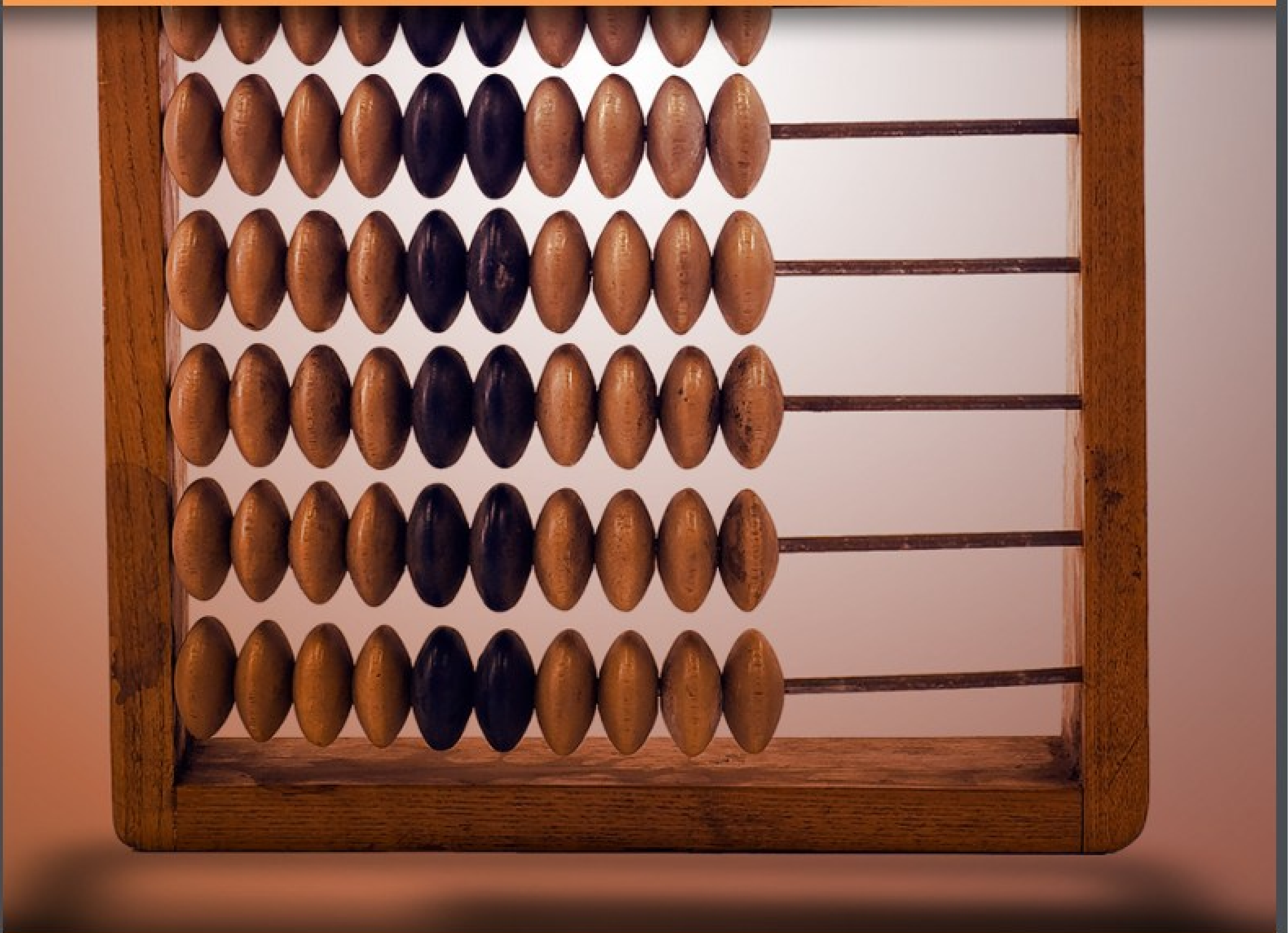
Real Functions of Several Variables - Plane Int...

Leif Mejlbro



Leif Mejlbro

Real Functions of Several Variables

Examples of Plane Integrals

Calculus 2c-5

Real Functions of Several Variables – Examples of Plane Integrals – Calculus 2c-5

ISBN 978-87-7681-255-3

Contents

Preface

In this volume I present some examples of *plane integrals*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
13th October 2007

1 Plane integrals, rectangular coordinates

Example 1.1 *Calculate in each of the following cases the given plane integral by applying the theorem of reduction for rectangular coordinates. Sketch first the domain of integration B.*

1) $\int_B \frac{1}{(x+y)^2}\, dS$, *where* $B = \{(x,y) \mid 1 \le x \le 2 \text{ og } 0 \le y \le x^3\}$.

2) $\int_B \frac{x}{1+xy}\, dS$, *where* $B = [0,1]\times[0,1]$.

3) $\int_B (x\sin y - ye^x)\, dS$, *where* $B = [-1,1]\times\left[0,\frac{\pi}{2}\right]$.

4) $\int_B \sqrt{|y-x^2|}\, dS$, *where* $B = [-1,1]\times[0,2]$.

5) $\int_B (x^2y^2+x)\, dS$, *where* $B = [0,2]\times[-1,0]$.

6) $\int_B |y|\cos\frac{\pi x}{4}\, dS$, *where* $B = [0,2]\times[-1,0]$.

7) $\int_B \frac{x^2}{(1+x+y)^2}\, dS$, *where* $B = \{(x,y) \mid 0 \le x,\, 0 \le y,\, x+y \le 1\}$.

8) $\int_B (4-y)\, dS$, *where* $B = \{(x,y) \mid 0 \le x,\, 0 \le y,\, x^2+y^2 \le 2\}$.

9) $\int_B (\sqrt{x}-y^2)\, dS$, *where B is the bounded set in the first quadrant, which is bounded by the curves* $y = x^2$ *and* $x = y^4$.

10) $\int_B x\cos(x+y)\, dS$, *where B is the triangle of the vertices* $(0,0)$, $(0,0)$, $(\pi,0)$ *and* (π,π).

11) $\int_B x\sqrt[3]{1+y-y^2+\frac{1}{3}y^3}\, dS$, *where* $B = \{(x,y) \mid 0 \le x,\, 0 \le y,\, x+y \le 1\}$.

12) $\int_B (3y^2+2xy)\, dS$, *where* $B = \{(x,y) \mid 0 \le x,\, 0 \le y,\, x+y \le 1\}$.

A Plan integrals in rectangular coordinates.

D Sketch the domain and apply the theorem of reduction.

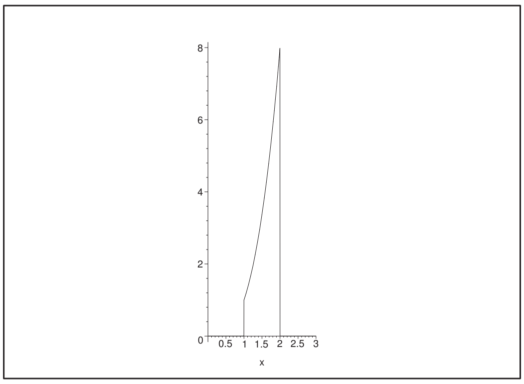


Figure 1: The domain B of **Example 1.1.1**.

I 1) We get by the theorem of reduction,

$$
\begin{aligned}
\int_B \frac{1}{(x+y)^2}\,dS &= \int_1^2 \left\{ \int_0^{x^3} \frac{1}{(x+y)^2}\,dy \right\} dx = \int_1^2 \left[-\frac{1}{x+y} \right]_{y=0}^{x^3} dx \\
&= \int_1^2 \left\{ -\frac{1}{x+x^3} + \frac{1}{x} \right\} dx = \int_1^2 \left\{ -\frac{1}{x} + \frac{x}{1+x^2} + \frac{1}{x} \right\} dx \\
&= \left[\frac{1}{2}\ln(1+x^2) \right]_1^2 = \frac{1}{2}\ln\left(\frac{5}{2}\right).
\end{aligned}
$$

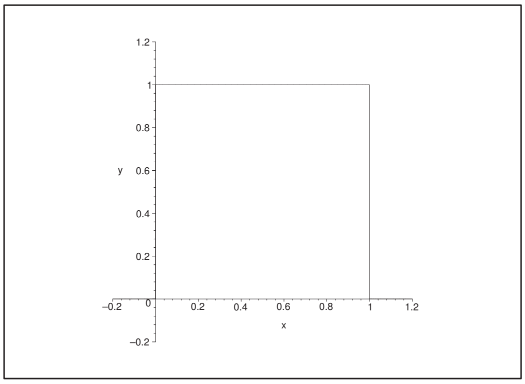


Figure 2: The domain B of **Example 1.1.2**.

2) We get by the theorem of reduction,

$$\begin{aligned}\int_B \frac{x}{1+xy}\,dS &= \int_0^1 \left\{\int_0^1 \frac{x}{1+xy}\,dy\right\} dx = \int_0^1 [\ln(1+xy)]_{y=0}^1\,dx \\ &= \int_0^1 1\cdot\ln(1+x)\,dx = [x\ln(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,dx \\ &= \ln 2 - \int_0^1 \left\{1-\frac{1}{1+x}\right\} dx = \ln 2 - 1 + \ln 2 \\ &= 2\ln 2 - 1.\end{aligned}$$

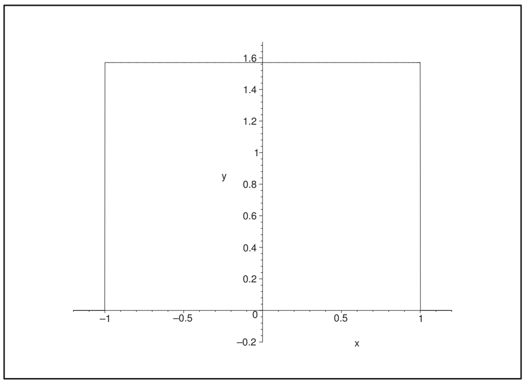


Figure 3: The domain B of **Example 1.1.3**.

3) We get by the theorem of reduktion,

$$\int_B (x\sin y - ye^x)\,dS = \int_0^{\frac{\pi}{2}} \left\{ \int_{-1}^1 (x\sin y - ye^x)\,dx \right\} dy$$

$$= 0 - \int_0^{\frac{\pi}{2}} y\cdot\left(e-\frac{1}{e}\right) dy = -\frac{1}{2}(e-e^{-1})\left[y^2\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi^2}{4}\sinh 1 \qquad \left(= -\frac{\pi^2(e^2-1)}{8e}\right),$$

where we first integrate with respect to x and then with respect to y.

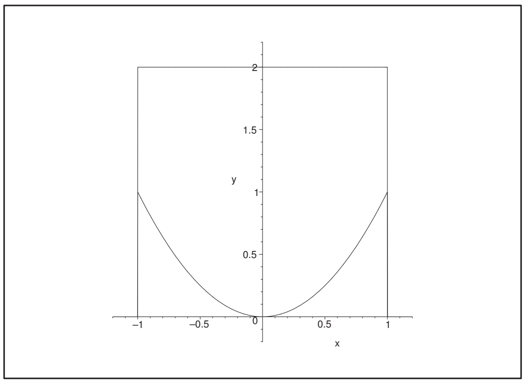


Figure 4: The domain B of **Example 1.1.4**.

4) Here, the curve $y = x^2$ may cause some troubles. For symmetric reasons

$$\int_B \sqrt{|y-x^2|}\,dS = \int_{-1}^1 \left\{ \int_0^2 \sqrt{|y-x^2|}\,dy \right\} dx$$

$$= 2\int_0^1 \left\{ \int_0^{x^2} \sqrt{x^2-y}\,dy + \int_{x^2}^2 \sqrt{y-x^2}\,dy \right\} dx$$

$$= 2\int_0^1 \left\{ \left[-\frac{2}{3}(x^2-y)^{\frac{3}{2}}\right]_{y=0}^{x^2} + \left[\frac{2}{3}(y-x^2)^{\frac{3}{2}}\right]_{y=x^2}^2 \right\} dx$$

$$= 2\int_0^1 \left\{ \frac{2}{3}(x^2)^{\frac{3}{2}} + \frac{2}{3}(2-x^2)^{\frac{3}{2}} \right\} dx = \frac{4}{3}\int_0^1 x^3\,dx + \frac{4}{3}\cdot 2\sqrt{2}\int_0^1 \left\{ 1 - \left(\frac{x}{\sqrt{2}}\right)^2 \right\}^{\frac{3}{2}} dx$$

$$= \frac{1}{3} + \frac{16}{3}\int_0^{\frac{\sqrt{2}}{2}} \{1-t^2\}^{\frac{3}{2}} dt = \frac{1}{3} + \frac{16}{3}\int_0^{\frac{\pi}{4}} \{1-\sin^2 u\}^{\frac{3}{4}} \cos u\,du$$

$$= \frac{1}{3} + \frac{16}{3}\int_+^{\frac{\pi}{3}4} \cos^4 u\,du = \frac{1}{3} + \frac{16}{3}\int_0^{\pi} 4\left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$= \frac{1}{3} + \frac{4}{3}\int_0^{\frac{\pi}{4}} \left\{ 1 + 2\cos 2u + \frac{1+\cos 4u}{2} \right\} du$$

$$= \frac{1}{3} + \frac{4}{3}\left[\frac{3}{2}u + \sin 2u + \frac{1}{8}\sin 4u\right]_0^{\frac{\pi}{4}} = \frac{1}{3} + \frac{\pi}{2} + \frac{4}{3} = \frac{5}{3} + \frac{\pi}{2}.$$

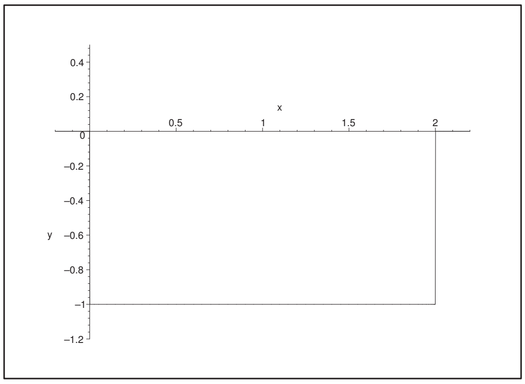


Figure 5: The domain B of **Example 1.1.5** and of **Example 1.1.6**.

5) Here,

$$\int_B (x^2y^2+x)\,dS = \int_0^2 \left\{\int_{-1}^0 (x^2y^2+x)\,dy\right\} dx$$
$$= \int_0^2 \left[\frac{1}{3}\,x^2y^3+xy\right]_{y=-1}^0 dx = \int_0^2 \left\{\frac{1}{3}\,x^2+x\right\} dx$$
$$= \left[\frac{1}{9}\,x^3+\frac{1}{2}\,x^2\right]_0^2 = \frac{8}{9}+\frac{4}{2} = 2+\frac{8}{9} = \frac{26}{9}.$$

6) The domain is identical with that of **Example 1.1.5**. It follows that

$$\int_B |y|\cos\frac{\pi x}{4}\,dS = \int_{-1}^0 (-y)\,dy \cdot \int_0^2 \cos\frac{\pi x}{4}\,dx = \left[-\frac{y^2}{2}\right]_{-1}^0 \cdot \frac{4}{\pi}\left[\sin\frac{\pi x}{4}\right]_0^2 = \frac{2}{\pi}.$$

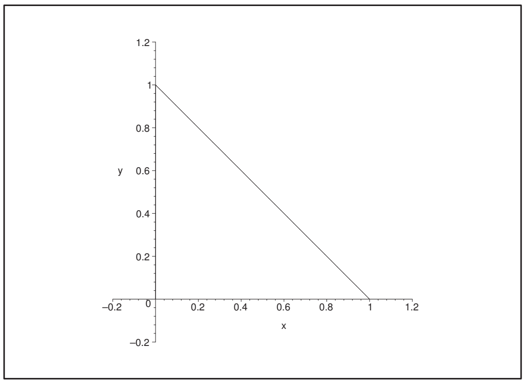


Figure 6: The domain B of **Example 1.1.7**.

7) Here,

$$\int_B \frac{x^2}{(1+x+y)^2}\,dS = \int_0^1 \left\{ \int_0^{1-x} \frac{x^2}{(1+x+y)^2}\,dy \right\} dx = \int_0^1 \left[-\frac{x^2}{1+x+y} \right]_{y=0}^{1-x} dx$$

$$= \int_0^1 \left\{ \frac{x^2}{1+x} - \frac{x^2}{2} \right\} dx = \int_0^1 \left\{ x - 1 + \frac{1}{x+1} - \frac{x^2}{2} \right\} dx$$

$$= \left[\frac{x^2}{2} - x + \ln(1+x) - \frac{x^3}{6} \right]_0^1 = \frac{1}{2} - 1 + 2\ln 2 - \frac{1}{6} = \ln 2 - \frac{2}{3}.$$

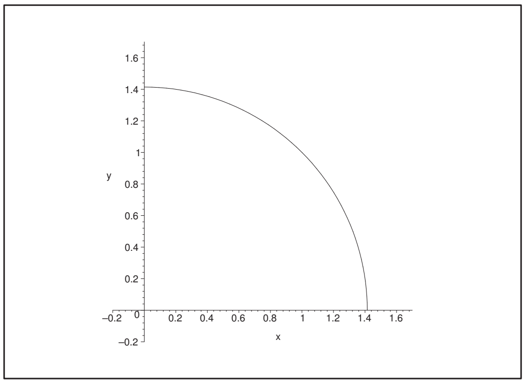


Figure 7: The domain B of **Example 1.1.8**.

8) The domain is a quarter of a disc in the first quadrant, hence by combining the method of identifying obvious areas and the theorem of reduction in rectangular coordinates,

$$\int_B (4-y)\,dS = 4\,\text{area}(B) - \int_0^{\sqrt{2}} \left\{ \int_0^{\sqrt{2-x^2}} y\,dy \right\} dx = 4\cdot\frac{1}{4}\pi(\sqrt{2})^2 - \int_0^{\sqrt{2}} \left[\frac{1}{2}\,y^2\right]_{y=0}^{\sqrt{2-x^2}} dx$$

$$= 2\pi - \frac{1}{2}\int_0^{\sqrt{2}} (2-x^2)dx = 2\pi - \sqrt{2} + \frac{1}{6}(\sqrt{2})^3 = 2\pi - \frac{2}{3}\sqrt{2}.$$

ALTERNATIVELY we get by using polar coordinates instead, cf. **Example 2.1.1**,

$$\int_B (4-y)\,dS = 4\,\text{area}(B) - \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\sqrt{2}} \varrho\sin\varphi\cdot\varrho\,d\varrho \right\} d\varphi$$

$$= 2\pi + [\cos\varphi]_0^{\frac{\pi}{2}} \cdot \left[\frac{\varrho^3}{3}\right]_0^{\sqrt{2}} = 2\pi - \frac{2}{3}\sqrt{2}.$$

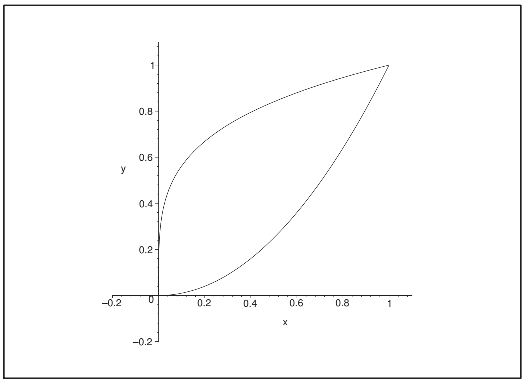


Figure 8: The domain B of **Example 1.1.9**.

9) When $x = y^4$ in the first quadrant, the inverse function is given by $y = \sqrt[4]{x}$, and it follows by

the theorem of reduction that

$$\int_B (\sqrt{x}-y^2)\,dS = \int_0^1 \left\{ \int_{x^2}^{\sqrt[4]{x}} (\sqrt{x}-y^2)\,dy \right\} dx = \int_0^1 \left[y\sqrt{x} - \frac{1}{3}\,y^3 \right]_{y=x^2}^{\sqrt[4]{x}} dx$$

$$= \int_0^1 \left\{ x^{\frac{3}{4}} - \frac{1}{3}\,x^{\frac{3}{4}} - x^{\frac{5}{2}} + \frac{1}{3}\,x^6 \right\} dx = \left[\frac{2}{3}\cdot\frac{4}{7}\,x^{\frac{7}{4}} - \frac{2}{7}\,x^{\frac{7}{2}} + \frac{2}{1}\,x^7 \right]_0^1 = \frac{8}{21} - \frac{2}{7} + \frac{1}{21}$$

$$= \frac{1}{7}.$$

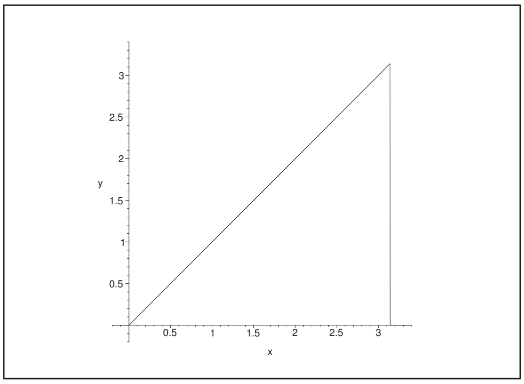


Figure 9: The domain B of **Example 1.1.10**.

10) The domain is the triangle bounded by the X-axis, the line $x=\pi$ and the line $y=x$. We get by the theorem of reduction,

$$\int_B x\cos(x+y)\,dS = \int_0^\pi \left\{ \int_0^x x\cos(x+y)\,dy \right\} dx = \int_0^\pi [x\sin(x+y)]_{y=0}^x dx$$

$$= \int_0^\pi \{x\sin 2x - x\sin x\}\,dx = \left[-x\cdot\frac{1}{2}\,\cos 2x + x\cos x \right]_0^\pi + \int_0^\pi \left\{ \frac{1}{2}\,\cos 2x - \cos x \right\} dx$$

$$= -\frac{\pi}{2} - \pi + \left[\frac{1}{4}\,\sin 2x - \sin x \right]_0^\pi = -\frac{3\pi}{2}.$$

11) Here, the idea of first (i.e. innermost) integrating with respect to y for fixed x is stillborn, so we interchange the order of integration. We shall therefore first (innermost) integrate with

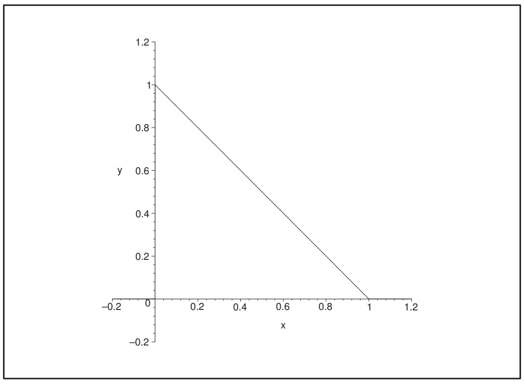


Figure 10: The domain B of **Example 1.1.11** and of **Example 1.1.12**.

respect to x and then outermost with respect to y.

$$\int_B x\sqrt[3]{1+y-y^2+\frac{1}{3}y^3}\,dS = \int_0^2\left\{\int_0^{1-y} x\left\{1+y-y^2+\frac{1}{3}y^3\right\}^{\frac{1}{3}}dx\right\}dy$$

$$= \frac{1}{2}\int_0^1\left\{1+y-y^2+\frac{1}{3}y^3\right\}^{\frac{1}{3}}(1-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1\left\{\frac{4}{3}+\frac{1}{3}(y^3-3y^2+3y-1)\right\}^{\frac{1}{3}}(y-1)^2\,dy$$

$$= \frac{1}{2}\int_{y=0}^1\left\{\frac{4}{3}+\frac{1}{3}(y-1)^3\right\}^{\frac{1}{3}}d\left(\frac{1}{3}(y-1)^3\right) = \frac{1}{2}\cdot\frac{3}{4}\left[\left(\frac{4}{3}+\frac{1}{3}(y-1)^3\right)^{\frac{4}{3}}\right]_{y=0}^1$$

$$= \frac{3}{8}\left\{\left(\frac{4}{3}\right)^{\frac{4}{3}}-\left(\frac{4}{3}-\frac{1}{3}\right)^{\frac{4}{3}}\right\} = \frac{3}{8}\left\{\left(\frac{4}{3}\right)^{\frac{4}{3}}-1\right\} = \frac{1}{2}\sqrt[3]{\frac{4}{3}}-\frac{3}{8}.$$

12) The sketch of B is identical with **Example 1.1.11**. We get by the theorem of reduction,

$$\int_B(3y^2+2xy)\,dS = \int_0^1\left\{\int_0^{1-y}(3y^2+2xy)\,dx\right\}dy$$

$$= \int_0^1\left\{3y^2(1-y)+y(1-y)^2\right\}dy = \int_0^1\left\{3y^2-3y^2+y-2y^2+y^3\right\}dy$$

$$= \int_0^1\left(y+y^2-2y^3\right)\,dy = \frac{1}{2}+\frac{1}{3}-2\cdot\frac{1}{4} = \frac{1}{3}.$$

Example 1.2 *Let B be the rectangle $[0,2\pi]\times\left[\frac{5}{4},\frac{5}{3}\right]$. Reduce the plane integral*

$$\int_B \frac{1}{y+\sin x}\,dS$$

in two ways, and then show the formula

$$\int_0^{2\pi} \ln\left(\frac{5+3\sin x}{5+4\sin x}\right)dx = 2\pi\,\ln\left(\frac{9}{8}\right).$$

A Plane integral.

D Reduce the plane integral in two different ways as double integrals, and then just compute.

I First note that the domain of integration is given by

$$y+\sin x>0 \qquad \text{and} \qquad y\geq\frac{5}{4}>1.$$

Then we reduce the plane integral in two different ways as double integrals,

$$\int_B \frac{1}{y+\sin x}\,dS = \int_0^{2\pi}\left\{\int_{\frac{5}{4}}^{\frac{5}{3}} \frac{1}{y+\sin x},dy\right\}dx = \int_{\frac{5}{4}}^{\frac{5}{3}}\left\{\int_0^{2\pi}\frac{1}{y+\sin x}\,dx\right\}dy.$$

By using that $\sin x$ is periodic, and then introducing the substitution $t = \tan \dfrac{x}{2}$, we get

$$\int_{\frac{5}{4}}^{\frac{5}{3}} \left\{ \int_0^{2\pi} \frac{1}{y+\sin x}\,dx \right\} dy = \int_{\frac{5}{4}}^{\frac{5}{3}} \left\{ \int_{-\pi}^{\pi} \frac{1}{y+\sin x}\,dx \right\} dy$$
$$= \int_{\frac{5}{4}}^{\frac{5}{3}} \left\{ \int_{-\pi}^{\pi} \frac{1}{y\sin^2 \frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + y\cos^2\frac{x}{2}}\,dx \right\} dy$$
$$= 2\int_{\frac{5}{4}}^{\frac{5}{3}} \left\{ \int_{-\infty}^{+\infty} \frac{1}{yt^2+2y+y}\,dt \right\} dy = 2\int_{\frac{5}{4}}^{\frac{5}{3}} \frac{1}{y} \left\{ \int_{-\infty}^{+\infty} \frac{1}{u^2+\frac{2}{y}+1}\,du \right\} dy$$
$$= 2\int_{\frac{5}{4}}^{\frac{5}{3}} \frac{1}{y} \left\{ \int_{-\infty}^{+\infty} \frac{1}{\left(u+\frac{1}{y}\right)^2 + 1 - \frac{1}{y^2}}\,du \right\} dy$$
$$= 2\int_{\frac{5}{4}}^{\frac{5}{3}} \frac{1}{y} \frac{1}{\sqrt{1-\frac{1}{y^2}}} \left[\operatorname{Arctan}\left(\frac{u+\frac{1}{y}}{\sqrt{1-\frac{1}{y^2}}} \right) \right]_{u=-\infty}^{+\infty} dy = 2\pi \int_{\frac{5}{4}}^{\frac{5}{3}} \frac{1}{\sqrt{y^2-1}}\,dy$$
$$= 2\pi \left[\ln\left(y + \sqrt{y^2-1} \right) \right]_{\frac{5}{4}}^{\frac{5}{3}} = 2\pi \left\{ \ln\left(\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 - 1} \right) - \ln\left(\frac{5}{4} + \sqrt{\left(\frac{5}{4}\right)^2 - 1} \right) \right\}$$
$$= 2\pi \left\{ \ln\left(\frac{5}{3} + \frac{4}{3} \right) - \ln\left(\frac{5}{4} + \frac{3}{4} \right) \right\} = 2\pi\{\ln 3 - \ln 2\} = 2\pi \ln\left(\frac{3}{2}\right).$$

On the other hand,

$$\int_0^{2\pi} \left\{ \int_{\frac{5}{4}}^{\frac{5}{3}} \frac{1}{y+\sin x}\,dy \right\} dx = \int_0^{2\pi} \left[\ln(y+\sin x) \right]_{y=\frac{5}{4}}^{\frac{5}{3}} dx = \int_0^{2\pi} \ln\left(\frac{\frac{5}{3}+\sin x}{\frac{5}{4}+\sin x} \right) dx$$
$$= \int_0^{2\pi} \left\{ \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5+3\sin x}{5+4\sin x} \right) \right\} dx = 2\pi \ln\left(\frac{4}{3}\right) + \int_0^{2\pi} \ln\left(\frac{5+3\sin x}{5+4\sin x} \right) dx.$$

As a conclusion we get

$$\int_B \frac{1}{y+\sin x}\,dS = 2\pi \ln\left(\frac{4}{3}\right) + \int_0^{2\pi} \ln\left(\frac{5+3\sin x}{5+4\sin x} \right) dx = 2\pi \ln\left(\frac{3}{2}\right).$$

Finally, by a rearrangement

$$\int_0^{2\pi} \ln\left(\frac{5+3\sin x}{5+4\sin x} \right) dx 2\pi \left\{ \ln\left(\frac{3}{2}\right) - \ln\left(\frac{4}{3}\right) \right\} = 2\pi \ln\left(\frac{9}{8}\right),$$

as required.

Example 1.3 *The unit square* $E = [0,1] \times [0,1]$ *is divided by the straight line of equation* $y = x$ *into two triangles:* T_1 *given by* $y \le x$*, and* T_2 *given by* $y > x$*. We define a function* $f : E \to \mathbb{R}$ *in the following way:*

$$f(x,y) = \begin{cases} x^2 + 2y, & (x,y) \in T_1, \\ \\ 1 + 3y^2, & (x,y) \in T_2. \end{cases}$$

Compute the plane integral $\int_E f(x,y)\,dS$.

A Plane integral.

D Reduce over each of the sets T_1 and T_2. The plane integral can be reduced to double integrals in $2 \times 2 = 4$ different ways, of which we only show one.

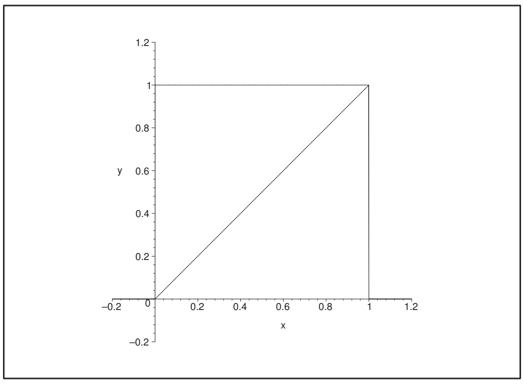


Figure 11: The triangle T_1 has an edge along the X-axis, and the triangle T_2 has an edge along the Y-axis.

I From

$$\begin{aligned} T_1 &= \{(x,y) \mid 0 \le x \le 1,\, 0 \le y \le x\} \\ T_2 &= \{(x,y) \mid 0 \le y \le 1,\, 0 \le x \le y\}, \end{aligned}$$

follows (note the two different successions of the order of integration)

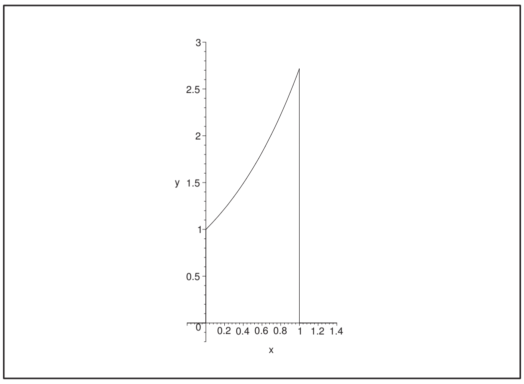
$$\begin{aligned} \int_E f(x,y)\,dS &= \int_{T_1} f(x,y)\,dS + \int_{T_2} f(x,y)\,dS \\ &= \int_0^1 \left\{ \int_0^x (x^2+2y)\,dy \right\} dx + \int_0^1 \left\{ \int_0^y (1+3y^2)\,dx \right\} dy \\ &= \int_0^1 \left[x^2 y + y^2 \right]_{y=0}^{x} dx + \int_0^1 \left[x + 3y^2 x \right]_{x=0}^{y} dy \\ &= \int_0^1 (x^3+x^2)\,dx + \int_0^1 (y+3y^3)\,dy \\ &= \int_0^1 (4t^3+t^2+t)\,dt = \left[t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 \right]_0^1 = 1 + \frac{1}{3} + \frac{1}{2} = \frac{11}{6}. \end{aligned}$$

Example 1.4 *Let D be the set which is bounded by the curve $y = e^x$, and the line $x = 1$, and the coordinate axes. Sketch D, and compute the plane integral*

$$\int_D \frac{1}{(1+y)^2 \cosh x}\, dS.$$

A Plane integral in rectangular coordinates.

D Sketch the domain and apply the theorem of reduction.



I When we reduce the plane integral, introduce the substitution $u = e^x$, and apply a decomposition, we get

$$\begin{aligned}
\int_D \frac{1}{(1+y)^2 \cosh x}\, dS &= \int_0^1 \frac{1}{\cosh x} \left\{ \int_+^{e^x} \frac{1}{(1+y)^2}\, dy \right\} dx = \int_0^1 \frac{2e^x}{e^{2x}+1} \left[-\frac{1}{1+y} \right]_{y=0}^{e^x} dx \\
&= \int_0^1 \left\{ \frac{2e^x}{e^{2x}+1} - \frac{2e^x}{e^{2x}+1} \cdot \frac{1}{e^x+1} \right\} dx = \int_1^e \left\{ \frac{2}{u^2+1} - \frac{2}{(u^2+1)(u+1)} \right\} du \\
&= \int_1^e \left\{ \frac{2}{u^2+1} - \frac{1}{u+1} - \frac{2}{(u^2+1)(u+1)} + \frac{1}{u+1} \right\} du \\
&= \int_1^e \left\{ \frac{2}{u^2+1} - \frac{1}{u+1} + \frac{u^2+1-2}{(u^2+1)(u+1)} \right\} du \\
&= \int_1^e \left\{ \frac{2}{u^2+1} - \frac{1}{u+1} + \frac{u}{u^2+1} - \frac{1}{u^2+1} \right\} du \\
&= \int_1^e \left\{ \frac{1}{u^2+1} + \frac{u}{u^2+1} - \frac{1}{u+1} \right\} du = \left[\text{Arctan}\, u + \frac{1}{2}\ln(u^2+1) - \frac{1}{2}\ln(u+1) \right]_1^e \\
&= \text{Arctan}\, e - \frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{e^2+1}{(e+1)^2} \right) - \frac{1}{2}\ln\left(\frac{1+1}{(1+1)^2} \right),
\end{aligned}$$

where we also can obtain the equivalent results

$$
\begin{aligned}
\int_D \frac{1}{(1+y)^2\cosh x}\,dS &= \text{Arctan } e - \frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{2(e^2+1)}{(e+1)^2}\right) \\
&= \text{Arctan } e - \frac{\pi}{4} + \ln\left(\frac{\cosh 1}{\cosh^2 \frac{1}{2}}\right) \\
&= \text{Arctan } e - \frac{\pi}{4} + \ln\left(\frac{2\cosh 1}{1+\cosh 1}\right).
\end{aligned}
$$

Example 1.5 *The function* $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *is given by*

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}.$$

Note that the domain of the function is the open *first quadrant. By the computations of integrals we shall whenever necessary use a continuous extension to the axes.*

1) Compute the double integrals

$$I_1 = \int_0^1 \left\{ \int_0^1 f(x,y)\,dy \right\} dx \quad og \quad I_2 = \int_0^1 \left\{ \int_0^1 f(x,y)\,dx \right\} dy.$$

2) It follows from 1) that $I_1 \neq I_2$. *Make a comment on this result by considering the plane integral of the function* f *over the unit square* $[0,1] \times [0,1]$.

A Double integrals.

D Compute I_1, and apply that $I_2 = -I_1$ by an argument of symmetry.

I 1) We get when $x \neq 0$,

$$\int_0^1 f(x,y)\,dy = \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy = \int_0^1 \frac{d}{dy}\left(\frac{y}{x^2+y^2}\right) dy = \frac{1}{1+x^2},$$

so

$$I_1 = \int_0^1 \left\{ \int_0^1 f(x,y)\,dy \right\} dx = \int_0^1 \frac{1}{1+x^2}\,dx = \text{ Arctan } 1 = \frac{\pi}{4}.$$

From $f(y,x) = -f(x,y)$ follows by interchanging the letters and by a small argument of symmetry that

$$\begin{aligned} I_2 &= \int_0^1 \left\{ \int_0^1 f(x,y)\,dx \right\} dy = \int_0^1 \left\{ \int_0^1 f(y,x)\,dy \right\} dx \\ &= -\int_0^1 \left\{ \int_0^1 f(x,y)\,dy \right\} = -I_1 = -\frac{\pi}{4} \neq I_1. \end{aligned}$$

2) The plane integral $\int_{[0,1]^2} f(x,y)\,dx\,dy$ is improper at $(0,0)$, and it is *not* convergent. If e.g.

$$D = \left\{ (x,y) \in [0,1]^2 \;\middle|\; y < \frac{1}{2}\,x \right\},$$

then

$$\begin{aligned} \int_D \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy &\geq \int_D \frac{x^2 - \frac{1}{4}\,x^2}{(x^2+\frac{1}{4}\,x^2)^2}\,dx\,dy \\ &= \int_0^1 \frac{\frac{3}{4}\,x^2}{(\frac{5}{4})^2 x^4} \cdot \frac{1}{2}\,x\,dx = \frac{3}{4} \cdot \frac{4^2}{5^2} \cdot \frac{1}{2} \int_0^1 \frac{1}{x}\,dx = +\infty, \end{aligned}$$

and $D \subset [0,1] \times [0,1]$.

Example 1.6 *Find the domain B for*

$$f(x,y) = \sqrt{1-x^2-y^2} + \sqrt{x^2y}.$$

Then find the range $f(B)$ and the plane *integral*

$$\int_B f(x,y)\,dS.$$

A Domain, range and plane integral.

D Use the standard methods. When we calculate the plane integral we neglect the zero set.

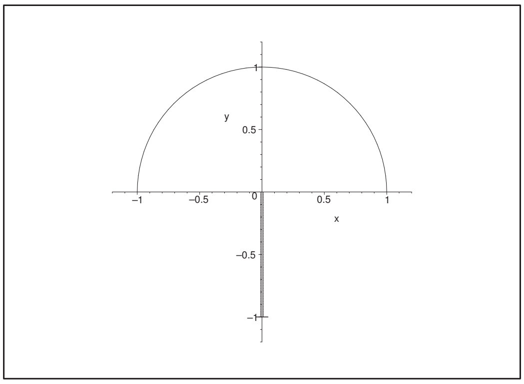


Figure 12: The domain B. Notice the interval on the negative Y-axis.

I The function is defined and continuous when $x^2+y^2 \le 1$ and $x^2y \ge 0$. From the first condition follows that B is contained in the closed unit disc. From the second condition follows that if $x \neq 0$, then $y \ge 0$; however, if $x = 0$, then $x^2y = 0$ for every y, so the latter term is defined in union of the closed upper half plane and the y-axis.

The domain is the intersection of these closed domains, i.e. union of the closed half disc in the upper half plane and the interval $[-1,0]$ on the y-axis, cf. the figure.

Since f is continuous in B, and B is closed and bounded and connected, then f has a maximum value S and a minimum value M in B (second main theorem), and by the first main theorem the range is connected, so $f(B) = [M,S]$.

We shall search the maximum and the minimum among:

1) the interior points, where f is not differentiable (the exceptional points: $x = 0$ and $0 < y < 1$),
2) the interior stationary points (i.e. inside the set $x^2+y^2 < 1$, $y > 0$, $x \neq 0$),
3) the boundary points.

1) The restriction of f to $x = 0$ and $y \in]0,1[$ is

$$\varphi(y) = \sqrt{1-y^2}, \qquad y \in]0,1[.$$

This function is decreasing and of the range $]0,1[$, so it has neither a minimum value nor a maximum value.

2) If (x,y) is a stationary point in the open quarter disc in the first quadrant, then $(-x,y)$ is clearly a stationary point in the open quarter disc in the second quadrant, and *vice versa*. Now, f only contains x in the form x^2, so the value is the same, $f(x,y)=f(-x,y)$. It will therefore suffice to consider the quarter disc

$$\{(x,y)\mid x>0,\, y>0,\, x^2+y^2<1\}$$

in the first quadrant. We have in this subdomain,

$$f(x,y)=\sqrt{1-x^2-y^2}+x\sqrt{y}.$$

The equations of possible stationary points are here

$$\begin{cases} \dfrac{\partial f}{\partial x}=-\dfrac{x}{\sqrt{1-x^2-y^2}}+\sqrt{y}=0, \\ \dfrac{\partial f}{\partial y}=-\dfrac{y}{\sqrt{1-x^2-y^2}}+\dfrac{1}{2}\,\dfrac{x}{\sqrt{y}}=0, \end{cases}$$

and it follows from $x>0$ and $y>0$ that

$$\frac{xy}{\sqrt{1-x^2-y^2}}=y\sqrt{y}=\frac{1}{2}\,\frac{x^2}{\sqrt{y}}.$$

Hence $y^2=\dfrac{1}{2}\,x^2$, so $y=+\dfrac{1}{\sqrt{2}}\,x$. But then

$$y=\frac{1}{\sqrt{2}}\,x=\frac{x^2}{1-x^2-y^2}=\frac{x^2}{1-\frac{3}{2}\,x^2},$$

hence by a rearrangement,

$$x^2+2\,\frac{\sqrt{2}}{3}\,x-\frac{2}{3}=0.$$

The solutions are $x=-\sqrt{2}$ (must be rejected because we are only considering points of the unit disc in the first quadrant) and $x=\dfrac{\sqrt{2}}{3}$, corresponding to $y=\dfrac{1}{\sqrt{2}}\,x=\dfrac{1}{3}$. Clearly, $\left(\dfrac{\sqrt{2}}{3},\dfrac{1}{3}\right)$ is an inner point of the domain, so it is a stationary point. Then by the above, $\left(-\dfrac{\sqrt{2}}{3},\dfrac{1}{3}\right)$ is also a stationary point, and these two points are the only stationary points. The value of the functions is here

$$f\left(\pm\frac{\sqrt{2}}{3},\frac{1}{3}\right)=\sqrt{1-\frac{2}{9}-\frac{1}{9}}+\frac{\sqrt{2}}{3}\sqrt{\frac{1}{3}}=\sqrt{\frac{2}{3}}+\frac{1}{3}\sqrt{\frac{2}{3}}=\frac{4}{3}\sqrt{\frac{2}{3}}.$$

3) The examination of the boundary is split into
   a) The circular arc, $x^2+y^2=1$, $x\in[-1,1]$, $y\in[0,1]$.
   b) The line segment on the X-axis, $y=0$, $x\in[-1,1]$.
   c) The line segment on the Y-axis, $x=0$, $y\in[-1,0]$.

a) Since $f(-x,y) = f(x,y)$, it suffices to consider the quarter circular arc $x^2 = 1 - y^2$, $x \geq 0$, $y \geq 0$. The restriction of f becomes

$$\varphi(y) = \sqrt{(1-y^2)y} = \sqrt{y - y^3}, \qquad y \in [0,1].$$

Since φ and $\Phi(y) = \varphi(y)^2 = y - y^3$ attain their maximum value and minimum value at the same points we compute

$$\Phi'(y) = 1 - 3y^2, \qquad \text{hence } \Phi'(y) = 0 \text{ for } y = \frac{1}{\sqrt{3}}.$$

Correspondingly, $x = \pm\sqrt{\frac{2}{3}}$, and

$$f\left(\pm\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = \sqrt{\frac{2}{3} \cdot \frac{1}{\sqrt{3}}} = \frac{1}{3}\sqrt{2\sqrt{3}}.$$

At the end points

$$[f(-1,0) =] \quad f(1,0) = 0 \quad \text{and} \quad f(0,1) = 0.$$

b) When $y = 0$ and $x \in [-1, 1]$, the restriction of f is given by

$$f(x, 0) = \sqrt{1 - x^2}, \qquad x \in [-1, 1],$$

which clearly has its maximum value $f(0, 0) = 1$ and its minimum value $f(-1, 0) = f(1, 0) = 0$.

c) When $x = 0$ and $y \in [-1, 0]$, we get

$$f(0, y) = \sqrt{1 - y^2}, \qquad y \in [-1, 0],$$

with the maximum value $f(0, 0) = 1$ and the minimum value $f(0, 1) = 0$.

It follows by a numerical comparison that the minimum value is attained at the boundary points

$$M = f(1, 0) = f(0, 1) = f(-1, 0) = f(+, -1) = 0,$$

and the maximum value is attained at the stationary points,

$$S = f\left(\frac{\sqrt{2}}{3}, \frac{1}{3}\right) = f\left(-\frac{\sqrt{2}}{3}, \frac{1}{2}\right) = \frac{4\sqrt{2}}{3\sqrt{3}}.$$

According to the first main theorem for continuous functions the range of the function is connected, thus

$$f(B) = [M, S] = \left[0, \frac{4\sqrt{2}}{3\sqrt{3}}\right].$$

We shall finally compute a plane integral. Since $f(x, y)$ is continuous on B, and the interval on the Y-axis in the lower half plane is a null set, the integral is zero over this part.

Let $\tilde{B}$ denote the closed half disc in the upper half plane. Then we get by reduction in polar coordinates

$$\begin{aligned}
\int_B f(x, y)\, dS &= \int_{\tilde{B}} \left\{\sqrt{1 - x^2 - y^2} + \sqrt{x^2 y}\right\} dS \\
&= \int_0^\pi \left\{\int_0^1 \left(\sqrt{1 - \varrho^2} + \sqrt{\varrho^2 \cos^2\varphi \cdot \varrho \sin\varphi}\right) \varrho\, d\varrho\right\} d\varphi \\
&= \frac{\pi}{2} \int_0^1 \left(1 - \varrho^2\right)^{\frac{1}{2}}\, 2\varrho\, d\varrho + 2\int_0^{\frac{\pi}{2}} |\cos\varphi| \sqrt{\sin\varphi}\, d\varphi \cdot \int_0^1 \varrho^{\frac{5}{2}}\, d\varrho \\
&= \frac{\pi}{2}\left[-\frac{2}{3}\left(1 - \varrho^2\right)^{\frac{3}{2}}\right]_0^1 + 2\left[\frac{2}{3}(\sin\varphi)^{\frac{3}{2}}\right]_{\varphi=0}^{\frac{\pi}{2}} \cdot \left[\frac{2}{7}\varrho^{\frac{7}{2}}\right]_{\varrho=0}^1 \\
&= \frac{\pi}{2} \cdot \frac{2}{3} + 2 \cdot \frac{2}{3} \cdot \frac{2}{7} = \frac{\pi}{3} + \frac{8}{21}.
\end{aligned}$$

Example 1.7 *Calculate the plane integral*

$$\int_B 3xy\,dx\,dy,$$

where B is the closed set in the first quadrant, which is bounded by the parabola of the equation $y = 4 - 4x^2$ and the coordinate axes.

A Plane integral.

D Sketch the domain and compute the plane integral.

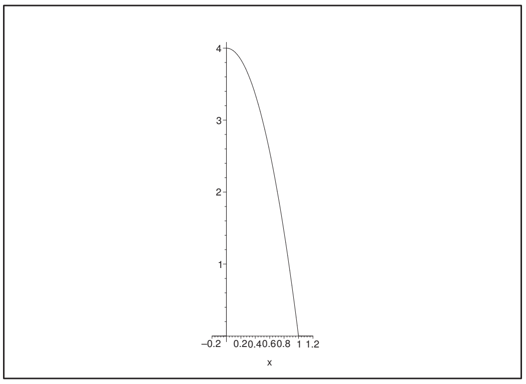


Figure 13: The domain of integration B.

I We get immediately,

$$\begin{aligned}\int_B 3xy\,dy\,dx &= 3\int_0^1 x\left\{\int_0^{4-4x^2} y\,dy\right\}dx = \frac{3}{2}\int_0^1 x\left(4-4x^2\right)^2\,dx \\ &= \frac{3}{2}\cdot 16\cdot\frac{1}{2}\int_0^1 (1-t)^2\,dt = 12\int_0^1 u^2\,du = 4.\end{aligned}$$

Example 1.8 *Let B denote the bounded set in the (X,Y)-plane, which is bounded by the line $y = x$ and the parabola $y = x^2$. Compute the plane integral*

$$\int_B x^2y\,dx\,dy.$$

A Plan integral.

D First sketch B.

I Since

$$B\left\{(x,y)\mid 0\le x\le 1,\, x^2\le y\le x\right\},$$

the plane integral is reduced to

$$\int_B x^2y\,dx\,dy = \int_0^1 x^2\left\{\int_{x^2}^x y\,dy\right\}dx = \frac{1}{2}\int_0^1 x^2\left[y^2\right]_{x^2}^x\,dx = \frac{1}{2}\int_0^1\left(x^4-x^6\right)\,dx = \frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right) = \frac{1}{35}.$$

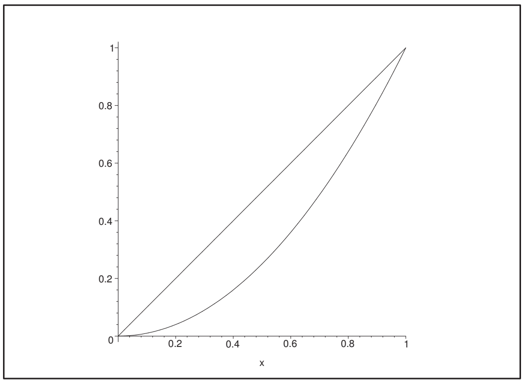


Figure 14: The domain B.

Example 1.9 *Let the set B be given by the inequalities*

$$x \geq 0, \qquad y \geq 0, \qquad \frac{x}{a} + \frac{y}{h} \leq 1.$$

where a and h are positive constants. Sketch B, and then compute the plane integral

$$J = \int_B x^3 y \, dS.$$

A Plane integral.

D Follow the guidelines and apply one of the theorems of reduction.

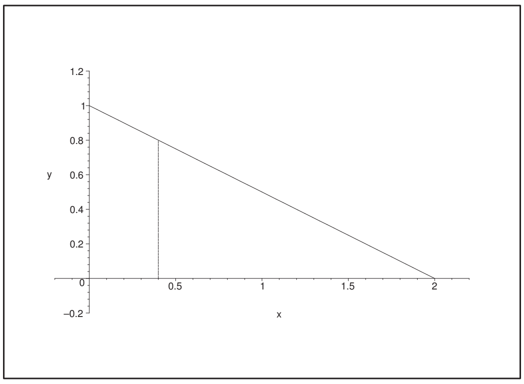


Figure 15: The domain B when $a = 2$ and $h = 1$.

I Since the integrand contains y of lower exponent than x, it will be easier first (i.e. innermost) to integrate vertically with respect to y, i.e. for fixed x,

$$0 \leq y \leq h\left(1 - \frac{x}{a}\right), \qquad 0 \leq x \leq a.$$

Then by means of the theorem of reduction in rectangular coordinates,

$$
\begin{aligned}
J &= \int_B x^3 y\, dS = \int_0^a x^3 \left(\int_0^{h\left(1-\frac{x}{a}\right)} y\, dy \right) = \int_0^a x^3 \cdot \frac{h^2}{2}\left(1-\frac{x}{a}\right)^2 dx \\
&= \frac{h^2}{2} \int_0^a x^3 \left(1 - \frac{2}{a}x + \frac{1}{a^2}x^2\right) dx = \frac{h^2}{2}\int_0^a \left(x^3 - \frac{2}{a}x^4 + \frac{1}{a^2}x^5\right) dx \\
&= \frac{h^2}{2}\left[\frac{x^4}{4} - \frac{2}{5a}x^5 + \frac{1}{6a^2}x^6\right]_0^a = \frac{h^2}{2}\left(\frac{a^2}{4} - \frac{2}{5}a^4 + \frac{1}{6}a^4\right) \\
&= \frac{h^2a^4}{2}\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = \frac{h^2a^4}{2}\cdot\frac{15-24+10}{60} = \frac{1}{120}h^2a^4.
\end{aligned}
$$

If we ALTERNATIVELY first integrate horizontally with respect to x, i.e.

$$0 \le x \le a\left(1-\frac{y}{h}\right), \qquad 0 \le y \le h,$$

then we get by another theorem of reduction in rectangular coordinates, where we apply the substitution $t = 1 - \dfrac{y}{h}$, $y = h(1-t)$ and $dy = -h\, dt$,

$$
\begin{aligned}
J &= \int_B x^3 y\, dS = \int_0^h y \left(\int_0^{a\left(1-\frac{y}{h}\right)} x^3\, dx \right) dy = \int_0^h y \cdot \frac{a^4}{4}\cdot\left(1-\frac{y}{h}\right)^4 dy \\
&= \int_0^1 \frac{a^4}{4}\cdot h(1-t)\cdot t^4 \cdot h\, dt = \frac{a^4h^2}{4}\int_0^1 \left(t^4 - t^5\right) dt = \frac{a^4h^2}{4}\left(\frac{1}{5}-\frac{1}{6}\right) = \frac{1}{120}a^4h^2.
\end{aligned}
$$

2 Plane integral, polar coordinates

Example 2.1 *Compute in each of the following cases the given plane integral by applying a theorem of reduction for polar coordinates. First sketch the domain of integration B.*

1) $\int_B (4-y)\,dS$, *where B is given by $x \geq 0$, $y \geq 0$, and $x^2+y^2 \leq 2$.*

2) $\int_B (a+y)\,dS$, *where B is given by $0 \leq \varphi \leq \dfrac{\pi}{2}$ and $0 \leq \varrho \leq a\cos\varphi$.*

3) $\int_B \sqrt{a^2-x^2-y^2}\,dS$, *where B is given by $-\dfrac{\pi}{2} \leq \varphi \leq \dfrac{\pi}{2}$ and $0 \leq \varrho \leq a\cos\varphi$.*

4) $\int_B xy\,dS$, *where B is given by $0 \leq \varphi \leq \dfrac{\pi}{3}$ and $2\cos\varphi \leq \varrho \leq \dfrac{4}{1+\cos\varphi}$.*

5) $\int_B \dfrac{x(x+y)}{(2x^2+y^2)(x^2+y^2)^{\frac{3}{2}}}\,dS$, *where B is given by $0 \leq \varphi \leq \dfrac{\pi}{4}$ and $\cos\varphi \leq \varrho \leq \cos\varphi+\sin\varphi$.*

6) $\int_B \dfrac{1}{\sqrt{a^2+x^2+y^2}}\,dS$, *where B is the disc $\overline{K}((0,0);a)$.*

7) $\int_B \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}\,dS$, *where B is given by $-\pi \leq \varphi \leq \pi$ and $b\exp(a\cos\varphi) \leq \varrho \leq 1$, and where furthermore $b < e^{-a}$.*

8) $\int_B \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}\,dS$, *where B is given by $-\pi \leq \varphi \leq \pi$ and $1 \leq \varrho \leq b\exp(a\cos\varphi)$, and where furthermore $b > e^{a}$.*

9) $\int_B (x^2-y^2)\,dS$, *where B is given by $-\dfrac{\pi}{4} \leq \varphi \leq \dfrac{\pi}{2}$ and $0 \leq \varrho \leq a$.*

10) $\int_B \sqrt{x^2+y^2}\,dS$, *where B is given by $-\dfrac{\pi}{2} \leq \varphi \leq \dfrac{\pi}{2}$ and $0 \leq \varrho \leq a\,\cos\varphi$.*

11) $\int_B xy\,dS$, *where B is given by $0 \leq \varphi \leq \dfrac{\pi}{4}$ and $a \leq \varrho \leq 2a\,\cos^2\varphi$.*

A Plane integral in polar coordinates.

D Sketch the domain and apply the theorem of reduction.

I 1) This example is the same as **Example 1.1.8**. We shall only use polar coordinates in the present case.

In polar coordinates B is described by

$$0 \leq \varphi \leq \frac{\pi}{2}, \qquad 0 \leq \varrho \leq \sqrt{2}.$$

From the theorem of reduction in polar coordinates follows that

$$\int_B (4-y)\,dS = 4\,\text{area}(B) - \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\sqrt{2}} \varrho\,\sin\varphi \cdot \varrho\,d\varrho \right\} d\varphi$$

$$= 4 \cdot \frac{1}{4}(\sqrt{2})^2 + [\cos\varphi]_0^{\frac{\pi}{2}} \cdot \left[\frac{1}{3}\varrho^3\right]_0^{\sqrt{2}} = 2\pi - \frac{2\sqrt{3}}{3}.$$

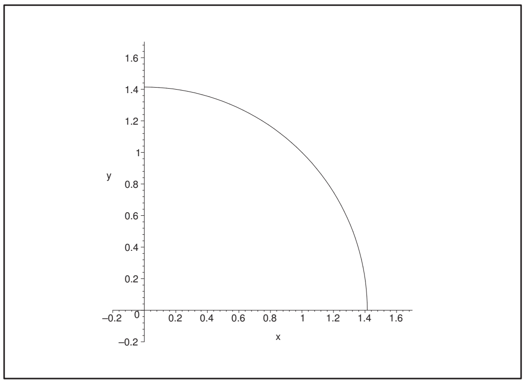


Figure 16: The domain B of **Example 2.1.1**.

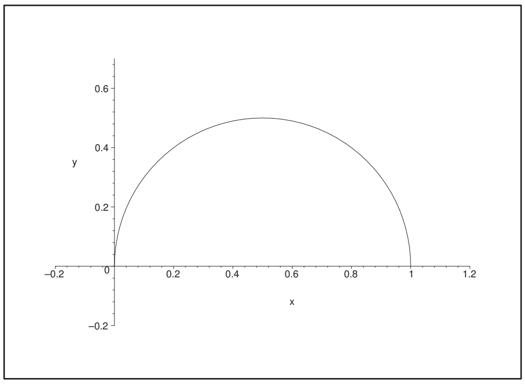


Figure 17: The domain B of **Example 2.1.2**.

2) From $0 \le \varrho \le a\,\cos\varphi$ follows that

$$0 \le \varrho^2 = x^2 + y^2 = a\varrho\cos\varphi = ax,$$

so the domain is a half disc in the first quadrant of centrum $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$. By the reduction formula in polar coordinates,

$$\begin{aligned}\int_B (a+y)\,dS &= a\cdot\text{area}(B) + \int_B y\,dS = a\cdot\frac{1}{2}\cdot\pi\left(\frac{a}{2}\right)^2 + \int_0^{\frac{\pi}{2}}\left\{\int_0^{a\,\cos\varphi}\varrho\,\sin\varphi\cdot\varrho\,d\varrho\right\}d\varphi \\ &= \pi\cdot\frac{a^3}{8} + \int_0^{\frac{\pi}{2}}\left[\frac{1}{3}\,\varrho^3\sin\varphi\right]_{\varrho=0}^{a\,\cos\varphi} d\varphi = \frac{a^3\pi}{8} + \frac{a^3}{3}\int_0^{\frac{\pi}{2}}\cos^3\varphi\cdot\sin\varphi\,d\varphi \\ &= \frac{\pi a^3}{8} - \frac{a^3}{12}\left[\cos^4\varphi\right]_0^{\frac{\pi}{2}} = a^3\left(\frac{\pi}{8}+\frac{1}{12}\right).\end{aligned}$$

3) Here B is the disc of centrum $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$, cf. **Example 2.1.2**. From the reduction

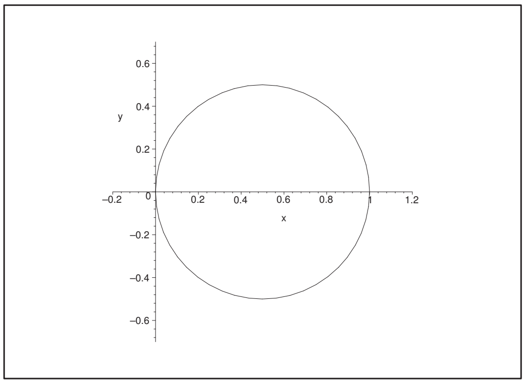


Figure 18: The domain B of **Example 2.1.3**.

formula in polar coordinates follows that

$$\int_B \sqrt{a^2-x^2-y^2}\,dS = \int_{-\frac{pi}{2}}^{\frac{\pi}{2}} \left\{\int_0^{a\ \cos\varphi} \sqrt{a^2-\varrho^2}\cdot\varrho\,d\varrho\right\} d\varphi$$

$$= 2\int_0^{\frac{\pi}{2}} \left[-\frac{1}{3}(a^2-\varrho^2)^{\frac{3}{2}}\right]_{\varrho=0}^{a\ \cos\varphi} d\varphi = \frac{2}{3}\int_0^{\frac{\pi}{2}} \left\{(a^2)^{\frac{3}{2}} - (a^2-a^2\cos^2\varphi)^{\frac{3}{2}}\right\} d\varphi$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} \left\{a^3 - a^3(1-\cos^2\varphi)\sin\varphi\right\} d\varphi = \frac{2}{3}\,a^3\left\{\frac{\pi}{2} + \int_{\varphi=0}^{\frac{\pi}{2}} (1-\cos^2\varphi)\,d\cos\varphi\right\}$$

$$= \frac{\pi a^3}{3} + \frac{2}{3}\,a^3\left[\cos\varphi - \frac{1}{3}\ \cos^3\varphi\right]_{\varphi=0}^{\frac{\pi}{2}} = \frac{\pi a^3}{3} - \frac{4}{9}\,a^3 = \frac{a^3}{9}(3\pi-4).$$

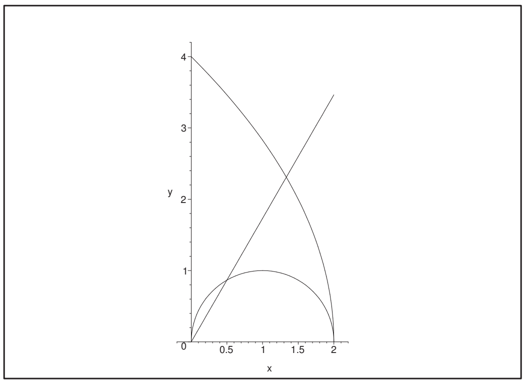


Figure 19: The domain B of **Example 2.1.4**.

4) From $2\cos\varphi \le \varrho$ follows

$$2x = 2\varrho\cos\varphi \le \varrho^2 = x^2+y^2,$$

which is rewritten as the inequality $(x-1)^2+y^2 \ge 1$ for the complementary set of the disc of centrum $(1,0)$ and radius 1.

From $\varrho \leq \dfrac{4}{1+\cos\varphi}$ follows $\varrho + \varrho\cos\varphi = \varrho + x \leq 4$, i.e. $\varrho \leq 4 - x$, so $x \leq 4$. Under this assumption we get by a squaring that $\varrho^2 = x^2 + y^2 \leq (4-x)^2$, hence

$$y^2 \leq (4-x)^2 - x^2 = 4(4-2x) = 8(2-x),$$

from which follows that we shall also require that $x \leq 2$, because $y^2 \geq 0$.
The domain is bounded by the parabola $y^2 = 16 - 8x$ and the circle $(x-1)^2 + y^2 = 1$ and the tow lines $\varphi = 0$ and $\varphi = \dfrac{\pi}{3}$.

Then by the theorem of reduction in polar coordinates followed by the substitution $u = \cos\varphi$,

$$\begin{aligned}
\int_B xy\,dS &= \int_0^{\frac{\pi}{3}} \left\{ \int_{2\cos\varphi}^{\frac{4}{1+\cos\varphi}} \varrho^3 \sin\varphi \cdot \cos\varphi \, d\varrho \right\} d\varphi \\
&= \frac{1}{4} \int_0^{\frac{\pi}{3}} \sin\varphi \cdot \cos\varphi \left\{ \frac{4^4}{(1+\cos\varphi)^4} - 2^4 \cos^4\varphi \right\} d\varphi \\
&= \int_0^{\frac{\pi}{3}} \left\{ \frac{64\cos\varphi}{(1+\cos\varphi)^4} - 4\cos^5\varphi \right\} \sin\varphi \, d\varphi = \int_{\frac{1}{2}}^1 \left\{ \frac{64(u+1-1)}{(u+1)^4} - 4u^5 \right\} du \\
&= \int_{\frac{1}{2}}^1 \left\{ \frac{64}{(u+1)^3} - \frac{64}{(u+1)^4} - 4u^5 \right\} du \\
&= \left[-\frac{1}{2} \cdot \frac{64}{(u+1)^2} + \frac{1}{3} \cdot \frac{64}{(u+1)^3} - \frac{4}{6} u^6 \right]_{\frac{1}{2}}^1 \\
&= -\frac{32}{4} + \frac{1}{3} \cdot \frac{64}{8} - \frac{2}{3} + \frac{32}{\frac{9}{4}} - \frac{1}{3} \cdot \frac{64}{\frac{27}{8}} + \frac{2}{3} \cdot \frac{1}{64} \\
&= -8 + \frac{8}{3} - \frac{2}{3} + \frac{128}{9} - \frac{512}{81} + \frac{1}{3 \cdot 32} = -6 + \frac{1252 - 512}{81} + \frac{1}{3 \cdot 32} \\
&= \frac{640 - 486}{81} + \frac{1}{3 \cdot 32} = \frac{154}{81} + \frac{1}{3 \cdot 32} = \frac{154 \cdot 32 + 27}{32 \cdot 81} = \frac{4955}{2592}.
\end{aligned}$$

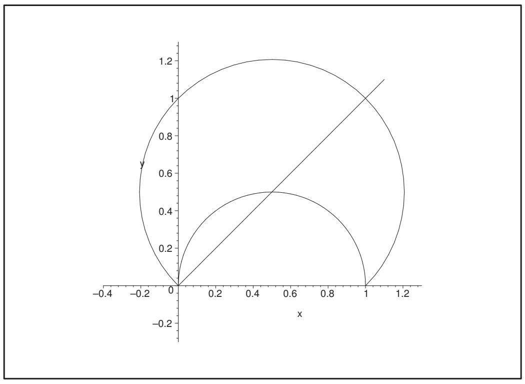


Figure 20: The domain B of **Example 2.1.5**.

5) Here the condition $\cos\varphi \le \varrho$ implies that

$$\varrho\cos\varphi = x \le \varrho^2 = x^2 + y^2,$$

which we rewrite as

$$\left(x - \frac{1}{2} \right)^2 + y^2 \ge \frac{1}{4} = \left(\frac{1}{2} \right)^2,$$

and we are describing the complementary set of a disc of centrum $\left(\frac{1}{2}, 0 \right)$ and radius $\frac{1}{2}$.

The condition $\varrho \le \cos\varphi + \sin\varphi$ means that

$$\varrho^2 = x^2 + y^2 \le \varrho\cos\varphi + \varrho\sin\varphi = x + y,$$

which is rewritten as

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \le \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2.$$

This inequality represents a disc of centrum $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$. As also $0 \le \varphi \le \frac{\pi}{4}$, it is now easy to sketch the domain B.

Then by the theorem of reduction in polar coordinates,

$$\begin{aligned}
&\int_B \frac{x(x+y)}{(2x^2+y^2)(x^2+y^2)^{\frac{3}{2}}}\,dS \\
&= \int_0^{\frac{\pi}{4}} \left\{ \int_{\cos\varphi}^{\cos\varphi+\sin\varphi} \frac{\varrho^2(\cos\varphi+\sin\varphi)\cos\varphi}{\varrho^2(2\cos^2\varphi+\sin^2\varphi)\varrho^3}\cdot\varrho\,d\varrho\right\} d\varphi \\
&= \int_0^{\frac{\pi}{4}} \frac{(\cos\varphi+\sin\varphi)\cos\varphi}{2\cos^2\varphi+\sin^2\varphi}\left[-\frac{1}{\varrho}\right]_{\varrho=\cos\varphi}^{\cos\varphi+\sin\varphi} d\varphi \\
&= \int_0^{\frac{\pi}{4}} \left\{\frac{\cos\varphi+\sin\varphi}{2\cos^2\varphi+\sin^2\varphi} - \frac{\cos\varphi}{2\cos^2\varphi+\sin^2\varphi}\right\} d\varphi = \int_0^{\frac{\pi}{4}} \frac{\sin\varphi}{\cos^2\varphi+1}\,d\varphi \\
&= \left[-\text{ Arctan}(\cos\varphi)\right]_0^{\frac{\pi}{4}} = \text{ Arctan } 1 - \text{ Arctan}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} - \text{ Arctan}\left(\frac{\sqrt{2}}{2}\right).
\end{aligned}$$

6) The disc $\overline{K}((0,0);a)$ is described in polar coordinates by

$$-\pi \le \varphi \le \pi, \qquad 0 \le \varrho \le a.$$

We shall here omit the sketch of the well-known disc of centrum $(0,0)$ and radius a.

Then by an application of the theorem of reduction in polar coordinates,

$$\int_B \frac{1}{\sqrt{a^2+x^2+y^2}}\,dS = \int_{-\pi}^{\pi}\left\{\int_0 \frac{\varrho}{\sqrt{a^2+\varrho^2}}\,d\varrho\right\} d\varphi = 2\pi\left[\sqrt{a^2+\varrho^2}\right]_0^1 = 2\pi a(\sqrt{2}-1).$$

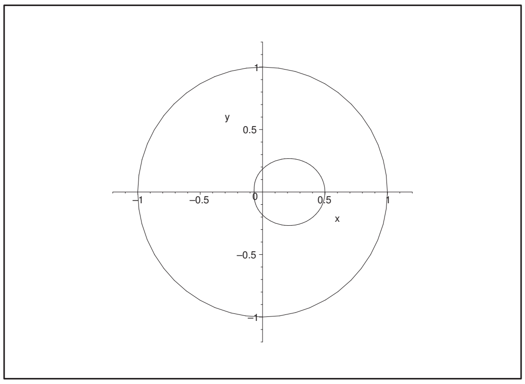


Figure 21: The domain B of **Example 2.1.7**, when $a=1$ and $b=\dfrac{1}{2e}$.

7) The set is an annulus shaped domain which is neither nice in a rectangular description nor in a polar description.

When we reduce the plane integral it is fairly simple to get

$$\int_B \frac{x}{(x^2+y^2)^{\frac{3}{2}}}\,dS = \int_{-\pi}^{\pi}\left\{\int_{b\exp(a\cos\varphi)}^{1} \frac{\varrho\cos\varphi}{\varrho^3}\cdot\varrho\,d\varrho\right\}d\varphi = \int_{-\pi}^{\pi}\cos\varphi\cdot[\ln\varrho]_{b\exp(a\cos\varphi)}^{1}d\varphi$$

$$= \int_{-\pi}^{\pi}\cos\varphi\{-\ln b - a\cos\varphi\}d\varphi = -a\int_{-\pi}^{\pi}\cos^2\varphi\,d\varphi = -a\pi.$$

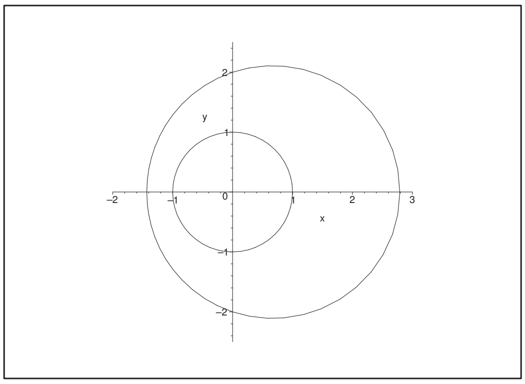


Figure 22: The domain B of **Example 2.1.8**, when $a = \dfrac{1}{3}$ and $b = 2$.

8) This case is similar to **Example 2.1.7**. We get

$$\int_B \frac{x}{(x^2+y^2)^{\frac{3}{2}}}\,dS = \int_{-\pi}^{\pi}\left\{\int_{1}^{b\exp(a\cos\varphi)} \frac{\cos\varphi}{\varrho}\,d\varrho\right\}d\varphi = +a\pi,$$

because, apart from the change of sign, the computations are formally the same as in **Example 2.1.7**.

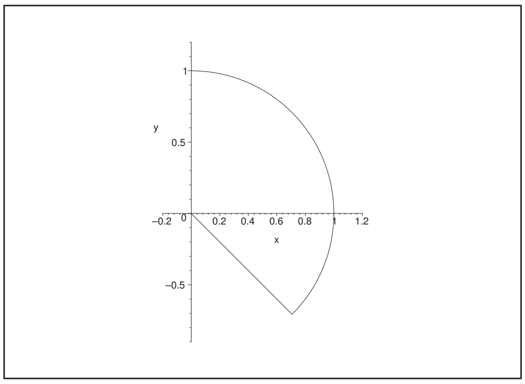


Figure 23: The domain B of **Example 2.1.9**.

9) The set B is a circular sector os shown on the figure.

Then by the theorem of reduction,

$$\int_B (x^2-y^2)\,dS = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0 \left\{\varrho^2\cos^2\varphi - \varphi^2\sin^2\varphi\right\}\varrho\,d\varrho\right)d\varphi$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^a \cos 2\varphi\cdot\varrho^3\,d\varrho\right)d\varphi = \left[\frac{1}{2}\,\sin 2\varphi\right]_{-\frac{\pi}{4}}^{\frac{\pi}{2}}\cdot\frac{a^4}{4} = \frac{1}{2}\{0-(-1)\}\cdot\frac{a^4}{4} = \frac{a^4}{8}.$$

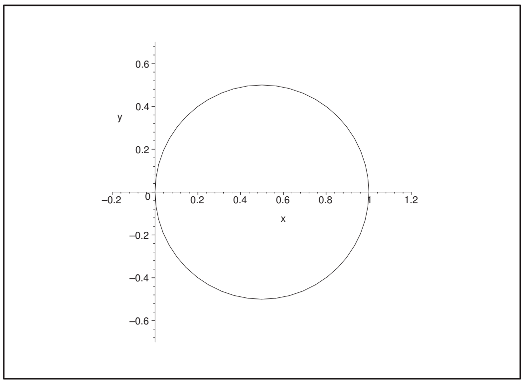


Figure 24: The domain B of **Example 2.1.10**.

10) From $0\le\varrho\le a\,\cos\varphi$ follows that

$$0\le\varrho^2 = x^2+y^2 = a\varrho\cos\varphi = ax,$$

so B is the closed disc of centrum $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$.

Then by the theorem of reduction,

$$\begin{aligned}\int_B\sqrt{x^2+y^2}\,dS &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_0^{a\cos\varphi}\varrho\cdot\varrho\,d\varrho\right\}d\varphi = \frac{a^3}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^3\varphi\,d\varphi\\ &= \frac{a^3}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}}(1-\sin^2\varphi)\cos\varphi\,d\varphi = \frac{a^3}{3}\left[\sin\varphi-\frac{1}{3}\,\sin^3\varphi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= \frac{a^3}{3}\cdot 2\left(1-\frac{1}{3}\right) = \frac{4a^3}{9}.\end{aligned}$$

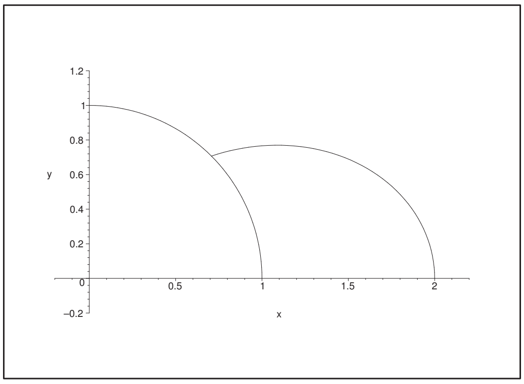


Figure 25: The domain B of **Example 2.1.11**.

11) There is no nice rectangular description of the domain. It follows by the theorem of reduction,

$$
\begin{aligned}
\int_B xy\,dS &= \int_0^{\frac{\pi}{4}}\left\{\int_1^{2a\cos^2\varphi} \varrho\cos\varphi\cdot\varrho\sin\varphi\cdot\varrho\,d\varrho\right\}d\varphi \\
&= \int_0^{\frac{\pi}{4}} \cos\varphi\sin\varphi\left\{\int_a^{2a\cos^2\varphi}\varrho^3\,d\varrho\right\}d\varphi, \\
&= \frac{1}{4}\int_0^{\frac{\pi}{4}}\cos\varphi\cdot\sin\varphi\left[(2a)^4\cos^8\varphi - a^4\right]d\varphi \qquad\qquad (t=\cos\varphi) \\
&= \frac{a^4}{4}\int_{\frac{1}{\sqrt{2}}}^1\left\{16t^9 - t\right\}dt = \frac{a^4}{4}\left[\frac{16}{10}t^{10}-\frac{1}{2}t^2\right]_{\frac{1}{\sqrt{2}}}^1 \\
&= \frac{a^4}{4}\left\{\frac{8}{5}-\frac{1}{2}-\frac{8}{5}\cdot\frac{1}{32}+\frac{1}{2}\cdot\frac{1}{2}\right\} = \frac{a^4}{4}\left\{\frac{31}{20}-\frac{1}{4}\right\}
\end{aligned}
$$

Example 2.2 *In each of the following cases a plane integral of a continuous function $f : B \to \mathbb{R}$ is written as a double integral. Sketch in each case the set B, and set up the double integral, or the sum of double integrals, which occur by interchanging the order or integration.*

1) $\int_0^1 \left\{\int_{x^2}^x f(x,y)\,dy\right\} dx.$

2) $\int_1^e \left\{\int_0^{\ln x} f(x,y)\,dy\right\} dx.$

3) $\int_1^2 \left\{\int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy\right\} dx.$

4) $\int_0^2 \left\{\int_{-\sqrt{2x-x^2}}^{0} f(x,y)\,dy\right\} dx.$

5) $\int_0^3 \left\{\int_{\frac{4y}{3}}^{\sqrt{25-y^2}} f(x,y)\,dx\right\} dy.$

6) $\int_{-6}^2 \left\{\int_{\frac{y^2-4}{4}}^{1-y} f(x,y)\,dx\right\} dy.$

7) $\int_0^1 \left\{\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dx\right\} dy.$

8) $\int_0^3 \left\{\int_0^{\sqrt{25-y^2}} f(x,y)\,dx\right\} dy.$

A Interchange of the order of integrations in double integrals.

D Sketch the set B and set up the double integral in the reverse order. Notice that a nice description in one case does not imply a nice description in the reverse order.

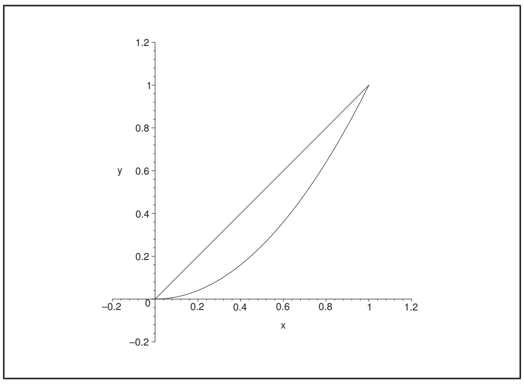


Figure 26: The domain B of **Example 2.2.1**.

I 1) The domain is given by

$$B = \{(x,y) \mid 0 \le x \le 1,\, x^2 \le y \le x\} = \{(x,y) \mid 0 \le y \le 1,\, y \le x \le \sqrt{y}\}.$$

In fact, it follows from the inner integral that $x^2 \le y \le x$, from which it is easy to derive

$$y \le x \le \sqrt{y}.$$

By interchanging the order of integration we get

$$\int_0^1 \left\{ \int_{x^2}^{x} f(x,y)\,dy \right\} dx = \int_0^1 \left\{ \int_y^{\sqrt{y}} f(x,y)\,dx \right\} dy.$$

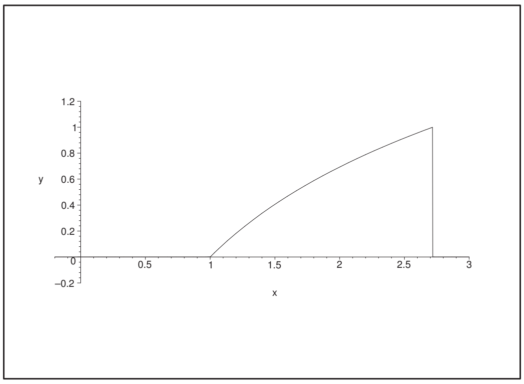


Figure 27: The domain B of **Example 2.2.2**.

2) The domain is found in the same way as in **Example 2.2.1**. It is given by

$$B = \{(x,y) \mid 1 \le x \le e,\, 0 \le y \le \ln x\} = \{(x,y) \mid 0 \le y \le 1,\, e^y \le x \le e\},$$

hence by interchanging the order of integration,

$$\int_1^e \left\{ \int_0^{\ln x} f(x,y)\, dy \right\} dx = \int_0^1 \left\{ \int_{e^y} f(x,y)\, dx \right\} dy.$$

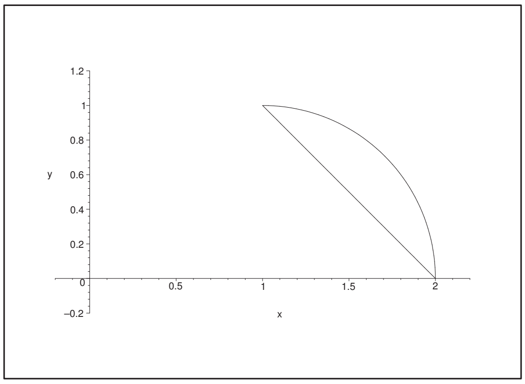


Figure 28: The domain B of **Example 2.2.3**.

3) This domain is bounded by the circle $(x-1)^2 + y^2 = 1$ and the straight line $y = 2 - x$, hence

$$\begin{aligned} B &= \{(x,y) \mid 1 \le x \le 2,\, 2-x \le y \le \sqrt{2x-x^2}\} \\ &= \{(x,y) \mid 0 \le y \le 1,\, 2-y \le x \le 1 + \sqrt{1-y^2}\}. \end{aligned}$$

When we interchange the order of integration we get

$$\int_1^2 \left\{ \int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\, dy \right\} dx = \int_0^1 \left\{ \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\, dx \right\} dy.$$

4) The domain is that part of the disc $(x-1)^2 + y^2 \le 1$ of centrum$(1,0)$ and radius 1, which lies in the fourth quadrant, thus below the X-axis, so

$$\begin{aligned} B &= \{(x,y) \mid 0 \le x \le 2,\, -\sqrt{2x-x^2} \le y \le 0\} \\ &= \{(x,y) \mid -1 \le y \le 0,\, 1 - \sqrt{1-y^2} \le x \le 1 + \sqrt{1-y^2}\}. \end{aligned}$$

When we interchange the order of integration we get

$$\int_0^2 \left\{ \int_{-\sqrt{2x-x^2}}^0 f(x,y)\, dy \right\} dx = \int_{-1}^0 \left\{ \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\, dx \right\} dy.$$

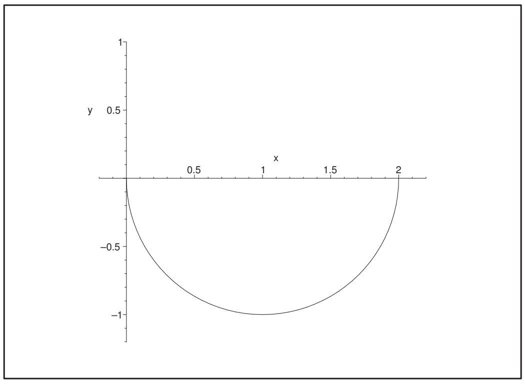


Figure 29: The domain B of **Example 2.2.4**.

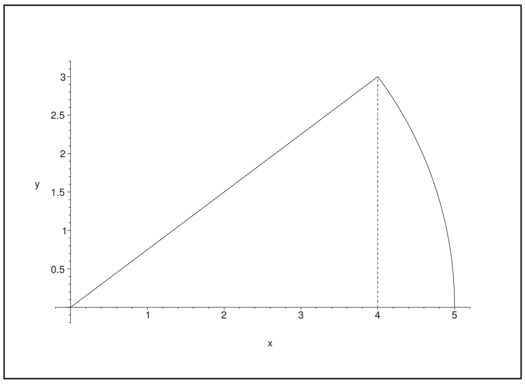


Figure 30: The domain B of **Example 2.2.5**.

5) The domain is bounded by the circle $x^2 + y^2 = 5^2$ and the lines $y = 0$ and $y = \dfrac{3}{4}\,x$. By the alternative description we must cut the domain by the dotted line $x = 4$. Then we get the two possible descriptions:

$$\begin{aligned} B &= \left\{(x,y) \;\middle|\; 0 \le y \le 3,\ \frac{4y}{3} \le x \le \sqrt{25-y^2}\right\} \\ &= \left\{(x,y) \;\middle|\; 0 \le x \le 4,\ 0 \le y \le \frac{3x}{4}\right\} \cup \{(x,y) \mid 4 \le x \le 5,\ 0 \le y \le \sqrt{25-x^2}\}. \end{aligned}$$

When we interchange the order of integration we obtain the following complicated expression

$$\int_0^3 \left\{ \int_{\frac{4y}{3}}^{\sqrt{25-y^2}} f(x,y)\,dx \right\} dy = \int_0^4 \left\{ \int_0^{\frac{3x}{4}} f(x,y)\,dy \right\} dx + \int_4^5 \left\{ \int_0^{\sqrt{25-x^2}} f(x,y)\,dy \right\} dx.$$

In this case we get the sum of two double integrals by interchanging the order of integration.

Remark. It follows from the form of the domain that it would be far more reasonable here to use polar coordinates, because B in these is described by

$$B = \left\{ (\varrho, \varphi) \mid 0 \leq \varrho \leq 5,\, 0 \leq \varphi \leq \text{ Arctan } \frac{3}{4} \right\},$$

and the integral is transformed into

$$\int_0^{\text{Arctan } \frac{3}{4}} \left\{ \int_0^5 \tilde{f}(\varrho, \varphi) \varrho \, d\varrho \right\} d\varphi. \qquad \Diamond$$

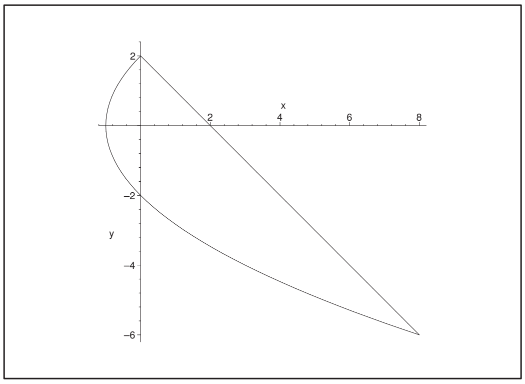


Figure 31: The domain B of **Example 2.2.6**.

6) By inspection of the integral we see that the domain is given by

$$B = \left\{ (x, y) \;\middle|\; -6 \leq y \leq 2,\, \frac{y^2 - 4}{4} \leq x \leq 2 - y \right\}.$$

It follows from the inequality $\dfrac{y^2-4}{4} \leq x$ that $y^2 \leq 4(x+1)$, and likewise we get from $x \leq 2 - y$ that $y \leq 2 - x$. Whenever the square root occurs (here by $|y| \leq 2\sqrt{x+1}$), we must be very careful! The figure shows that we have to split by the line $x = 0$, so B is written as a union of two sets which do not have the same structure,

$$\begin{aligned} B &= \{(x, y) \mid -1 \leq x \leq 0,\, -2\sqrt{x+1} \leq y \leq 2\sqrt{x+1}\} \\ &\quad \cup \{(x, y) \mid 0 \leq x \leq 8,\, -2\sqrt{x+1} \leq y \leq 2 - x\}. \end{aligned}$$

When we interchange the order of the integration we get a sum of two double integrals,

$$\int_{-6}^{2} \left\{ \int_{\frac{y^2-4}{4}}^{2-y} f(x, y)\, dx \right\} dy = \int_{-1}^{0} \left\{ \int_{-2\sqrt{x+1}}^{2\sqrt{x+1}} f(x, y)\, dy \right\} dx + \int_0^8 \left\{ \int_{-2\sqrt{x+1}}^{2-x} f(x, y)\, dy \right\} dx.$$

7) The domain is bounded by the unit circle in the second quadrant, by the X-axis and by the line $y + x = 1$. It is natural to split in the two subdomains along the Y-axis, thus

$$\begin{aligned} B &= \{(x, y) \mid 0 \leq y \leq 1,\, -\sqrt{1 - y^2} \leq x \leq 1 - y\} \\ &= \{(x, y) \mid -1 \leq x \leq 0,\, 0 \leq y \leq \sqrt{1 - x^2}\} \cup <, \{(x, y) \mid 0 \leq x \leq 1,\, 0 \leq y \leq 1 - x\}. \end{aligned}$$

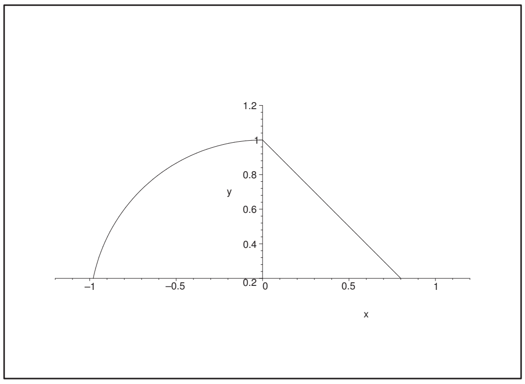


Figure 32: The domain B of **Example 2.2.7**.

Then by interchanging the order of integration,

$$\int_0^1 \left\{ \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dx \right\} dy = \int_{-1}^0 \left\{ \int_0^{\sqrt{1-x^2}} f(x,y)\,dy \right\} dx + \int_0^1 \left\{ \int_0^{1-x} f(x,y)\,dy \right\} dx.$$

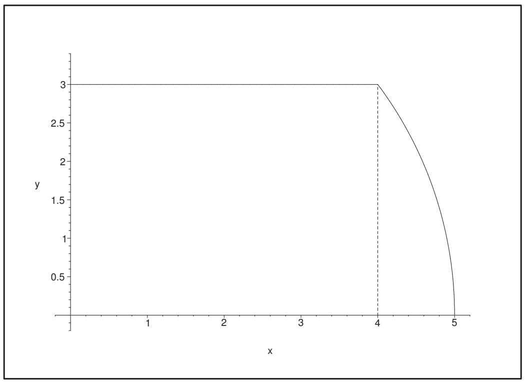


Figure 33: The domain B of **Example 2.2.8**.

8) The domain is described by

$$B = \{(x,y) \mid 0 \le x \le \sqrt{25-y^2},\, 0 \le y \le 3\},$$

thus B is that part of the quarter disc in the first quadrant of centrum $(0,2,)$ and radius 5, which also lies below the line $y = 3$. When we interchange the coordinates we must cut the domain by the line $x = 4$. Then B is written as the union of the two sets,

$$B = \{(x,y) \mid 0 \le y \le \sqrt{25-x^2},\, 4 \le x \le 5\} \cup \{(x,y) \mid 0 \le x \le 4,\, 0 \le y \le 3\}.$$

Then by interchanging the order of integration,

$$\int_0^3 \left\{ \int_0^{\sqrt{25-y^2}} f(x,y)\,dx \right\} dy = \int_0^4 \left\{ \int_0^3 f(x,y)\,dy \right\} dx + \int_4^5 \left\{ \int_0^{\sqrt{25-x^2}} f(x,y)\,dy \right\} dx.$$

Example 2.3 *Sketch the point sets*

$$B = \{(x,y) \mid 0 \le x \le 2,\, 0 \le y \le 2,\, xy \ge 2\}$$

and

$$D = \{(x,y) \mid 1 \le x,\, 1 \le y,\, xy \le 2\}.$$

Then compute the plane integrals

$$\int_B \frac{1}{xy}\, dS \qquad \text{and} \qquad \int_D \frac{1}{xy}\, dS.$$

A and D Sketch of a domain; computation of a plane integral.

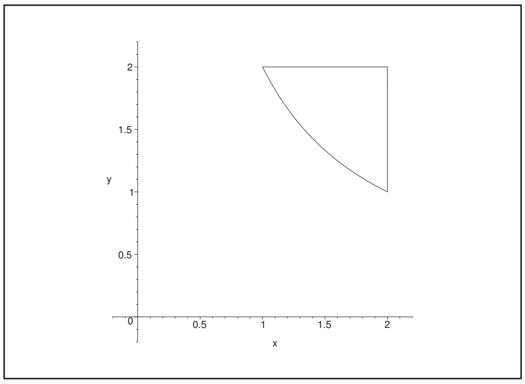


Figure 34: The domain B.

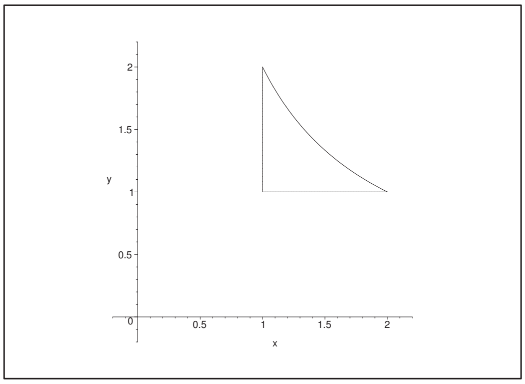


Figure 35: The domain D.

I The domains are sketched on the two figures. We see that

$$B \cup D = [1,2] \times [1,2],$$

which may be exploited in one of the variants, because B and D have just one boundary curve in common and are otherwise disjoint; cf. the alternative below.

From

$$B = \left\{ (x,y) \;\middle|\; 1 \le x \le 2,\, \frac{2}{x} \le y \le 2 \right\},$$

follows that

$$\int_B \frac{1}{xy}\, dS = \int_1^2 \left\{ \int_{\frac{2}{x}}^2 \frac{1}{xy}\, dy \right\} dx = \int_1^2 \frac{1}{x}\, [\ln y]_{\frac{2}{x}}^2\, dx = \int_1^2 \frac{1}{x}\, \ln x\, dx = \frac{1}{2}(\ln 2)^2.$$

From

$$D = \left\{(x,y) \;\middle|\; 1 \le x \le 2,\, 1 \le y \le \frac{2}{x}\right\},$$

we get analogously

$$\begin{aligned}\int_D \frac{1}{xy}\,dS &= \int_1^2 \left\{\int_1^{\frac{2}{x}} \frac{1}{xy}\,dy\right\} dx = \int_1^2 \frac{1}{x}\,[\ln y]_1^{\frac{2}{x}}\,dx \\ &= \int_1^2 \frac{1}{x}\{\ln 2 - \ln x\}dx = \left[\ln 2\cdot \ln x - \frac{1}{2}(\ln x)^2\right]_1^2 = (\ln 2)^2 - \frac{1}{2}(\ln 2)^2 = \frac{1}{2}(\ln 2)^2.\end{aligned}$$

Alternatively,

$$\int_{B\cup D} \frac{1}{xy}\,dS = \int_1^2 \frac{dx}{x}\cdot\int_1^2 \frac{dy}{y} = (\ln 2)^2 = \int_B \frac{1}{xy}\,dS + \int_D \frac{1}{xy}\,dS = \frac{1}{2}(\ln 2)^2 + \int_D \frac{1}{xy}\,dS,$$

hence

$$\int_D \frac{1}{xy}\,dS = (\ln 2)^2 - \frac{1}{2}(\ln 2)^2 = \frac{1}{2}(\ln 2)^2.$$

Example 2.4 *Let B be the domain in the first quadrant, which is bounded by the curves of the equations*

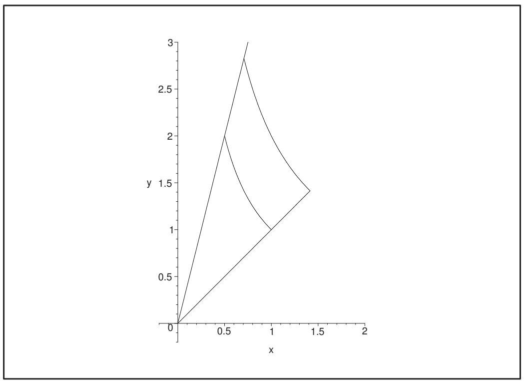
$$y = x, \qquad y = 4x, \qquad xy = 1, \qquad xy = 2.$$

Describe B in polar coordinates and then compute the plane integral

$$\int_B x^2 \exp(xy)\ln\left(\frac{y}{x}\right) dS.$$

A Plane integral reduced by polar coordinates.

D Sketch B. Then describe B in polar coordinates.



I In polar coordinates the line $y = 4x$ is described by $\varphi = \text{ Arctan } 4$, and the line $y = x$ by

$$\varphi = \text{ Arctan } 1 = \frac{\pi}{4}.$$

Since

$$xy = \varrho^2 \sin\varphi \cos\varphi,$$

the hyperbola $xy = 1$ is described by

$$\varrho = \frac{1}{\sqrt{\sin\varphi\cos\varphi}}, \qquad \varphi \in \left[\frac{\pi}{4}, \text{Arctan } 4\right],$$

and the hyperbola $xy = 2$ by

$$\varrho = \frac{2}{\sqrt{\sin\varphi\cos\varphi}}, \qquad \varphi \in \left[\frac{\pi}{4}, \text{Arctan } 4\right].$$

Summarizing we get by the reduction of the plane integral in polar coordinates that

$$\begin{aligned}
&\int_B x^2 \exp(xy) \ln\left(\frac{y}{x}\right) dS \\
(1) \qquad &= \int_{\text{Arctan }1}^{\text{Arctan }4} \left\{ \int_{1/\sqrt{\sin\varphi\cos\varphi}}^{2/\sqrt{\sin\varphi\cos\varphi}} \varrho^2 \cos^2\varphi \cdot \exp\left(\varrho^2 \sin\varphi\cos\varphi\right) \ln(\tan\varphi) \varrho \, d\varrho \right\} d\varphi.
\end{aligned}$$

First compute the inner integral by using the substitution $t = \varrho^2 \sin\varphi \cos\varphi$, where φ is kept fixed. This gives

$$\begin{aligned}
&\int_{1/\sqrt{\sin\varphi\cos\varphi}}^{2/\sqrt{\sin\varphi\cos\varphi}} \varrho^2 \cos^2\varphi \cdot \exp\left(\varrho^2 \sin\varphi\cos\varphi\right) \ln(\tan\varphi) \varrho \, d\varrho \\
&\quad = \frac{1}{2} \frac{\ln(\tan\varphi)}{\sin^2\varphi} \int_1^2 t\, e^t \, dt = \frac{1}{2} \frac{\ln(\tan\varphi)}{\sin^2\varphi} \left[t\, e^t - e^t\right]_1^2 = \frac{e^2}{2} \frac{\ln(\tan\varphi)}{\sin^2\varphi}.
\end{aligned}$$

When this result is put into (1), it follows by the substitution $u = \tan\varphi$ that

$$\begin{aligned}
\int_B x^2 \exp(xy) \ln\left(\frac{y}{x}\right) dS &= \frac{e^2}{2} \int_{\text{Arctan }1}^{\text{Arctan }4} \frac{\ln(\tan\varphi)}{\sin^2\varphi} \, d\varphi \\
&= \frac{e^2}{2} \int_{\text{Arccot }1}^{\text{Arccot }\frac{1}{4}} (+\ln(\cot\varphi)) \cdot \frac{-1}{\sin^2\varphi} \, d\varphi \\
&= \frac{e^2}{2} \int_1^{\frac{1}{4}} \ln u \, du = \frac{e^2}{2} \left[u \ln u - u\right]_1^{\frac{1}{4}} \\
&= \frac{e^2}{2} \left(\frac{1}{4} \cdot 2 \ln \frac{1}{2} - \frac{1}{4} + 1 \right) = \frac{e^2}{8} (3 - 2\ln 2).
\end{aligned}$$

ALTERNATIVELY one may introduce the new variables

$$(u, v) = \left(xy, \frac{y}{x}\right).$$

This transformation is considered in all details in Example 5.2, so we shall just mention the main points, namely

$$D = \{(u,v) \mid 1 \le u \le 2,\, 1 \le v \le 4\} = [1,2] \times [1,4],$$

and

$$x(u,v) = \sqrt{\frac{u}{v}} \qquad \text{and} \qquad y(u,v) = \sqrt{uv},$$

and that the Jacobian is $\dfrac{1}{2v}$.

By the transformation of the plane integral

$$\int_B x^2 \exp(xy) \ln\left(\frac{y}{x}\right) dS = \int_D \frac{u}{v} \cdot e^u \ln v \cdot \frac{1}{2v}\, dudv = \frac{1}{2}\int_1^2 ue^u du \cdot \int_1^4 \frac{1}{v^2} \ln v\, dv$$

$$= \frac{1}{2}\left[ue^u - e\right]_1^2 \cdot \left[-\frac{\ln v}{v} - \frac{1}{v}\right]_1^4 = \frac{1}{2}\, e^2 \left\{1 - \frac{\ln 4}{4} - \frac{1}{4}\right\} = \frac{e^2}{8}\,(3 - 2\ln 2),$$

which is far easier than the method above. $\Diamond$

Example 2.5 *Find the domain D of the function*

$$f(x,y)=\sqrt{a^2-x^2-y^2},$$

where a is a positive constant. Then compute the plane integral

$$\int_D \{f(x,y)\}^2\,dx\,dy.$$

A Domain of a function, plane integral.

D Analyze f. Compute the plane integral by using polar coordinates.

I It follows immediately that

$$D=\{(x,y)\mid x^2+y^2\le a^2\}=\overline{K}(\mathbf{0};a),$$

and

$$\begin{aligned}\int_D \{f(x,y)\}^2\,dx\,dy &= \int_{\overline{K}(\mathbf{0};a)}\{a^2-x^2-y^2\}\,dx\,dy\\ &= a^2\cdot\text{area}(\overline{K}(\mathbf{0};a))-2\pi\int_0^a \varrho^2\cdot\varrho\,d\varrho = a^2\cdot\pi a^2-2\pi\cdot\frac{a^4}{4}=\frac{\pi}{2}\,a^4.\end{aligned}$$

Example 2.6 *Let the point set B be given by*

$$B=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; 0\le x\le\frac{\pi}{4},\, x\le y\le\frac{1}{\cos x}\right\}.$$

Find the value of the plane integral

$$\int_B y\,dS.$$

A Plane integral.

D Sketch the domain B and reduce to a double integral.

I By the reduction to a double integral we get

$$\begin{aligned}\int_B y\,dS &= \int_0^{\frac{\pi}{4}}\left\{\int_x^{1/\cos x} y\,dy\right\}dx=\int_0^{\frac{\pi}{4}}\left[\frac{1}{2}\,y^2\right]_x^{1/\cos x}dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}\left\{\frac{1}{\cos^2 x}-x^2\right\}dx\\ &= \frac{1}{2}\left[\tan x-\frac{1}{3}\,x^3\right]_0^{\frac{\pi}{4}}=\frac{1}{2}\left\{1-\frac{1}{3}\cdot\frac{\pi^3}{64}\right\}=\frac{1}{2}-\frac{\pi^3}{384}.\end{aligned}$$

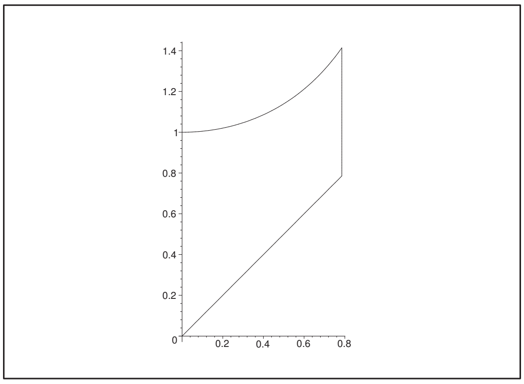


Figure 36: The domain B.

Example 2.7 *Compute the plane integral*

$$\int_B yx^2\, dS,$$

where B er is the quarter disc given by the inequalities

$$1 \le x, \qquad 0 \le y, \qquad x^2 + y^2 \le 2x.$$

A Plane integral.

D There are at least three different solutions:

1) Reduction in rectangular coordinates.
2) Reduction in polar coordinates.
3) Reduction in a translated polar coordinate system.

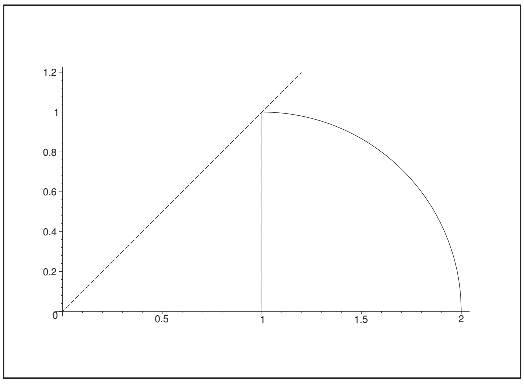


Figure 37: The quarter disc B.

I First method. *Reduction in rectangular coordinates.*

The set B is described in rectangular coordinates by

$$B = \{(x,y) \mid 0 \le y \le \sqrt{2x-x^2},\, x \in [1,2]\}.$$

Hence

$$\begin{aligned}\int_B yx^2\,dS &= \int_1^2 \left\{\int_0^{\sqrt{2x-x^2}} yx^2\,dy\right\} dx = \frac{1}{2}\int_1^2 x^2\{2x-x^2\}\,dx = \frac{1}{2}\int_1^2 \left\{2x^3-x^4\right\}\,dx \\ &= \frac{1}{2}\left[\frac{x^4}{2}-\frac{x^5}{5}\right]_1^2 = \frac{1}{4}\{16-1\} - \frac{1}{10}\{32-1\} = \frac{75-62}{20} = \frac{13}{20}.\end{aligned}$$

Second method. *Reduction in polar coordinates.*

It follows from the figure that every point in B lies in the angular space $\varphi \in \left[0,\frac{\pi}{4}\right]$ (den dotted oblique line). We get the lower ϱ-limit from $a \le x = \varrho\cos\varphi$,

$$\frac{1}{\cos\varphi} \le \varrho.$$

From $\varrho^2 = x^2+y^2 \le 2x = 2\varrho\,\cos\varphi$ we get the upper ϱ-limit $\varrho \le 2\cos\varphi$.

Summarizing, B is described in polar coordinates by

$$\left\{(\varrho,\varphi)\;\middle|\; \frac{1}{\cos\varphi} \le \varrho \le 2\cos\varphi,\, \varphi\in\left[0,\frac{\pi}{4}\right]\right\}.$$

Hence by reduction in polar coordinates,

$$\begin{aligned}\int_B yx^2\,dS &= \int_0^{\frac{\pi}{4}}\left\{\int_{\frac{1}{\cos\varphi}}^{2\cos\varphi} \varrho\sin\varphi\cdot\{\varrho\cos\varphi\}^2\cdot\varrho\,d\varrho\right\}d\varphi \\ &= \int_0^{\frac{\pi}{4}} \sin\varphi\cdot\cos^2\varphi\left\{\int_{\frac{1}{\cos\varphi}}^{2\cos\varphi}\varrho^4\,d\varrho\right\}d\varphi = \int_0^{\frac{\pi}{4}}\sin\varphi\cdot\cos^2\varphi\left[\frac{1}{5}\varrho^5\right]_{\frac{1}{\cos\varphi}}^{2\cos\varphi}d\varphi \\ &= \frac{1}{5}\int_0^{\frac{\pi}{4}}\left\{32\cos^7\varphi - \frac{1}{\cos^3\varphi}\right\}\sin\varphi\,d\varphi = \frac{1}{5}\left[-32\cdot\frac{1}{8}\cos^8\varphi - \frac{1}{2}\cdot\frac{1}{\cos^2\varphi}\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{5}\left\{4\left(-\cos^8\frac{\pi}{4}+1\right)+\frac{1}{2}\left(-\frac{1}{\cos^2\frac{\pi}{2}}+1\right)\right\} \\ &= \frac{1}{5}\left\{4\left(-\frac{1}{16}+1\right)+\frac{1}{2}(-2+1)\right\} = \frac{1}{5}\left\{\frac{15}{4}-\frac{1}{2}\right\} = \frac{1}{5}\cdot\frac{13}{4} = \frac{13}{20}.\end{aligned}$$

Third method. *Translated polar coordinate system.*

As $x^2+y^2 \le 2x$ can also be written $(x-1)^2+y^2 \le 1$, the set B can be described by

$$\left\{(x,y)\;\middle|\; x = 1+\varrho\cos\varphi,\, y=\varrho\sin\varphi,\, \varrho\in[0,1],\, \varphi\in\left[0,\frac{\pi}{2}\right]\right\},$$

where the pole lies in $(x, y) = (1, 0)$. Then we get the plane integral

$$\begin{aligned}
\int_B yx^2\,dS &= \int_0^{\frac{\pi}{2}} \left\{ \int_0^1 \varrho \sin\varphi \cdot \{1+\varrho\cos\varphi\}^2 \varrho\,d\varrho \right\} d\varphi \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_0^1 \varrho^2 \left\{1 + 2\varrho\cos\varphi + \varrho^2\cos^2\varphi\right\} d\varrho \right\} \sin\varphi\,d\varphi \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{\varrho^3}{3} + \frac{\varrho^4}{2}\cos\varphi + \frac{\varrho^5}{5}\cos^2\varphi \right]_{\varrho=0}^1 \sin\varphi\,d\varphi \\
&= \int_0^{\frac{\pi}{2}} \left\{ \frac{1}{3} + \frac{1}{2}\cos\varphi + \frac{1}{5}\cos^2\varphi \right\} \sin\varphi\,d\varphi \\
&= \left[-\frac{1}{3}\cos\varphi - \frac{1}{4}\cos^2\varphi - \frac{1}{15}\cos^3\varphi \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{3} + \frac{1}{4} + \frac{1}{15} = \frac{1}{4} + \frac{6}{15} = \frac{1}{4} + \frac{2}{5} = \frac{13}{20}.
\end{aligned}$$

3 Area

Example 3.1 *Let A be the plane point set which in polar coordinates is bounded by the inequalities*

$$-\pi \le \varphi \le \pi, \qquad 0 \le \varrho \le 1+\cos\varphi;$$

the boundary curve ∂A is a cardioid. Let B be the disc which is bounded by $0 \le \varrho \le 1$. Find the area of the intersection $A \cap B$.

A Area of a set given in polar coordinates.

D Sketch the boundary curves. Then set up the integrals of the area and compute.

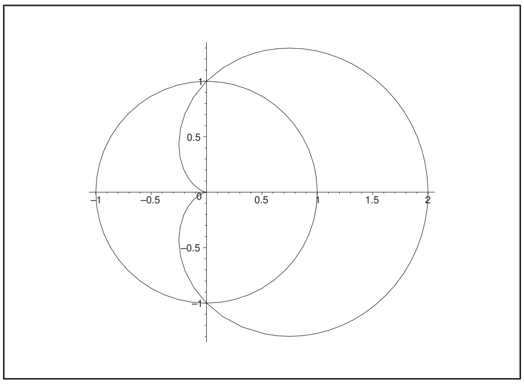


Figure 38: The intersection of the unit disc and the cardioid.

I By examining the figure we set up the formula of the area where we have a half disc in the right half plane,

$$\begin{aligned}
\text{area}(A\cap B) &= \frac{1}{2}\,\pi\cdot 1^2 + 2\int_{\frac{\pi}{2}}^{\pi}\left\{\int_0^{1+\cos\varphi}\varrho\,d\varrho\right\}d\varphi = \frac{\pi}{2}+2\int_{\frac{\pi}{2}}^{\pi}\frac{1}{2}\,(1+\cos\varphi)^2\,d\varphi \\
&= \frac{\pi}{2}+\int_{\frac{\pi}{2}}^{\pi}\left\{1+2\cos\varphi+\frac{1}{2}\,(1+\cos 2\varphi)\right\}\,d\varphi \\
&= \frac{\pi}{2}+\frac{3\pi}{2}\cdot\frac{1}{2}+\left[2\sin\varphi\right]_{\frac{\pi}{2}}^{\pi}+\frac{1}{4}\left[\sin 2\varphi\right]_{\frac{\pi}{2}}^{\pi} = \frac{5\pi}{4}-2.
\end{aligned}$$

Example 3.2 *In each of the following cases a plane and bounded point set B is given by the boundary curve ∂B given in polar coordinates. Sketch B and find the area of B.*

1) The cardiod,

$$\varrho = a(1+\cos\varphi), \qquad \varphi \in [-\pi, \pi].$$

2) (A part of) Descartes's leaf,

$$\varrho = \frac{3a\sin\varphi\cos\varphi}{\sin^3\varphi + \cos^3\varphi}, \qquad \varphi \in \left[0, \frac{\pi}{2}\right].$$

3) (Part of) Maclaurin's trisectrix,

$$\varrho = 4a \cdot \cos\varphi - \frac{1}{\cos\varphi}, \qquad \varphi \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right].$$

A Sketches of curves given in polar coordinates. Area by a plane integral.

D Sketch the boundary curve. Then apply the theorem of reduction.

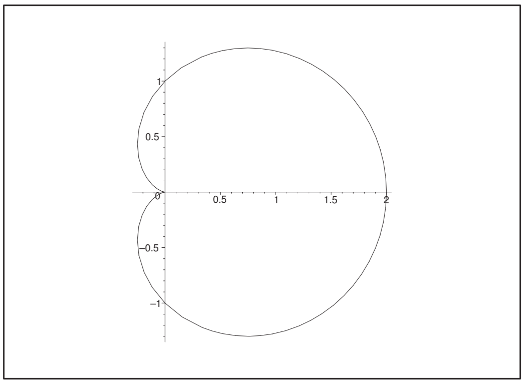


Figure 39: The cardioid.

I 1) Cardioid, from Greek "$\eta\ \kappa\alpha\varrho\delta\iota\alpha$ = the heart", because the curve has the shape of a heart. The area is given by

$$\begin{aligned}\int_B dS &= \int_{-\pi}^{\pi}\left\{\int_0^{a(1+\cos\varphi)} \varrho\, d\varrho\right\} d\varphi = \int_{-\pi}^{\pi} \frac{1}{2}\, a^2(1+\cos\varphi)^2\, d\varphi \\ &= \frac{1}{2}\, a^2 \int_{-\pi}^{\pi}\left\{1 + 2\cos\varphi + \frac{1+\cos 2\varphi}{2}\right\} d\varphi = \frac{1}{2}\, a^2 \cdot \frac{3}{2} \cdot 2\pi = \frac{3}{2}\, a^2\pi.\end{aligned}$$

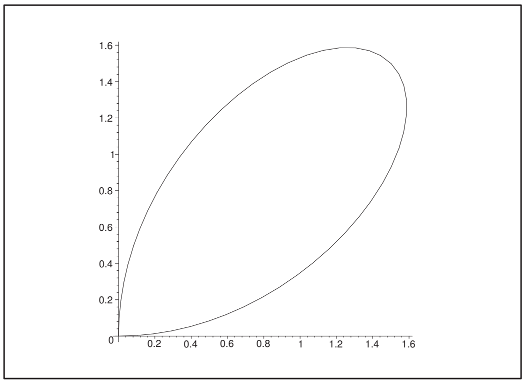


Figure 40: Part of Descartes's leaf.

2) The area is here computed in the following way

$$\begin{aligned}\int_B dS &= \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\frac{3a \sin\varphi \cos\varphi}{\sin^3\varphi + \cos^3\varphi}} \varrho \, d\varrho \right\} d\varphi = \frac{1}{2} \int_0^{\frac{\pi}{2}} 9a^2 \cdot \frac{\sin^2\varphi \cos^2\varphi}{(\sin^3\varphi + \cos^3\varphi)^2} \, d\varphi \\ &= \frac{9}{2} a^2 \int_0^{\frac{\pi}{2}} \frac{\tan^2\varphi \cdot \cos^4\varphi}{\cos^6\varphi (1 + \tan^3\varphi)^2} \, d\varphi = \frac{9a^2}{2} \int_{u=\tan\varphi=0}^{+\infty} \frac{u^2}{(1+u^3)^2} \, du \\ &= \frac{3}{2} a^2 \left[-\frac{1}{1+u^3} \right]_0^{+\infty} = \frac{3}{2} a^2.\end{aligned}$$

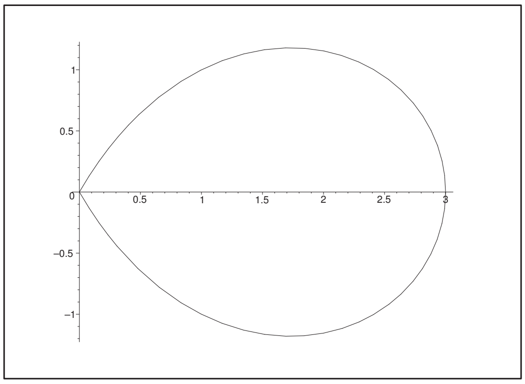


Figure 41: A part of Maclaurin's trisectrix.

3) By the usual reduction the area is here computed in the following way,

$$\begin{aligned}\int_B dS &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2} \left\{ \frac{a}{\cos\varphi} - 4a\cos\varphi \right\}^2 d\varphi = \frac{a^2}{2} \cdot 2 \int_0^{\frac{\pi}{3}} \left\{ \frac{1}{\cos^2\varphi} - 8 + 8 + 8\cos 2\varphi \right\} d\varphi \\ &= a^2 [\tan t + 4\sin 2\varphi]_0^{\frac{\pi}{3}} = a^2 \left\{ \tan\frac{\pi}{3} + 4\sin\frac{2\pi}{3} \right\} = a^2 \left(\sqrt{3} + 4 \cdot \frac{\sqrt{3}}{2} \right) = 3\sqrt{3}\, a^2.\end{aligned}$$

Example 3.3 *Find the area of the plane domain B, which is bounded by (i) a part of Archimedes's spiral given in polar coordinates by*

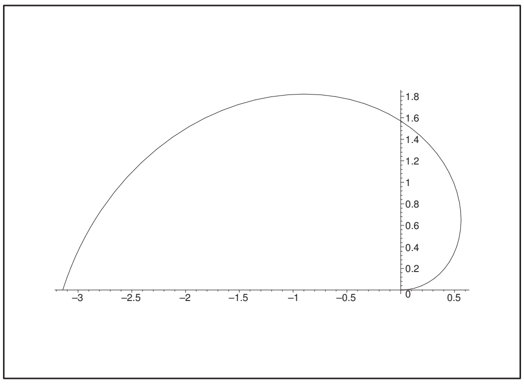
$$\varrho = a\varphi, \qquad \varphi \in [0, \pi],$$

and (ii) the part of the negative X-axis given by

$$(y = 0 \text{ and } x \in [-\pi a, 0]), \quad \text{or} \quad (\varphi = \pi \text{ and } \varrho \in [0, \pi a]).$$

A Area in polar coordinates.

D Sketch the domain; compute the area by reduction in polar coordinates.



I The area is

$$\int_B dS = \int_0^{\pi} \left\{ \int_0^{a\varphi} \varrho \, d\varrho \right\} d\varphi = \int_0^{\pi} \frac{1}{2}\, a^2\varphi^2 \, d\varphi = \frac{1}{6}\, a^2\pi^3.$$

4 Improper plane integral

Example 4.1 *Check in each of the following cases if the given improper plane integral over the bounded set B is convergent or divergent; indicate the value of the plane integral in case of convergency.*

1) $\int_B \frac{x+1}{\sqrt{x^2+y^2}}\,dS$, *where B is $\overline{K}((0,0);1)$.*

2) $\int_B \frac{\ln(x+2y)}{x^2}\,dS$, *where B is given by $0 \le x \le 1$ and $0 \le y \le \frac{x}{2}$.*

3) $\int_B \frac{1}{(1-x)(1+x+y)^2}\,dS$, *where B is given by $0 \le x$, $0 \le y$, $x+y \le 1$.*

4) $\int_B \frac{1}{x+y}\,dS$, *where B is $[0,1]\times[0,1]$.*

5) $\int_B \frac{1}{x+y}\,dS$, *where B is given by $0 \le x \le 1$ and $x \le y \le \sqrt{1+x^2}$.*

6) $\int_B \frac{y}{x^2+y^2}\,dS$, *where B is given by $0 \le x \le 1$ and $x \le y \le \sqrt{1+x^2}$.*

7) $\int_B \frac{x^2-y^2}{(x^2+y^2)^2}\,dS$, *where B is $\overline{K}((0,0);1)$.*

8) $\int_B \frac{1}{\sqrt{1-xy}}\,dS$, *where B is $[0,1]\times[0,1]$.*

9) $\int_B \frac{1}{x^2+y}\,dS$, *where B is given by $0 \le x \le 1$, and $x^2 \le y \le x$.*

10) $\int_B \ln\left(\frac{1}{x^2+y^2}\right)dS$, *where B is $\overline{K}((0,0);1)$.*

11) $\int_B \ln(1-x^2-y^2)\,dS$, *where B is $\overline{K}((0,0);1)$.*

12) $\int_B \frac{1}{\sqrt{x+y-1}}\,dS$, *where B is the triangle of the vertices $(1,0)$, $(1,1)$ and $(0,1)$.*

A Improper plane integrals.

D Sketch the domain. Indicate where the integrand is not defined. Then check where the function is positive and where it is negative. Truncate in a suitable way and check if the limit exists. Notice that if the integrand has the same sign everywhere in a dotted neighbourhood of a critical point, then the truncation can be very simple, which does not have to be "small" in geometry (but of course in area). If this is not the case one must be far more careful and split into the positive and the negative part of the function.

I 1) The domain is the closed unit disc. The integrand is not defined at $(0,0)$, and it is otherwise positive (or zero) in $B \setminus \{(0,0)\}$.
Let $B_r = \overline{K}((0,0);1) \setminus K((0,0);r)$, where $0 < r < 1$. Then B_r is described in polar coordinates by

$$0 \le \varphi \le 2\pi, \qquad 0 < r \le \varrho \le 1.$$

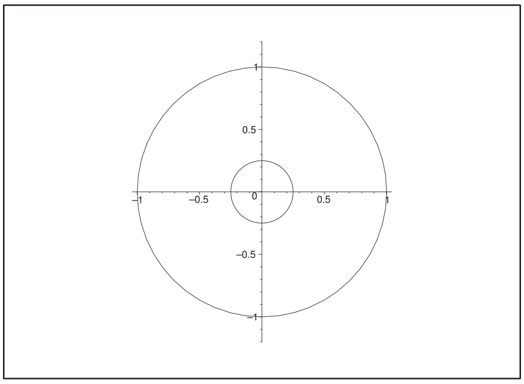


Figure 42: The truncation of **Example 4.1.1**.

Then by the theorem of reduction in polar coordinates followed by interchanging the order of integration,

$$\begin{aligned}\int_{B_r} \frac{x+1}{\sqrt{x^2+y^2}}\,dS &= \int_0^{2\pi}\left\{\int_r^1 \frac{\varrho\cos\varphi+1}{\varrho}\,\varrho\,d\varrho\right\}d\varphi = \int_r^1\left\{\int_0^{2\pi}(\varrho\cos\varphi+1)d\varphi\right\}d\varrho \\ &= 2\pi\int_r^1 d\varrho = 2\pi(1-r),\end{aligned}$$

which clearly has a limit for $r \to 0+$, thus the improper plane integral exists and it has the value

$$\int_B \frac{x+1}{\sqrt{x^2+y^2}}\,dS = \lim_{r\to 0+}\int_{B_r}\frac{x+1}{\sqrt{x^2+y^2}}\,dS = \lim_{r\to 0+} 2\pi(1-r) = 2\pi.$$

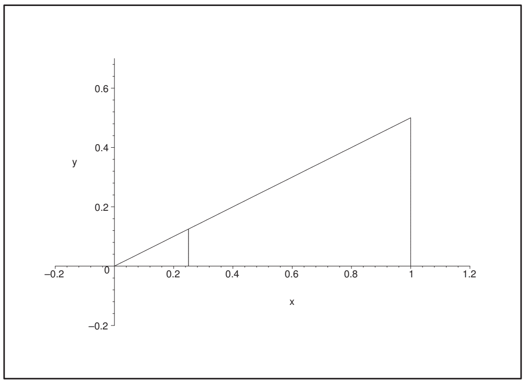


Figure 43: The truncation of **Example 4.1.2**.

2) The integrand is not defined at $(0,0) \in B$. Note that the integrand is negative elsewhere, i.e. it has a fixed sign when $(x,y) \in B$ and $0 < x < \dfrac{1}{2}$. Thus we can choose the truncation

$$B_t = \{(x,y) \in B \mid x \geq t\}, \qquad 0 < t < \frac{1}{2}.$$

We then get by integration over B_t,

$$\int_{B_t} \frac{\ln(x+2y)}{x^2}\,dS = \int_t^1 \frac{1}{x^2}\left\{\int_0^{\frac{x}{2}} \ln(x+2y)\,dy\right\}dx = \int_t^1 \frac{1}{2x^2}\left\{\int_0^x \ln(x+u)du\right\}\,dx$$

$$= \int_t^1 \frac{1}{2x^2}[(x+u)\ln(x+u)-(x+u)]_{u=0}^x dx$$

$$= \int_t^1 \frac{1}{2x^2}\{3x\ln(2x)-2x-x\ln x+x\}dx = \int_t^1 \frac{1}{2x^2}\{2x\ln 2 + x\ln x - x\}dx$$

$$= \int_t^1 \left\{\frac{\ln 2}{x} - \frac{1}{2x} + \frac{\ln x}{2x}\right\}dx = -\frac{1}{2}(2\ln 2 - 1)\ln t - \frac{1}{4}(\ln t)^2,$$

which tends to $-\infty$ for $t \to 0+$, and the improper integral does not exist.

3) The domain is the well-known triangle in the first quadrant. This time the integrand is not defined at $(1,0) \in B$. It is positive in the rest of B. We choose the truncation

$$B_t = \{(x,y) \in B \mid x \leq t\}, \qquad 0 < t < 1.$$

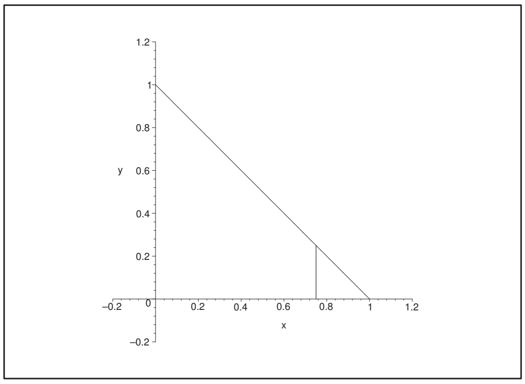


Figure 44: The truncation of **Example 4.1.3**.

Then by integration over B_t,

$$\int_{B_t} \frac{1}{(1+x+y)^2} \cdot \frac{1}{1-x}\, dS = \int_0^t \frac{1}{1-x} \left\{ \int_0^{1-x} \frac{1}{(1+x+y)^2}\, dy \right\} dx$$

$$= \int_0^t \frac{1}{1-x} \left[-\frac{1}{1+x+y} \right]_{y=0}^{1-x} dx = \int_0^t \frac{1}{1-x} \left\{ \frac{1}{1+x} - \frac{1}{2} \right\} dx$$

$$= \int_0^t \left\{ \frac{1}{2} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{1+x} - \frac{1}{2} \cdot \frac{1}{1-x} \right\} dx = \frac{1}{2} [\ln(1+x)]_0^t = \frac{1}{2}\, \ln(1+t).$$

The improper plane integral exists and it has the value

$$\int_B \frac{1}{(1+x+y)^2} \cdot \frac{1}{1-x}\, dS = \lim_{t \to 1-} \int_{B_t} \frac{1}{(1+x+y)^2} \cdot \frac{1}{1-x}\, dS = \lim_{t \to 1-} \frac{1}{2} \ln(1+t) = \frac{1}{2} \ln 2.$$

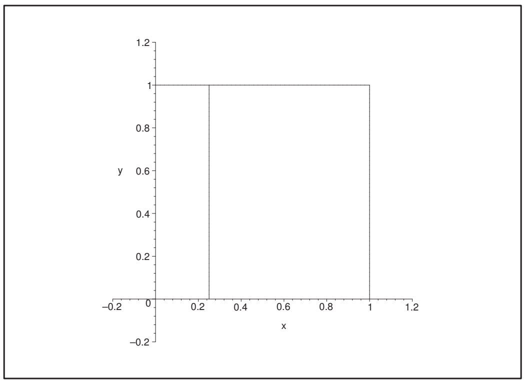


Figure 45: The truncation of **Example 4.1.4**.

4) The domain is the unit square. The integrand is not defined at $(0,0)$, and it is positive in the rest of B. We can therefore choose the geometrical rather “large” truncation

$$B_t = \{(x,y) \in B \mid x \geq t\}, \qquad 0 < t < 1.$$

The essential here is that the *area* of the removed domain tends to 0 for $t \to 0+$.

We get by integration over B_t,

$$\begin{aligned}\int_{B_t} \frac{1}{x+y}\,dS &= \int_t^1 \left\{\int_0^1 \frac{1}{x+y}\,dy\right\} dx = \int_t^1 \{\ln(1+x)-\ln x\}dx \\ &= [(x+1)\ln(1+x)-x-x\ln x+x]_t^1 = 2\ln 2-(t+1)\ln(t+1)+t\ln t.\end{aligned}$$

Due to the rules of magnitudes, $t\ln t \to 0$ for $t \to 0+$. Hence the improper plane integral exists, and it has the value

$$\int_B \frac{1}{x+y}\,dS = \lim_{t\to 0+}\int_{B_t}\frac{1}{x+y}\,dS = \lim_{t\to 0+}\{2\ln 2-(t+1)\ln(1+t)+t\ln t\} = 2\ln 2.$$

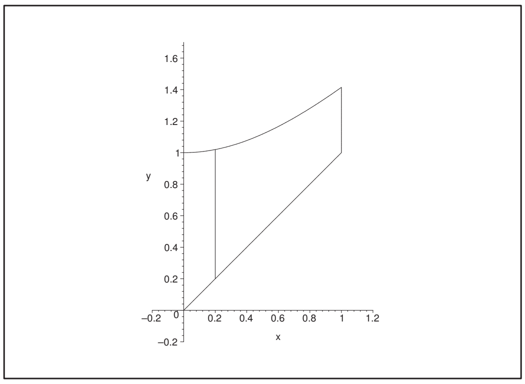


Figure 46: The truncation of **Example 4.1.5** and of **Example 4.1.6**.

5) We consider the same integrand (and the same singularity) as in **Example 4.1.4**. We can therefore choose the truncation

$$B_t = \{(x,y) \mid t \le x \le 1,\, x \le y \le \sqrt{1+x^2}\}, \qquad 0<t<1,$$

where we remove a strip along the Y-axis. Then by integrating over B_t,

$$\begin{aligned}\int_{B_t} \frac{1}{x+y}\,dS &= \int_t^1 \left\{\int_x^{\sqrt{1+x^2}} \frac{1}{x+y}\,dy\right\} dx = \int_t^1 \{\ln(x+\sqrt{1+x^2})-\ln 2-\ln x\}dx \\ &= \left[x\ln(x+\sqrt{1+x^2})\right]_t^1 - \int_t^1 \frac{x}{\sqrt{1+x^2}}\,dx - [\ln 2\cdot x + x\ln x - x]_t^1 \\ &= \ln 2 - t\ln(t+\sqrt{1+t^2}) - \left[\sqrt{1+t^2}\right]_t^1 + (1-\ln 2)\cdot(1-t) - t\ln t \\ &= 1-\sqrt{2}+\sqrt{1+t^2}-t\ln(t+\sqrt{1+t^2})-t\ln t - t(1-\ln 2).\end{aligned}$$

Due to the rules of magnitudes, $t\cdot\ln t \to 0-$ for $t \to 0+$, so we conclude that the improper plane integral exists and it has the value

$$\int_B \frac{1}{x+y}\,dS = \lim_{t\to 0+}\int_{B_t}\frac{1}{x+y}\,dS = 2-\sqrt{2}.$$

6) The domain is the same as in **Example 4.1.5**, and the integrand is not defined at $(0,0)$. The integrand is positive elsewhere, so we can choose the same truncation as in **Example 4.1.5**. Thus,

$$B_t = \{(x,y) \mid t \le x \le 1,\, x \le y \le \sqrt{1+x^2}\}, \qquad 0 < t < 1.$$

Then

$$\int_{B_t} \frac{y}{x^2+y^2}\, dS = \int_t^1 \left\{ \int_x^{\sqrt{1+x^2}} \frac{y}{x^2+y^2}\, dy \right\} dx = \int_t^1 \left[\frac{1}{2} \ln(x^2+y^2) \right]_{y=x}^{\sqrt{1+x^2}} dx$$

$$= \frac{1}{2} \int_t^1 \{\ln(1+2x^2) - \ln 2 - 2\ln x\} dx$$

$$= \frac{1}{2} \left[x \ln(1+2x^2) \right]_t^1 - \frac{1}{2} \int_t^1 \frac{4x^2}{1+2x^2}\, dx - \frac{1}{2} \ln 2 \cdot (1-t) - [x \ln x - x]_t^1$$

$$= \frac{1}{2} \ln 3 - \frac{1}{2}\, t \ln(1+2t^2) - \int_t^1 \frac{2x^2+1-1}{1+2x^2}\, dx - \frac{1}{2} \ln 2 \cdot (1-t) + 1 + t \ln t - t$$

$$= \frac{1}{2} \ln \frac{3}{2} + 1 - \frac{1}{2}\, t \ln(1+2t^2) + \frac{1}{2} \ln 2 \cdot t + t \ln t - t - 1 + t + \left[\frac{1}{\sqrt{2}} \operatorname{Arctan}(\sqrt{2}x) \right]_t^1$$

$$= \frac{1}{2} \ln \frac{3}{2} + \frac{1}{\sqrt{2}} \operatorname{Arctan} \sqrt{2} - \frac{1}{2}\, t \ln(1+2t^2) + \frac{1}{2} \ln 2 \cdot t + t \cdot \ln t - \frac{1}{\sqrt{2}} \operatorname{Arctan}(\sqrt{2}t).$$

It follows by taking the limit that the improper plane integral exists and that it has the value

$$\int_B \frac{y}{x^2+y^2}\,dS = \lim_{t\to 0+}\int_{B_t}\frac{y}{x^2+y^2}\,dS = \frac{1}{2}\ln\frac{3}{2}+\frac{1}{\sqrt{2}}\,\text{Arctan}\,\sqrt{2}.$$

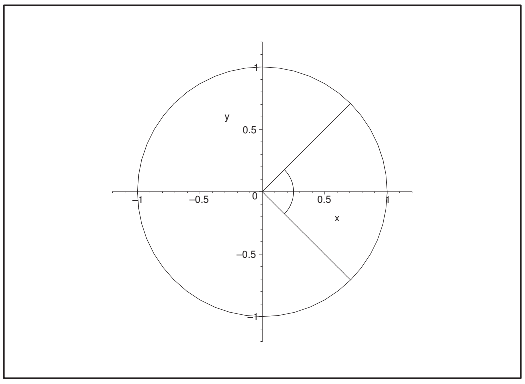


Figure 47: A subdomain with truncation in **Example 4.1.7**.

7) This is a vicious example which is constructed to mislead the reader to an erroneous conclusion.

The domain is the closed unit disc. The integrand is not defined at $(0,0)$. The integrand is both positive and negative in any dotted neighbourhood of $(0,0)$, so we *shall split it into a positive part and a negative part!* We shall here only consider the truncated quarter disc

$$B_r = \left\{(\varrho,\varphi)\;\middle|\; -\frac{\pi}{4}\le\varphi\le\frac{\pi}{4},\, r\le\varrho\le 1\right\},\qquad 0<r<1,$$

where the integrand is nonnegative. Then by a reduction in polar coordinates,

$$\begin{aligned}\int_{B_r}\frac{x^2-y^2}{(x^2+y^2)^2}\,dS &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left\{\int_r^1\frac{\varrho^2\cos 2\varphi}{\varrho^4}\,\varrho\,d\varrho\right\}d\varphi = \left[\frac{1}{2}\sin 2\varphi\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cdot[\ln\varrho]_r^1\\ &= \ln\frac{1}{r}\to+\infty\qquad\text{for } r\to 0+,\end{aligned}$$

so the improper plane integral does *not* exist.

WARNING. The careless reader may give the following solution: Choose the truncation

$$B_r' = \{(\varrho,\varphi)\mid 0\le\varphi\le 2\pi,\, r\le\varrho\le 1\},\qquad 0<r<1.$$

Then we have the following *correct* computation,

$$\int_{B_r'}\frac{x^2-y^2}{(x^2+y^2)^2}\,dS = \int_0^{2\pi}\cos 2\varphi\,d\varphi\cdot\int_r^1\frac{d\varrho}{\varrho} = 0,$$

for every $r\in\,]0,1[$.

However, by taking the limit we only get what may be called the *principal value*,

$$\text{vp.} \int_B \frac{x^2-y^2}{(x^2+y^2)^2}\,dS = \lim_{r\to 0+} \int_{B_r'} \frac{x^2-y^2}{(x^2+y^2)^2}\,dS = 0.$$

The improper plane integral does not exist as proved above, but the principal value does. In some cases one may even give this a physical interpretation. We shall, however, not pursue this aspect here. $\Diamond$

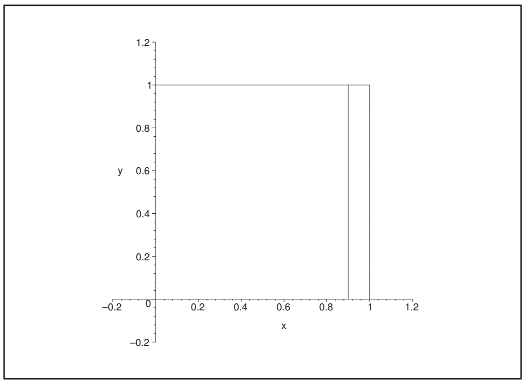


Figure 48: The truncation of **Example 4.1.8**.

8) The domain is the unit square, and the singular point is $(1,1)$. The integrand is positive elsewhere, so we may choose the truncation

$$B_t = \{(x,y) \mid 0 \le x \le t,\, 0 \le y \le 1\}, \qquad 0 < t < 1.$$

Then

$$\begin{aligned}\int_{B_t} \frac{1}{\sqrt{1-xy}}\,dS &= \int_0^t \left\{\int_0^1 \frac{1}{\sqrt{1-xy}}\,dy\right\}dx = \int_0^t \left[-\frac{2}{x}\sqrt{1-xy}\right]_{y=0}^1 dx \\ &= \int_0^t \frac{2}{x}\{1-\sqrt{1-x}\}dx = \int_0^t \frac{2}{1+\sqrt{1-x}}\,dx,\end{aligned}$$

which clearly has a limit for $t \to 1-$, because the latter integrand is continuous in all of $[0,1]$. Hence the improper integral exists. By introducing the substitution

$$u = \sqrt{1-x}, \qquad x = 1-u^2,$$

we finally obtain the value

$$\begin{aligned}\int_B \frac{1}{\sqrt{1-xy}}\,dS &= \int_0^1 \frac{2}{1+\sqrt{1-x}}\,dx = -\int_1^0 \frac{4u}{1+u}\,du \\ &= \int_0^1 \left\{4 - \frac{4}{1+u}\right\}du = 4 - 4\ln 2.\end{aligned}$$

9) The domain is bounded by the parabola $y = x^2$ and the line $y = x$. The integrand is not defined at $(0,0)$, and it is positive elsewhere in B. Choose the truncation

$$B_t = \{(x,y) \mid t \le x \le 1,\, x^2 \le y \le x\}, \qquad 0 < t < 1.$$

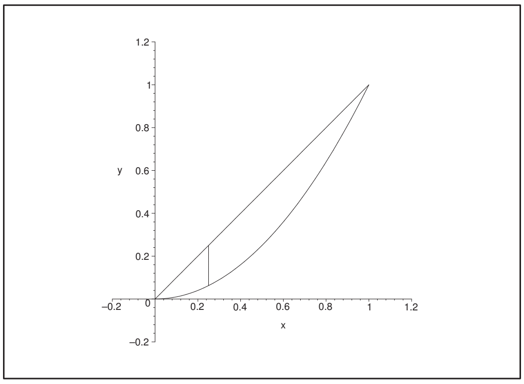


Figure 49: The truncation of **Example 4.1.9**.

Then

$$\begin{aligned}\int_{B_t} \frac{1}{x^2+y}\,dS &= \int_t^1 \left\{\int_{x^2}^x \frac{1}{x^2+y}\,dy\right\}dx = \int_t^1 \left[\ln(x^2+y)\right]_{y=x^2}^x dx \\ &= \int_t^1 \{\ln(x^2+x)-\ln(2x^2)\}\,dx = \int_t^1 \{\ln x+\ln(1+x)-\ln 2-2\ln x\}\,dx \\ &= \int_t^1 \{\ln(1+x)-\ln x-\ln 2\}\,dx = [(1+x)\ln(1+x)-x-x\ln x+x-\ln 2\cdot x]_t^1 \\ &= 2\ln 2-\ln 2-(1+t)\ln(1+t)+t\ln t+\ln 2\cdot t.\end{aligned}$$

This expression has a limit for $t \to 0+$, so we conclude that the improper plane integral exists and its value is given by

$$\int_B \frac{1}{x^2+y}\,dS = \lim_{t\to 0+}\int_{B_t}\frac{1}{x^2+y}\,dS = \ln 2.$$

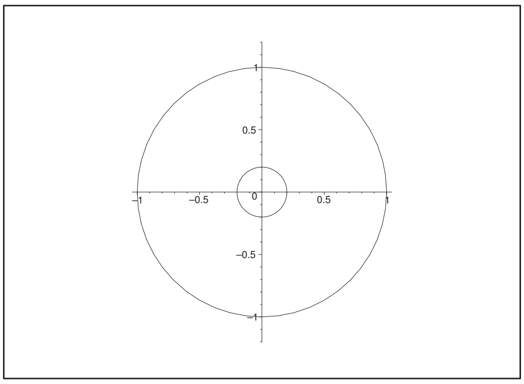


Figure 50: The truncation of **Example 4.1.10**.

10) The domain is the closed unit disc, where the integrand is not defined at $(0,0)$. The integrand

is otherwise positive. Choose the truncation

$$B_r = \{(\varrho, \varphi) \mid 0 \leq \varphi \leq 2\pi,\, r \leq \varrho \leq 1\}, \qquad 0 < r < 1.$$

Then

$$\begin{aligned} \int_{B_r} \ln\left(\frac{1}{x^2+y^2}\right) dS &= \int_0^{2\pi} \left\{ \int_r^1 -\ln(\varrho^2)\,\varrho\, d\varrho \right\} d\varphi = 2\pi \left[\frac{1}{2} \left\{ -\varrho^2 \ln(\varrho^2) + \varrho^2 \right\} \right]_r^1 \\ &= \pi + \pi r^2 \ln(r^2) - r^2. \end{aligned}$$

The improper plane integral exists and its value is

$$\int_B \ln\left(\frac{1}{x^2+y^2}\right) dS = \lim_{r \to 0+} \int_{B_r} \ln\left(\frac{1}{x^2+y^2}\right) dS = \pi.$$

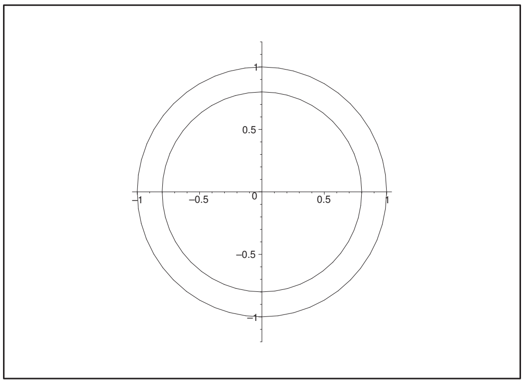


Figure 51: The truncation of **Example 4.1.11**.

11) The domain is again the closed unit disc. Here the integrand is only defined in the open unit disc, where it is ≤ 0 (fixed sign). Therefore we can choose the truncation in polar coordinates

$$B_r = \{(\varrho, \varphi) \mid 0 \leq \varphi \leq 2\pi,\, 0 \leq \varrho \leq r\}, \qquad 0 < r < 1.$$

Then

$$\begin{aligned}\int_{B_r} \ln(1-x^2-y^2)\,dS &= \int_0^{2\pi}\left\{\int_0^r \ln(1-\varrho^2)\cdot\varrho\,d\varrho\right\}d\varphi \\ &= 2\pi\int_{\varrho=0}^{r}\left\{-\frac{1}{2}\ln(1-\varrho^2)\right\}d(1-\varrho^2) \\ &= -\pi\left[(1-\varrho^2)\ln(1-\varrho^2)-(1-\varrho^2)\right]_0^r \\ &= -\pi\{(1-r^2)\ln(1-r^2)-(1-r^2)+1\}.\end{aligned}$$

This expression has a limit for $1 - r^2 \to 0+$. We conclude that the improper plane integral exists and it has the value

$$\int_B \ln(1-x^2-y^2)\,dS = \lim_{r\to 1-}\int_{B_r}\ln(1-x^2-y^2)\,dS = -\pi.$$

12) The integrand is positive everywhere in the interior of B. It tends to $+\infty$, when we are approaching the line $x + y = 1$ from above. Choose the truncation

$$B_t = \{(x, y) \mid 0 \leq x \leq 1,\, 0 \leq y \leq 1,\, x + y \geq t + 1\}, \qquad 0 < t < 1.$$

Then

$$\begin{aligned}\int_{B_t}\frac{1}{\sqrt{x+y-1}}\,dS &= \int_t^1\left\{\int_{1-y+t}^1\frac{dx}{\sqrt{x+y-1}}\right\}dy = 2\int_t^1\left[\sqrt{x+y-1}\right]_{x=1-y+t}^1 dy \\ &= 2\int_t^1\left\{\sqrt{y}-\sqrt{t}\right\}dy = 2\left[\frac{2}{3}\,y\sqrt{y}-y\sqrt{t}\right]_{x=1-y+t}^1 \\ &= 2\left\{\frac{2}{3}-\sqrt{t}-\frac{2}{3}\,t\sqrt{t}+t\sqrt{t}\right\} \to \frac{4}{3}\end{aligned}$$

for $t \to 0+$. We conclude that the improper plane integral is convergent with the value

$$\int_B\frac{1}{\sqrt{x+y-1}}\,dS = \lim_{t\to 0+}\int_{B_t}\frac{1}{\sqrt{x+y-1}}\,dS = \frac{4}{3}.$$

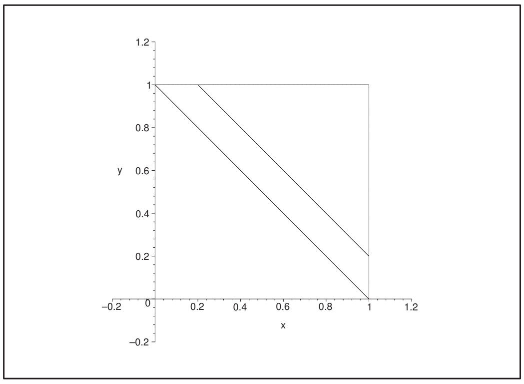


Figure 52: The truncation of **Example 4.1.12**.

Example 4.2 *Check in each of the following cases if the given improper plane integral over the unbounded point set B is convergent or divergent. In case of convergency find the value of the plane integral.*

1) $\int_B \frac{y}{x^2+x}\,dS$, *where B is given by $x \geq 1$ and $x \leq y \leq \sqrt{x^2+1}$.*

2) $\int_B \frac{1}{x+y}\,dS$, *where B is given by $x \geq 1$ and $\frac{1}{x} \leq y \leq x$.*

3) $\int_B \frac{y}{(1+x^2+y^2)^2}\,dS$, *where B is given by $y \geq 0$.*

4) $\int_B \ln(x^2+y^2)\,dS$, *where B is given by $x^2+y^2 \geq \frac{1}{4}$.*

5) $\int_B xy\exp(-x^2-y^2)\,dS$, *where $B = \mathbb{R}^2$.*

6) $\int_B xy\,dS$, *where $B = \mathbb{R}^2$.*

7) $\int_B x\exp(-(x+y))\,dS$, *where B is given by $0 \leq x < +\infty$ and $x \leq y$.*

8) $\int_B \exp(-|x|-y)\,dS$, *where B is given by $-\infty < x < +\infty$ and $y > |x|$.*

9) $\int_B x^2\exp(-yx^2-x)\,dS$, *where B is given by $x \geq 1$ and $0 \leq y \leq \frac{1}{x}$.*

10) $\int_B \frac{y^2}{1+x^2}\,dS$, *where B is given by $0 \leq x < +\infty$ and $0 \leq y \leq$ Arctan x.*

11) $\int_B \frac{x}{y(1+x^2)}\,dS$, *where B is given by $1 \leq y < +\infty$ and $0 \leq x \leq \sqrt{y-1}$.*

12) $\int_B \frac{1}{\sqrt{x+y-1}}\,dS$, *where B is the triangle of the vertices $(1,0)$, $(1,1)$ and $(0,1)$.*

A Improper plane integrals where the integrand is continuous and the domain is unbounded.

Note that **Example 4.2.12** is not at the right place, because the domain in this example is bounded, while the integrand is unbounded. For the same reason it is also given as **Example 4.1.12**.

D The integrands are defined in the given unbounded domains (sketch these). If the integrand is of fixed sign for $x^2 + y^2$ sufficiently large, we may choose an easy truncation. However, if the integrand is both positive and negative when $x^2 + y^2$ tends to infinity, we shall be more careful in our choice of truncation. At last check the limit $x^2 + y^2 \to +\infty$.

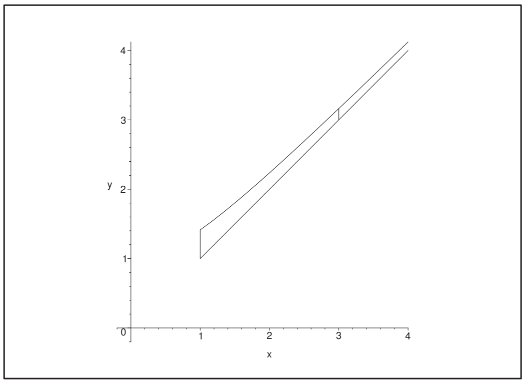


Figure 53: The truncation of **Example 4.2.1**.

I 1) The integrand is positive in all of the domain B, which lies in the first quadrant. We choose the truncation

$$B_t = \{(x,y) \mid 1 \le x \le t,\, x \le y \le \sqrt{x^2+1}\}, \qquad \text{for } t > 1.$$

Then

$$\begin{aligned}\int_{B_t} \frac{y}{x^2+x}\,dS &= \int_1^t \frac{1}{x(x+1)} \left\{\int_x^{\sqrt{x^2+1}} y\,dy\right\} dx = \int_1^t \frac{1}{x(x+1)} \left[\frac{1}{2}\,y^2\right]_{y=x}^{\sqrt{x^2+1}} dy \\ &= \frac{1}{2}\int_1^t \left\{\frac{1}{x} - \frac{1}{x+1}\right\} dx = \frac{1}{2}\left[\ln\left(\frac{x}{x+1}\right)\right]_1^t \\ &= \frac{1}{2}\ln 2 - \frac{1}{2}\ln\left(1+\frac{1}{t}\right).\end{aligned}$$

We conclude by taking the limit that the improper plane integral exists and that its value is given by

$$\int \frac{y}{x^2+x}\,dS = \lim_{t\to+\infty} \int_{B_t} \frac{y}{x^2+x}\,dS = \frac{1}{2}\ln 2.$$

2) The integrand is positive in all of the domain. We choose the truncation

$$B_t = \left\{(x,y) \;\middle|\; 1 \le x \le t,\, \frac{1}{x} \le y \le x\right\}, \qquad t > 1.$$

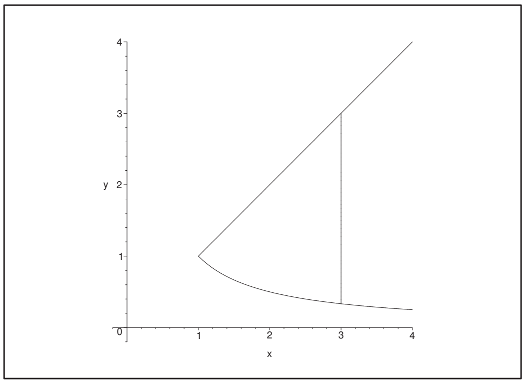


Figure 54: The truncation of **Example 4.2.2**.

When we integrate over B_t we get

$$\begin{aligned}
\int_{B_t} \frac{1}{x+y}\,dS &= \int_1^t \left\{\int_{\frac{1}{x}}^{x} \frac{1}{x+y}\,dy\right\} dx = \int_1^t [\ln(x+y)]_{y=\frac{1}{x}}^{x}\,dx \\
&= \int_1^t \left\{\ln(2x) - \ln\left(x+\frac{1}{x}\right)\right\} dx = \int_1^t \{\ln 2 + 2\ln x - \ln(1+x^2)\}\,dx \\
&= \ln 2\cdot(t-1) + 2[x\ln x - x]_1^t - \left[x\ln(1+x^2)\right]_1^t + 2\int_1^t \frac{x^2}{1+x^2}\,dx \\
&= \ln 2\cdot(t-1) + 2t\ln t - 2t + 2 - t\ln(1+t^2) \\
&\qquad + \ln 2 + 2\int_1^t \left(1 - \frac{1}{1+x^2}\right) dx \\
&= 2 + t\cdot\ln 2 + 2t\ln t - 2t - t\ln(1+t^2) + 2t - 2 - 2[\text{Arctan}\, x]_1^t \\
&= t\cdot\ln 2 - t\cdot\ln\left(1+\frac{1}{t^2}\right) - 2\,\text{Arctan}\, t + \frac{\pi}{2}.
\end{aligned}$$

Here

$$\text{Arctan}\, t \to \frac{\pi}{2} \quad \text{og} \quad t\ln\left(1+\frac{1}{t^2}\right) = t\left(\frac{1}{t^2} + \frac{1}{t^2}\varepsilon\left(\frac{1}{t}\right)\right) \to 0$$

for $t \to +\infty$, while $t\cdot\ln 2 \to +\infty$. Hence we conclude that the improper integral does not exist.

3) In this case the domain of integration is the upper half plane, and the integrand is ≥ 0 everywhere. Choose the truncation

$$B_{s,t} = \{(x,y) \mid -s \leq x \leq s,\, 0 \leq y \leq t\}, \qquad s,\, t > 0.$$

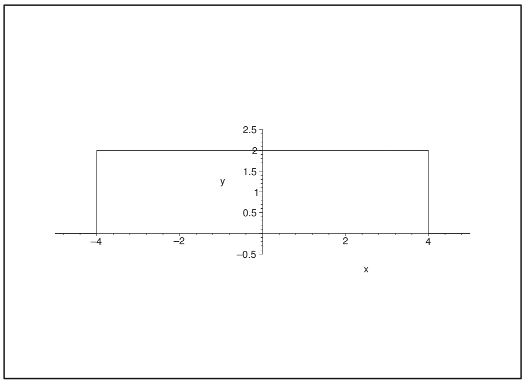


Figure 55: Example of a truncation in **Example 4.2.3**.

Then

$$\begin{aligned}\int_{B_{s,t}} \frac{y}{(1+x^2+y^2)^2}\,dS &= \int_{-s}^{s}\left\{\int_0^t \frac{y}{(1+x^2+y^2)^2}\,dy\right\} = 2\cdot\frac{1}{2}\int_0^s\left[-\frac{1}{1+x^2+y^2}\right]_{y=0}^{t} dx\\ &= \int_0^s\left\{\frac{1}{1+x^2}-\frac{1}{(1+t^2)+x^2}\right\}dx = \left[\text{Arctan}\, x - \frac{1}{\sqrt{1+t^2}}\,\text{Arctan}\left(\frac{x}{\sqrt{1+t^2}}\right)\right]_0^s\\ &= \text{Arctan}\, s - \frac{1}{\sqrt{1+t^2}}\,\text{Arctan}\left(\frac{s}{\sqrt{1+t^2}}\right).\end{aligned}$$

Here $\text{Arctan}\left(\dfrac{s}{\sqrt{1+t^2}}\right)$ is bounded, so we conclude that the improper plane integral exists and it has the value

$$\int_B \frac{y}{(1+x^2+y^2)^2}\,dS = \lim_{s\to+\infty}\lim_{t\to+\infty}\int_{B_{s,t}} \frac{y}{(1+x^2+y^2)^2}\,dS = \lim_{s\to+\infty}\ \text{Arctan}\, s = \frac{\pi}{2}.$$

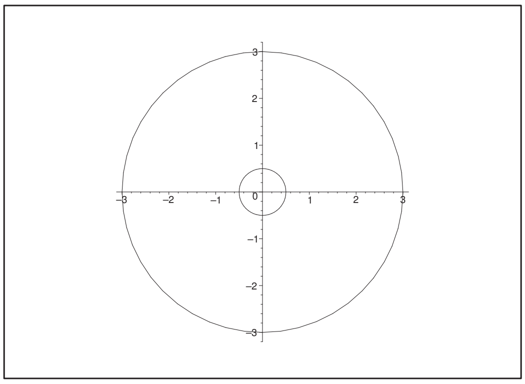


Figure 56: The truncation of **Example 4.2.4**.

4) The domain is the complementary set of a disc, and the integrand is positive for $x^2+y^2>1$.

We choose the following truncation (in polar coordinates)

$$B_r = \left\{ (\varrho, \varphi) \;\middle|\; 0 \le \varphi \le 2\pi,\, \frac{1}{2} \le \varrho \le r \right\}, \qquad r > 1.$$

Then

$$\begin{aligned}\int_{B_r} \ln(x^2+y^2)\, dS &= \int_0^{2\pi} \left\{ \int_{\frac{1}{2}}^{r} \ln(\varrho^2) \cdot \varrho \, d\varrho \right\} d\varphi = 2\pi \left[\frac{1}{2} \left(\varrho^2 \ln(\varrho^2) - \varrho^2 \right) \right]_{\frac{1}{2}}^{2} \\ &= \pi \left\{ r^2 \ln(r^2) - r^2 - \frac{1}{4} \ln \frac{1}{4} + \frac{1}{4} \right\} \to +\infty,\end{aligned}$$

for $x \to +\infty$, so the improper integral does not exist.

5) This is another vicious example. The point is that we can separate the variables, so (roughly speaking)

$$\begin{aligned}\int_B xy \exp(-x^2-y^2)\, dS &= \int_{-\infty}^{+\infty} x \exp(-x^2)\, dx \cdot \int_{-\infty}^{+\infty} y \exp(-y^2)\, dy \\ &= \left\{ \int_{-\infty}^{+\infty} t \exp(-t^2)\, dt \right\}^2 .\end{aligned}$$

Since

$$\int_{-\infty}^{\infty} t\,\exp(-t^2)\,dt = \int_{-\infty}^{0} t\exp(-t^2)\,dt + \int_{0}^{+\infty} t\exp(-t^2)\,dt,$$

where e.g.

$$\int_{0}^{+\infty} t\exp(-t^2)\,dt = \left[-\frac{1}{2}\,\exp(-t^2)\right]_0^{+\infty} = \frac{1}{2},$$

and similarly over the negative X-axis, the improper integral exists an it follows by the symmetry that

$$\int_B xy\exp(-x^2-y^2)\,dS = 0.$$

Remark. Note that we must argue of the convergency of the improper integral, otherwise the treatment of the example is not correct, even if one formally obtains the correct result 0. $\Diamond$

6) If we restrict the truncation $B_n^+ = [0,n]\times[0,n]$ only to the first quadrant, then clearly $xy \geq 0$ on B_n^+. The integral over B_n^+,

$$\int_{B_n^+} xy\,dS = \int_0^n x\,dx \cdot \int_0^n y\,dy = \left(\frac{n^2}{2}\right)^2 = \frac{n^4}{4},$$

tends to $+\infty$ for $n \to +\infty$. Hence it follows that the improper plane integral is divergent.

Remark. Notice that if one e.g uses the pocket calculator TI-92 with the command

$$\texttt{limit(}\int(\int(x*y,x,-n,n),y,-n,n),n,\infty),$$

then one gets the *wrong* result 0. In this case one cannot trust one's pocket calculator! $\Diamond$

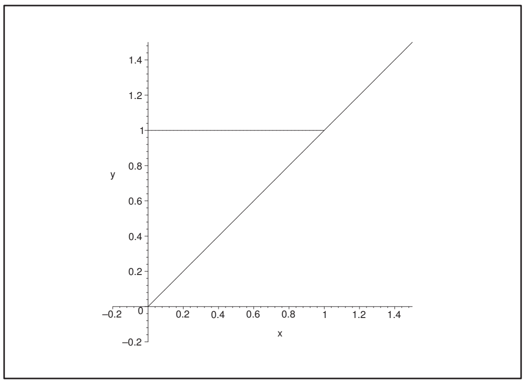


Figure 57: The truncated triangle of **Example 4.2.7**.

7) The integrand is positive everywhere, so we can use the truncation

$$B_t = \{(x,y) \mid 0 \leq y \leq t,\, 0 \leq x \leq y\}, \qquad t > 0.$$

The by integration,

$$\int_{B_t} x\exp(-(x+y))\,dS = \int_0^t \left\{\int_0^y xe^{-x}e^{-y}\,dx\right\}dy = \int_0^t e^{-y}\left\{\int_0^y xe^{-x}\,dx\right\}dy$$
$$= \int_0^t e^{-y}\left[-xe^{-x}-e^{-x}\right]_{x=0}^{y}dy = \int_0^t e^{-y}\left\{-ye^{-y}-e^{-y}+1\right\}dy$$
$$= \int_0^t \left\{e^{-y}-e^{-2y}-ye^{-2y}\right\}dy = \left[-e^{-y}+\frac{1}{2}\,e^{-2y}+\frac{1}{2}\,ye^{-2y}+\frac{1}{4}\,e^{-2y}\right]_0^t$$
$$= 1-\frac{1}{2}-\frac{1}{4}-e^{-t}-\frac{3}{4}\,e^{-2t}-\frac{1}{2}\,te^{-2t} = \frac{1}{4}-e^{-t}\left\{1+\frac{3}{4}\,e^{-t}+\frac{1}{2}\,te^{-t}\right\}.$$

We conclude by taking the limit $t \to +\infty$ that the improper integral exists and its value is given by

$$\int_B x\exp(-(x+y))\,dS = \lim_{t\to+\infty}\int_{B_t} x\exp(-(x+y))\,dS = \frac{1}{4}.$$

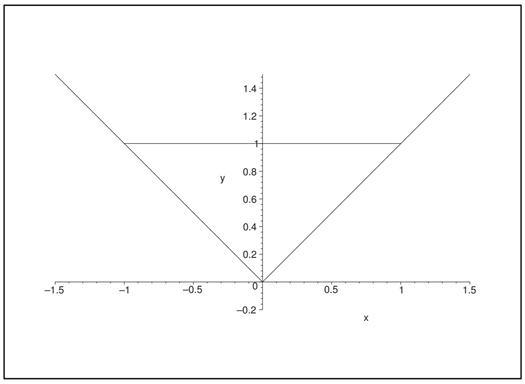


Figure 58: The truncated triangle of **Example 4.2.8**.

8) The integrand is positive in the domain of integration, so we can choose the truncation

$$B_t = \{(x,y) \mid |x| < y \le t\}, \qquad t > 0.$$

Then by integration over B_t,

$$\begin{aligned}\int_{B_t} \exp(-|x|-y)\,dS &= \int_0^t \left\{\int_{-y}^y e^{-|x|}e^{-y}\,dx\right\}dy = 2\int_0^t e^{-y}\left\{\int_0^y e^{-x}\,dx\right\}dy \\ &= 2\int_0^t e^{-y}(1-e^{-y})\,dy = 2\int_0^t (e^{-t}-e^{-2y})\,dy \\ &= 2(1-e^{-t})-(1-e^{-2t}) = 1-2e^{-t}+e^{-2t}.\end{aligned}$$

Taking the limit $t \to +\infty$ we conclude that the improper plane integral exists and that its value is

$$\int_B \exp(-|x|-y)\,dS = \lim_{t\to+\infty}\int_{B_t} \exp(-|x|-y)\,dS = 1.$$

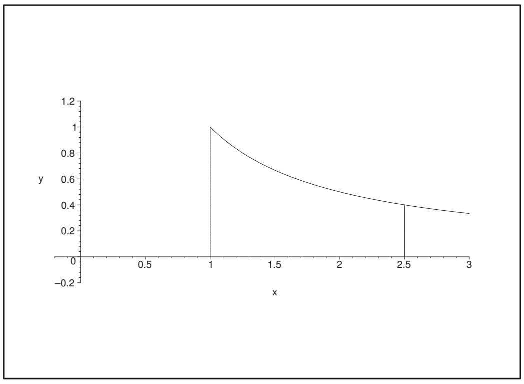


Figure 59: The truncation of **Example 4.2.9**.

9) The integrand is defined and ≥ 0 all over $\mathbb{R}^2$. We can choose the truncation

$$B_t = \left\{ (x,y) \;\middle|\; 1 \leq x \leq t,\, 0 \leq y \leq \frac{1}{x} \right\}, \qquad t > 1.$$

Then by the theorem of reduction for $t > 1$,

$$\begin{aligned}
\int_{B_t} x^2 \exp(-yx^2 - x)\, dS &= \int_1^t \left\{ \int_0^{\frac{1}{x}} x^2 \exp(-yx^2)\, e^{-x} dy \right\} dx \\
&= \int_1^t e^{-x} \left\{ \int_0^{\frac{1}{x}} \exp(-yx^2) x^2\, dy \right\} dx = \int_1^t e^{-x} \left[-\exp(-yx^2) \right]_{y=0}^{\frac{1}{x}} dx \\
&= \int_1^t e^{-x}(1 - e^{-x})\, dx = \int_1^t (e^{-x} - e^{-2x})\, dx = \left[-e^{-x} + \frac{1}{2}\, e^{-2x} \right]_1^t \\
&= \frac{1}{e} - \frac{1}{2e^2} - e^{-t} + \frac{1}{2}\, e^{-2t}.
\end{aligned}$$

Taking the limit $t \to +\infty$ we get

$$\lim_{t \to +\infty} \int_{B_t} x^2 \exp(-yx^2 - x)\, dS = \frac{1}{e} - \frac{1}{2e^2} = \frac{2e-1}{2e^2},$$

so we conclude that the improper plane integral is convergent with the value

$$\int_B x^2 \exp(-yx^2 - x)\, dS = \frac{2e-1}{2e^2}.$$

10) The integrand is positive everywhere in the unbounded domain. If we choose the truncation

$$B_t = \{(x,y) \mid 0 \leq x \leq t,\, 0 \leq y \leq \text{ Arctan } x\}, \qquad t > 0,$$

we get the integral

$$\begin{aligned}
\int_{B_t} \frac{y^2}{1+x^2}\, dS \quad &= \quad \int_0^t \frac{1}{1+x^2} \left\{ \int_0^{\text{Arctan } x} y^2\, dy \right\} dx = \frac{1}{3} \int_0^t \frac{\{\text{Arctan} x\}^3}{1+x^2}\, dx \\
&= \frac{1}{12} \{\text{Arctan } t\}^4 \to \frac{1}{12} \cdot \left(\frac{\pi}{2} \right)^4 = \frac{\pi^4}{192},
\end{aligned}$$

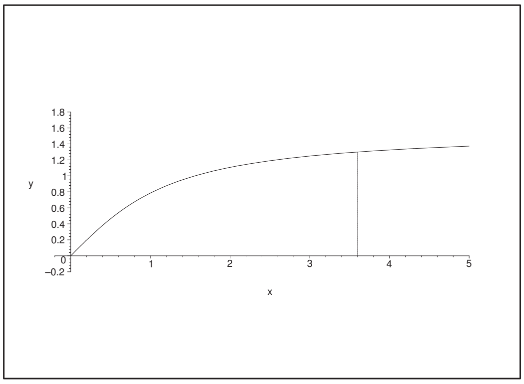


Figure 60: The truncation of the domain of **Example 4.2.10**.

and the improper plane integral is convergent with the value

$$\int_B \frac{y^2}{1+x^2}\,dS = \lim_{t\to+\infty}\int_{B_t}\frac{y^2}{1+x^2}\,dS = \frac{\pi^4}{192}.$$

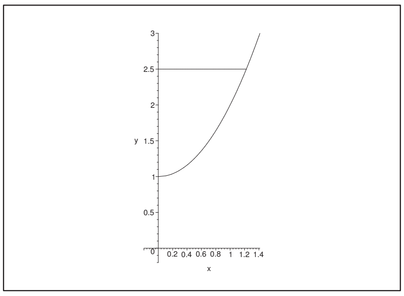


Figure 61: The truncation of **Example 4.2.11**.

11) The integrand is positive everywhere in the open and unbounded domain. We choose the truncation

$$B_t = \{(x,y) \mid 1 \le y \le t,\, 0 \le x \le \sqrt{y-1}\},$$

so

$$\begin{aligned}\int_{B_t}\frac{x}{y(1+x^2)}\,dS &= \int_1^t \frac{1}{y}\left\{\int_0^{\sqrt{y-1}}\frac{x}{1+x^2}\,dx\right\}dy = \frac{1}{2}\int_1^t\frac{1}{y}\left[\ln(1+x^2)\right]_0^{\sqrt{y-1}}dy \\ &= \frac{1}{2}\int_1^t \frac{1}{y}\ln y\,dy = \frac{1}{4}\{\ln t\}^2 \to +\infty\end{aligned}$$

for $t \to +\infty$. We conclude that the improper plane integral is divergent.

Example 4.3 *There is in each of the following cases given a plane integral, in which there enters a parameter $\alpha \in \mathbb{R}$. The integral is improper for some or every value of the parameter α. Let M_C and $M_D = \mathbb{R} \setminus M_C$ be sets of real numbers, such that the integral is convergent (or proper) for $\alpha \in M_C$ and divergent for $\alpha \in M_D$. Find in each of the cases M_C and M_D and the value of the integral for $\alpha \in M_C$.*

1) $\int_B \dfrac{\ln(x^2+y^2)}{\left(\sqrt{x^2+y^2}\right)^\alpha}\, dS$, *where* $B = \overline{K}((0,0);e)$.

2) $\int_{B(\alpha)} xy\, dS$, *where* $B(\alpha) = \{(x,y) \mid 1 \le x < +\infty,\, 0 \le y \le x^{-\alpha}\}$.

3) $\int_B y^\alpha\, dS$, *where* $B = \left\{(x,y) \;\middle|\; 1 \le x < +\infty,\, \dfrac{1}{x^2} \le y \le \dfrac{1}{x}\right\}$.

4) $\int_{B(\alpha)} \dfrac{1}{x}\, dS$, *where* $B(\alpha) = \{(x,y) \mid 0 \le x \le 1,\, 0 \le y \le x^{|\alpha|}\}$.

5) $\int_B y^\alpha\, dS$, *where* $B = \left\{(x,y) \;\middle|\; 1 \le x < +\infty,\, \dfrac{1}{x} \le y \le \dfrac{2}{x}\right\}$.

6) $\int_B \exp(-\alpha(x^2+y^2))\, dS$, *where* $B = \mathbb{R}^2$.

7) $\int_B (1-\cos(\alpha xy))\, dS$, *where* $B = \mathbb{R} \times [0,1]$.

A Improper plane integrals.

D Sketch the domain. Analyze where "there is something wrong". This happens typically when

1) the integrand is not defined,
2) the domain is unbounded,

but it may principally also be of another kind.

Calculate the following improper plane integrals, and find M_C. The the rest follows easily.

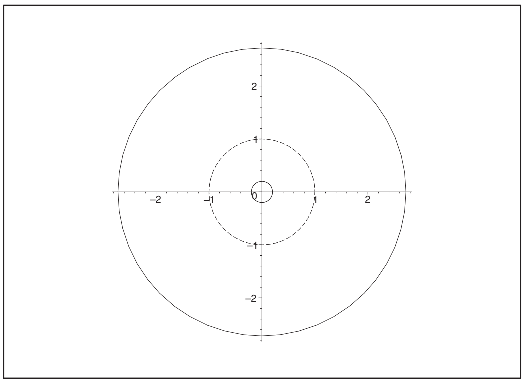


Figure 62: The truncated domain of **Example 4.3.1**.

I 1) The domain is the closed disc of centrum $(0,0)$ and radius e. The integrand is not defined at $(0,0)$. It is negative in every dotted neighbourhood of $(0,0)$ of radius < 1. Since the positive part, corresponding to $1 < \varrho \le e$, is finite, we may use the truncation

$$B_r = \{(\varrho,\varphi) \mid 0 \le \varphi \le 2\pi,\, r \le \varrho \le e\}, \qquad 0 < r < e.$$

In the computation of the following plane integral over B_r, we use a reduction in polar coordinates and the substitution $u = \ln \varrho$. Then

$$\begin{aligned}\int_{B_r} \frac{\ln(x^2+y^2)}{\left(\sqrt{x^2+y^2}\right)^{\alpha}}\, dS &= \int_0^{2\pi}\left\{\int_r^e \frac{\ln(\varrho^2)}{\varrho^{\alpha}}\cdot \varrho\, d\varrho\right\} d\varphi = 4\pi \int_r^e \frac{\ln \varrho}{\varrho^{\alpha-2}}\cdot\frac{1}{\varrho}\, d\varrho \\ &= 4\pi \int_{\ln r}^1 u\cdot e^{(2-\alpha)u}\, du.\end{aligned}$$

As $\ln r \to -\infty$ for $r \to 0+$, and as the integrand is monotonous for

$$u < \min\left\{0, \frac{1}{\alpha-2}\right\},$$

we must at least require that the integrand tends to 0 for $u \to -\infty$, which according to the rules of magnitudes gives the condition $2-\alpha > 0$. We conclude that

$$M_C \subseteq]-\infty, 2[.$$

Then let $\alpha < 2$. It follows from (2) that

$$\int_{B_r} \frac{\ln(x^2+y^2)}{\left(\sqrt{x^2+y^2}\right)^{\alpha}}\, dS = 4\pi \int_{\ln r}^{1} u \cdot e^{2-\alpha)u}\, du = 4\pi \left[\frac{1}{2-\alpha}\, u\, e^{(2-\alpha)u} - \frac{1}{(2-\alpha)^2}\, e^{(2-\alpha)u} \right]_{\ln r}^{1}$$

$$= 4\pi \left\{ \frac{(2-\alpha)-1}{(2-\alpha)^2}\, e^{2-\alpha} - \frac{1}{2-\alpha}\, r^{2-\alpha} \ln r - \frac{1}{(2-\alpha)^2}\, r^{2-\alpha} \right\}.$$

Using the rules of magnitudes once more we see that this expression converges for $r \to 0+$, hence

$$M_C =]-\infty, 2[\quad \text{og} \quad M_D = [2, +\infty[.$$

When $\alpha \in M_C =]-\infty, 2]$, we get the value of the improper plane integral

$$\int_B \frac{\ln(x^2+y^2)}{(\sqrt{x^2+y^2})^{\alpha}}\, dS = \lim_{r \to 0+} \int_{B_r} \frac{\ln(x^2+y^2)}{(\sqrt{x^2+y^2})^{\alpha}}\, dS = 4\pi \cdot \frac{1-\alpha}{(2-\alpha)^2}\, e^{2-\alpha}.$$

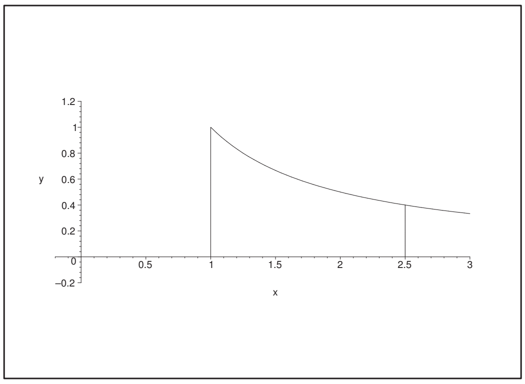


Figure 63: The truncated domain of **Example 4.3.2** for $\alpha = 1$.

2) The domain is unbounded. Since $xy \geq 0$ in $B(\alpha)$, we consider the truncation

$$B_t(\alpha) = \{(x,y) \mid 1 \leq x \leq t,\, 0 \leq y \leq x^{-\alpha}\}, \quad 1 < t < +\infty.$$

The corresponding plane integral is

$$\int_{B_t(\alpha)} xy\, dS = \int_1^t \left\{ \int_0^{x^{-\alpha}} xy\, dy \right\} dx = \frac{1}{2} \int_1^t x^{1-2\alpha}\, dx.$$

It is well-known that this integral converges for $t \to +\infty$, if and only if $1 - 2\alpha < -1$, i.e. if and only if $\alpha > 1$. It follows that

$$M_C =]1, +\infty[\quad \text{and} \quad M_D =]-\infty, 1].$$

For $\alpha \in M_C$, i.e. for $\alpha > 1$,

$$\int_{B(\alpha)} xy\,dS = \lim_{t\to+\infty}\int_{B_t(\alpha)} xy\,dS = \lim_{t\to+\infty}\left[\frac{x^{2-2\alpha}}{2(2-2\alpha)}\right]_1^t = \frac{1}{4(\alpha-1)}.$$

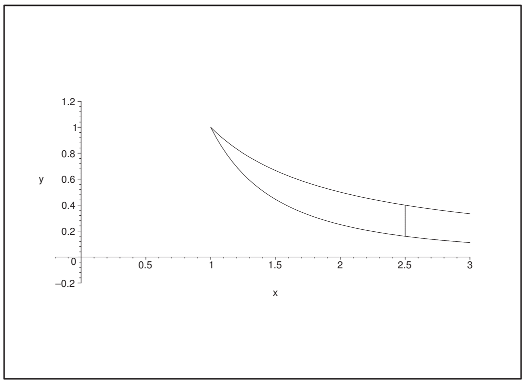


Figure 64: The truncation of **Example 4.3.3**.

3) In this case α enters the integrand and the domain is unbounded. The integrand is positive in B, so we can choose the truncation

$$B_t = \left\{(x,y)\;\middle|\; 1 \le x \le t,\, \frac{1}{x^2} \le y \le \frac{1}{x}\right\}, \qquad t > 1.$$

When $\alpha \neq 1$ we get

$$\begin{aligned}\int_{B_t} y^\alpha\,dS &= \int_1^t \left\{\int_{\frac{1}{x^2}}^{\frac{1}{x}} y^\alpha\,dy\right\}dx = \int_1^t \left[\frac{1}{\alpha+1}\,y^{\alpha+1}\right]_{y=\frac{1}{x^2}}^{\frac{1}{x}} dx \\ &= \frac{1}{\alpha+1}\int_1^t \left\{x^{-\alpha-1} - x^{-2\alpha-2}\right\}dx.\end{aligned}$$

Furthermore, if $\alpha \neq 0$ and $\alpha \neq -\dfrac{1}{2}$, then

$$\begin{aligned}\int_{B_t} y^\alpha\,dS &= \frac{1}{\alpha+1}\left[-\frac{1}{\alpha}\,x^{-\alpha} + \frac{1}{2\alpha+1}\,x^{-2\alpha-1}\right]_1^t \\ &= \frac{1}{\alpha+1}\left\{\frac{1}{\alpha} - \frac{1}{2\alpha+1}\right\} + \frac{1}{\alpha+1}\left\{-\frac{1}{\alpha}\cdot\frac{1}{t^\alpha} + \frac{1}{2\alpha+1}\cdot\frac{1}{t^{2\alpha+1}}\right\}.\end{aligned}$$

This expression converges for $t \to +\infty$, if and only if $\alpha > 0$ (as $\alpha \neq 0$ was assumed in advance) and $2\alpha+1 > 0$, i.e. if and only if $\alpha > 0$. In this case,

$$\int_B y^\alpha\,dS = \lim_{t\to+\infty}\int_{B_t} y^\alpha\,dS = \frac{1}{\alpha+1}\left\{\frac{1}{\alpha} - \frac{1}{2\alpha+1}\right\} = \frac{2\alpha+1-\alpha}{\alpha(\alpha+1)(2\alpha+1)} = \frac{1}{\alpha(2\alpha+1)}.$$

In the exceptional case $\alpha = 0$,

$$\begin{aligned}\int_{B_t} y^\alpha\,dS &= \int_{B_t} dS = \int_1^t \left\{\int_{\frac{1}{x^2}}^{\frac{1}{x}} dy\right\}dx = \int_1^t \left\{\frac{1}{x} - \frac{1}{x^2}\right\}dx = \left[\ln x + \frac{1}{x}\right]_1^t \\ &= \ln t + \frac{1}{t} - 1 \to +\infty, \quad t \to +\infty.\end{aligned}$$

We get in the other two exceptional cases $\alpha = -1$ and $\alpha = -\dfrac{1}{2}$,

$$y^\alpha \geq y^0 = 1,$$

because $0 < y < 1$ in B. Since already the case $\alpha = 0$ is divergent, we conclude that we have divergence in all three exceptional cases $\alpha = -1$, $-\frac{1}{2}$ and 0.

Summarizing,

$$M_C =]0, +\infty[\quad \text{and} \quad M_D =]-\infty, 0].$$

If $\alpha \in M_C$, i.e. $\alpha > 0$, then the value of the improper plane integral is

$$\int_B y^\alpha \, dS = \frac{1}{\alpha(2\alpha+1)}.$$

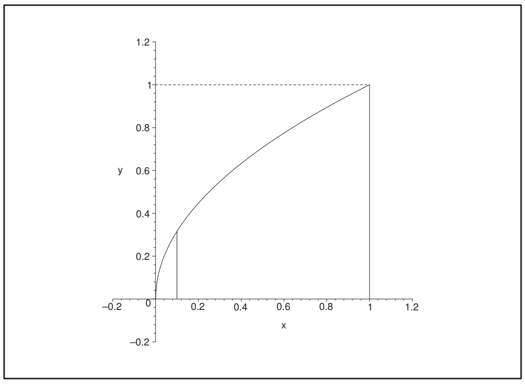


Figure 65: The truncation of **Example 4.3.4** in case of $\alpha = \pm 2$.

4) The domain is bounded for all $\alpha \in \mathbb{R}$. The integrand is positive in $B(\alpha) \setminus \{(0,0)\}$ and it is not defined at $(0,0)$. Therefore, we can choose the truncation

$$B_t(\alpha) = \{(x,y) \mid t \le x \le 1,\, 0 \le y \le x^{|\alpha|}\}, \qquad 0 < t < 1.$$

Then by a computation,

$$\int_{B_t(\alpha)} \frac{1}{x}\,dS = \int_t^1 \left\{ \int = 0^{x^{|\alpha|}} \frac{1}{x}\,dy \right\} dx = \int_t^1 x^{|\alpha|-1}\,dx = \begin{cases} \ln \dfrac{1}{t}, & \text{for } \alpha = 0, \\[2mm] \dfrac{1}{|\alpha|}(1 - t^{|\alpha|}), & \text{for } \alpha \neq 0. \end{cases}$$

When we take the limit $t \to +$ we see that this is divergent for $\alpha = 0$ and convergent for $\alpha \neq 0$. Then it follows that

$$M_C = \mathbb{R} \setminus \{0\} \quad \text{and} \quad M_D = \{0\}.$$

We have for $\alpha \in M_C$, i.e. for $\alpha \neq 0$,

$$\int_{B(\alpha)} \frac{1}{x}\,dS = \lim_{t \to 0+} \int_{B_t(\alpha)} \frac{1}{x}\,dS = \lim_{t \to 0+} \frac{1}{|\alpha|}(1 - t^{|\alpha|}) = \frac{1}{|\alpha|}.$$

5) Here B is unbounded and the integrand is defined and positive in B. Choosing the truncation

$$B_t = \left\{ (x,y) \;\middle|\; 1 \le x \le t,\, \frac{1}{x} \le y \le \frac{2}{x} \right\}, \qquad t > 1,$$

it follows by the theorem of reduction that the integral over B_t is given by

$$\int_{B_t} y^{\alpha}\,dS = \int_1^t \left\{ \int_{\frac{1}{x}}^{\frac{2}{x}} y^{\alpha}\,dy \right\} dx.$$

We get in particular for $\alpha = -1$,

$$\int_{B_t} y^{-1}\,dS = \int_1^t \left\{ \int_{\frac{1}{x}}^{\frac{2}{x}} \frac{1}{y}\,dy \right\} dx = \int_1^t \left(\ln \frac{1}{x} - \ln \frac{1}{x} \right) dx = (t-1)\ln 2,$$

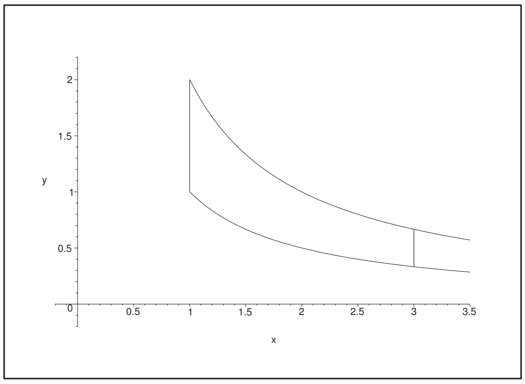


Figure 66: The truncation of **Example 4.3.5**.

which is divergent for $t \to +\infty$, hence $\alpha = -1 \in M_D$.

If $\alpha \neq -1$, then we get instead

$$\int_{B_t} y^\alpha \, dS = \int_1^t \left\{ \int_{\frac{1}{x}}^{\frac{2}{x}} y^\alpha \, dy \right\} dx = \int_1^t \frac{1}{\alpha+1} \left[y^{\alpha+1} \right]_{\frac{1}{x}}^{\frac{2}{x}} dx = \frac{2^{\alpha+1}-1}{\alpha+1} \int_1^t x^{-\alpha-1} \, dx.$$

If $\alpha = 0$, then in particular

$$\int_{B_t} dS = \frac{1}{0+1} \left(2^{0+1} - 1 \right) \int_1^t \frac{1}{x} \, dx = \ln t,$$

which is divergent for $t \to +\infty$, hence $\alpha = 0 \in M_D$.

If $\alpha \neq -1$ and $\alpha \neq 0$, then

$$\int_{B_t} y^\alpha \, dS = \frac{2^\alpha - 1}{\alpha+1} \int_1^t x^{-\alpha-1} \, dx = \frac{2^{\alpha+1}-1}{\alpha(\alpha+1)} \left(1 - t^{-\alpha}\right).$$

When $t \to +\infty$, this is divergent for $\alpha < 0$ and convergent for $\alpha > 0$.

Summarizing we see that $M_D = \mathbb{R} \setminus \mathbb{R}_+$ and $M_C = \mathbb{R}_+$.

If $\alpha \in M_C = \mathbb{R}_+$, then we get the value of the improper plane integral

$$\int_B y^\alpha \, dS = \lim_{t \to +\infty} \int_{B_t} y^\alpha \, dS = \frac{2^{\alpha+1}-1}{\alpha(\alpha+1)}, \qquad \alpha > 0.$$

6) When $\alpha < 0$, the integrand tends to $+\infty$ for $\sqrt{x^2+y^2} \to +\infty$, and when $\alpha = 0$, the integrand is a constant $= 1$. It follows that the improper integral is divergent for $\alpha \leq 0$.

Let $\alpha > 0$. Then the integrand is > 0 everywhere. By choosing the truncation

$$B_t = \{(x,y) \mid x^2+y^2 \leq t^2\} = B[\mathbf{0}, t], \qquad t > 0,$$

and using polar coordinates and the change of variables $u = \varrho^2$ we get

$$\begin{aligned}\int_{B_t} \exp(-\alpha(x^2+y^2))\,dS &= 2\pi \int_0^t \exp(-\alpha\varrho^2)\cdot \varrho\,d\varrho \\ &= \pi \int_0^{t^2} e^{-\alpha u}\,du = \frac{\pi}{\alpha}\left\{1-\exp(-\alpha t^2)\right\} \to \frac{\pi}{\alpha} \qquad \text{for } t \to +\infty.\end{aligned}$$

We conclude that

$$M_C =]0,+\infty[\qquad \text{and} \qquad M_D =]-\infty, 0].$$

If $\alpha \in M_C$, i.e. $\alpha > 0$, then

$$\int_{\mathbb{R}^2} \exp(-\alpha(x^2+y^2))\,dS = \frac{\pi}{\alpha}.$$

7) If $\alpha = 0$, the integrand is 0, so the improper plane integral is convergent for $\alpha = 0 \in M_C$. As $\cos(-\alpha xy) = \cos(\alpha xy)$ we may in the following restrict ourselves to $\alpha > 0$. The integrand is ≥ 0 everywhere, so it suffices with the truncation

$$B_t = [-t,t]\times[0,1], \qquad t > 0.$$

Then

$$\int_{B_t} \{1-\cos(\alpha xy))\,dS = 2t - \int_{-t}^{t} \left\{ \int_0^1 \cos(\alpha xy)\,dy \right\} dx = 2t - \int_{-t}^{t} \left[\frac{\sin(\alpha xy)}{\alpha x} \right]_0^1 dx$$

$$= 2t - 2\int_0^t \frac{\sin(\alpha xy)}{\alpha x}\,dx = 2t - \frac{2}{\alpha}\int_0^{\frac{t}{\alpha}} \frac{\sin x}{x}\,dx.$$

Note that if $t > \pi\alpha$, then

$$\left| \int_0^{\frac{t}{\alpha}} \frac{\sin x}{x}\,dx \right| \le \int_0^{\pi} \frac{\sin x}{x}\,dx < \int_0^{\pi} 1\,dx = \pi,$$

hence

$$\int_{B_t} (1-\cos(\alpha xy))\,dS \ge 2t - \frac{2\pi}{\alpha} \to +\infty \qquad \text{for } t \to +\infty$$

for every $\alpha > 0$, and thus also for every $\alpha < 0$. We conclude that

$$M_C = \{0\} \qquad \text{and} \qquad M_D = \mathbb{R} \setminus \{0\},$$

and that

$$\int_B (1-\cos(\alpha xy))\,dS = 0 \qquad \text{for } \alpha = 0.$$

Example 4.4 *Let* $B = \{(x,y) \mid 0 \le x \le y \le 1\}$*, and let*

$$f(x,y) = \frac{1}{1-x}.$$

Compute the plane integrals

$$\int_B y\,f(x,y)\,dS \qquad \text{and} \qquad \int_B y^2\,f(x,y)\,dS.$$

Which order of integration will give the shortest calculations?

A Improper plane integrals.

D Sketch the domain. Truncate it. Set up the double integrals and compute. Finally, take the limits.

I The domain B can be described in two ways,

$$B = \{(x,y) \mid 0 \le x \le 1,\, x \le y \le 1\} = \{(x,y) \mid 0 \le y \le 1,\, 0 \le x \le y\}.$$

The integrand is not defined at the point $(1,1) \in B$. It is positive in the rest of B. We have (at least) two possibilities of truncation,

$$B_t = \{(x,y) \mid 0 \le x \le t,\, x \le y \le 1\} \quad 0 < t < 1,$$

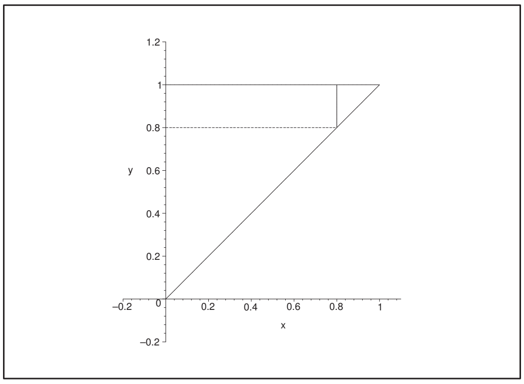


Figure 67: The domain with two possible truncations.

and

$$B_t^* = \{(x,y) \mid 0 \le y \le t,\, 0 \le x \le y\}, \quad 0 < t < 1.$$

Formally we have the following two possibilities for the improper double integrals,

$$(2) \quad \int_B y\, f(x,y)\, dS = \int_0^1 \frac{1}{1-x} \left\{ \int_x^{1-} y\, dy \right\} dx = \int_0^{1-} y \left\{ \int_0^y \frac{1}{1-x}\, dx \right\} dy.$$

By the computation we shall strictly speaking apply the truncation B_t in the first double integral, and the truncation B_t^* in the second one. However, if we write $1-$ in the bound of integration we indicate that the integral is calculated by taking a limit. The integrand is positive where it is defined, so the only thing which may go wrong is that the value of (2) becomes $+\infty$, which will immediately be seen. Therefore, we shall allow ourselves to be careless in the following and only write $1-$ instead of t with $\lim_{t\to 1-}$ in front of the integral.

When we consider the two possible double integrals of (2), the former one looks like the easiest one. Of course the same order of integrations in the double integral, when y is replaced by y^2, because it is the integral $\int_0^y \frac{1}{1-x}\, dx = -\ln(1-y)$, which is troublesome in the computations of the y-integral. We shall below demonstrate both possibilities.

1) The easy double integral:

$$\begin{aligned}
\int_0^{1-} \frac{1}{1-x} \left\{ \int_x^1 y\, dy \right\} dx &= \frac{1}{2} \int_0^{1-} \frac{1}{1-x} \left[y^2 \right]_{y=x}^1 dx = \frac{1}{2} \int_0^{1-} \frac{1-x^2}{1-x}\, dx \\
&= \frac{1}{2} \int_0^1 (1+x)\, dx = \frac{1}{2} \left[x + \frac{1}{2} x^2 \right]_0^1 = \frac{3}{4}.
\end{aligned}$$

2) The difficult double integral:

$$\int_0^{1-} y\left\{\int_0^y \frac{1}{1-x}\,dx\right\}dy = \int_0^{1-} y\{-\ln(1-y)\}dy$$

$$= -\lim_{y\to 1-}\left\{\frac{y^2}{2}\ln(1-y)+\int_0^y \frac{t^2}{2}\cdot\frac{1}{1-t}\,dt\right\}$$

$$= -\lim_{y\to 1-}\left\{\frac{y^2}{2}\ln(1-y)+\frac{1}{2}\int_0^y\left(-1-t+\frac{1}{1-t}\right)dt\right\}$$

$$= -\lim_{y\to 1-}\left\{\frac{y^2}{2}\ln(1-y)-\frac{1}{2}\,y-\frac{1}{4}\,y^2-\frac{1}{2}\ln(1-y)\right\}$$

$$= \lim_{y\to 1-}\left\{-\frac{(y-1)(y+1)\ln(1-y)}{2}\right\}+\frac{1}{2}+\frac{1}{4}=\frac{3}{4}.$$

Anyway, we get the same value by both methods.

Similarly, when we only show the easiest method,

$$\int_B y^2 f(x,y)\,dS = \int_0^{1-}\frac{1}{1-x}\left\{\int_x^1 y^2\,dy\right\}dx = \frac{1}{3}\int_0^{1-}\frac{1}{1-x}\left[y^3\right]_{y=x}^1 dx$$

$$= \frac{1}{3}\int_0^{1-}\frac{1-x^3}{1-x}\,dx = \frac{1}{3}\int_0^1\{1+x+x^2\}\,dx = \frac{1}{3}\left[x+\frac{x^2}{2}+\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right)=\frac{11}{18}.$$

Example 4.5 *It is well-known that*

$$\int_0^{+\infty} t^n e^{-t}\,dt = n!, \qquad n \in \mathbb{N}_0.$$

Now, let

$$B = \{(x,y) \mid 0 \le x \le y < +\infty\}$$

and

$$I = \int_B x^n (y-x)^m\, e^{-y}\,dS, \qquad m,\, n \in \mathbb{N}_0.$$

1) Prove that I is convergent with the value $n!\,m!$. (Truncate B by the lines $x=T$ and $y=T+x$, and then let T tend to plus infinity).

2) Truncate B by the line $y=T$. Let $T \to +\infty$ and find an expression of the integral

$$J = \int_0^1 t^n (1-t)^m\,dt, \qquad m,\, n \in \mathbb{N}_0.$$

In order to secure that the two integrals over $[0,+\infty[$ and $[0,1]$ exist it is not necessary to require that m and n are integers. It suffices to require that they are bigger than -1. On this basis one introduced the gamma function *and the* beta function *in the following way:*

$$\Gamma(\xi) = \int_0^{+\infty} t^{\xi-1}\, e^{-y}\,dt, \qquad \xi > 0,$$

and

$$B(\xi,\eta) = \int_0^1 t^{\xi-1}(1-t)^{\eta-1}\,dt, \qquad \xi>0, \quad \eta>0.$$

Notice that the exponents are written differently from the original integrals. In particular, $\Gamma(n+1)=n!$.

3. *Use the result of 2) to express $B(\xi,\eta)$ by means of the gamma function.*

A Improper plane integral; the gamma function and the beta function.

D Follow the guidelines.

I Just in case, we first prove that

$$(3) \quad \int_0^{+\infty} t^n\, e^{-t}\,dt = n!, \qquad n \in \mathbb{N}_0.$$

If $n=0$, then

$$\int_0^{+\infty} e^{-t}\,dt = \left[-e^{-t}\right]_0^{+\infty} = 1 = 0!.$$

If $n=1$, then

$$\int_0^{+\infty} t\,e^{-t}\,dt = \left[-t\,e^{-t} - e^{-t}\right]_0^{+\infty} = 1 = 1!.$$

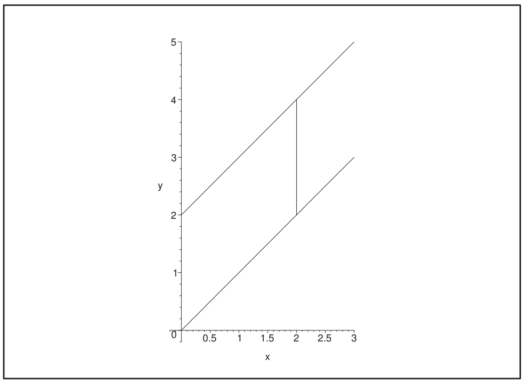


Figure 68: The truncation B_T.

Assume that (3) holds for some $n \in \mathbb{N}$. Then we get by partial integration

$$\int_0^{+\infty} t^{n+1}e^{-t}\,dt = \left[-t^{n+1}e^{-t}\right]_0^{+\infty} + (n+1)\int_0^{+\infty} t^n e^{-t}dt = 0 + (n+1)n! = (n+1)!$$

according to the assumption of induction. Then (3) follows by induction.

The domain

$$B = \{(x,y) \mid 0 \leq x \leq y < +\infty\} = \{(x,y) \mid 0 \leq x < +\infty,\, x \leq y < +\infty\}$$

is unbounded. The integrand $x^n(y-x)^m e^{-y}$ is ≥ 0 in B, so we can use the truncation

$$B_T = \{(x,y) \mid 0 \leq x \leq T,\, x \leq y \leq x+T\},$$

which catches every point of B, when $T \to +\infty$. In fact, every $(x,y) \in B$ lies in B_T, if only $T \geq T_0 = 2x$.

1) The shape of the domain invites one first to integrate with respect to y and then with respect to x. The plane integral over B_T becomes

$$\begin{aligned}\int_{B_T} x^n(y-x)^m e^{-y}\,dS &= \int_0^T x^n \left\{\int_x^{x+T} (y-x)^m e^{-y}\,dy\right\} dx \\ &= \int_0^T x^n e^{-x}\left\{\int_x^{x+T}(y-x)^m e^{-(y-x)}\,dy\right\} dx = \int_0^T x^n e^{-x}\left\{\int_0^T t^m e^{-t}\,dt\right\} dx.\end{aligned}$$

This implies according to (3) that

$$\begin{aligned} I &= \int_B x^n(y-x)^m e^{-y}\,dS = \lim_{T\to+\infty}\int_{B_T} x^n(y-x)^m e^{-y}\,dS \\ &= \lim_{T\to+\infty}\left\{\int_0^T x^n e^{-x}\,dx\right\}\left\{\int_0^T t^m e^{-t}\,dt\right\} = n!\,m!.\end{aligned}$$

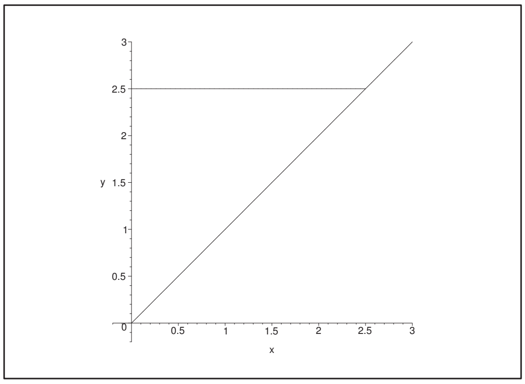


Figure 69: The truncation B_T'.

2) Next we truncate in the following way

$$B_T' = \{(x,y) \mid 0 \le x \le T,\, x \le y \le T\} = \{(x,y) \mid 0 \le y \le T,\, 0 \le x \le y\}.$$

Applying the substitution $y = ty$, $t \in [0,1]$, we get

$$\begin{aligned}\int_{B_T'} x^n(y-x)^m e^{-y}\,dS &= \int_0^T e^{-y}\left\{\int_0^y x^n(y-x)^m\,dx\right\}dy\\ &= \int_0^T e^{-y}y^{n+m+1}\left\{\int_0^1 t^n(1-t)^m\,dt\right\}dy = \int_0^1 t^n(1-t)^m\,dt\cdot\int_0^T e^{-y}y^{n+m+1}\,dy.\end{aligned}$$

By taking the limit $T \to +\infty$, followed by the result of 1) and (3), we get

$$\begin{aligned} n!\,m! &= \int_B x^n(y-x)^m e^{-y}\,dS = \lim_{T\to+\infty}\int_{B_T'} x^n(y-x)^m e^{-y}\,dS\\ &= (n+m+1)!\int_0^1 t^n(1-t)^m\,dt,\end{aligned}$$

hence

$$(4)\quad J = \int_0^1 t^n(1-t)^m\,dt = \frac{n!\,m!}{(n+m+1)!}.$$

3) A small consideration shows that the proofs of 1) and 2) carry over unchanged if only n, $m \ge 0$ are real. If instead $n > -1$ or $m > -1$, we need an extra standard consideration concerning the existence of the improper plane integral (truncation of the domain around $(0,0)$, etc.). The details are skipped here.

All this means that the result of 2) can now be written,

$$J = B(n+1,m+1) = \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)}, \qquad m,\, n > -1.$$

Putting $x = n+1$ and $y = m+1$ we get $n+m+2 = x+y$, thus

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x,\, y > 0.$$

Example 4.6 *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function, where the limit*

$$f(+\infty) = \lim_{x \to +\infty} f(x),$$

exists and is finite, and which also satisfies the condition

$$\int_1^{+\infty} \frac{f(x) - f(+\infty)}{x}\, dx \qquad \text{is convergent.}$$

Show by reducing the plane integral of the function $g(x,y) = f'(xy)$ over the rectangle $[0,c] \times [a,b]$ in two ways and then let c tend towards $+\infty$ that

$$\int_+^{+\infty} \frac{f(ax) - f(bx)}{x}\, dx = \{f(0) - f(+\infty)\} \ln\left(\frac{a}{b}\right).$$

Note that there do not exist a theorem which makes it possible to let c tend towards $+\infty$ under the sign of integration. *However, the result can be obtained by some rearrangements, by which one finally can apply the given condition.*

A Improper integral computed by means of an improper plane integral.

D Follow the guidelines. Be in particular careful with the limit.

I By the reduction of the plane integral we get on one hand that

$$\int_{[0,c]\times[a,b]} f'(xy)\, dS = \int_0^c \left\{ \int_a^b f'(xy)\, dy \right\} dx = \int_0^c \left[\frac{f(xy)}{x} \right]_{y=a}^b dx = \int_0^c \frac{f(bx) - f(ax)}{x}\, dx.$$

We note that the integrand of the latter integral can be extended continuously to $x = 0$, e.g. by L'Hôpital's rule,

$$\lim_{x \to 0} \frac{f(bx) - f(ax)}{x} = \lim_{x \to 0} \frac{bf'(bx) - af'(ax)}{1} = (b-a)f'(0),$$

which can be used as the value of the integrand at $x = 0$.

On the other hand, we get by interchanging the order of integration,

$$\int_{[0,c]\times[a,b]} f'(xy)\, dS = \int_a^b \left\{ \int_0^c f'(xy)\, dx \right\} dy = \int_a^b \left[\frac{f(xy)}{y} \right]_{x=0}^c dy = \int_a^b \frac{f(ct) - f(0)}{y}\, dy.$$

When we identify the two expressions of change their signs we get

$$\int_0^c \frac{f(ac) - f(bx)}{x}\, dx = \int_a^b \frac{f(0) - f(cy)}{y}\, dy.$$

Here we cannot take the limit $c \to +\infty$ on the right hand side. Instead we rewrite it in the following way,

$$\begin{aligned} \int_a^b \frac{f(0) - f(cy)}{y}\, dy &= \int_a^b \frac{f(0) - f(+\infty)}{y}\, dy + \int_a^b \frac{f(+\infty) - f(cy)}{y}\, dy \\ &= \{f(0) - f(+\infty)\} \ln \frac{b}{a} - \int_a^b \frac{f(cy) - f(+\infty)}{cy}\, c\, dy \qquad \text{(where } t = cy\text{)} \\ &\{f(0) - f(+\infty)\} \ln \frac{b}{a} - \int_{ca}^{cb} \frac{f(t) - f(+\infty)}{t}\, dt, \end{aligned}$$

i.e.

$$(5)\quad \int_0^c \frac{f(ax)-f(bx)}{x}\,dx = \{f(0)-f(+\infty)\}\ln\frac{b}{a} - \int_{ca}^{cb}\frac{f(t)-f(+\infty)}{t}\,dt.$$

We shall now prove that the right hand side of (5) is convergent for $c \to +\infty$. The first term is constant, so we shall consider the latter integral. By the assumptions,

$$\int_1^{+\infty}\frac{f(t)-f(+\infty)}{t}\,dt \qquad \text{convergent.}$$

This means that

$$\lim_{k\to+\infty}\int_k^{+\infty}\frac{f(t)-f(+\infty)}{t}\,dt = 0.$$

By the definition of convergency there exists to every $\varepsilon > 0$ a $k(\varepsilon) \geq 1$, such that

$$\left|\int_k^{+\infty}\frac{f(t)-f(+\infty)}{t}\,dt\right| < \frac{\varepsilon}{2} \qquad \text{for every } k \geq k(\varepsilon).$$

If c satisfies $k(\varepsilon) \le ca < cb$, then we get the estimate

$$\left|\int_{ca}^{cb} \frac{f(t)-f(+\infty)}{t}\,dt\right| = \left|\int_{ca}^{+\infty} \frac{f(t)-f(+\infty)}{t}\,dt - \int_{cb}^{+\infty} \frac{f(t)-f(+\infty)}{t}\,dt\right|$$
$$\le \left|\int_{ca}^{+\infty} \frac{f(t)-f(+\infty)}{t}\,dt\right| + \left|\int_{cb}^{+\infty} \frac{f(t)-f(+\infty)}{t}\,dt\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This holds for every $\varepsilon > 0$, hence it follows that the right hand side of (5) is convergent for $c \to +\infty$, and the same must then be the case of the left hand side. Finally, by taking the limit.

$$\int_0^{+\infty} \frac{f(ax)-f(bx)}{x}\,dx = \{f(0) - f(+\infty)\}\ln\left(\frac{b}{a}\right)$$

as required.

Example 4.7 *Let*

$$B = \{(x,y) \mid x^2 + y^2 \le 4,\ y \ge |x|\}$$

and

$$f(x,y) = \frac{x^n + y^2}{y^4 + x^2y^2}, \qquad (x,y) \in B \setminus \{(0,0)\},$$

where n is an integer.

1. *Prove the inequality $f(x,y) \ge 0$ for all $(x,y) \in B \setminus \{(0,0)\}$.*

Then consider the improper plane integral

$$I = \int_B f(x,y)\,dx\,dy.$$

2. *Find the values of n, for which I is convergent.*

3. *Compute the value of I for $n = 3$.*

A Improper plane integral.

D Sketch B. Estimate $f(x,y)$. Check the improper plane integral.

I 1) From $y \ge |x|$ follows that $y^2 \ge |x|^2 \ge -x^n$, hence $x^n + y^2 \ge 0$, and we clearly have

$$y^4 + x^2y^2 = y^2(x^2+y^2) > 0 \qquad \text{for } (x,y) \in B \setminus \{(0,0)\}.$$

2) The domain B is best described in polar coordinates by

$$B = \left\{(\varrho,\varphi) \;\middle|\; 0 \le \varrho \le 2,\ \varphi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]\right\}.$$

Put for $\varepsilon \in]0,2[$,

$$B_\varepsilon = \left\{(\varrho,\varphi) \;\middle|\; \varepsilon \le \varrho \le 2,\ \varphi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]\right\}.$$

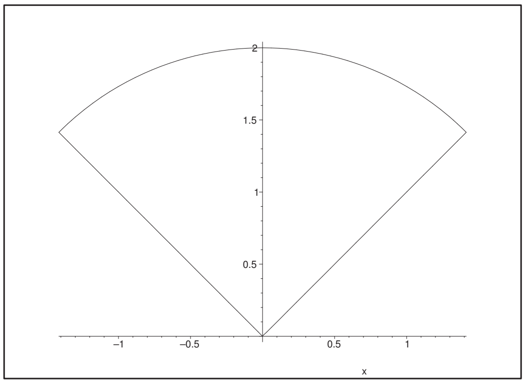


Figure 70: The domain B.

Then

$$\begin{aligned}\int_{B_\varepsilon} f(x,y)\,dx\,dy &= \int_{B_\varepsilon} \frac{\varrho^n}{\varrho^4}\cdot\frac{\cos^n\varphi+\sin^n\varphi}{\sin^4\varphi+\cos^2\varphi\,\sin^2\varphi}\,\varrho\,d\varphi\,d\varrho \\ &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\cos^n\varphi+\sin^n\varphi}{\sin^2\varphi}\,d\varphi\cdot\int_\varepsilon^2 \varrho^{n-3}\,d\varrho.\end{aligned}$$

The former integral exists for every $n \in \mathbb{N}$ independently of $\varepsilon > 0$, because $\sin^2\varphi \geq \dfrac{1}{2}$ for $\varphi \in \left[\dfrac{\pi}{4},\,\dfrac{3\pi}{4}\right]$.

Furthermore, the limit

$$\lim_{\varepsilon\to 0}\int_\varepsilon^2 \varrho^{n-3}\,d\varrho$$

exists and has a finite value if and only if $n-3 > -1$, i.e. if and only if $n > 2$, or put in another way, if and only if $n \in \mathbb{N}\setminus\{1,2\}$, since we require that $n \in \mathbb{N}$.

3) If $n = 3$, then it follows from the above that

$$\int_{B_\varepsilon} f(x,y)\,dx\,dy = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\cos^3\varphi+\sin^3\varphi}{\sin^2\varphi}\,d\varphi\cdot\int_\varepsilon^2 d\varrho,$$

thus

$$\begin{aligned}\int_B f(x,y)\,dx\,dy &= 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\cos^2\varphi}{\sin^2\varphi}\cdot\cos\varphi\,d\varphi + 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\varphi\,d\varphi \\ &= 0 + 2[-\cos\varphi]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 2\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}.\end{aligned}$$

Example 4.8 *Let B be the rectangle $[0,1]\times[-1,1]$. Show that the integral*

$$I=\int_B \frac{1}{1+xy}\,dS$$

is convergent and has the value $\dfrac{1}{4}\,\pi^2$.
HINT. *Replace y the new variable of integration u given by*

$$\sin u = y+\frac{1}{2}\,x\left(y^2-1\right), \qquad -\frac{\pi}{2}\le u\le\frac{\pi}{2}.$$

First integrate with respect to x and apply the formula

$$\tan\left(\frac{\pi}{4}+\frac{u}{2}\right)=\frac{1+\sin u}{\cos u}.$$

A Improper plane integral.

D Check that the integrand is >0, and the points in which it not defined. Prove that using (x,u) instead of (x,y) is a legal change of parameters. Explain the trigonometric formula, and apply the trick.

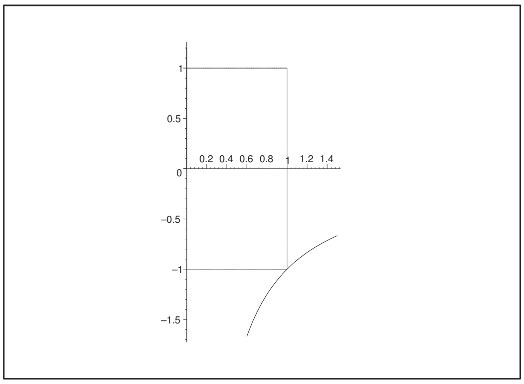


Figure 71: The domain B with the curve of singularities $y=-\dfrac{1}{x}$.

Clearly, $1+xy\ge 0$ i B, where $1+xy=0$ only at the point $(1,-1)\in B$. The integrand is >0 in $B\setminus\{(1,-1)\}$, so we can allow ourselves carelessly to skip the truncation. In fact, if the integral is divergent, this can only happen by getting

$$\int_B f(x,y)\,dS=+\infty,$$

which is immediately seen. This convention eases the solution of the task.

REMARK 1. By the traditional procedure one would e.g. get

$$I=\int_B\frac{1}{1+xy}\,dS=\int_{-1}^{1}\left\{\int_0^1\frac{1}{1+xy}\,dx\right\}dy=\int_{-1}^{1}\frac{1}{y}\,\ln(1+y)\,dy,$$

which does not look promising. It is here of no help to interchange the order of integration, because than one would get the even more incalculable expression

$$I = \int_0^1 \frac{1}{x} \ln\left(\frac{1+x}{1-x}\right) dx.$$

Therefore, we choose to use the hint. $\Diamond$

Remark 2. For the sake of completeness we here prove the trigonometric formula which is given in the hint. When $-\frac{\pi}{2} < u < \frac{\pi}{2}$, then

$$\begin{aligned}\tan\left(\frac{\pi}{4}+\frac{u}{2}\right) &= \frac{1+\tan\frac{u}{2}}{1-\tan\frac{u}{2}} = \frac{\cos\frac{u}{2}+\sin\frac{u}{2}}{\cos\frac{u}{2}-\sin\frac{u}{2}} = \frac{\left(\cos\frac{u}{2}+\sin\frac{u}{2}\right)^2}{\cos^2\frac{u}{2}-\sin^2\frac{u}{2}} \\ &= \frac{\cos^2\frac{u}{2}+\sin^2\frac{u}{2}+2\sin\frac{u}{2}\cdot\cos\frac{u}{2}}{\cos u} = \frac{1+\sin u}{\cos u},\end{aligned}$$

and the formula is proved. $\Diamond$

First notice that for every fixed $x \in [0,1]$,

$$\varphi(x,y) = y + \frac{1}{2}\,x\left(y^2-1\right), \qquad y \in [-1,1],$$

is increasing in y, because

$$\frac{\partial \varphi}{\partial y} = 1 + xy \geq 0 \qquad \text{i } B,$$

and this expression if only $= 0$ for $(x,y) = (1,-1) \in B$.

Since φ is continuous, the range is connected (first main theorem for continuous functions). Now, $\varphi(x,-1) = -1$ and $\varphi(x,1) = 1$, so the range is again B. This shows the legacy of introducing the transform of the coordinate,

$$u = \operatorname{Arcsin}\left(y + \frac{1}{2}\,x\left(y^2-1\right)\right), \qquad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}.$$

Remark 3. We shall not use this expression in the rest of the example. The essential thing here is only to assure that we can make the given change of variable. $\Diamond$

We now put

$$\sin u = y + \frac{1}{2}\,x\left(y^2-1\right), \qquad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}.$$

If $x = 0$ (a null set), then $y = \sin u$.
If $x > 0$ and $(x,y) \in B$, then

$$\frac{2}{x}\,\sin u = \frac{2}{x}\,y + y^2 - 1 = \left(y + \frac{1}{x}\right)^2 - \left(1 + \frac{1}{x^2}\right),$$

hence

$$y = -\frac{1}{x} + \frac{1}{x}\,\sqrt{1 + 2x\sin u + x^2} = \frac{\sqrt{1+2x\sin u + x^2}-1}{x},$$

because $|y| \leq 1$ and $0 < x \leq 1$ imply that we can only use one sign in front of the square root. Notice that

$$1 + 2x\sin u + x^2 = (x + \sin u)^2 + \cos^2 u,$$

so the square root is defined.

Now,

$$\frac{\partial y}{\partial u} = \frac{1}{x}\cdot\frac{1}{2}\,\frac{2x\cos u}{\sqrt{1+2x\sin u+x^2}} = \frac{\cos u}{\sqrt{1+2x\sin u + x^2}},$$

and the Jacobian of the change of variables is

$$\begin{vmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial u} \\[2mm] \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \star & \dfrac{\cos u}{\sqrt{1+2x\sin u+x^2}} \end{vmatrix} = \frac{\cos u}{\sqrt{1+2x\sin u + x^2}},$$

where $\star$ indicates that there is no need to compute $\dfrac{\partial y}{\partial x}$, because it shall later be multiplied by 0.

Finally,

$$1 + xy = 1 + \sqrt{1 + 2x \sin u + x^2} - 1 = \sqrt{1 + 2x \sin u + x^2}.$$

By insertion into the formula of changing variables we get(NOTICE: the integrals are still improper with a positive integrand),

$$\begin{aligned}
\int_B \frac{1}{1+xy}\,dS &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^1 \frac{1}{\sqrt{1+2x\sin u + x^2}} \cdot \frac{\cos u}{\sqrt{1+2x\sin u+x^2}}\,dx \right\} du \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^1 \frac{\cos u}{1+2x\sin u+x^2}\,dx \right\} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^1 \frac{\cos u}{(x+\sin u)^2+\cos^2}\,dx \right\} du \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \frac{1}{\cos u} \int_0^1 \frac{1}{1+\left(\dfrac{x+\sin u}{\cos u}\right)^2}\,dx \right\} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\operatorname{Arctan}\left(\frac{x+\sin u}{\cos u} \right) \right]_{x=0}^{1} du \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \operatorname{Arctan}\left(\frac{1+\sin u}{\cos u}\right) - \operatorname{Arctan}(\tan u) \right\} du.
\end{aligned}$$

If $\psi \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$, then $\operatorname{Arctan}(\tan\psi) = \psi$, hence the latter term of the integrand is $-u$.

If $u \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$, then $\dfrac{\pi}{4} + \dfrac{u}{2} \in \left]-\dfrac{\pi}{2}\right[$, hence

$$\operatorname{Arctan}\left(\frac{1+\sin u}{\cos u}\right) = \operatorname{Arctan}\left(\tan\left(\frac{\pi}{4}+\frac{u}{2}\right)\right) = \frac{\pi}{4}+\frac{u}{2}.$$

Then by insertion,

$$\int_B \frac{1}{1+xy}\,dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \frac{\pi}{4} + \frac{u}{2} - u \right\} du = \frac{\pi}{4}\cdot\pi - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\,du = \frac{\pi^2}{4},$$

as required.

REMARK 4. If we above first had integrated with respect to u (which could be tempting, considering the integrand), then we would get

$$\begin{aligned}
\int_B \frac{1}{1+xy}\,dS &= \int_0^1 \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos u}{1+2x\sin u+x^2}\,du \right\} dx = \int_0^1 \left[\frac{1}{2x} \ln\left(1+2x\sin u+x^2\right) \right]_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \\
&= \int_0^1 \frac{1}{2x} \ln\left(\frac{1+2x+x^2}{1-2x+x^2} \right) dx = \int_0^1 \frac{1}{x} \ln\left(\frac{1+x}{1-x}\right) dx,
\end{aligned}$$

and we had ended in the same dead end as before.

Note however, that if we compare the results of the "impossible" rearrangements of the expression which we have found, we have unawares proved that

$$\int_{-1}^1 \frac{1}{x}\ln(1+x)\,dx = \int_0^1 \frac{1}{x} \ln\left(\frac{1+x}{1-x}\right) dx = \frac{\pi^2}{4},$$

which is a result which usually cannot be obtained in Calculus. $\diamond$

Example 4.9 *Let E be the square $[0,1]\times[0,1]$. Find the integrals*

$$J_1 = \int_E \frac{1}{1-xy}\,dS, \qquad J_2 = \int_E \frac{1}{1+xy}\,dS,$$

bo forming $J_1 - J_2$ and $J_1 + J_2$ and then apply the result of the previous example.

A Computation of "impossible" plane integrals, of which one is improper.

D Apply the hint as well as results from **Example 4.8**, supplied by an attempt to calculate J_1 directly.

I The integrands are > 0, thus we shall not need to be too careful with the improper plane integrals. If one of them should be divergent, this will show up naturally as the value $+\infty$.

First notice that (an improper plane integral)

$$\begin{aligned} J_1 &= \int_E \frac{1}{1-xy}\,dS = \int_0^1 \left\{ \int_0^1 \frac{1}{1-xy}\,dy \right\} dx \\ &= \int_0^1 \left[-\frac{1}{x}\ln(1-xy) \right]_{y=0}^1 dx = -\int_0^1 \frac{1}{x}\ln(1-x)\,dx. \end{aligned}$$

Then (an improper plane integral of non-negative integrand)

$$\begin{aligned} J_1 - J_2 &= \int_E \left\{ \frac{1}{1-xy} - \frac{1}{1+xy} \right\} dS = \int_E \frac{2xy}{1-x^2y^2}\,dS = \int_0^1 \left\{ \int_0^1 \frac{2xy}{1-x^2y^2}\,dy \right\} dx \\ &= -\int_0^1 \frac{1}{x} \left[\ln\left(1-x^2y^2\right) \right]_{y=0}^1 dy = -\int_0^1 \frac{1}{x}\ln(1-x^2)\,dx = -\frac{1}{2}\int_0^1 \frac{1}{t}\ln(1-t)\,dt = \frac{1}{2}J_1, \end{aligned}$$

hence $J_2 = \dfrac{1}{2}J_1$.

Finally, it follows from **Example 4.8**,

$$\begin{aligned} J_1 + J_2 &= \frac{3}{2}J_1 = \int_0^1 \left\{ \int_0^1 \frac{1}{1-xy}\,dy \right\} dx + \int_0^1 \left\{ \int_0^1 \frac{1}{1+xy}\,dy \right\} dx \\ &= \int_0^1 \left\{ \int_{-1}^0 \frac{1}{1+xy}\,dy \right\} dx + \int_0^1 \left\{ \int_0^1 \frac{1}{1+xy}\,dy \right\} dx = \int_0^1 \left\{ \int_{-1}^1 \frac{1}{1+xy}\,dy \right\} dx = \frac{\pi^2}{4}, \end{aligned}$$

thus

$$J_1 = -\int_0^1 \frac{1}{x}\ln(1-x)\,dx = \frac{2}{3}\cdot\frac{\pi^2}{4} = \frac{\pi^2}{6} \quad \text{and} \quad J_2 = \frac{1}{2}J_1 = \frac{\pi^2}{12}.$$

Example 4.10 *Let B be the disc of centrum $(0,0)$ and radius a. Prove that the improper integrals*

$$I = \int_B \frac{1}{\sqrt{a^2-x^2-y^2}}\,dS \quad \textit{and} \quad J = \int_B \frac{1}{\sqrt{a^2-x^2-y^2}\cdot\sqrt{x^2+y^2}}\,dS$$

are convergent, and find their values.

A Improper integrals.

D The integrands are positive, where they are defined, hence it suffices to truncate in polar coordinates, followed by taking the limit.

I 1) The integrand is defines and positive in the interior of the disc B. When we use polar coordinates and integrate over $B_{a-\varepsilon}(\mathbf{0})$, i.e. the disc of centrum $(0,0)$ and radius $a-\varepsilon$, then

$$\begin{aligned} I_\varepsilon &= \int_{B_{a-\varepsilon}(\mathbf{0})} \frac{1}{\sqrt{a^2-x^2-y^2}}\,dS = 2\pi\int_0^{a-\varepsilon}\frac{r}{\sqrt{a^2-r^2}}\,dt = \pi\left[-\sqrt{a^2-r^2}\right]_0^{a-\varepsilon} \\ &= \pi\left\{\sqrt{a^2}-\sqrt{a^2-(a-\varepsilon)^2}\right\} = \pi\left\{a-\sqrt{2a\varepsilon-\varepsilon^2}\right\} \to \pi\,a \qquad \text{for } \varepsilon \to 0+\,. \end{aligned}$$

We conclude that the improper integral is convergent and it has the value

$$I = \int_B \frac{1}{\sqrt{a^2-x^2-y^2}}\,dS = \pi\, a.$$

2) The integrand is defined and positive in $B^\circ \setminus \{\mathbf{0}\}$. We choose the truncation as

$$B_\varepsilon := B_{a-\varepsilon}(\mathbf{0}) \setminus B_\varepsilon(\mathbf{0}).$$

The integrand is positive everywhere in B_ε, so

$$J_\varepsilon = \int_{B_\varepsilon} \frac{1}{\sqrt{a^2-x^2-y^2}\cdot\sqrt{x^2+y^2}}\,dS = 2\pi\int_\varepsilon^{a-\varepsilon} \frac{r}{\sqrt{a^2-r^2}\cdot r}\,dr = 2\pi\int_\varepsilon^{a-\varepsilon}\frac{dr}{\sqrt{a^2-r^2}}.$$

By choosing the substitution $r = a\cdot\sin t$ we get

$$J_\varepsilon = 2\pi\int_{\mathrm{Arcsin}\left(\frac{\varepsilon}{a}\right)}^{\mathrm{Arcsin}\left(1-\frac{\varepsilon}{a}\right)} \frac{a\cdot\cos t}{a\cdot\cos t}\,dt = 2\pi\left\{\mathrm{Arcsin}\left(1-\frac{\varepsilon}{a}\right) - \mathrm{Arcsin}\left(\frac{\varepsilon}{a}\right)\right\}.$$

It follows that the improper integral is convergent, and that is has the value

$$J = \int_B \frac{1}{\sqrt{a^2-x^2-y^2}\cdot\sqrt{x^2+y^2}}\,dS = \lim_{\varepsilon\to 0} J_\varepsilon = 2\pi\{\mathrm{Arcsin}\,1 - \mathrm{Arcsin}\,0\} = \pi^2.$$

Example 4.11 *Let B be the triangle given by the inequalities*

$$0 \le y \le x \quad \textit{and} \quad 0 \le x \le 1.$$

Show that the improper plane integral

$$\int_B \left(y - \frac{\ln x}{x}\right) dS$$

is convergent, and find its value.

A Improper plane integral.

D First prove that the integrand is ≥ 0, whenever it is defined. Then truncate; compute the plane integral over the truncated domain and finally, take the limit.

I The integrand is not defined at $(0,0) \in B$.

If $(x,y) \in B \setminus \{(0,0)\}$, then $0 \le y \le 1$ and $0 < x \le 1$, så $-\infty < \ln x \le 0$, and hence

$$y - \frac{\ln x}{x} \ge 0 \qquad \text{for } (x,y) \in B \setminus \{(0,0)\}.$$

The integrand is positive or zero elsewhere, so we can choose the following truncation,

$$B_\varepsilon: \quad 0 \le y \le x \quad \text{og} \quad \varepsilon \le x \le 1, \qquad \text{for } 0 \le \varepsilon < 1.$$

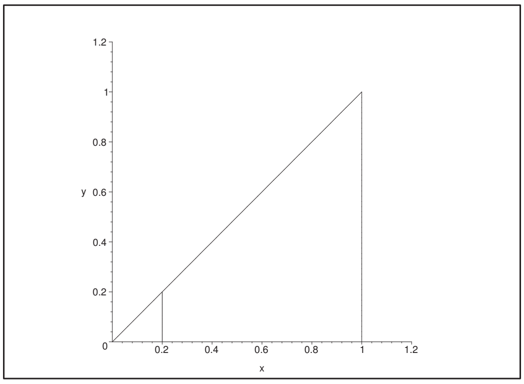


Figure 72: The domains B and $B_{0,2}$.

When we integrate over this truncated domain, we get

$$\begin{aligned}\int_{B_\varepsilon}\left(y-\frac{\ln x}{x}\right)dS &= \int_\varepsilon^1\left\{\int_0^x\left(y-\frac{\ln x}{x}\right)dy\right\}dx=\int_\varepsilon^1\left(\frac{x^2}{2}-\ln x\right)dx\\ &= \left[\frac{x^3}{6}-(x\ln x-x)\right]_\varepsilon^1=\frac{7}{6}-\frac{\varepsilon^3}{6}+\varepsilon\ln\varepsilon-\varepsilon.\end{aligned}$$

It follows from the rules of magnitudes that

$$\varepsilon\ln\varepsilon=-\frac{\ln\left(\dfrac{1}{\varepsilon}\right)}{\dfrac{1}{\varepsilon}}\to 0 \qquad \text{for } \varepsilon\to 0+,$$

hence it follows by taking the limit that the improper plane integral exists, and it has the value

$$\int_B\left(y-\frac{\ln x}{x}\right)dS=\lim_{\varepsilon\to 0+}\int_{B_\varepsilon}\left(y-\frac{\ln x}{x}\right)dS=\frac{7}{6}.$$

Example 4.12 . *Let B be the triangle given by the inequalities*

$$0\le y\le x,\qquad 0\le x\le 1.$$

Prove that the improper integral

$$\int_B\frac{2y}{x^2}\,dS$$

is convergent, and find its value.

A Improper plane integral.

D Sketch a figure. Truncate the domain and compute.

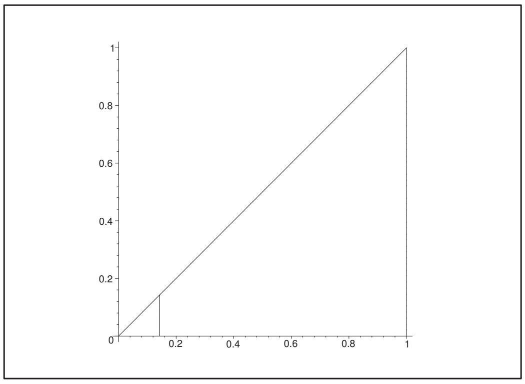


Figure 73: The domain B truncated in the neighbourhood of $(0,0)$.

I The integrand is not defined at $(0,0)$. It is positive everywhere in the remaining set $B \setminus \{(0,0)\}$. Hence it suffices to truncate by an $\varepsilon \in]0,1[$ as on the figure, thereby obtaining the domain

$$B_\varepsilon = \{(x,y) \mid 0 \le y \le x,\, \varepsilon \le x \le 1\}.$$

Then by integration over B_ε,

$$\int_{B_\varepsilon} \frac{2y}{x^2}\,dS = \int_\varepsilon^1 \left\{\int_0^x \frac{2y}{x^2}\,dy\right\} dx = \int_\varepsilon^1 \left[\frac{y^2}{x^2}\right]_{y=0}^x dx = \int_\varepsilon^1 1\,dx = 1-\varepsilon.$$

This expression tends to 1 for $\varepsilon \to 0+$, so we conclude that the improper integral is convergent and its value is

$$\int_B \frac{2y}{x^2}\,dS = \lim_{\varepsilon\to 0+} \frac{2y}{x^2}\,dS = \lim_{\varepsilon\to 0+}(1-\varepsilon) = 1.$$

5 Transformation of a plane integral

Example 5.1 *Let B be the trapeze which is bounded by the coordinate axes and the lines given by the equations $x+y=1$ and $x+y=\frac{1}{2}$. Compute the plane integral*

$$\int_B \exp\left(\frac{y}{x+y}\right) dx\,dy$$

by introducing the new variable $(u,v)=(x+y,x-y)$.

A Transformation of a plane integral.

D Compute the Jacobian and find the new domain D.

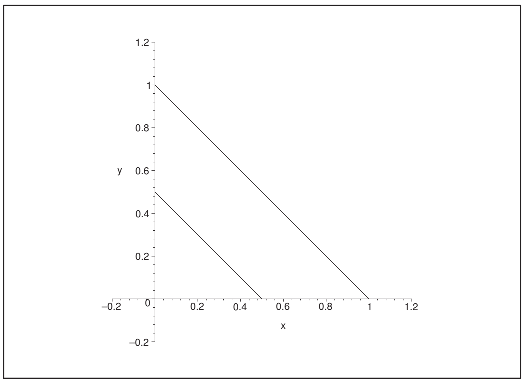


Figure 74: The domain B in the XY-plane.

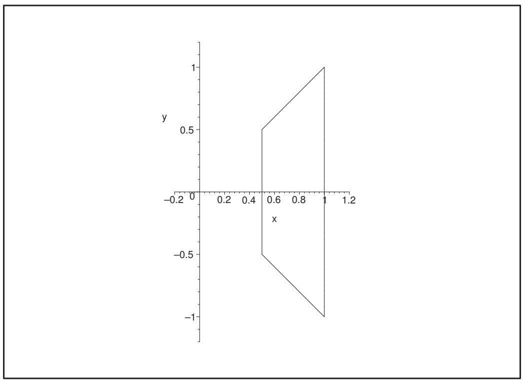


Figure 75: The domain D after the transformation to the UV-plane.

I From

$$(x,y)=\mathbf{\Phi}(u,v)=\left(\frac{u+v}{2},\frac{u-v}{2}\right),$$

follows that

$$J_{\Phi} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & d\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2},$$

and

$$D = \left\{ (u,v) \;\middle|\; \frac{1}{2} \leq u \leq 1, \; -u \leq v \leq u \right\}.$$

Then by the formula of transformation,

$$\begin{aligned} \int_B \exp\left(\frac{y}{x+y}\right) dxdy &= \int_D \exp\left(\frac{u-v}{2u}\right) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \frac{1}{2}\int_{\frac{1}{2}}^{1} \sqrt{e}\left\{\int_{-u}^{u} \exp\left(-\frac{v}{2u}\right) dv\right\} du \\ &= \frac{\sqrt{e}}{2}\int_{\frac{1}{2}}^{1} (-2u)\left[\exp\left(-\frac{v}{2u}\right)\right]_{v=-u}^{u} du = -\sqrt{e}\int_{\frac{1}{2}}^{1} u \cdot \left(\frac{1}{\sqrt{e}} - \sqrt{e}\right) du \\ &= (e-1)\int_{\frac{1}{2}}^{1} u\,du = \frac{3}{8}\,(e-1). \end{aligned}$$

Example 5.2 *Let B denote set in the first quadrant, which is bounded by the curves $xy = 1$ and $xy = 2$ and by the lines $y = x$ and $y = 4x$. Sketch B and compute the plan integral*

$$\int_B x^2y^2\,dx\,dy$$

by introducing the new variables $(u,v) = \left(xy, \dfrac{y}{x}\right)$.

A Transformation of a plane integral.

D Sketch B. Find den inverse function

$$(x,y) = (x(u,v), y(u,v)) = \mathbf{\Phi}(u,v),$$

and find the corresponding domain D in the UV-plane. Compute the Jacobian and finally transform the plane integral.

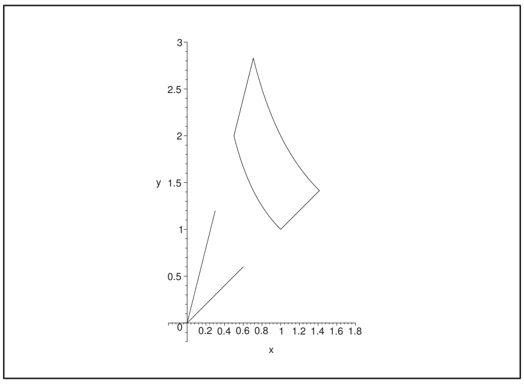


Figure 76: The domain B in the XY-plane.

I If $u = xy$ and $v = \dfrac{y}{x}$ and $x,\, y > 0$, then $u,\, v > 0$, and

$$x(u,v) = \sqrt{\frac{u}{v}}, \qquad y(u,v) = \sqrt{uv}.$$

The domain D is given by

$$1 \le xy = u \le 2 \quad \text{and} \quad 1 \le \frac{y}{x} = v \le 4,$$

hence

$$D = \{(u,v) \mid 1 \le u \le 2,\, 1 \le v \le 4\} = [1,2] \times [1,4],$$

i.e. a rectangle in the UV-plane, which it is no need to sketch.

Finally, the Jacobian is

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2\sqrt{uv}} & -\dfrac{1}{2}\cdot\dfrac{1}{v}\sqrt{\dfrac{u}{v}} \\[2mm] \dfrac{1}{2}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2}\cdot\dfrac{1}{u}\sqrt{\dfrac{u}{v}} \end{vmatrix} = \frac{1}{4}\left\{\frac{1}{\sqrt{uv}}\sqrt{\frac{u}{v}} + \sqrt{\frac{v}{u}}\cdot\frac{1}{v}\sqrt{\frac{u}{v}}\right\} = \frac{1}{4}\left\{\frac{1}{v}+\frac{1}{v}\right\} = \frac{1}{2v}.$$

We get by the transformation formula of the plane integral

$$\begin{aligned}\int_B x^2y^2\,dxdy &= \int_D u^2\cdot\frac{1}{2v}\,dudv = \frac{1}{2}\int_1^2 u^2\,du\cdot\int_1^4\frac{1}{v}\,dv\\ &= \frac{1}{2}\left[\frac{1}{3}u^3\right]_1^2\cdot[\ln v]_1^4 = \frac{1}{6}(8-1)\ln 4 = \frac{7}{3}\ln 2.\end{aligned}$$

Example 5.3 *Find the area of the set in the first quadrant, which is bounded by the curves*

$$xy=4,\qquad xy=8,\qquad xy^3=5,\qquad xy^3=15,$$

by introducing the new variables $u = xy$ *and* $v = xy^3$.

A Area of a set computed by a transformation of a plane integral.

D Find the transformed domain D in the UV-plane and the inverse functions $x(u,v)$ and $y(u,v)$ by this transformation. Calculate the Jacobian and apply the transformation formula to find the area.

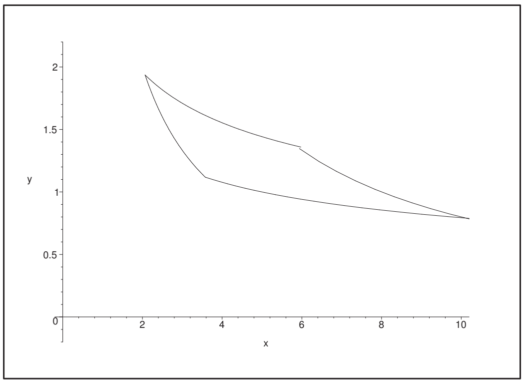


Figure 77: The domain D in the XY-plane. (Different scales on the axes).

I Let B be the given set in the first quadrant. Then x, $y > 0$ for $(x,y) \int B$. It follows immediately that we by the transformation get the domain

$$D = [4,8]\times[5,15].$$

From $u = xy$, $v = xy^3$, $u > 0$ and $v > 0$ follows $y^2 = \dfrac{v}{u}$ and $x^2 = \dfrac{u^3}{v}$, i.e.

$$y = +\sqrt{\frac{v}{u}},\quad\text{and}\quad x = +\sqrt{\frac{u^3}{v}}.$$

Then we get the Jacobian,

$$J(u,v) = \begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}\dfrac{3}{2}\sqrt{\dfrac{u}{v}} & -\dfrac{1}{2}\sqrt{\dfrac{u^3}{v^3}}\\[2mm] -\dfrac{1}{2}\sqrt{\dfrac{v}{u^3}} & \dfrac{1}{2}\sqrt{\dfrac{1}{uv}}\end{vmatrix} = \frac{3}{4}\,\frac{1}{v} - \frac{1}{4}\,\frac{1}{v} = \frac{1}{2v} > 0.$$

Hence the area is

$$\text{areal}(B) = \int_B dx\,dy = \int_D J(u,v)\,du\,dv = \int_4^8 du \cdot \int_4^1 5\frac{1}{2v}\,dv = \frac{4}{2}[\ln v]_5^{15} = 2\ln 3.$$

Example 5.4 *Find the area of the set in the first quadrant, which is bounded by the curves*

$$y = x^3, \qquad y = 4x^3, \qquad x = y^3, \qquad x = 4y^3,$$

by introducing the new variables

$$u = \frac{y}{x^3}, \qquad v = \frac{x}{y^3}.$$

A Area of a set by a transformation of a plane integral.

D Sketch the domain B. Then find D and $x(u,v)$ and $y(u,v)$ by the transformation. Compute the Jacobian and apply the transformation formula to find the area.

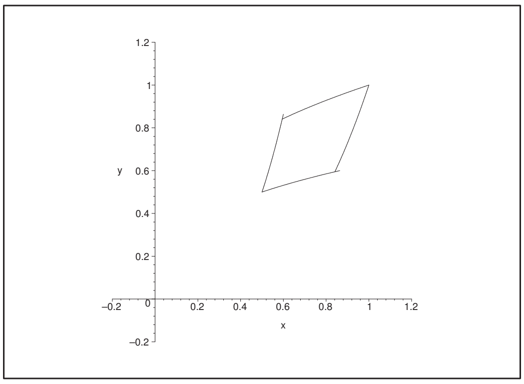


Figure 78: The domain B in the XY-plane.

I The curves $y = x^3$ and $x = y^3$ intersect at $(x,y) = (1,1)$. The curves $y = 4x^3$ and $x = 4y^3$ intersect at $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$. It follows that if the transformation exists and is bijective, then

$$D = [1,4] \times [1,4].$$

Clearly, $x > 0$ and $y > 0$, and hence $u > 0$ and $v > 0$. We shall now try to solve the equations

$$u = \frac{y}{x^3} \quad \text{and} \quad v = \frac{x}{y^3} \qquad \text{for } u,\, v \in [1,4].$$

From

$$u^3 v = \frac{y^3}{x^9} \cdot \frac{x}{y^3} = \frac{1}{x^8}$$

follows that

$$x = u^{-\frac{3}{8}}\, v^{-\frac{1}{8}}, \quad \text{and similarly } y = u^{-\frac{1}{8}}\, v^{-\frac{3}{8}}.$$

The Jacobian is

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2mm] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{3}{8}\, u^{-\frac{11}{8}}\, v^{-\frac{1}{8}} & -\dfrac{1}{8}\, u^{-\frac{9}{8}}\, v^{-\frac{3}{8}} \\[2mm] -\dfrac{1}{8}\, u^{-\frac{3}{8}}\, v^{-\frac{9}{8}} & -\dfrac{3}{8}\, u^{-\frac{1}{8}}\, v^{-\frac{11}{8}} \end{vmatrix}$$

$$= \frac{9}{64}\, u^{-\frac{3}{2}}\, v^{-\frac{3}{2}} - \frac{1}{64}\, u^{-\frac{3}{2}}\, v^{-\frac{3}{2}} = \frac{1}{8}\, u^{-\frac{3}{2}}\, v^{-\frac{3}{2}}.$$

We get the area by applying the transformation formula

$$\operatorname{area}(B) = \int_B dS = \frac{1}{8}\int_1^4 u^{-\frac{3}{2}}\, du \cdot \int_1^4 v^{-\frac{3}{2}}\, dv = \frac{1}{8}\left\{\int_1^4 t^{-\frac{3}{2}}\, dt\right\}^2$$

$$= \frac{1}{8}\left\{\left[-\frac{2}{\sqrt{t}}\right]_1^4\right\}^2 = \frac{1}{8}\,(2-1)^2 = \frac{1}{8}.$$

Example 5.5 *Let $B \subset \mathbb{R}^2$ be given by*

$$0 \le x, \qquad 0 \le y, \qquad \sqrt{x}+\sqrt{y} \le 1.$$

find the area of B and the plane integral

$$I = \int_B \exp\left[\left(\sqrt{x}+\sqrt{y}\right)^4\right] dx\, dy$$

by introducing the new variables

$$u = \sqrt{x}+\sqrt{y}, \qquad v = \sqrt{x}-\sqrt{y}.$$

A Transformation of a plane integral.

D Sketch B; find x and y as functions of u and v; compute the Jacobian; find the domain of the parameters $(u,v) \in A$; finally, apply the transformation theorem.

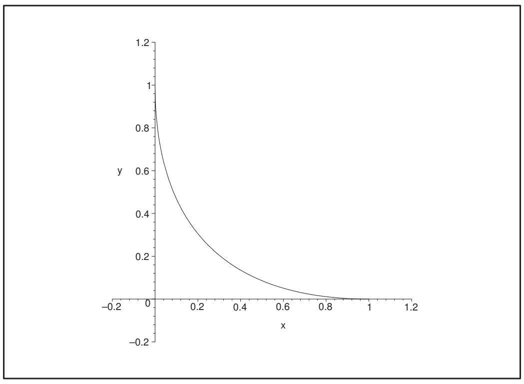


Figure 79: The domain A in the (X,Y)-plane.

I If we put $u = \sqrt{x}+\sqrt{y}$ and $v = \sqrt{x}-\sqrt{y}$, then

$$2\sqrt{x} = u+v \qquad \text{and} \qquad 2\sqrt{y} = u-v,$$

thus

$$x = \frac{1}{4}\,(u+v)^2 \qquad \text{and} \qquad y = \frac{1}{2}\,(u-v)^2.$$

Then we get the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \left|\begin{array}{cc} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{array}\right| = \left|\begin{array}{cc} \frac{1}{2}\,(u+v) & \frac{1}{2}\,(u+v) \\[2mm] \frac{1}{2}\,(u-v) & -\frac{1}{2}\,(u-v) \end{array}\right|$$

$$= \frac{1}{2}\,(u+v)\cdot\frac{1}{2}\,(u-v)\left|\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right| = -\frac{1}{2}\,(u^2-v^2).$$

We shall now find the domain of the new parameters A:

1) The boundary part $x = 0$ corresponds to $u + v = 0$.
2) The boundary part $y = 0$ corresponds to $u - v = 0$.
3) The boundary part $\sqrt{x} + \sqrt{y} = 1$ corresponds to $u = 1$.

Since a closed and bounded set by the second main theorem of continuous functions is mapped into a closed and bounded set by this continuous change of variables, the new domain is the triangle A on the figure.

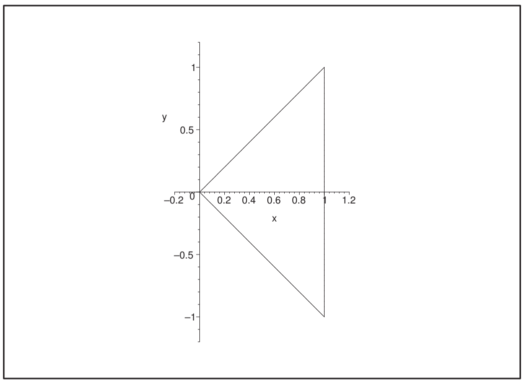


Figure 80: The domain A in the (U, V)-plane.

Note that the Jacobian is *negative* on A, so this time we shall need the absolute values in the formula.

By the transformation theorem,

$$\begin{aligned}
\operatorname{area}(B) &= \int_B dx\,dy = \int_A \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{2}\int_A (u^2 - v^2)\,du\,dv \\
&= \frac{1}{2}\int_0^1 \left\{\int_{-u}^{u} (u^2 - v^2)\,dv\right\} du = \frac{1}{2}\int_0^1 \left[u^2 v - \frac{1}{3}v^3\right]_{-u}^{u} du \\
&= \frac{1}{2}\int_0^1 \left(2u^3 - \frac{2}{3}u^3\right) du = \frac{2}{3}\int_0^1 u^3\,du = \frac{1}{6},
\end{aligned}$$

and

$$\begin{aligned}
I &= \int_B \exp\left[\left(\sqrt{x} + \sqrt{y}\right)^4\right) dx\,dy = \frac{1}{2}\int_A \exp\left(u^4\right) \cdot \left(u^2 - v^2\right)\,du\,dv \\
&= \frac{1}{2}\int_0^1 \left\{\int_{-u}^{u} \exp\left(u^4\right) \cdot \left(u^2 - v^2\right)\,dv\right\} du = \frac{2}{3}\int_0^1 \exp\left(u^4\right) \cdot u^3\,du \\
&= \frac{1}{6}\int_0^1 e^t\,dt = \frac{e-1}{6}.
\end{aligned}$$

Example 5.6 *Define a vector field* $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^2$ *in the following way,*

$$\mathbf{r}(u,v) = \left(e^u \cos v, e^u \sin v\right).$$

Prove that the Jacobian $J_{\mathbf{r}}$ *is different from zero almost everywhere, and that* $\mathbf{r}$ *is not injective.*

A Jacobian and a non-injective transformation.

D Compute and exploit the periodicity.

I The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\[2mm] e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u} \neq 0.$$

Then note that $(u,v) = (u_0, v_0 + 2p\pi)$, $p \in \mathbb{Z}$, are all mapped into the same point

$$(x,y) = \left(e^{u_0} \cos v_0, e^{u_0} \sin v_0\right),$$

so the transformation is not injective.

Remark. We may add that $\mathbb{R}^2$ by $\mathbf{r}$ is mapped (infinitely often) onto $\mathbb{R}^2 \setminus \{(0,0)\}$.

Example 5.7 *Define a vector field* $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^2$ *as follows:*

$$\mathbf{r}(u,v) = \left(u^2 - v^2, 2uv\right).$$

Prove that the Jacobian $J_{\mathbf{r}}$ *is different from zero almost everywhere ant that* $\mathbf{r}$ *is not injective.*

A Jacobian and a non-injective transformation.

D Calculate the Jacobian and find two different (u,v)-points which are mapped into the same (x,y).

I The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\[2mm] 2v & 2u \end{vmatrix} = 4\left(u^2+v^2\right) \neq 0 \qquad \text{for } (u,v) \neq (0,0).$$

Clearly, (u,v) and $(-u,-v)$ are mapped into the same point,

$$(x,y) = \left(u^2 - v^2, 2uv\right),$$

so the map is not injective for $(u,v) \neq (0,0)$.

Example 5.8 *Let B be the parallelogram of vertices $(0,0)$, $(1,-1)$, $(2,1)$ and $(3,0)$. Compute the plane integral*

$$I = \int_B \frac{\cos\left(\frac{1}{2}\,\pi(x+y)\right)}{1+x-2y}\,dx\,dy$$

by introducing the new variables

$$u = x+y, \qquad v = x-2y.$$

A Plane integral by a change of variables and the transformation formula.

D Sketch B and find the domain D. Compute the Jacobian and insert into the formula.

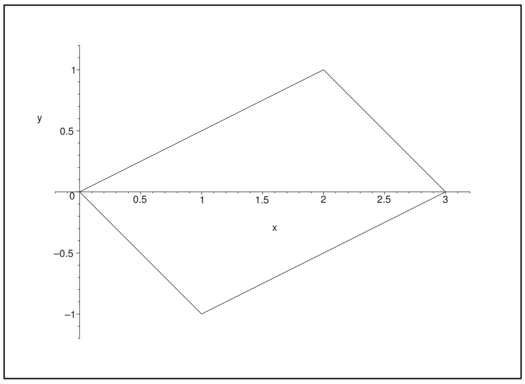


Figure 81: The parallelogram B.

I It follows from the figure that

$$u = x + y \in [0,3] \qquad \text{and} \qquad v = x - 2y \in [0,3],$$

and the new domain is the square $D = [0,3] \times [0,3]$.

From

$$x = \frac{2}{3}\,u + \frac{1}{3}\,v \qquad \text{and} \qquad y = \frac{1}{3}\,u - \frac{1}{3}\,v,$$

follows that the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9}.$$

When we finally put everything into the transformation formula, then

$$\begin{aligned} I &= \int_B \frac{\cos(\frac{1}{2}\,\pi(x+y))}{1+x-2y}\,dx\,dy = \int_D \frac{\cos(\frac{1}{2}\,\pi\cdot u)}{1+v}\left|-\frac{1}{9}\right|\,du\,dv \\ &= \frac{1}{9}\int_0^3 \cos\left(\frac{\pi}{2}\,u\right)du \cdot \int_0^3 \frac{dv}{1+v} = \frac{1}{9}\cdot\frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\,u\right)\right]_0^3 \cdot [\ln(1+v)]_0^3 \\ &= \frac{2}{9\pi}\left\{\sin\left(\frac{3\pi}{2}\right) - 0\right\}\cdot\{\ln 4 - \ln 1\} = -\frac{4}{9\pi}\,\ln 2. \end{aligned}$$

Example 5.9 *Let B be the plane set which is bounded by the X-axis and the line of equation $y = x$ and an arc of the parabola given by*

$$5x = 4 + y^2, \qquad y \in [0,1].$$

Calculate the plane integral

$$I = \int_B \cos\left[\left(\sqrt{\frac{5}{4}\,x + y} + \sqrt{\frac{5}{4}\,x - y}\right)^4\right] dx\,dy$$

by introducing the new variables (u,v) given by

$$5x = u^2 + v^2, \qquad 2y = uv, \qquad -u \le v \le u.$$

A Plane integral by a change of variables and the transformation formula.

D Sketch B and find the new domain D. Compute the Jacobian and put everything into the formula.

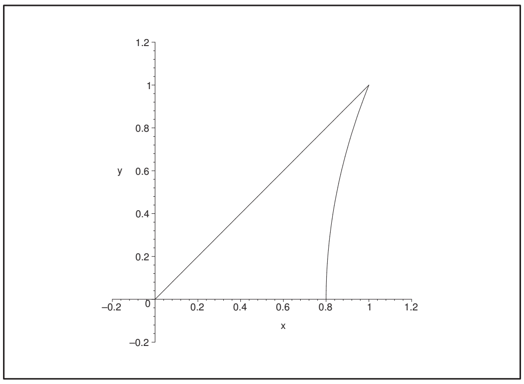


Figure 82: The point set B.

I It follows from $5x = u^2 + v^2$ and $2y = uv$ that

$$5x + 4y = u^2 + v^2 + 2uv = (u+v)^2, \qquad 5x - 4y = u^2 + v^2 - 2uv = (u-v)^2.$$

Since $|v| \le u$, we get from this

$$u + v = +\sqrt{5x+4y} \quad \text{and} \quad u - v = +\sqrt{5x-4y},$$

hence

$$u = \frac{\sqrt{5x+4y} + \sqrt{5x-4y}}{2} \quad \text{and} \quad v = \frac{\sqrt{5x+4y} - \sqrt{5x-4y}}{2}.$$

Then we determine the boundary curves of the new domain.

1) If $y=x$, $x\in[0,1]$, then

$$u=\frac{\sqrt{9x}+\sqrt{x}}{2}=2\sqrt{x} \quad\text{and}\quad v=\frac{\sqrt{9x}-\sqrt{x}}{2}=\sqrt{x},$$

so this boundary curve is transformed into $v=\dfrac{1}{2}\,u$. Then by a small consideration, $u\in[0,2]$.

2) If $y=0$, $x\in\left[0,\dfrac{4}{5}\right]$ on the X-axis, then $v=0$ and $u=\sqrt{5x}\in[0,2]$.

3) If finally $5x=4+y^2$, $y\in[0,1]$, then

$$4+y^2=u^2+v^2 \qquad\text{and}\qquad 4y=2uv,$$

i.e.

$$(u+v)^2=(y+2)^2 \qquad\text{og}\qquad (u-v)^2=(2-y)^2,$$

thus

$$u+v=y+2\geq 0 \qquad\text{and}\qquad u-v=2-y\geq 0,$$

or $u=2$ and $v=y\in[0,1]$. Then we can sketch the new domain (a triangle).

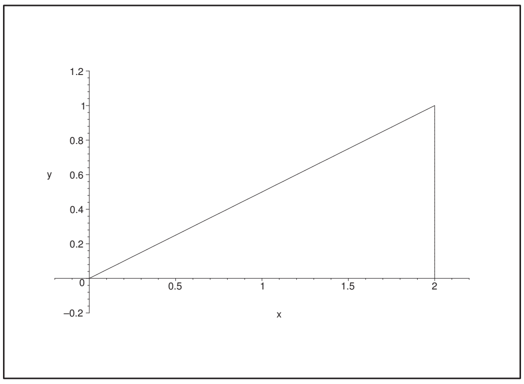


Figure 83: The new domain D.

Since

$$x=\frac{1}{5}\,(u^2+v^2), \qquad y=\frac{1}{2}\,uv,$$

we get the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{2}{5}\,u & \frac{2}{5}\,v \\ \frac{1}{2}\,v & \frac{1}{2}\,u \end{vmatrix}=\frac{1}{5}\,\left(u^2-v^2\right)\geq 0.$$

Finally, since

$$\frac{5}{4}\,x+y=\frac{1}{4}\,(5x+4y)=\frac{1}{4}\,\left(u^2+v^2+2uv\right)=\left(\frac{u+v}{2}\right)^2,$$

and similarly,

$$\frac{5}{4}\,x - y = \left(\frac{u-v}{2}\right)^2,$$

we get the plane integral

$$\begin{aligned}
I &= \int_B \cos\left[\left(\sqrt{\frac{5}{4}\,x+y}+\sqrt{\frac{5}{4}\,x-y}\right)^4\right] dx\,dy \\
&= \int_D \cos\left[\left(\frac{u+v}{2}+\frac{u-v}{2}\right)^4\right]\cdot\frac{1}{5}\left(u^2-v^2\right)\,du\,dv = \int_D \cos\left(u^4\right)\cdot\frac{1}{5}\left(u^2-v^2\right)\,du\,dv \\
&= \int_0^2 \cos\left(u^4\right)\left\{\int_0^{\frac{u}{2}}\frac{1}{5}\left(u^2-v^2\right)\,dv\right\}du = \int_0^2 \cos\left(u^4\right)\cdot\frac{1}{5}\left\{u^2\cdot\frac{u}{2}-\frac{1}{3}\cdot\left(\frac{u^3}{8}\right)\right\}du \\
&= \frac{1}{5}\cdot\frac{11}{24}\int_0^2 \cos\left(u^4\right)\,u^3\,du = \frac{11}{480}\left[\sin\left(u^4\right)\right]_0^2 = \frac{11}{480}\,\sin 16.
\end{aligned}$$

Example 5.10 *Let B be the plane point set which is bounded by the X-axis and the line of equation $y = \frac{1}{2}\,x$, and the branches of the hyperbola,*

$$x^2 - y^2 = 1, \quad x > 0, \quad \textit{and} \quad x^2 - y^2 = 4, \quad x > 0.$$

Compute the plane integral

$$I = \int_B \frac{x+y}{x-y} \exp\left(x^2 - y^2\right)\, dx\, dy$$

by introducing the new variables (u, v) given by

$$x = u \cosh v, \qquad y = u \sinh v.$$

A Transformation of a plane integral.

D Sketch the domain B, and find the domain D of the new variables, and compute the Jacobian. Finally, insert everything into the transformation formula.

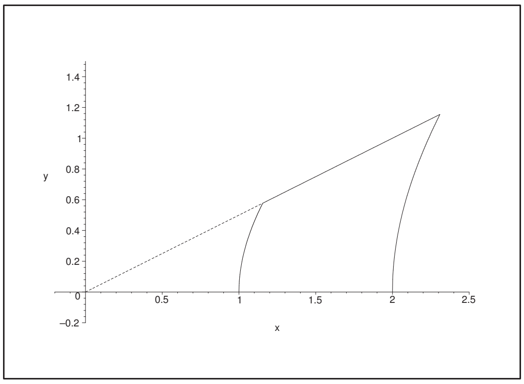


Figure 84: The domain B.

I If $y = 0$, $x > 0$, then $v = 0$ and $x = u$, hence the segment on the X-axis is transformed onto a segment on the U-axis.

If $y = \frac{1}{2}\,x$, then $u \sinh v = \frac{1}{2}\,u \cosh v$, i.e.

$$\tanh v = \frac{1}{2} = \frac{e^v - e^{-v}}{e^v + e^{-v}} = \frac{e^{2v} - 1}{e^{2v} + 1},$$

or $e^{2v} + 1 = 2\,e^{2v} - 2$, thus $e^{2v} = 3$, and hence $v = \frac{1}{2}\,\ln 3$, and u is a "free" variable.

If $x^2 - y^2 = 1$ $x > 0$, then $u^2 = 1$, and since $u > 0$, we must have $u = 1$.

If $x^2 - y^2 = 4$, $x > 0$, then $u^2 = 4$, and since $u > 0$, we must have $u = 2$.

Summarizing, the new domain is the rectangle

$$D = [1,2] \times \left[0, \frac{1}{2}\,\ln 3\right].$$

Then the Jacobian is computed,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cosh v & u\,\sinh v \\ \sinh v & u\,\cosh v \end{vmatrix} = u > 0.$$

By the transformation formula,

$$\begin{aligned} I &= \int_B \frac{x+y}{x-y}\,\exp\left(x^2-y^2\right)\,dx\,dy = \int_D \frac{u(\cosh v + \sinh v)}{u\cosh v - \sinh v)}\,\exp\left(u^2\right)\cdot u\,du\,dv \\ &= \int_D e^{2v}\,\exp\left(u^2\right)\,u\,du\,dv = \int_1^2 \exp\left(u^2\right)\,u\,du \cdot \int_0^{\frac{1}{2}\,\ln 3} e^{2v}\,dv \\ &= \frac{1}{2}\left[\exp\left(u^2\right)\right]_1^2 \cdot \frac{1}{2}\left[e^{2v}\right]_0^{\frac{1}{2}\,\ln 3} = \frac{1}{4}\,\left(e^4-e\right)\cdot(3-1) = \frac{e}{2}\,\left(e^3-1\right). \end{aligned}$$

Example 5.11 *A triangle B in the (X,Y)-plane is given by the inequalities*

$$x+y \geq 1, \qquad 2y-x \leq 2, \qquad y-2x \geq -2.$$

By introducing

(6) $u = x+y, \qquad v = x-y,$

we get a map from the (X,Y)-plane onto the (U,V)-plane.

1) Prove that the image D in the (U,V)-plane of B by this map is given by

$$1 \leq u \leq 4, \qquad u-4 \leq 3v \leq 4-u,$$

and sketch D.

2) Compute the plane integral

$$\int_B \frac{3}{x+y}\,dx\,dy$$

by introducing the new variables given by (6).

A Transformation of a plane integral.

D Find the domain of the new variables D and compute the Jacobian, and then finally insert into the formula.

I 1) It follows from (5.11) that

$$x = \frac{u+v}{2} \qquad \text{og} \qquad y = \frac{u-v}{2}.$$

Hence

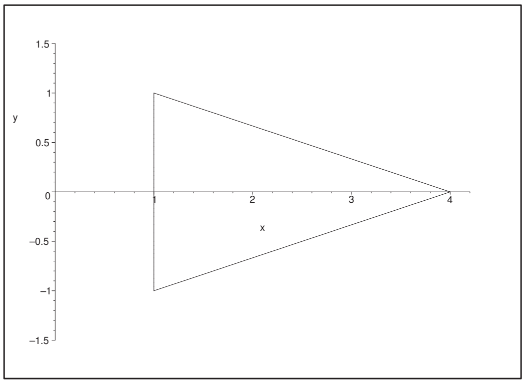


Figure 85: The new domain D.

a) $x + y \geq 1$ is transformed into $u \geq 1$,
b) $2y - x \leq 2$ is transformed into $u - 3v \leq 4$,
c) $y - 2x \geq -2$ is transformed into $u + 3v \leq 4$.

We get by a rearrangement, $u - 4 \leq 3v \leq 4 - u$, hence $u \leq 4$, and

$$D = \{(u,v) \mid 1 \leq u \leq 4,\ u - 4 \leq 3v \leq 4 - u\}.$$

We can here exploit that it is given that B is a triangle and thus bounded. The transformation (5.11) is continuous, so D is connected an bounded, and then we can sketch the three boundary lines and identify the image as the bounded part.

2) The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Then by the transformation formula,

$$\begin{aligned} \int_B \frac{3}{x+y}\,dx\,dy &= \int_D \frac{3}{u}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{3}{2}\int_1^4 \left\{\int_{-\frac{4-u}{3}}^{\frac{4-u}{3}} dv\right\} du \\ &= \frac{3}{2}\int_1^4 \frac{1}{u}\cdot 2\cdot\frac{4-u}{3}\,du = \int_1^4 \left(\frac{4}{u} - 1\right) du = 4\ln 4 - 3 = 8\ln 2 - 3. \end{aligned}$$

Example 5.12 *Let B be the bounded domain which is given by the inequalities*

$$e^{-x} \le y \le 2e^{-x}, \qquad e^{x} \le y \le^{2} e^{x}.$$

1. *Sketch B.*

If we put

(7) $u = y\,e^{x}, \qquad v = y\,e^{-x},$

we get a map of the (X,Y)-plane into the (U,V)-plane.

2. *Prove that the image of B by this map is the square $[1,2] \times [1,2]$.*

3. *Compute the plane integral*

$$I = \int_B 4y^2 \exp\left(y^2 + x\right)\,dx\,dy$$

by introducing the new variables given by (7).

A Transformation of a plane integral.

D Follow the guidelines supplied by a computation of the Jacobian before everything is put into the transformation formula.

ALTERNATIVELY, one can actually in this case compute the plane integral directly.

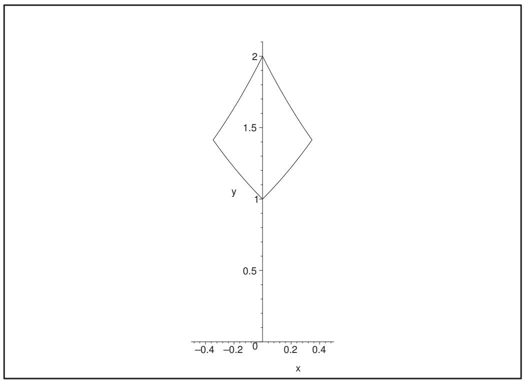


Figure 86: The domain B.

I 1) Let us first find the intersection point of the boundary curves of B.

a) If $y = e^{-x} = 2e^{x}$, then $x = -\dfrac{1}{2}\ln 2$ and hence $y = \sqrt{2}$.

b) If $y = e^{x} = 2e^{-x}$, then $x = \dfrac{1}{2}\ln 2$ and hence $y = \sqrt{2}$.

c) The remaining two intersection points are immediately seen to be $(0,1)$ and $(0,2)$.

Then it is easy to sketch the domain B, even if one does not have MAPLE at hand.

2) By the change of variables $u = y\,e^{x}$ and $v = y\,e^{-x}$,

a) $y = e^{x}$ and $x \in \left[0, \dfrac{1}{2}\ln 2\right]$ is transformed into $v = 1$ and $u = e^{2x} \in [1,2]$,

b) $y = 2e^{-x}$ and $x \in \left[0, \dfrac{1}{2}\ln 2\right]$ is transformed into $u = 2$ and $v = 2e^{2x} \in [1,2]$,

c) $y = 2e^{x}$ and $x \in \left[-\dfrac{1}{2}\ln 2, 0\right]$ is transformed into $v = 2$ and $u = 2e^{2x} \in [1,2]$,

d) $y = e^{-x}$ and $x \in \left[-\dfrac{1}{2}\ln 2, 0\right]$ is transformed into $u = 1$ and $v = e^{-2x} \in [1,2]$.

Thus we get the new domain $D = [1,2] \times [1,2]$ in the (U,V)-plane.

3) Then we find x and y as functions of u and v:

From $y \geq 1$ and u, $v \geq 1$, follows that

$$\frac{u}{v} = e^{2x}, \quad \text{dvs.} \quad x = \frac{1}{2}\ln\left(\frac{u}{v}\right) = \frac{1}{2}\ln u - \frac{1}{2}\ln v.$$

From $v = y\,e^{-x}$, follows that

$$y = v\,e^{x} = v\sqrt{\frac{u}{v}} = \sqrt{uv}.$$

This gives the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2u} & -\frac{1}{2v} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix} = \frac{1}{4}\left(\frac{1}{\sqrt{uv}} + \frac{1}{\sqrt{uv}}\right) = \frac{1}{2}\,\frac{1}{\sqrt{uv}} > 0.$$

When we insert into the transformation formula, we get

$$\begin{aligned}
I &= \int_B 4y^2 \exp\left(y^2+x\right)\, dx\, dy = \int_D 4\,uv\, \exp\left(uv + \frac{1}{2}\,\ln\left(\frac{u}{v}\right)\right) \frac{1}{2}\,\frac{1}{\sqrt{uv}}\, du\, dv \\
&= \int_D 4uv\, \exp(uv)\cdot\sqrt{\frac{u}{v}}\cdot\frac{1}{2}\,\frac{1}{\sqrt{uv}}\, du\, dv = \int_D 2u\, \exp(uv)\, du\, dv \\
&= 2\int_1^2 \left\{\int_1^2 u\, \exp(uv)\, dv\right\} du = 2\int_1^2 [\exp(uv)]_{v=1}^2\, du \\
&= 2\int_1^2 \left(e^{2u}-e^u\right)\, du = \left[e^{2u}-2e^u\right]_1^2 = e^4-2e^2-e^2+2e = e^4-3e^2+2e.
\end{aligned}$$

Alternatively, it is actually possible to compute the plane integral directly without using the transformation theorem. First write $B = B_1 \cup B_2$, as an (almost) disjoint union where

$$B_1 = \left\{(x,y) \;\middle|\; \sqrt{2} \le y \le 2,\, \ln\left(\frac{y}{2}\right) \le x \le \ln\left(\frac{2}{y}\right)\right\}$$

and

$$B_2 = \left\{(x,y) \;\middle|\; 1 \le y \le \sqrt{2},\, \ln\left(\frac{1}{y}\right) \le x \le \ln y\right\}.$$

We have the following natural splitting,

$$\int_B 4y^2\, \exp\left(y^2+x\right)\, dx\, dy = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \int_{B_1} 4y^2\, \exp\left(y^2+x\right)\, dx\, dy = \int_{\sqrt{2}}^2 \left\{\int_{\ln\left(\frac{y}{2}\right)}^{\ln\left(\frac{2}{y}\right)} 4y^2\, \exp\left(y^2\right)\, e^x\, dx\right\} dy \\
&= \int_{\sqrt{2}}^2 4y^2\, \exp\left(y^2\right)\cdot\left(\frac{2}{y}-\frac{y}{2}\right) dy = \int_{\sqrt{2}}^2 \left(8y-2y^3\right)\, \exp\left(y^2\right)\, dy \\
&= \int_{\sqrt{2}} 2\left(4-y^2\right)\, \exp\left(y^2\right)\, 2y\, dy = \int_{t=2}^4 (4-t)e^t\, dt = \left[(5-t)e^t\right]_2^4 = e^4-3e^2,
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_{B_2} 4y^2\, \exp\left(y^2+x\right)\, dx\, dy = \int_1^{\sqrt{2}} \left\{\int_{\ln\left(\frac{1}{y}\right)}^{\ln y} 4y^2\, \exp\left(y^2\right)\, e^x\, dx\right\} dy \\
&= \int_1^{\sqrt{2}} 4y^2\, \exp\left(y^2\right)\cdot\left(y-\frac{1}{y}\right)\, dy = \int_1^{\sqrt{2}} \left(4y^3-4y\right)\, \exp\left(y^2\right)\, dy \\
&= \int_1^{\sqrt{2}} \left(2y^2-2\right)\, \exp\left(y^2\right)\cdot 2y\, dy = 2\int_1^2 (t-1)e^t\, dt = 2\left[(t-2)e^t\right]_1^2 = 2e.
\end{aligned}$$

Summarizing we get

$$\int_B 4y^2\, \exp\left(y^2+x\right)\, dx\, dy = I_1 + I_2 = e^4-3e^2+2e.$$