these cannot be extrema.

Since f is of class C^∞, and since the two closed half discs are bounded with the value 0 of the function on the boundaries, it follows from the second main theorem that we have a minimum somewhere in the interior of the upper half disc and a maximum somewhere in the interior of the lower half disc. These must necessarily be attained at stationary points. There is exactly one stationary point in each of these half discs, so we conclude that the local minimum is

$$f\left(\frac{5}{4},\frac{\sqrt{5}}{4}\right)=\frac{5}{4}\cdot\frac{\sqrt{5}}{4}\left\{\frac{25}{16}-\frac{5}{2}+\frac{5}{16}\right\}=\frac{5\sqrt{5}}{16}\left(-\frac{10}{16}\right)=-\frac{25\sqrt{5}}{128},$$

and the local maximum is

$$f\left(\frac{5}{4},-\frac{\sqrt{5}}{4}\right)=\frac{25\sqrt{5}}{128}.$$


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ALTERNATIVELY one may try the (r,s,t)-method. First calculate

$$\begin{aligned} r &= \frac{\partial^2 f}{\partial x^2} = 6xy - 4y = 2y(3x-2), \\ s &= \frac{\partial^2 f}{\partial x \partial y} = 3x^2 - 4x + 3y^2, \\ t &= \frac{\partial^2 f}{\partial y^2} = 6xy. \end{aligned}$$

For $(0,0)$ we cannot conclude anything, because $r = s = t = 0$.

For $(2,0)$ we get $r = 0$, $s = 12 - 8 = 4$ and $t = 0$, thus $rt < s^2$, and we have no extremum.

For $\left(\frac{5}{4}, \frac{\sqrt{5}}{4}\right)$ we get

$$\begin{aligned} r &= \frac{\sqrt{5}}{4}\left(\frac{15}{4} - 2\right) = \frac{7\sqrt{5}}{8}, \\ s &= \frac{75}{16} - 5 + \frac{15}{16} = \frac{5}{8}, \\ t &= 6 \cdot \frac{5}{4} \cdot \frac{\sqrt{5}}{4} = \frac{15\sqrt{5}}{8}. \end{aligned}$$

Since $r > 0$, $t > 0$ and $rt = \frac{525}{64} > \frac{25}{64} = s^2$, we conclude that we have a proper minimum at the point.

For $\left(\frac{5}{4}, -\frac{\sqrt{5}}{4}\right)$ we get [cf. the above]

$$r = -\frac{7\sqrt{5}}{8}, \quad s = \frac{5}{8}, \quad t = -\frac{15\sqrt{5}}{8},$$

so $r < 0$, $t < 0$ and $rt > s^2$, and we have a proper maximum at the point.

Finally, let us consider the point $(0,0)$. By using polar coordinates we get

$$\begin{aligned} f(x,y) &= \varrho^4 \cos^3\varphi \sin\varphi - 2\varrho^3 \cos^2\varphi \sin\varphi + \varrho^4 \cos\varphi \cdot \sin^3\varphi \\ &= \varrho^2\{-2\cos^2\varphi\, \sin\varphi + \varphi \cos\varphi \cdot \sin\varphi\}. \end{aligned}$$

When $\varrho \to 0+$, the first term dominates in most cases, and since the first term can take on both positive and negative values, we conclude that $(0,0)$ is not an extremum.

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Eksempel 4.2 *Let α be a constant. Check for each value of α if the function*

$$f(x,y) = x^2 + y^2 + \alpha xy + (x-y)^4$$

has an extremum at $(0,0)$. Whenever this is the case, check also if it is a proper extremum.

A Investigation of extrema.

D First method. Rewrite $f(x,y)$ and give a direct argument.

Second method. Apply the (r,s,t)-method.

I As $f(x,y)$ is a polynomial, $f(x,y)$ is of class C^∞. We note that $f(0,0)=0$.

1) First method. This method is similar to (though not identical with) the procedure of finding the approximating polynomial of second degree. By rewriting the terms of second degree we get

$$\begin{aligned} f(x,y) &= x^2 + y^2 + \alpha xy + (x-y)^4 \\ &= \left\{x^2 + 2x \cdot \frac{\alpha}{2}\, y + \left(\frac{\alpha}{2}\, y\right)^2\right\} + \left\{1 - \frac{\alpha^2}{4}\right\} y^2 + (x-y)^4 \\ &= \left((x + \frac{\alpha}{2}\, y\right)^2 + \left\{1 - \left(\frac{\alpha}{2}\right)^2\right\} y^2 + (x-y)^4. \end{aligned} \tag{4}$$

Then we split the investigation into various cases.

a) If $|\alpha| < 2$, then all three terms of (4) are bigger than or equal to 0, and when $(x,y) \neq (0,0)$, then at least one of them is bigger than zero. Thus we conclude that $(0,0)$ is a proper minimum.

b) If $\alpha = 2$, then (4) reduces to

$$f(x,y) = (x+y)^2 + (x-y)^4 \quad [\geq 0].$$

If $(x,y) \neq (0,0)$, then $x+y$ and $x-y$ cannot both be 0 and we conclude as above that we have a proper minimum at $(0,0)$.

c) If $\alpha = -2$ (the difficult case) we write (4) in the form

$$f(x,y) = (x-y)^2 + (x-y)^4 = (x-y)^2\{1 + (x-y)^2\} \quad [\geq 0].$$

It follows that the minimum 0 is attained on the line $y = x$, so we conclude that we have a minimum, but not a proper minimum at $(0,0)$.

d) If $|\alpha| > 2$, then $0 > 1 - \left(\frac{\alpha}{2}\right)^2 = -\beta^2$, where $\beta > 0$, and the function can be written

$$f(x,y) = \left(x + \frac{\alpha}{2}\, y\right)^2 - (\beta y)^2 + (x-y)^4,$$

and the approximating polynomial of second degree is

$$P_2(x,y) = \left(x + \frac{\alpha}{2}\, y\right)^2 - (\beta y)^2.$$

Since

$$P_2\left(x, -\frac{2}{\alpha}\, x\right) = 0 - \frac{4\beta^2}{\alpha^2}\, x^2 = -\left(\frac{2\beta x}{\alpha}\right)^2 < 0 \quad \text{for } x \neq 0,$$

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and

$$P_2(x,0) = x^2 > 0 \qquad \text{for } x \neq 0,$$

[in fact also $f(x,0) > 0$ for $x \neq 0$], we conclude that $f(x,y)$ attains both positive and negative values in any neighbourhood of $(0,0)$, and we do not have an extremum.

Summarizing we have

$$\begin{array}{ll} \alpha < -2: & (0,0) \text{ is not an extremum,} \\ \\ \alpha = -2: & (0,0) \text{ is a (non-proper) minimum,} \\ \alpha \in]-2,2]: & (0,0) \text{ is a proper minimum,} \\ \\ \alpha > 2: & (0,0) \text{ is not an extremum.} \end{array}$$

2) Second method, the (r,s,t)-method. First notice that the (r,s,t)-method can *only/* be applied if $f \in C^2$ and if $(0,0)$ *indeed is* a stationary point. Therefore, we must first show that $(0,0)$ is a stationary point.

It follows from

$$f(x,y) = x^2 + y^2 + \alpha xy + (x-y)^4, \qquad\qquad f(0,0) = 0,$$

that

$$\frac{\partial f}{\partial x} = 2x + \alpha y + 4(x-y)^3, \qquad\qquad \frac{\partial f}{\partial x}(0,0) = 0,$$

$$\frac{\partial f}{\partial y} = 2y + \alpha x - 4(x-y)^3, \qquad\qquad \frac{\partial f}{\partial y}(0,0) = 0,$$

so $(0,0)$ *is* a stationary point.

Furthermore,

$$\frac{\partial^2 f}{\partial x^2} = 2 + 12(x-y)^2, \qquad\qquad r = \frac{\partial^2 f}{\partial x^2}(0,0) = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \alpha - 12(x-y)^2, \qquad\qquad s = \frac{\partial^2 f}{\partial x \partial y}(0,0) = \alpha,$$

$$\frac{\partial^2 f}{\partial y^2} = 2 + 12(x-y)^2, \qquad\qquad t = \frac{\partial^2 f}{\partial y^2}(0,0) = 2.$$

A *sufficient* condition for extremum is that $rt > s^2$, i.e. $\alpha < 4$, which is fulfilled for $\alpha \in]-2,2[$. Since r, $t > 0$, the stationary point $(0,0)$ is a proper minimum, when $\alpha \in]-2,2[$.

If $|\alpha| > 2$, then $rt = 4 < \alpha^2 = s^2$, and we do *not* have an extremum in this case.

If $\alpha^2 = 4$, i.e. $\alpha = \pm 2$, then $s^2 = rt$, and we cannot conclude anything by the (r,s,t)-method.

Notice that this negative result is no reason to stop. We have only demonstrated that one particular *method* does not work. Let us now investigate each of the cases $\alpha = 2$ and $\alpha = -2$ one by one.

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a) If $\alpha = 2$, then $f(x,y)$ is written

$$f(x,y) = x^2 + y^2 + 2xy + (x-y)^4 = (x+y)^2 + (x-y)^4 \geq 0.$$

When $(x,y) \neq (0,0)$, at least one of the terms $(x+y)^2$ is $(x-y)^4$ positive, and we conclude that $(0,0)$ is a proper minimum.

b) Analogously, if $\alpha = -2$,

$$f(x,y) = x^2 + y^2 - 2xy + (x-y)^4 = (x-y)^2 + (x-y)^4 = (x-y)^2\{1 + (x-y)^2\}.$$

We conclude that $f(x,y) \geq 0$ and $f(x,x) = 0$ for every x, i.e. on the line. This shows that $(0,0)$ is a non-proper minimum. Summarizing we get

$|\alpha| > 2:$ $(0,0)$ is not an extremum.

$|\alpha| < 2:$ $(0,0)$ is a proper minimum,

$\alpha = 2:$ $(0,0)$ is a proper minimum,

$\alpha = -2:$ $(0,0)$ is a (non-proper) minimum.

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Eksempel 4.3 *Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x,y) = 3x^4 - 4x^2y + y^2.$$

Show that the restriction of f to any straight line through $(0,0)$ has a proper minimum at the point $(0,0)$. Then prove that f does not have a minimum at this point.
(Write $f(x,y)$ as a product of two factors and check the sign of f in discs with centrum at the origo).

A Extremum.

D Insert $x = 0$ and $y = \alpha x$ into $f(x,y)$, and prove that there is a minimum at $(0,0)$ on each of these lines.

Consider possibly $f(x,y)$ as a polynomial in y of degree 2 for every x which will help to factorize $f(x,y)$. Use this factorization to discuss the sign of $f(x,y)$.

I 1) When $x = 0$ we get $f(0,y) = y^2$, which clearly has a minimum for $y = 0$, i.e. at $(0,0)$.

2) When we restrict ourselves to the line $y = \alpha x$ we get

$$\begin{aligned} f(x,\alpha x) &= 3x^4 - 4\alpha x^3 + \alpha^2 x^2 = x^2(\alpha^2 - 4\alpha x + 3x^2) \\ &= x^2\{(\alpha - 2x)^2 - x^2\} = x^2(\alpha - x)(\alpha - 3x). \end{aligned}$$

If $\alpha \neq 0$, then $(\alpha - x)(\alpha - 3x) > 0$ for $|x| < \left|\dfrac{\alpha}{3}\right|$, i.e.

$$f(x,\alpha x) > 0 \qquad \text{for } 0 < |x| < \frac{1}{3}|x|,$$

and $f(0,0) = 0$, proving that $(0,0)$ is a minimum.

If $\alpha = 0$, then $f(x,0) = 3x^4$, which clearly has a minimum for $x = 0$, i.e. at the point $(0,0)$.

Summarizing we get that the restriction of $f(x,y)$ to any straight line through $(0,0)$ has a minimum at $(0,0)$.

3) If we consider $f(x,y) = y^2 - 4x^2y + 3x^4$ as a polynomial of second degree in y for every fixed x, we get the roots

$$y = x^2 \qquad \text{and} \qquad y = 3x^2.$$

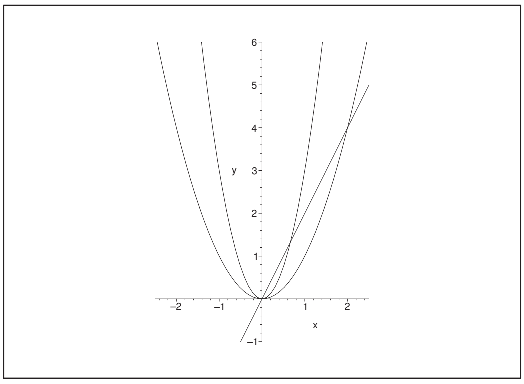
Then we conclude that we have the factorization

$$f(x,y) = 3x^4 - 4x^2y + y^2 = (y - x^2)(y - 3x^2).$$

When we sketch the zero curves, the plane is divided as shown on the figure into four regions, where $f(x,y) > 0$ inside both parabolas or outside both parabolas, and $f(x,y) < 0$ between the two parabolas.

It follows from the figure that $f(x,y)$ is both positive and negative in any neighbourhood of the point $(0,0)$, where $f(0,0) = 0$.

Remark. When we consider the line on the figure we see that it will always lie totally in a "positive" region, when we are close to $(0,0)$. The trick of the example is that the zero curves



Figur 33: The function is positive inside both parabolas, and outside both parabolas, and negative between them.

are so "flat" in the neighbourhood of $(0,0)$ that they cannot be "caught" by any straight line. $\Diamond$

Alternatively we see that if we take the restriction to the curve given by the equation $y = 2x^2$, then

$$f(x, 2x^2) = 3x^4 - 8x^4 + 4x^4 = -x^4,$$

which obviously has a (local) *maximum* for $x = 0$, i.e. at $(0,0)$, so $(0,0)$ cannot be an extremum. The construction of this alternative solution relies on that we can choose a curve in the "negative" region between the two zero curves approaching $(0,0)$.

Eksempel 4.4 *Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x,y) = -x^2 + 2xy^2 - y^4 + y^5.$$

Show that the restriction of f to any straight line through $(0,0)$ has a proper maximum at this point. Then prove that f dos not have an extremum at this point.
(Find a restriction $f(g(y), y)$, such that only the latter term on the right hand side remains).

A Extremum.

D Insert $x = 0$, $y = 0$ and $y = \alpha x$ i $f(x,y)$. Then exploit that $-x^2 + 2xy^2 - y^4 = -(x - y^2)^2$.

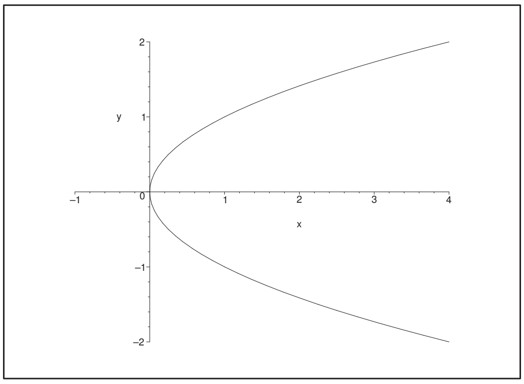
I Clearly, $f(0,0) = 0$.

If $x = 0$, then $f(0,y) = -y^4 + y^5 = -y^4(1-y)$, which clearly has a maximum for $y = 0$, i.e. at$(0,0)$.

If $y = 0$, then $f(x,0) = -x^2$, which clearly has a maximum for $x = 0$, i.e. at $(0,0)$.

If $y = \alpha x$, $\alpha \neq 0$, then

$$f(x, \alpha x) = -x^2 + 2\alpha^2 x^3 - \alpha^4 x^4 + \alpha^5 x^5 = -x^2 \left\{1 - 2\alpha^2 x + \alpha^4 x^2 - \alpha^5 x^3\right\}.$$



Figur 34: The parabola $x = y^2$.

When $|x|$ is small, then the latter factor is positive, and it follows that $f(x, \alpha x)$ has a maximum for $x = 0$.

Finally, we can write $f(x, y)$ in the form

$$f(x, y) = -(x^2 - 2xy^2 + y^4) + y^5 = -(x - y^2)^2 + y^5.$$

When we restrict ourselves to the parabola $x = y^2$, $y \in \mathbb{R}$, then

$$f(y^2, y) = y^2$$

has the same sign as y, so $f(x, y)$ attains both positive and negative values in any neighbourhood of $(0, 0)$. It follows that $(0, 0)$ is not an extremum.

Eksempel 4.5 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x, y) = x^2y + xy^2 + x^2 + xy + 1.$$

check if f *has an extremum at* $(0, 0)$.

A Extremum.

D Here we have several possible methods:

1) First variant. The (r, s, t)-method.
2) Second variant. Factorize $x^2 + xy^2 + x^2 + xy$, and discuss the sign of it.
3) Third variant. Put $y = \alpha x$ into $f(x, y)$ for some choice of α.

I 1) First variant. This time (almost an exception) the (r, s, t)-method is the simplest one, so we shall start with it. From

$$\frac{\partial f}{\partial x} = 2xy + y^2 + 2x + y, \qquad \frac{\partial f}{\partial y} = x^2 + 2xy + x,$$

follows

$$\frac{\partial f}{\partial x}(0,0) = 0 \qquad \text{og} \qquad \frac{\partial f}{\partial y}(0,0) = 0,$$

so $(0,0)$ *is* a stationary point. (Notice that this *must* be checked before we continue with the (r,s,t)-method).

Then it is quite easy to see that we have at $(0,0)$

$$r = 2, \qquad s = 1 \quad \text{and} \quad t = 0,$$

so $rt = 0 < 1 = s^2$. We conclude that there is no extremum at $(0,0)$.

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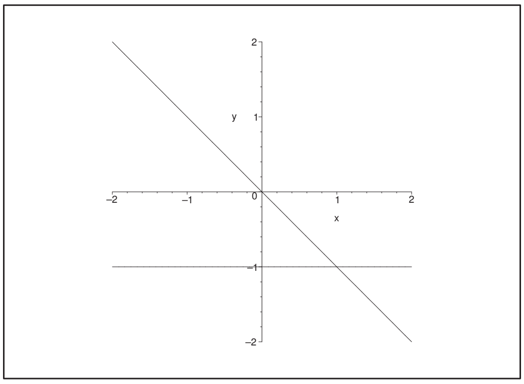

Figur 35: The zero curves are the lines $x + y = 0$, $x = 0$ and $y = -1$. In the subregions the product $(y + 1)x(x + y)$ is positive and negative every second time.

2) Second variant. We get by a factorization,

$$f(x, y) = x^2y + xy^2 + x^2 + x + 1 = xy(x + y) + x(x + y) + 1.$$

The value of the function is $f(0, 0) = 1$, and the “disturbance”

$$(y + 1)x(x + y)$$

has the zero curves as shown on the figure. The sign of the “disturbance” changes whenever one passes a zero curve, so if one moves around $(0, 0)$, one will always pass through both positive and negative regions for $f(x, y) - f(0, 0)$. This means that $f(x, y)$ cannot have an extremum at $(0, 0)$, because $f(x, y)$ attains both values $> f(0, 0)$ and $< f(0, 0)$ in any neighbourhood of $(0, 0)$.

3) Third variant Along the line $y = \alpha x$ we get

$$\varphi_\alpha(x) = f(x, \alpha x) = (\alpha + \alpha^2)x^3 + (1 + \alpha)x^2 + 1 = 1 + (1 + \alpha)x^2 + \cdots ,$$

where the dots indicate terms of higher degree which are small in a neighbourhood of $(0, 0)$ compared to x^2.

a) If $\alpha > -1$, then $\varphi_\alpha(x) \geq 1$ in a neighbourhood of $x = 0$.
b) If $\alpha < -1$, then $\varphi_\alpha(x) \leq 1$ in a neighbourhood of $x = 0$.

In both cases we have a strict inequality sign in a dotted neighbourhood, and we conclude that there is no extremum at $(0, 0)$.

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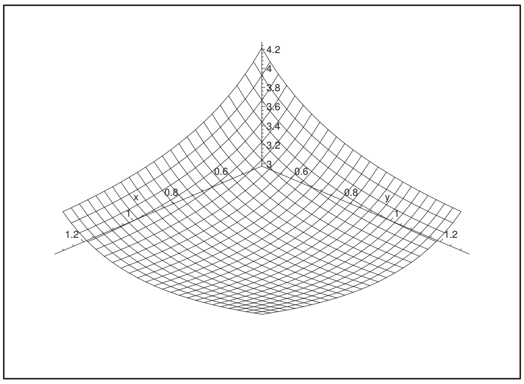
Eksempel 4.6 *Let $\alpha \neq 0$ be a constant. Consider the function*

$$f(x,y) = \alpha^3 xy + \frac{1}{x} + \frac{1}{y}, \qquad xy \neq 0.$$

Find the extremum of the function, and indicate for every value of α the type of the extremum.

A Extremum.

D Find the stationary points, if any, and check if they are extrema.



Figur 36: The graph of $f(x,y)$ for $\alpha = 1$. It is difficult to see on this figure that we have a minimum at $(1,1)$. The graph of $f(x,y)$ for $\alpha = -1$ gives even less information.

I The coordinates of the possible stationary points are the solutions of the equations

$$\frac{\partial f}{\partial x} = \alpha^3 y - \frac{1}{x^2} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = \alpha^3 x - \frac{1}{y^2} = 0,$$

accordingly,

$$\alpha^3 x^2 y = 1 \qquad \text{og} \qquad \alpha^3 xy^2 = 1 = \alpha^3 x^2 y,$$

so $y = x$.

We get by insertion $x = y = \dfrac{1}{\alpha}$, so there is just one stationary point, $\left(\dfrac{1}{\alpha}, \dfrac{1}{\alpha}\right)$, and the value of the function is here

$$f\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right) = 3\alpha.$$

Furthermore,

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}, \quad \frac{\partial^2 f}{\partial x \partial y} = \alpha^3, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3},$$

hence

$$r = 2\alpha^3, \qquad s = \alpha^3, \qquad t = 2\alpha^3,$$

and whence

$$rt - s^2 = 4\alpha^6 - \alpha^6 = 3\alpha^6 > 0,$$

showing that we have an extremum.

1) If $\alpha > 0$, then $r = 2\alpha^3 > 0$ and $t = 2\alpha^3 > 0$, and we have a proper minimum.
2) If instead $\alpha < 0$, we get analogously a proper maximum.

Eksempel 4.7 *Check if the function*

$$f(x,y) = x^3 + xy^2 + 4xy - 3x - 4y, \qquad (x,y) \in \mathbb{R}^2,$$

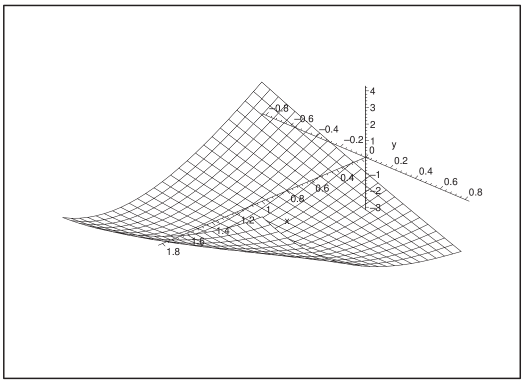
has an extremum at $(1,0)$.

A Extremum.

D Here we have two variants:

First variant. Show that $(1,0)$ is a stationary point, and then apply the (r,s,t)-method.

Second variant. Translate the coordinate system to $(1,0)$ and argue directly.



Figur 37: The surface in a neighbourhood of $(1,0)$. The figure does not give any hint of the type of the stationary point.

I First variant. It follows from

$$\frac{\partial f}{\partial x} = 3x^2 + y^2 + 4y - 3, \qquad \frac{\partial f}{\partial x}(1,0) = 0,$$
$$\frac{\partial f}{\partial y} = 2xy + 4x - 4, \qquad \frac{\partial f}{\partial y}(1,0) = 0,$$

that $(1,0)$ is a stationary point.

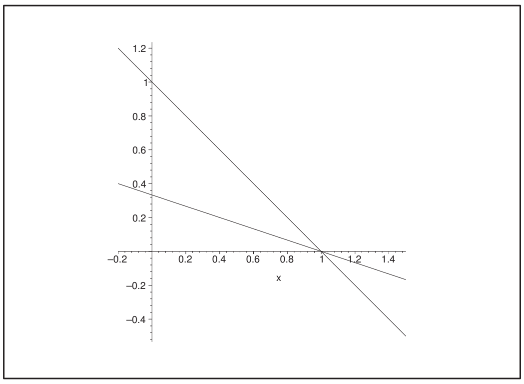
Now,

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial x \partial y} = 2y + 4, \qquad \frac{\partial^2 f}{\partial y^2} = 2x,$$

so we get at the point $(1,0)$ that

$$r=6, \qquad s=4, \qquad t=2.$$

Since $rt = 12 < 16 = s^2$, there is no extremum at $(1,0)$.



Figur 38: The zero curves of $(x+y-1)(x+3y-1)$.

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Second variant. If we put $x = t + 1$, then we shall check what happens in the neighbourhood of $(t, y) = (0, 0)$. A simple computation gives

$$\begin{aligned} f(t+1,y) &= (t+1)^3 + (t+1)y^2 + 4(t+1)y - 3(t+1) - 4y \\ &= t^3 + 3t^2 + 3t + 1 + ty^2 + y^2 + 4ty + 4y - 3t - 3 - 4y \\ &= -2 + 3t^2 + 4ty + y^2 + t(t^2 + y^2) \\ &= -2 + (t+y)(t+3y) + t(t^2 + y^2). \end{aligned}$$

The term of second order

$$(t + y)(t + 3y) = (x + y - 1)(x + 3y - 1)$$

is both positive and negative in any neighbourhood of the point $(x, y) = (1, 0)$, and since it in general dominates the term of third order $t(t^2 + y^2) = (x - 1)\{(x - 1)^2 + y^2\}$, there is no extremum at $(1, 0)$.

Eksempel 4.8 *Given the function*

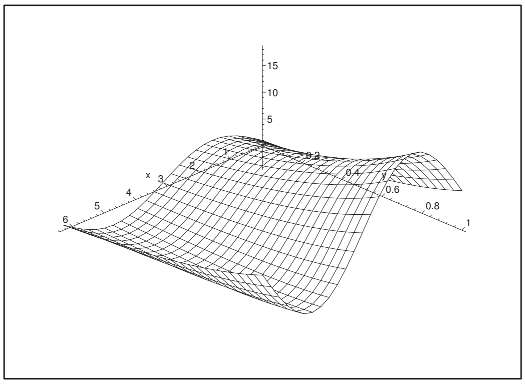
$$f(x, y) = e^{2y} + 4e^y \sin x, \qquad (x, y) \in \mathbb{R}^2.$$

1) Find the stationary points of f; check if f has a proper extremum.

2) Explain why f has both a maximum S and a minimum M on the rectangle $[0, 2\pi] \times [0, 1]$; Finally, find S and M.

A Extrema.

D Find the stationary points; check the values on the boundary.



Figur 39: The graph of $f(x, t)$ for $(x, y) \in [0, 2\pi] \times [0, 1]$.

I 1) The equations of the stationary points are

$$\frac{\partial f}{\partial x} = 4e^y \cos x = 0 \quad \text{og} \quad \frac{\partial f}{\partial y} = 2e^{2y} + 4e^y \sin x = 0.$$

It follows from the former equation that $\cos x = 0$, thus $x = \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$.

If $x = \dfrac{\pi}{2} + 2p\pi$, $p \in \mathbb{Z}$, then

$$\frac{\partial f}{\partial y} = 2e^{2y} + 4e^{y} > 0,$$

and these x-values do not correspond to stationary points.

If $x = \dfrac{3\pi}{2} + 2p\pi$, $p \in \mathbb{Z}$, then

$$\frac{\partial f}{\partial y} = 2e^{2y} - 4e^{y} = 2e^{y}(e^{y} - 2) = 0$$

for $y = \ln 2$. Thus the stationary points are

$$\left\{ \left(\frac{3\pi}{2} + 2p\pi, \ln 2 \right) \;\middle|\; p \in \mathbb{Z} \right\}.$$

Furthermore,

$$\frac{\partial^2 f}{\partial x^2} = -4e^{y}\sin x, \quad \frac{\partial^2 f}{\partial x \partial y} = 4e^{y}\cos x, \quad \frac{\partial^2 f}{\partial y^2} = 4e^{2y} + 4e^{y}\sin x.$$

At the points $\left(\dfrac{3\pi}{2} + 2p\pi, \ln 2 \right)$ we get $r = 8$, $s = 0$, $t = 16 - 8 = 8$, so $rt - s^2 = 64 > 0$, and we have an extremum. As $r > 0$ and $t > 0$, these are all minima.
The values of the function at these points are

$$f\left(\frac{3\pi}{2} + 2p\pi, \ln 2 \right) = 4 - 8 = -4.$$

2) Since f is continuous (it is of class C^{∞}), and the rectangle

$$D = [0, 2\pi] \times [0, 1]$$

is closed and bounded (compact), it follows from the second main theorem for continuous functions that f has both a maximum and a minimum in D.

These can only be attained at a stationary point

$$\left(\frac{3\pi}{2}, \ln 2 \right) \in D,$$

or on the boundary.

a) For the stationary point we get

$$f\left(\frac{3\pi}{2}, \ln 2 \right) = -4.$$

b) Taking the restriction of f to the boundary curve $(x, y) = (x, 0)$, $x \in [0, 2\pi]$, we get:

$$\varphi_1(x) = 1 + 4\sin x, \qquad x \in [0, 2\pi],$$

with the maximum and minimum, respectively,

$$\varphi_1\left(\frac{\pi}{2} \right) = 1 + 4 = 5, \qquad \varphi_1\left(\frac{3\pi}{2} \right) = 1 - 4 = -3.$$

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c) On the boundary curve $(x,y) = (0,y)$, $y \in [0,1]$, or the boundary curve $(x,y) = (2\pi, y)$, $y \in [0,1]$, we get the restriction of f:

$$\varphi_2(y) = e^{2y}, \qquad y \in [0,1],$$

with the maximum and minimum, respectively,

$$\varphi_2(1) = e^2, \qquad \varphi_2(0) = 1.$$

d) On the boundary curve $(x,y) = (x,1)$, $x \in [0, 2\pi]$, we get the restriction of f:

$$\varphi_3(x) = e^2 + 4e \sin x, \qquad x \in [0, 2\pi],$$

with the maximum and minimum, respectively,

$$\varphi_3\left(\frac{\pi}{2}\right) = e^2 + 4y = (e+2)^2 - 4 > 10 > e^2$$

and

$$\varphi_3\left(\frac{3\pi}{2}\right) = e^2 - 4e = -e(4-e) = (e-2)^2 - 4 > -4.$$

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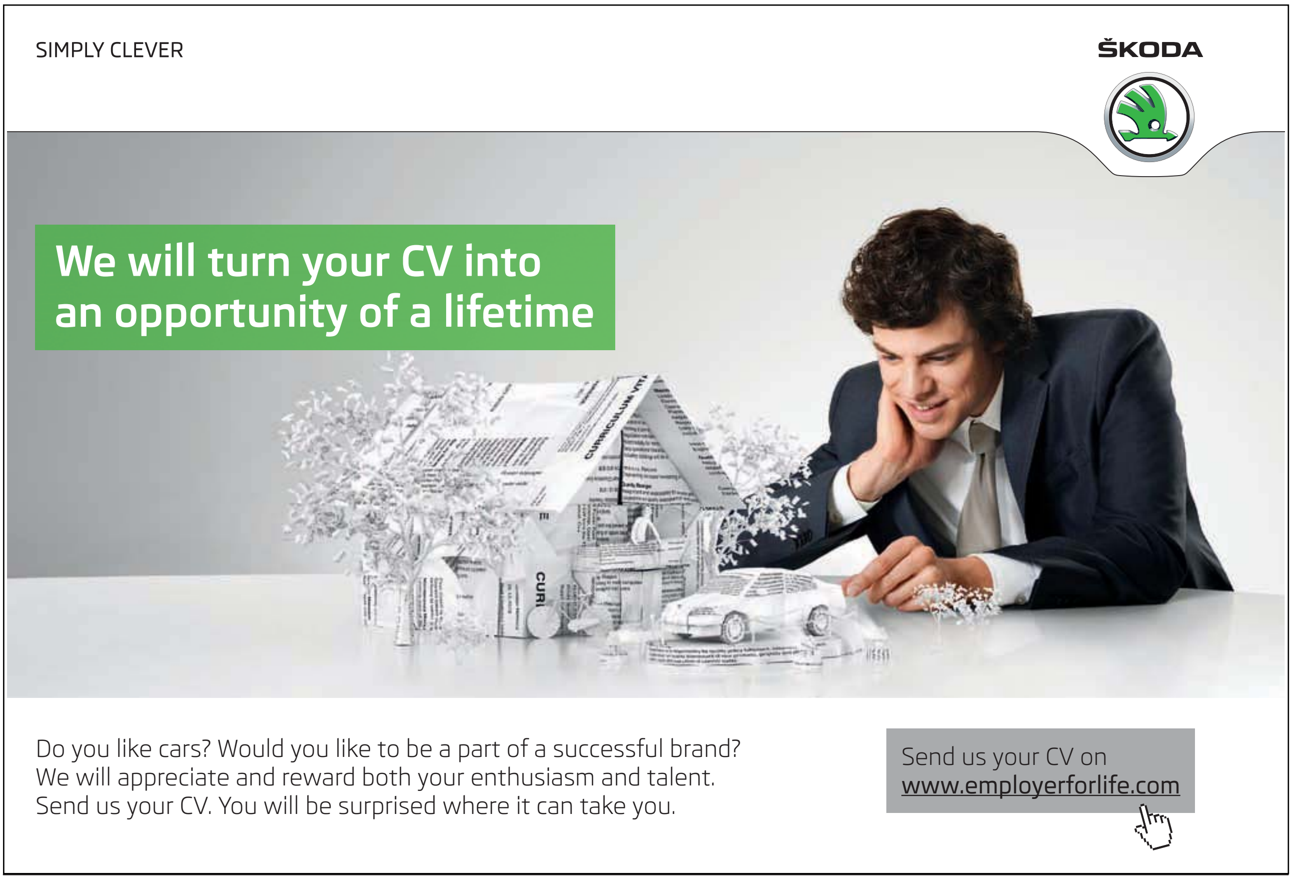

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Finally, by a numerical comparison of

a)

$$f\left(\frac{3\pi}{2}, \ln 2\right) = -4,$$

b)

$$f\left(\frac{\pi}{2}, 0\right) = 5, \qquad f\left(\frac{3\pi}{2}, 0\right) = -3,$$

c)

$$f(0,1) = f(2\pi, 1) = e^2, \qquad f(0,0) = f(2\pi, 0) = 1,$$

d)

$$f\left(\frac{\pi}{2}, 1\right) = e^2 + 4eAe^2, \qquad f\left(\frac{3\pi}{2}, 1\right) = e^2 - 4e > -4,$$

we conclude that the maximum is

$$S = f\left(\frac{\pi}{2}, 1\right) = e^2 + 4e,$$

and the minimum is

$$M = f\left(\frac{3\pi}{2}, \ln 2\right) = -4.$$

Eksempel 4.9 *Check if the function*

$$f(x,y) = 2\cosh(x+y) - e^{xy}, \qquad (x,y) \in \mathbb{R}^2,$$

has an extremum and $(0,0)$*, and indicate its type if there is an extremum.*

A Extremum.

D Either use known series expansions, or compute the Taylor coefficients.

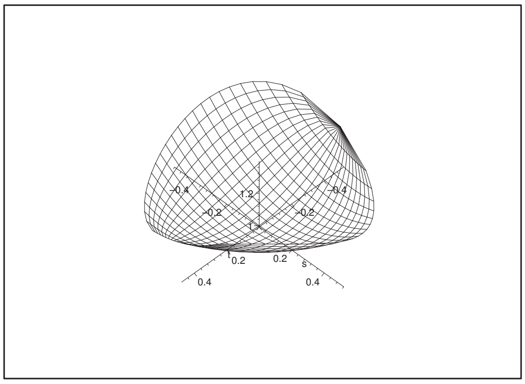
I First method. We get by well-known series expansions from $(0,0)$,

$$\begin{aligned} f(x,y) &= 2\cosh(x+y) - e^{xy} \\ &= 2\left\{1 + \frac{1}{2}(x+y)^2 + \cdots\right\} - \{1 + xy + \cdots\} \\ &= 1 + (x+y)^2 - xy + \cdots, \end{aligned}$$

so the approximating polynomial of at most second degree is

$$P_2(x,y) = 10x^2 + xy + y^2 = 1 + \left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2.$$

It follows from the latter expression that $f(x,y)$ has a local minimum at $(0,0)$.



Figur 40: The graph in a neighbourhood of the point $(0,0)$.

Second method. As $f \in C^\infty$, we get by computation,

$$f(x,y) = 2\cosh(x+y) - e^{xy}, \qquad f(0,0) = 1,$$

$$f'_x(x,y) = 2\sinh(x+y) - y\,e^{xy}, \quad f'_x(0,0) = 0,$$

$$f'_y(x,y) = 2\sinh(x+y) - x\,e^{xy}, \quad f'_y(0,0) = 0.$$

It follows that $(0,0)$ is a stationary point. Furthermore,

$$f''_{xx}(x,y) = 2\cosh(x+y) - y^2 e^{xy}, \qquad f''_{xx}(0,0) = 2,$$

$$f''_{xy}(x,y) = 2\cosh(x+y) - xye^{xy} - e^{xy}, \quad f''_{xy}(0,0) = 1,$$

$$f''_{yy}(x,y) = 2\cosh(x+y) - x^2 e^{xy}, \qquad f''_{yy}(0,0) = 2.$$

Since $rt = 4 > 1 = s^2$, and r, $t > 0$, we conclude that we have a local minimum.

Remark. Because

$$f(x,x) = 2\cosh(2x) - \exp(x^2) \to -\infty \qquad \text{for } x \to +\infty,$$

the minimum above cannot be global.

We have e.g.

$$f(x,0) = 2\cosh x - 1 \to +\infty \qquad \text{for } x \to +\infty,$$

and since f is continuous on the connected set $\mathbb{R}^2$, the range is $f(\mathbb{R}^2) = \mathbb{R}$.

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Eksempel 4.10 *Given the function*

$$f(x,y) = xy - y^2 - 2\ln x, \qquad (x,y) \in D,$$

where

$$D = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}.$$

1) *Find the approximating polynomial of at most second degree of the function f, where $(x_0, y_0) = (2,1)$ is used as point of expansion.*

2) *Check if the function f has an extremum at the point $(2,1)$, and indicate its type if it is an extremum.*

A Approximating polynomials; extremum.

D Apply a series expansion from $(2,1)$, or alternatively the standard method by calculating the Taylor coefficients.

I First variant. Translate the coordinate system by

$$(x,y) = (x_1 + 2, y_1 + 1).$$

Then by insertion and a series expansion for $\ln(1+u)$,

$$\begin{aligned}
f(x,y) &= xy - y^2 - 2\ln x \\
\\
&= (x_1+2)(y_1+1) - (y_1+1)^2 - 2\ln(2+x_1) \\
&= 2 + x_1 + 2y_1 + x_1y_1 - y_1^2 - 2y_1 - 1 - 2\ln 2 - 2\ln\left(1 + \frac{x_1}{2}\right) \\
&= 1 - 2\ln 2 + x_1 + x_1y_1 - y_1^2 - 2\left\{\frac{x_1}{2} - \frac{1}{2}\,\frac{x_1 * 2}{4} + \cdots\right\} \\
&= 1 - 2\ln 2 + \left(\frac{x_1}{2}\right)^2 + 2\cdot\frac{x_1}{2}\cdot y_1 + y_1^2 - 2y_1^2 + \cdots \\
&= 1 - 2\ln 2 + \left\{\frac{1}{2}\,x_1 + y_1\right\}^2 - 2y_1^2 + \cdots .
\end{aligned}$$

The approximating polynomial of at most second degree is

$$\begin{aligned}
P_2(x,y) &= 1 - 2\ln 2 + \frac{1}{4}\,(x-2)^2 + (x-2)(y-1) - (y-1)^2 \\
&= 1 - 2\ln 2 + \frac{1}{4}\,\{(x-2) + 2(y-1)\}^2 - (y-1)^2.
\end{aligned}$$

As the first derivatives are zero, $(2,1)$ is a stationary point. It follows from the terms of second degree that $(2,1)$ is not an extremum.

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Second variant. By straightforward computations,

$$
\begin{array}{ll}
f(x,y) = xy - y^2 - 2\ln x, & f(2,1) = 1 - 2\ln 2, \\[2mm]
f_x'(x,y) = y - \dfrac{2}{x}, & f_x'(2,1) = 0, \\[2mm]
f_y'(x,y) = x - 2y, & f_y'(2,1) = 0, \\[2mm]
f_{xx}''(x,y) = \dfrac{2}{x^2}, & r = f_{xx}''(2,1) = \dfrac{1}{2}, \\[2mm]
f_{xy}''(x,y) = 1, & s = f_{xy}''(2,1) = 1, \\[2mm]
f_{yy}''(x,y) = -2, & t = f_{yy}''(2,1) = -2.
\end{array}
$$

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The approximating polynomial of at most second degree is

$$P_2(x,y) = 1 - 2\ln 2 + \frac{1}{4}\,(x-2)^2 + (x-2)(y-1) - (y-1)^2.$$

As

$$f'_x(2,1) = f'_y(2,1) = 0,$$

we see that $(2,1)$ is a stationary point.

As $rt < 0$, there is no extremum at $(2,1)$.

Eksempel 4.11 *Given the function*

$$f(x,y) = 2x^3 + 27x^2 - 60xy + 75y^2, \qquad (x,y) \in \mathbb{R}^2.$$

1) Find the stationary points of the function.

2) Check for each of the stationary points if f has an extremum; when this is the case one should indicate its type.

3) Find the range $f(\mathbb{R}^2)$ of the function.

A Stationary points; extremum; range.

D Solve the equations of the stationary points. Check the behaviour of the function in a neighbourhood of the stationary points. Finally, consider the restriction of f to the X axis.

I 1) The equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = 6x^2 + 54x - 60y = 0, & \text{dvs.} \quad x^2 + 9x - 10y = 0, \\[2ex] \dfrac{\partial f}{\partial y} = -60x + 150y = 0, & \text{dvs.} \quad -4x + 10y = 0. \end{cases}$$

We get by an addition that $x^2 + 5x = 0$, so either $x = 0$ or $x = -5$. By insertion into the latter equation we obtain the points $(0,0)$ and $(-5,-2)$. These are also satisfying the former equation, so the stationary points are $(0,0)$ and $(-5,-2)$.

2) a) The value at $(0,0)$ is $f(0,0) = 0$, and the approximating polynomial of at most second degree is

$$\begin{aligned} P_2(x,y) &= 27x^2 - 60xy + 75y^2 \\ &= 2x^2 + 25x^2 - 60xy + 36y^2 + 39y^2 \\ &= 2x^2 + (5x-6y)^2 + 39y^2 > 0 \qquad \text{for } (x,y) \neq (0,0), \end{aligned}$$

hence $f(x,y)$ has a local minimum at $(0,0)$.

Alternatively,

$$\begin{aligned} f''_{xx}(x,y) &= 12x + 54, & r &= f''_{xx}(0,0) = 54, \\ f''_{xy}(x,y) &= -60, & s &= f''_{xy}(0,0) = -60, \\ f''_{yy}(x,y) &= 150, & t &= f''_{yy}(0,0) = 150. \end{aligned}$$

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We conclude from $rt = 54 \cdot 150 > (-60)^2 = s^2$, and $r,\ t > 0$, that $f(x,y)$ has a proper minimum at$(0,0)$.

b) At $(-5,-2)$ we get

$$r = 12 \cdot (-5) + 54 = -6, \qquad s = -60, \qquad t = 150.$$

It follows from $rt < 0 < s^2$ that there is no extremum at $(-5,-2)$.

Alternatively we put $(x,y) = (h-5, k-2)$, from which

$$\begin{aligned} f(x,y) &= 2x^3 + 27x^2 - 60xy + 75y^2 \\ &= 2(h-5)^3 + 27(h-5)^2 - 60(h-5)(k-2) + 75(k-2)^2 \\ &= 2\left\{h^3 - 15h^2 + 75h - 125\right\} + 27\{h^2 - 10h + 25\} \\ &\qquad -60\{hk - 2h - 5k + 10\} + 75\left\{k^2 - 4k + 4\right\} \\ &= 2h^3 + \left\{-3h^2 - 60hk + 75k^2\right\} + 125. \end{aligned}$$

The approximating polynomial of at most second degree in (h,k) is

$$P_2(h,k) = -3h^2 - 60hk + 75k^2 + 125.$$

Since this expression attains values both $>$ and $<$ 125 in any neighbourhood of $(h,k) = (0,0)$, it follows that $(-5,-2)$ is not an extremum.

3) By taking the restriction of f to the X axis we get

$$\varphi(x) = f(x,0) = 2x^3 + 27x^2 = x^2\{2x + 27\}, \qquad x \in \mathbb{R}.$$

Since already $\varphi(\mathbb{R}) = \mathbb{R}$, we conclude that

$$f(\mathbb{R}^2) = \mathbb{R}.$$

Eksempel 4.12 *Given the point set*

$$A = \{(x,y) \in \mathbb{R}^2 \mid -4 \le x \le 4,\ 16 - 6y \le x^2 + y^2 \le 16\}.$$

1. *Sketch A, and show that the boundary ∂A consists of two circular arcs.*

We shall also consider the function

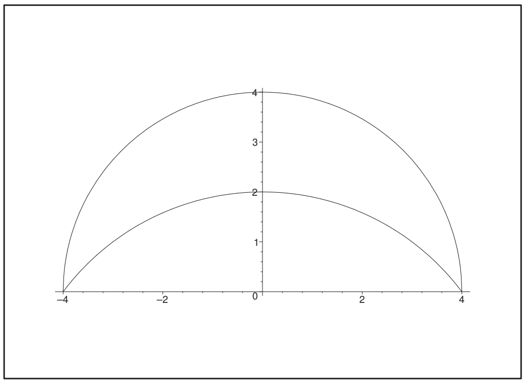
$$f(x,y) = \frac{y+8}{x^2 + y^2 + 12}, \qquad (x,y) \in A.$$

2. *Show that the function f does not have a stationary point in the interior of A.*

3. *Find the range $f(A)$ of the function.*

A Range of a continuous function over a closed and bounded (i.e. compact) and connected set.

D Follow the guidelines and apply the main theorems.



Figur 41: The domain A.

I 1) It follows immediately from $x^2 + y^2 \leq 16 = 4^2$ that A lies inside the disc of centrum $(0,0)$ and radius 4.

We rearrange $16 - 6y \leq x^2 + y^2$ as

$$25 = x^2 + y^2 + 6y + 9 = z^2 + (y+3)^2.$$

Then we can see that A lies outside the disc of centrum $(0,-3)$ and radius 5.

The domain is the half moon shaped region on the figure. It is closed and bounded and connected. The boundary clearly consists of two circular arcs which intersect at the points $(-4,0)$ and $(4,0)$.

2) The stationary points, if any, shall fulfil the equations

$$\frac{\partial f}{\partial x} = -\frac{2x(y+8)}{(x^2+y^2+12)^2} = 0, \quad \text{i.e. } x = 0 \text{ or } y = -8,$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2+12)\cdot 1 - 2y(y+8)}{(x^2+y^2 12)^2} = \frac{x^2-y^2-16y+12}{(x^2+y^2+12)^2} = 0.$$

Here $y = -8$ is not possible in A, so we get $x = 0$ from the first equation. When this put into the second equation we get after a reduction that

$$y^2 + 16y - 12 = 0, \quad \text{from which} \quad y = -8 \pm \sqrt{76}.$$

The line $x = 0$ intersects A in the interval $[2,4]$ on the Y axis. Since

$$-8 \pm \sqrt{76} \leq -8 + \sqrt{76} < -8 + \sqrt{81} = -8 + 9 = 1 < 2,$$

neither of the two candidates $(0, -8 \pm \sqrt{76})$ (on the Y axis) lie in A, and f does not have a stationary point in A.

3) Since f is continuous on A, and A is closed and bounded, it follows from the second main theorem for continuous functions that the range $f(A)$ is closed and bounded.

Since A is also connected, cf. the figure, if follows from the first main theorem for continuous functions, that the range is connected.

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Since $f(A) \subset \mathbb{R}$, it follows from the above that $f(A)$ is a closed interval, i.e.

$$f(A) = [M, S],$$

where M denotes the minimum and S denotes the maximum of f in A, because these exist according to the second main theorem.

The maximum and the minimum must be found among the values at

a) the exceptional points (where f is not differentiable; there are none of them here),

b) the stationary points (none of the either),

c) the points on the boundary.

It follows that both the maximum and the minimum shall be found on the boundary.

INVESTIGATION OF THE BOUNDARY. The boundary is naturally split up into

a) $x^2 + y^2 = 16$, where $y \in [0, 4]$, (cf. the figure),

b) $x^2 + (y+3)^2 = 25$, i.e. $x^2 = 25 - (y+3)^2$, where $y \in [0, 2]$.

Now, x only occurs in the form x^2 in $f(x, y)$. It is therefore natural to eliminate x^2 and use y as a parameter.

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a) The restriction of $f(x,y)$ to $x^2+y^2=16$, $y \in [0,4]$, is given by

$$g(y) = \frac{y+8}{28}, \qquad y \in [0+,4].$$

Clearly, $g(y)$ is increasing in this interval, so the candidates are

$$g(0) = f(\pm 4,0) = \frac{8}{28} = \frac{2}{7}, \qquad g(4) = f(0,4) = \frac{12}{28} = \frac{3}{7}.$$

b) The restriction of $f(x,y)$ to $x^2+(y+3)^2=25$, $y \in [0,2]$, is

$$h(y) = \frac{y+8}{37+y^2-(y+3)^2} = \frac{y+8}{28-6y}, \qquad y \in [0,2].$$

Both the numerator and the denominator are positive in the given interval. Furthermore, when y increases, then the numerator increases too, while the denominator decreases. Hence, $h(y)$ is increasing with the values at the end points

$$h(0) = f(\pm 4,0) = \frac{2}{7}, \qquad h(2) = \frac{10}{16} = \frac{5}{8}.$$

Alternatively,

$$h'(y) = \frac{(28-y^2-6y)\cdot 1-(y+8)(-2y-6)}{(28-y^2-6y)^2} = \frac{y^2+28y+76}{(28-y^2-6y)^2} > 0 \qquad \text{for } y \in [0,2],$$

thus $h(y)$ is increasing.

By a numerical comparison we get $M = \dfrac{2}{7}$ and $S = \dfrac{5}{8}$, so the range is

$$f(A) = \left[\frac{2}{7}, \frac{5}{8}\right].$$

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5 Extrema (three or more variables)

Eksempel 5.1 *Find in each of the following cases the stationary points of the given function*

$f : \mathbb{R}^3 \to \mathbb{R}$.

Then check if f in these points has an extremum; whenever this is the case check if it is a maximum or a minimum.

1) $f(x,y,z) = x^2 + y^2 + z^2 + xyz$.

2) $f(x,y,z) = x^3 + y^3 + z^3 + xyz$.

3) $f(x,y,z) = x^4 + y^4 + z^4 - 4xyz$.

4) $f(x,y,z) = x \cos z + y^2$.

5) $f(x,y,z) = \exp(xy + yz + zx)$.

6) $f(x,y,z) = y^3 + \ln(1 + x^2 + z^2)$.

A Stationary points; extrema in three variables.

D Compute $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$; then find the stationary points; finally check if there are any extrema.

I 1) The equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + yz = 0, \\[2mm] \dfrac{\partial f}{\partial y} = 2y + xz = 0, \\[2mm] \dfrac{\partial f}{\partial z} = 2z + xy = 0, \end{cases}$$

i.e.

$$(5) \quad \begin{cases} x = -\dfrac{yz}{2}, \\[2mm] y = -\dfrac{xz}{2}, \\[2mm] z = -\dfrac{xy}{2}. \end{cases}$$

When we multiply the equations of (5) we get a *necessary condition* of stationary points,

$$xyz = -\frac{1}{8}(xyz)^2, \qquad \text{i.e.} \qquad xyz\{xyz + 8\} = 0.$$

Then either $xyz = 0$ or $xyz = -8 = (-2)^3$.

a) If $xyz = 0$, then one of the factors must be 0. Assume that $x = 0$. Then it follows from (5) that $y = z = 0$.
Analogously, if we assume that $y = 0$ or $z = 0$.

Summarizing we get in this case that $(0,0,0)$ is a stationary point.

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b) If $xyz \neq 0$, then all three factors are $\neq 0$. By insertion of the latter equation into (5), we get

$$y = -\frac{1}{2}\,xz = +\frac{1}{4}\,x^2 y,$$

hence $x^2 = 4$.
Analogously we get $y^2 = 4$ and $z^2 = 4$, so the candidates shall be found among $(\pm 2, \pm 2, \pm 2)$ with all possible combinations of the signs. By a simple test in (5) we see that we in this case get the stationary points

$$(2, 2, -2), \quad (2, -2, 2), \quad (-2, 2, 2), \quad (-2, -2, -2).$$

Summarizing we have the five stationary points

$$(0, 0, 0), \quad (2, 2, -2), \quad (2, -2, 2), \quad (-2, 2, 2), \quad (-2, -2, -2).$$

i) The point $(0, 0, 0)$ is a proper minimum, because

$$P_2(x, y, z) = x^2 + y^2 + z^2$$

is positive in any dotted neighbourhood of $(0, 0, 0)$.

INSERTION. Notice that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = z, \qquad \frac{\partial^2 f}{\partial y \partial z} = x, \qquad \frac{\partial^2 f}{\partial z \partial x} = y,$$

so the approximating polynomial $P_2(x, y, z)$ from an expansion point (x_0, y_0, z_0) is

$$\begin{aligned} P_2(x, y, z) &= f(x_0, y_0, z_0) + (x - x_0)^2 + (y - y_0)^2 + (x - z_0)^2 \\ &\quad + z_0(x - x_0)(y - y_0) + x_0(y - y_0)(z - z_0) \\ &\quad + y_0(z - z_0)(x - x_0). \end{aligned}$$

When $|x_0| = |y_0| = |z_0| = 2$ and $x_0 y_0 z_0 = -8$, then either one or three of the factors are negative. $\Diamond$

ii) Assume that only one of the factors is negative. Due to the symmetry we can assume that $z_0 = -2$, hence $x_0 = y_0 = 2$. Then

$$\begin{aligned} P_2(x, y, z) &= f(x_0, y_0, z_0) + (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \\ &\quad + 2(x - x_0)(y - y_0) + 2(y - y_0)(z - z_0) \\ &\quad + 2(x - x_0)(z - z_0) - 4(x - x_0)(y - y_0) \\ &= f(x_0, y_0, z_0) + \{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)\}^2 \\ &\qquad -4(x - x_0)(y - y_0). \end{aligned}$$

In the plane $z - z_0 = -(x - x_0) - (y - y_0)$ the term

$$-4(x - x_0)(y - y_0)$$

is both positive and negative in any neighbourhood of (x_0, y_0), so $(x_0, y_0, z_0) = (2, 2, -2)$ is not an extremum.
It follows from the symmetry that neither $(2, -2, 2)$ nor $(-2, 2, 2)$ are extrema.

iii) If $(x_0, y_0, z_0) = (-2, -2, -2)$, then

$$\begin{aligned}
P_2(x,y,z) &= f(-2,-2,-2) + (x+2)^2 + (y+2)^2 + (z+2)^2 \\
&\quad -2(x+2)(y+2) - 2(y+2)(z+2) - 2(z+2)(x+2) \\
&= f(-2,-2,-2) + (x+2)^2 + (y+2)^2 + (z+2)^2 \\
&\quad -2(x+2)(y+2) + 2(y+2)(z+2) - 2(z+2)(x+2) - 4(y+2)(z+2) \\
&= f(-2,-2,-2) + \{-(x+2) + (y+2) + (z+2)\}^2 - 4(y+2)(z+2).
\end{aligned}$$

We see that in the plane $x + 2 = (y+2) + (z+2)$ the term $-4(y+2)(z+2)$ is both positive and negative in any neighbourhood of $(y, z) = (-2, -2)$, so $(-2, -2, -2)$ is not an extremum.

The conclusion is that only $(0, 0, 0)$ is an extremum (a proper minimum).

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2) In this case,

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3x^2 + yz = 0, \\[2mm] \dfrac{\partial f}{\partial y} = 3y^2 + xz = 0, \\[2mm] \dfrac{\partial f}{\partial z} = 3z^2 + xy = 0, \end{cases}$$

i.e.

$$(6) \quad \begin{cases} yz = -3x^2 \le 0, \\[2mm] xz = -3y^2 \le 0, \\[2mm] xy = -3z^2 \le 0. \end{cases}$$

From (6) we get the *necessary condition*

$$(yz) \cdot (zx) \cdot (xy) = (xyz)^2 = -27(xyz)^2$$

for a stationary point. The only possibility is $xyz = 0$. Since e.g. $x = 0$ implies that $y = z = 0$, and analogously for $y = 0$ and $z = 0$, it follows that $(0, 0, 0)$ is the only stationary point.

There is no extremum at$(0, 0, 0)$, because e.g. $f(x, 0, 0) = x^3$ is both positive and negative in any neighbourhood of $x_0 = 0$.

3) Here

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 - 4yz = 0, \\[2mm] \dfrac{\partial f}{\partial y} = 4y^3 - 4xz = 0, \\[2mm] \dfrac{\partial f}{\partial z} = 4z^3 - 4xyz = 0, \end{cases}$$

so

$$(7) \quad \begin{cases} yz = x^3, \\ xz = y^3, \\ xy = z^3. \end{cases}$$

We get the following *necessary condition* for the stationary points

$$(yz) \cdot (xz) \cdot (xy) = (xyz)^2 = (xyz)^3,$$

i.e. either $xyz = 0$ or $xyz = 1$.

a) If $xyz = 0$, then e.g. $x = 0$, which immediately implies that $y = z = 0$. Analogously, if we assume $y = 0$ or $z = 0$. In this case we get the stationary point $(0, 0, 0)$.

b) If $xyz = 1$, it follows from the first equation of (7) that

$$xyz = 1 = x^4,$$

i.e. $x = \pm 1$. Analogously we get $y = \pm 1$ and $z = \pm 1$. Therefore, the stationary points should be searched among $(\pm 1, \pm 1, \pm 1)$ with all possible eight combinations of the signs. By insertion into (7), i.e. testing these points, we find the stationary points

$$(1,1,1), \quad (1,-1,1), \quad (-1,1,-1), \qquad (-1,-1,1).$$

Summarizing we find that the function has five stationary points,

$$(0,0,0), \quad (1,1,1), \quad (1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1).$$

We shall now check each of them considering extremum.

a) The point $(0,0,0)$ is not an extremum, because $-4xyz$ is the dominating term in a dotted neighbourhood of $(0,0,0)$, and $-4xyz$ is both positive and negative in any neighbourhood of $(0,0,0)$.

Insertion. It follows from

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2, \qquad \frac{\partial^2 f}{\partial z^2} = 12z^2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4z, \qquad \frac{\partial^2 f}{\partial y \partial z} = -4x, \qquad \frac{\partial^2 f}{\partial x \partial z} = -4y,$$

that the approximating polynomial at a stationary point $(x_0, y_0, z_0) \neq (0,0,0)$ [of course also at $(0,0,0)$, but this is not relevant here] is given by

$$\begin{aligned} P_2(x,y,z) &= f(x_0,y_0,z_0)+6\{x_0^2(x-x_0)^2+y_0^2(y-y_0)^2+z_0^2(z-z_0)^2\} \\ &\quad -4z_0(x-x_0)(y-y_0)-4x_0(y-y_0)(z-z_0)-4y_0(x-x_0)(z-z_0). \end{aligned}$$

When $|x_0| = |y_0| = |z_0| = 1$, this is written

$$\begin{aligned} P_2(x,y,z) &= f(x_0,y_0,z_0)+2\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\} \\ &\quad +2\left\{[z_0(x-x_0)]^2-2z_0(x-x_0)(y-y_0)+(y-y_0)^2\right\} \\ &\quad +2\left\{[x_0(y-y_0)]^2-2x_0(y-y_0)(z-z_0)+(z-z_0)^2\right\} \\ &\quad +2\left\{[y_0(x-x_0)]^2-2y_0(x-x_0)(z-z_0)+(x-x_0)^2\right\}, \end{aligned}$$

where we have used that. By a rearrangement,

$$\begin{aligned} &P_2(x,y,z)-f(x_0,y_0,z_0) \\ (8) \quad &= 2\left\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right\}+2\{z_0(x-x_0)-(y-y_0)\}^2 \\ &\quad +2\{x_0(y-y_0)-(z-z_0)\}^2+2\{y_0(x-x_0)-(z-z_0)\}^2. \end{aligned}$$

b) In the latter four stationary points the approximating polynomial is given by (8). It follows from this expression that they are all proper minima.

Summarizing we get that

$$(1,1,1), \quad (1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1)$$

are all proper minima, while $(0,0,0)$ is not an extremum.

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4) The equations of the stationary points are

$$\frac{\partial f}{\partial x} = \cos z = 0, \quad \frac{\partial f}{\partial y} = 2y = 0, \quad \frac{\partial f}{\partial z} = -x \sin z = 0,$$

so accordingly $z = \frac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$, $y = 0$ and $x = 0$, and the stationary points are

$$\left(0, 0, \frac{\pi}{2} + p\pi\right), \qquad p \in \mathbb{Z}.$$

In all of these points, $f\left(0, 0, \frac{\pi}{2} + p\pi\right) = 0$. Since the restriction

$$f(x, 0, z) = x \cos z$$

is both positive and negative in any neighbourhood of any such point, none of them is an extremum.

5) If $f(x, y, z) = \exp(xy + yz + zx)$, then $f(x, y, z) > 0$, and the equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = (y + z) f(x, y, z) = 0, \\[2ex] \dfrac{\partial f}{\partial y} = (z + x) f(x, y, z) = 0, \\[2ex] \dfrac{\partial f}{\partial z} = (x + y) f(x, y, z) = 0, \end{cases}$$

i.e.

$$(9) \quad \begin{cases} y + z = 0, \\ z + x = 0, \\ x + y = 0. \end{cases}$$

The system (9) has only the solution $x = y = z = 0$, so $(0, 0, 0)$ is the only stationary point.

By a Taylor expansion,

$$\begin{aligned} f(x, y, z) &= \exp(xy + yz + zx) \\ &= 1 + xy + yz + xz + (x^2 + y^2 + z^2)\varepsilon(x, y, z), \end{aligned}$$

where $\varepsilon(x, y, z) \to 0$ for $(x, y, z) \to (0, 0, 0)$. Hence

$$P_2(x, y, z) = 1 + xy + yz + zx,$$

where e.g.

$$P_2(x, y, 0) - 1 = xy$$

attains both positive and negative values in any neighbourhood of $(x, y) = (0, 0)$. Thus there is no extremum at $(0, 0, 0)$.

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6) Ifr $f(x,y,z) = y^3 + \ln(1+x^2+z^2)$, the wquations of the stationary points are

$$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+z^2} = 0, \quad \frac{\partial f}{\partial y} = 3y^2 = 0, \quad \frac{\partial f}{\partial z} = \frac{2z}{1+x^2+z^2} = 0.$$

It follows that $(0,0,0)$ is the only stationary point. The restriction $f(0,y,0) = y^3$ is both positive and negative in any neighbourhood of $y_0 = 0$, so there is no extremum at $(0,0,0)$.

Eksempel 5.2 *Examine in the same way as in* **Example 5.1** *the function*

$$f(x,y,z) = xyz(4a-x-y-z), \qquad (x,y,z) \in \mathbb{R}_+^3.$$

A Stationary points; extrema.

D Find the possible stationary points; check if they are extrema.

I The function can by continuity be extended to its zero set. This is the surface of the tetrahedron on the figure, and it is obvious that $f(x,y,z) > 0$ in the open tetrahedron. The function must have a maximum in the tetrahedron, according to the second main theorem for continuous functions, and because $f(x,y,z)$ is of class C^∞, this maximum can only be attained at a stationary point in the interior of the tetrahedron.

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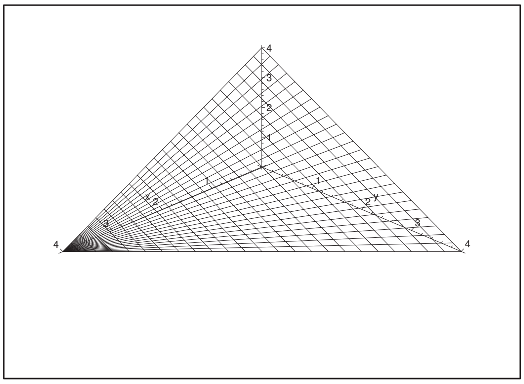

The equations of the stationary points are

$$\begin{cases} \dfrac{\partial f}{\partial x} = yz(4a-x-y-z)-xyz = yz(4a-2x-y-z)=0, \\[2mm] \dfrac{\partial f}{\partial y} = xz(4a-x-y-z)-xyz = xz(4a-x-2y-z)=0, \\[2mm] \dfrac{\partial f}{\partial z} = xy(4a-x-y-2z)=0. \end{cases}$$

It follows from the assumptions $x>0$, $y>0$ and $z>0$ that these equations are equivalent to

$$\begin{cases} x = 4a-x-y-z, \\ y = 4a-x-y-z, \\ z = 4a-x-y-z, \end{cases}$$

and it follows immediately that

$$x = 4a-x-y-z = y = z = a.$$

Hence, (a,a,a) is the only stationary point in the first octant.

It follows from the application of the second main theorem above that we have a maximum at (a,a,a), and the value of the function is here

$$f(a,a,a) = a^4.$$

Alternatively we compute

$$\frac{\partial^2 f}{\partial x^2} = -2yz, \qquad \frac{\partial^2 f}{\partial y^2} = -2xz, \qquad \frac{\partial^2 f}{\partial z^2} = -2xy,$$

$$\frac{\partial^2 f}{\partial x \partial y} = z(4a-2x-y-z)-yz = z(4a-2x-2y-z),$$

$$\frac{\partial^2 f}{\partial y \partial z} = x(4a-x-2y-2a), \qquad \frac{\partial^2 f}{\partial z \partial x} = y(4a-2x-y-2z),$$

hence

$$\frac{\partial^2 f}{\partial x^2}(a,a,a) = \frac{\partial^2 f}{\partial y^2}(a,a,a) = \frac{\partial^2 f}{\partial z^2}(a,a,a) = -2a^2,$$

$$\frac{\partial^2 f}{\partial y \partial z}(a,a,a) = \frac{\partial^2 f}{\partial x \partial z}(a,a,a) = \frac{\partial^2 f}{\partial x \partial z}(a,a,a) = -a^2.$$

Now $f(a,a,a) = a^4$, so

$$\begin{aligned} P_2(x,y,z) &= a^4 - a^2\left\{(x-a)^2 + (y-a)^2 + (z-a)^2\right. \\ &\qquad \left. +(x-a)(y-a) + (y-a)(z-a) + (z-a)(x-a)\right\} \\ &= a^4 - \frac{a^2}{2}\left\{(x-a)^2 + (y-a)^2 + (z-a)^2\right. \\ &\qquad \left. +[(x-a)+(y-a)+(z-a)]^2\right\}, \end{aligned}$$

and we see that (a,a,a) is a maximum.

Eksempel 5.3 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a polynomial of second degree. Prove that f has none or one or infinitely many stationary points.*
Then give for $k = 2$ examples of all three possibilities.

A Stationary points.

D Use some Linear Algebra on the system of equations of the stationary points.

I A general polynomial of second degree in $\mathbb{R}^k$ is of the form

$$f(\mathbf{x}) = \sum_{i,j=1}^{k} a_{ij} x_i x_j + \sum_{i=1}^{k} b_i x_i + c.$$

The equations of the stationary points are

$$\frac{\partial f}{\partial x_m} = \sum_{i=1}^{k} (a_{im} + a_{mi}) x_i + b_m = 0, \qquad m = 1, \dots, k,$$

i.e. a system of k linear equations in k unknowns. It is known from Linear Algebra that such a system of equations has none or one or an infinity of solutions, and the claim is proved.

Let $k = 2$, and denote the variables by (x,y).

If $f(x,y) = x^2 + y^2$, we clearly have $(0,0)$ as the only stationary point.

If $f(x,y) = y^2$, then we clearly have infinitely many stationary points, namely $(x,0)$ for $x \in \mathbb{R}$.

Finally, let $f(x,y) = (x+y)^2 + x - y$. Then the equations of the de stationary points are

$$\frac{\partial f}{\partial x} = 2(x+y) + 1 = 0, \qquad \frac{\partial f}{\partial y} = 2(x+y) - 1 = 0.$$

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This system of equations clearly has no solution.

A simpler system is

$$f(x,y) = x^2 + y,$$

where $\dfrac{\partial f}{\partial y} = 1 \neq 0$ for every (x,y). The structure is the same as in the example above. The difference is that both x^2 and y^2 occur in the former example, so the polynomial is of second degree in both x and y. This is not the case in the latter example.

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