Series on Knots and Everything – Vol. 16

DELTA

A Paradox Logic

N S K Hellerstein

World Scientific

DELTA

A Paradox Logic

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A Paradox Logic

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Introduction

There once was a poet from Crete
who performed a remarkable feat
He announced to the wise
"Every Cretan tells lies"
thus ensuring their logic's defeat.

"It cannot be too strongly emphasized that the logical paradoxes are not idle or foolish tricks. They were not included in this volume to make the reader laugh, unless it be at the limitations of logic. The paradoxes are like the fables of La Fontaine which were dressed up to look like innocent stories about fox and grapes, pebbles and frogs. For just as all ethical and moral concepts were skillfully woven into their fabric, so all of logic and mathematics, of philosophy and speculative thought, is interwoven with the fate of these little jokes."

— Kasner and Newman, "Paradox Lost and Paradox Regained"
from volume 3, "The World of Mathematics"

This book is about "delta", a paradox logic. In delta, a statement can be true yet false; an "imaginary" state, midway between being and non-being. Delta's third value solves many logical paradoxes unsolvable in two-valued logic.

The purpose of this book is not to bury Paradox but to praise it. I do not intend to explain these absurdities away; instead I want them to blossom to their full mad glory.

I gather these riddles together here to see what they have in common. Maybe they'll reveal some underlying unity, perhaps even a kind of fusion energy! They display many common themes; reverse logic, self-reference, diagonality, nonlinearity, chaos, system failure, tactics versus strategy, and transcendence of former reference frames. Although these paradoxes are truly insoluble as posed, they do in general allow this (fittingly paradoxical!) resolution; namely through loss of resolution! To demand precision is to demand partial vision. These paradoxes define, so to speak, sharp vagueness.

A sense of humor is the best guide to these wild regions. The alternative seems to be a kind of grim defensiveness. There exists a strange tendency for scholars to denigrate these paradoxes by giving them derogatory *names*. Paradoxes have been dubbed "absurd" and "imaginary" and even (O horror!) "irrational". Worse than such bitter insults are the hideously morbid *stories* which the guardians of rationality tell about these agents of Chaos. All too many innocuous riddles have been associated with frightening fables of imprisonment and death; quite gratuitously, I think. It is as if the discoverers of these little jokes hated them and wanted them dead. Did these jests offend some pedant's pride?

Paradox is free. It overthrows the tyranny of logic and thus undermines the logic of tyranny. This book's paradoxes are more subversive than spies, more explosive than bombs, more dangerous than armies, and more trouble than even the President of the United States. They are the weak points in the status quo; they threaten the security of the State. These paradoxes are why the pen is mightier than the sword; a fact which is itself a paradox.

This book is divided into three parts: "inner delta logic", "outer delta logic", and "beyond delta logic". In the first part, the three logic values are three in a row; in the second part, the three values are three in a loop; and in the third part, the three values are three out of four.

The "inner logic" section covers: classic paradoxes of mathematical logic; Kleenean "inner" logic; Brownian forms and bracket algebra; DeMorgan equational laws; completeness theorems; the "inner order" semi-lattice; proof that inner logic resolves all self- referential systems; classic paradoxes resolved; the Halting Theorem; inner logic plus splice embeds the continuum; "Zeno's theorem", and "Fuzzy Chaos". The "outer delta logic" section covers: permutating inner logic; non-Kleenean ternary operators; the three perpendicular logics; "pivot"; "loop"; Z mod 3; boolean mappings of delta; "cyclic distribution" and voter's paradox; non-Aristotelean "voter's logic"; "banker's dilemma"; and the "Chairman's Paradox". The "beyond delta" section connects delta with "diamond", a four-valued wave logic, and "dilemma", a non-zero-sum game.

The general reader will probably prefer these chapters and sections: **1, 2AB, 7, 8ACEF, 13, 14, 16ABE,** and the **Notes**. I encourage all readers to attempt the "exercises for the reader"; for doing teaches better than reading.

Readers familiar with World Scientific Publishing's "Series On Knots And Everything" may recognize some similarities between "Delta" and "Diamond", also published by this author in this series. Indeed, for the first half of both books, the similarity is near identity. In effect, the first half of Delta is the second edition of the first half of Diamond, with material added, revised, and improved.

In their second halves, though, Diamond and Delta diverge. Delta is a smaller logic, so there's less *of it* to analyze. Hence Delta has a closure, a compactness, and a unity unlike Diamond's. It seems to me that Delta is more "organic", or "holistic", than Diamond, which is the more "analytic" of the two. Delta is just big enough — and just small enough — to maximize paradox.

I would be a liar indeed not to acknowledge my many friends and colleagues. These include Douglas Hofstadter, Louis Kauffman, Tarik Peterson, Sylvia Rippel, Rudy Rucker, Dick Shoup, Raymond Smullyan, Stan Tenen, and Francisco Varela; their vital imput over many years helped make this book possible. Love and thanks go to my parents, Earl and Marjorie, who helped make *me* possible. Special thanks go to my dear wife Sherri, without whom I would not have published this. Finally, due credit (and blame!) go to myself, for boldly rushing in where logicians fear to tread.

Said a monk to a man named Joshu
"Is that dog really God?" He said "Mu."
This answer is vexing
And highly perplexing
And that was the best he could do.

Part One

Inner Delta Logic

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Chapter 1

Paradox

A. *The Liar*

Epimenides the Cretan said that all Cretans lie; did he tell the truth, or not? Let us assume, for the sake of argument, that every Cretan, except possibly Epimenides himself, was in fact a liar; but what then of Epimenides?

In effect, he says he himself lies; but if he is lying, then he is telling the truth; and if he is telling the truth, then he is lying! Which then is it?

The same conundrum arises from the following sentence:

"This sentence is false."

That sentence is known as the "Liar Paradox", or "pseudomenon".

The pseudomenon obeys this equation:

L = not L .

It's true if false, and false if true. Which then is it?

That little jest is King of the Contradictions. They all seem to come back to that persistent riddle. If it is false then it is true, by its own definition; yet if it is true then it is false, for the exact same reason! So which is it, true or false? It seems to undermine dualistic reason itself. Dualists fear this paradox; they would banish it if they could.

Since it is, so to speak, the leader of the Opposition Party, it naturally bears a nasty name; the "Liar" paradox. Don't trust it, say the straight thinkers; and it agrees with them! They denigrate it, but it denigrates *itself*; it admits that it is a liar, and thus it is not *quite* a liar! It is straightforward in its deviation, accurate in its errors, and honest in its lies! Does that make sense to you, dear reader? I must admit that it has never quite made sense to me.

The name "Liar" paradox is nonetheless a gratuitous insult. The pseudomenon merely denies its truth, not its intentions. It may be false innocently, out of lack of ability or information. It may be contradicting itself, not bitterly, as the name "Liar" suggests, but in a milder tone.

Properly speaking, the Liar paradox goes:

"This statement is a lie."

"I am lying."

"I am a liar."

But consider these statements:

"This statement is wrong."

"I am mistaken."

"I am a fool."

This is the Paradox of the Fool; for the Fool is wise if and only if the Fool is foolish! The underlying logic is identical, and rightly so. For whom, after all, does the Liar fool best but the Liar? And whom else does the Fool deceive except the Fool? The Liar is nothing but a Fool, and vice versa!

Therefore I sometimes call the pseudomenon (or Paradox of Self-Denial) the "Fool Paradox", or "Fool's Paradox", or even "Fool's Gold". The mineral "fool's gold" is iron pyrite; a common ore. This fire-y and ironic little riddle is also a common 'ore, with a thousand wry offspring.

For instance:

"I am not a Marxist" — Karl Marx

"Everything I say is self-serving" — Richard Nixon

Tell me, dear reader; would you believe either of these politicos?

Compare the Liar to the following quarrel:

Tweedledee: "Tweedledum is a liar."

Tweedledum: "Tweedledee is a liar."

— *two* calling *each other* liars rather than *one* calling *itself* a liar! This dispute, which I call "Tweedle's Quarrel", is also known as a "toggle".

Its equations are:

EE = not UM

UM = not EE

This system has two boolean solutions: (true, false) and (false, true). The brothers, though symmetrical, create a difference between them; a memory circuit! It seems that paradox, though chaotic, contains order within it.

Now consider this three-way quarrel:

Moe: "Larry and Curly are liars."

Larry: "Curly and Moe are liars."

Curly: "Moe and Larry are liars."

M = not L nor K

L = not K nor M

K = not M nor L

This system has three solutions: (true, false, false), (false, true, false), and (false, false, true). *One* of the Stooges is honest; but which one?

B. *The Anti-Diagonal*

Here are two paradoxes of mathematical logic, generated by an "anti-diagonal" process:

Grelling's Paradox. Call an adjective 'autological' if it applies to itself, 'heterological' if it does not: "A" is heterological = "A" is not A.

Thus, 'short' and 'polysyllabic' are autological, but 'long' and 'monosyllabic' are heterological.

Is 'heterological' heterological?

"Heterological" is heterological = "Heterological" is not heterological.

It is to the extent that it isn't!

Quine's Paradox. Let "quining" be the action of preceding a sentence fragment by its own quotation. For instance, when you quine the fragment "is true when quined", you get:

"Is true when quined" is true when quined.

— a sentence which declares itself true.

In general the sentence:

"Has property P when quined" has property P when quined.

is equivalent to the sentence:

"This sentence has property P."

Now consider the sentence:

"Is false when quined" is false when quined.

That sentence declares itself false. Is it true or false?

C. *Russell's Paradox*

Let R be the set of all sets which do not contain themselves:

$$R = \{ x \mid x \notin x \}$$

R is an anti-diagonal set. Is it an element of itself?

In general: $x \in R = x \notin x$

and therefore: $R \in R = R \notin R$.

Therefore R is paradoxical. Does R exist?

Here's a close relative of Russell's set; the "Short-Circuit Set":

$$S = \{x : S \notin S\}.$$

S is a constant-valued set, like the universal and null sets:

For all x, $(x \in S) = (S \notin S) = (S \in S)$.

All sets are paradox elements for S.

Bertrand Russell told a story about the barber of a Spanish village. Being the only barber in town, he boasted that he shaves all those — and only those — who do not shave themselves. Does the barber shave himself?

To this legend I add a political postscript. That very village is guarded by the watchmen, whose job is to watch all those, and only those, who do not watch themselves. But who shall watch the watchmen?

(Thus honesty in government is truly imaginary!)

The town is also guarded by the trusty watchdog, whose job is to watch all houses, and only those houses, that are not watched by their owners. Does the watchdog watch the doghouse?

Not too long ago that village sent its men off to fight the Great War, which was a war to end all wars, and only those wars, which do not end themselves. Did the Great War end itself?

That village's priest often ponders this theological riddle:

God is worshipped by all those, and only those, who do not worship themselves. Does God worship God?

D. *Parity of Infinity*

What is the *parity* of infinity? Is infinity odd or even? In the standard Cantorian theory of infinity, $\aleph_0$ equals its own successor:

$$\aleph_0 = \aleph_0 + 1$$

But therefore $\aleph_0$ is even if and only if it is odd! Since $\aleph_0$ is a *counting* number, it is presumably an integer; but any integer is even *or else* odd!

Infinity has paradoxical parity. We encounter this paradox when we try to define the limit of an infinite oscillation. Consider the sequence $\{x_0, x_1, x_2, \ldots\}$:

$$x_0 = \text{true};$$

$$x_{n+1} = \text{not}(x_n) \quad , \quad \text{for all n.}$$

The x's make an oscillation: {T,F,T,F,...} Now, can we define a *limit* of this sequence? $\text{Lim}(x_n) = ?$ If we cannot define this limit, in what sense does infinity have a parity at all? And if no parity, why other arithmetical properties?

We can illuminate the Parity of Infinity paradox with a fictional lamp; the Thompson Lamp, capable of infinitely making many power-toggles in a finite time. The Thompson Lamp clicks on for one minute, then off for a half-minute, then back on for a quarter-minute; then off for an eighth-minute; and so on, in geometrically decreasing intervals until the limit at two minutes, at which point the Lamp stops clicking. After the second minute, is the Lamp on or off?

E. *Santa Sentences*

Suppose that a young child were to proclaim:

"If I'm not mistaken, then Santa Claus exists."

If one assumes that Boolean logic applies to this sentence, then its mere existence would imply the existence of Santa Claus!

Why? Well, let the child's statement be symbolized by 'R', and the statement "Santa exists" be symbolized by 'S'. Then we have the equation:

$$R \quad = \quad \text{if R then S} \quad = \quad R \Rightarrow S \quad = \quad (\text{not R}) \text{ or } S\ .$$

Then we have this line of argument:

R = (R ⇒ S); assume that R is either true or false.

If R is false, then R = (false ⇒ S) = (true or S) = true.

R = false implies that R = true;

therefore (by contradiction) R must be true.

Since R = (R ⇒ S), (R ⇒ S) is also true.

R is true, (R ⇒ S) is true; so S is true.

Therefore Santa Claus exists!

This proof uses proof by contradiction; an indirect method, suitable for avoiding overt mention of paradox. Here is another argument, one which confronts the paradox directly:

S is either true or false. If it's true, then so is R:

R = (not R) or true = true.

No problem. But if S is false, then R becomes a liar paradox:

R = (not R) or false = not R .

If S is false, then R is non-boolean.

therefore: If R is boolean, then S is true.

Note that both arguments work equally well to prove any other statement besides S to be true; one need merely display the appropriate "santa sentence". Thus, for instance, if some skeptic were to declare:

"If I'm not mistaken, then Santa Claus does not exist."

— then by identical arguments we can prove that Santa Claus does *not* exist!

Given two opposite Santa sentences:

$R_1 = (R_1 \Rightarrow S)$; $R_2 = (R_2 \Rightarrow \text{not } S)$

then at least one of them must be paradoxical.

We can create Santa sentences by Grelling's method. Let us call an adjective "Santa-logical" when it applies to itself only if Santa Claus exists;

"A" is Santa-logical = If "A" is A, then Santa exists.

Is "Santa-logical" Santa-logical?

"Santa-logical" is Santa-logical =

If "Santa-logical" is Santa-logical, then Santa exists.

Here is a Santa sentence via quining:

"Implies that Santa Claus exists when quined" implies that Santa Claus exists when quined.

If that statement is boolean, then Santa Claus exists.

Here's the "Santa Set for sentence G":

$$S_G \quad = \quad \{ x \mid (x \in x) \Rightarrow G \}$$

S_G is the set of all sets which contain themselves only if sentence G is true:

$$x \in S_G \ = \ ((x \in x) \Rightarrow G) .$$

Then "S_G is an element of S_G" equals a Santa sentence for G:

$$S_G \in S_G \ = \ ((S_G \in S_G) \Rightarrow G) .$$

"$S_G \in S_G$", if boolean, makes G equal true; another one of Santa's gifts. If G is false, then "$S_G \in S_G$" is paradoxical.

One could presumably tell Barber-like stories about Santa sets. For instance, in another Spanish village, the barber takes weekends off; so he shaves all those, and only those, who shave themselves only on the weekend:

B shaves M = If M shaves M, then it's the weekend.

One fine day someone asked: does the barber shave himself?

B shaves B = If B shaves B, then it's the weekend.

Has it been weekends there ever since?

That village is watched by the watchmen, who watch all those, and only those, who watch themselves only when fortune smiles:

W watches C = if C watches C, then fortune smiles.

One fine day someone asked: who watches the watchmen?

W watches W = if W watches W, then fortune smiles.

Does fortune smile on that village?

Recently that village saw the end of the Cold War, which ended all wars, and only those wars, which end themselves only if money talks:

CW ends W = if W ends W, then money talks.

Did the Cold War end itself?

CW ends CW = if CW ends CW, then money talks.

Does money talk?

That village's priest proclaimed this theological doctrine:

God blesses all those, and only those, who bless themselves only when there is peace:

G blesses S = If S blesses S, then there is peace.

One fine day someone asked the priest: Does God bless God?

G blesses G = If G blesses G, then there is peace.

Is there peace?

Finally, consider the case of Promenides the Cretan, who always disagrees with Epimenides. Recall that Epimenides the Cretan accused all Cretans of being liars, including himself. If we let E = Epimenides, P = Promenides, and H = "honest Cretans exist", then:

$$E = (\text{not } E) \text{ and } (\text{not } H)$$

$$P = \text{not } E = \text{not } ((\text{not } E) \text{ and } (\text{not } H))$$

$$= E \text{ or } H = (\text{not } P) \text{ or } H = (P \Rightarrow H)$$

Thus we get this dialog:

Epimenides: All Cretans are liars.
Promenides: You're a liar.
Epimenides: All Cretans are liars, and I am a liar.
Promenides: Either some Cretan is honest, or you're honest.
Epimenides: You're a liar.
Promenides: Either some Cretan is honest, or I'm a liar.
Epimenides: All Cretans are liars, including myself.
Promenides: If I am honest, then some Cretan is honest.

Promenides is the Santa Claus of Crete; for if his statement is boolean, then some honest Cretan exists.

F. *Antistrephon*

That is, "The Retort". This is a tale of the law-courts, dating back to Ancient Greece. Protagoras agreed to train Euathius to be a lawyer, on the condition that his fee be paid, or not paid, according as Euathius win, or lose, his first case in court. (That way Protagoras had an incentive to train his pupil well; but it seems that he trained him too well!) Euathius delayed starting his practice so long that Protagoras lost patience and brought him to court, suing him for the fee. Euathius chose to be his own lawyer; this was his first case.

Protagoras said, "If I win this case, then according to the judgement of the court, Euathius must pay me; if I lose this case, then according to our contract he must pay me. In either case he must pay me."

Euathius retorted, "If Protagoras loses this case, then according to the judgement of the court I need not pay him; if he wins, then according to our contract I need not pay him. In either case I need not pay."

How should the judge rule?

Here's another way to present this paradox:

According to the contract, Euathius will avoid paying the fee — that is, win this lawsuit — exactly if he loses his first case; and Protagoras will get the fee — that is, win this lawsuit — exactly if Euathius wins his first case. But this lawsuit *is* Euathius's first case, and he will win it exactly if Protagoras loses. Therefore Euathius wins the suit if and only if he loses it; ditto for Protagoras.

G. *Size Paradoxes*

The Heap. Surely one grain of sand does not constitute a heap of sand. Surely adding another grain will not make it a heap. Nor will adding another, or another, or another. In fact, it seems absurd to say that adding one single grain of sand will turn a non-heap into a heap. By adding enough ones, we can reach any finite number; therefore no finite number of grains of sand will form a sand heap. Yet sand heaps exist; and they contain a finite number of grains of sand!

Let's take it in the opposite direction. Let us grant that a finite sand heap exists. Surely removing one grain of sand will not make it a non-heap. Nor will removing another, nor another, nor another. By subtracting enough ones, we can reduce any finite number to one. Therefore one grain of sand makes a heap!

What went wrong?

Let's try a third time. Grant that one grain of sand forms no heap; but that some finite number of grains do form a heap. If we move a single grain at a time from the heap to the non-heap, then they will eventually become indistinguishable in size. Which then will be the heap, and which the nonheap?

The First Boring Number. This is closely related to the paradox of the Heap. For let us ask the question: are there any boring (that is, uninteresting) numbers? If there are, then surely that collection has a *smallest* element; the *first* uninteresting number. How interesting!

Thus we find a contradiction; and this seems to imply that there are no uninteresting numbers!

But in practice, most persons will agree that most numbers are stiflingly boring, with no interesting features whatsoever! What then becomes of the above argument?

Simply this; that the *smallest* boring number is inherently paradoxical. If being the first boring number were a number's only claim to our interest, then we would find it interesting if and only if we do *not* find it interesting.

Which then is it?

Berry's Paradox. What is "the smallest number that cannot be defined in less than twenty syllables"? If this defines a number, then we have done so in nineteen syllables! So this defines a number if and only if it does not.

Presumably Berry's number equals the first boring number, if your boredom threshold is twenty syllables.

These paradoxes connect to the paradox of the Heap by simple psychology. If, for some mad reason, you actually *did* try to count the number of grains in a sand heap, then you will eventually get bored with such an absurd task. Your attention would wander; you would lose track of all those sand grains; errors would accumulate, and the number would become indefinite.

The Heap arises at the onset of uncertainty. In practice, the Heap contains a boring number of sand grains; and the smallest Heap contains the smallest boring number of sand grains!

Finitude. Finite is the opposite of infinite; but in paradox-land, that's no excuse! In fact the concept of finiteness is highly paradoxical; for though finite numbers are finite individually and in finite groups, yet they form an infinity.

Let us attempt to *evaluate* finiteness. Let F = 'finitude', or 'finity'; the *generic* finite expression. You may replace it with any finite expression.

Is Finity finite?

If F is finite, then you can replace it by F+1, and thus by F+2, F+3, etc. But such a substitution, indefinitely prolonged, yields an infinity.

If F is not finite, then you may not replace F by F, nor by any expression involving F; you must replace F by a well-founded finite expression, which will then be limited.

Therefore F is finite if and only if it is not finite.

Finitude is *just short* of infinity! It is infinity seen from underneath. You may think of it as that mysterious 'large finite number' N, larger than any number you care to mention.

Call a number "large" if it is bigger than any number you care to mention; that is, bigger than any interesting number. Call a number "medium" if it is bigger than some boring number but less than some interesting number. Call a number "small" if it is less than any boring number. Presumably Finitude is the smallest large number; that is, the smallest number greater than any interesting number. (How interesting!)

Finitude is dual to the Heap, which is the largest number less than any uninteresting number. The Heap is the lower limit of boredom; Finitude is the upper limit of interest.

We get these inequalities:

small interesting numbers
< The Heap = first boring number = last small number
< medium numbers
< Finitude = last interesting number = first large number
< large boring numbers

Finally, consider this Berry-like definition:

"One plus the largest number defineable in less than twenty syllables."

If this defines a number, then it has done so in only nineteen syllables, and therefore is its own successor. If your boredom threshold is 20 syllables, then this number = "one plus the last interesting number" = Finitude.

H. *Game Paradoxes*

Hypergame and the Mortal

Let "Hypergame" be the game whose initial position is the set of all "short" games — that is, all games that end in a finite number of moves. For one's first move in Hypergame, one may move to the initial position of any short game.

Is Hypergame short?

If Hypergame is short, then the first move in Hypergame can be to — Hypergame! But this implies an endless loop, thus making Hypergame no longer a short game!

But if Hypergame is *not* short, then its first move must be into a short game; thus play is bound to be finite, and Hypergame a short game.

The Hypergame paradox resembles the paradox of Finitude. Presumably Hypergame lasts Finitude moves; one plus the largest number definable in less than twenty syllables.

Dear reader, allow me to dramatize this paradox by means of a fictional story about a mythical being. This entity I shall dub "the Mortal"; an unborn spirit who must now make this fatal choice; to choose some mortal form to incarnate as, and thus be be doomed to certain death.

The Mortal has a choice of dooms. Is the Mortal doomed?

Normalcy and the Rebels

Define a game as "normal" if and only if it does not offer the option of moving to its own starting position:

G is normal = the move G ⇨ G is not legal.

Let "Normalcy" be the game of all normal games. In it one can move to the initial position of any normal game:

The move N ⇨ G is legal = the move G ⇨ G is not legal.

Is Normalcy normal? Let G = N:

The move N ⇨ N is legal = the move N ⇨ N is not legal.

Normalcy is normal if and only if it is *ab*normal!

That was Russell's paradox for game theory. Now consider this:

The Rebel is a being who must become one who changes. The Rebel may become all those, and only those, who do not remain themselves:

R may become B = B may not become B .

Can the Rebel remain a Rebel?

R may become R = R may not become R .

A Santa Rebel may become all those, and only those, who remain themselves only if Santa Claus exists:

SR may become B = ((B may become B) ⇒ Santa exists)

Therefore: SR may become SR = ((SR may become SR) ⇒ Santa exists)

If the pivot bit is boolean, then Santa Claus exists!

I. *Cantor's Paradox*

Cantor's proof of the "uncountability" of the continuum relies on an "anti-diagonalization" process. Suppose we had a countable list of the real numbers between 0 and 1:

$R_1 = 0 . D_{11}, D_{12}, D_{13}, D_{14} \dots$

$R_2 = 0 . D_{21}, D_{22}, D_{23}, D_{24} \dots$

$R_3 = 0 . D_{31}, D_{32}, D_{33}, D_{34} \dots$

$R_4 = 0 . D_{41}, D_{42}, D_{43}, D_{44} \dots$

...

where D_{NM} is the Mth binary digit of the Nth number.

Then we define Cantor's "anti-diagonal" number:

$C = 0 . \text{ not } D_{11}, \text{ not } D_{22}, \text{ not } D_{33}, \text{ not } D_{44} \dots$

If $C = R_N$ for any N, then $D_{NX} = \text{not } D_{XX}$;

Therefore $D_{NN} = \text{not } D_{NN}$; the pivot bit buzzes.

From this paradox, Cantor deduced that the continuum has too many points to be counted, and thus is of a "higher order" of infinity. Thus a single buzzing bit implies infinities beyond infinities! Was more ever made from less?

I say, why seek "transfinite cardinals", whatever those are? Why not ask for Santa Claus? In this spirit, I introduce the *Santa*-diagonal number:

$S = 0 . (D_{11} \Rightarrow \text{Santa}), (D_{22} \Rightarrow \text{Santa}), (D_{33} \Rightarrow \text{Santa}) \dots$

If $S = R_M$ for any M, then $D_{MX} = (D_{XX} \Rightarrow \text{Santa exists})$;

Therefore $D_{MM} = (D_{MM} \Rightarrow \text{Santa exists})$.

If the pivot bit is boolean, then Santa Claus exists!

J. *Paradox of the Boundary*

The continuum is paradoxical because it is *continuous*, and boolean logic is discontinuous. This topological difference yields a logical riddle which I call the Paradox of the Boundary.

The paradox of the boundary has many formulations, such as:

What day is midnight?

Is noon A.M. or P.M.?

Is dawn day or night? Is dusk?

Which side of the mirror is Alice on?

Which country owns the border?

Is zero plus or minus?

If a statement is true at point A and false at point B, then somewhere in-between lies a boundary. At any point on the boundary, is the statement true, or is it false?

(If line segment AB spanned the island of Crete, then somewhere in the middle we should, of course, find Epimenides!)

Chapter 2

Ternary Logic

The Third Value
Inner functions; not, and, or, yet, others
Ternary circuits; phased-delay, dual-rail
Brownian forms and laws
Bracket forms

A. *The Third Value*

Chapter 1 posed the problem of paradox but left it undecided. Is the Liar true or false? Boolean logic cannot answer. All these paradoxes point to a *third value*, equal to its own negation. So let there be paradox:

$$I \quad = \quad \sim I$$

In this equation, "~" denotes "not", or negation; and "I" denotes "intermediate", or "inner", or "imaginary", or "indeterminate", or "inconsistent"; all equally valid interpretations of paradox.

If we want a logic with three truth values (F for false, I for intermediate, and T for true), then we need to know how the truth-functional operators "and", "or", "not", "iff", and so on are defined. So far we have "not"'s truth table:

$$\sim F = T \quad ; \quad \sim I = I \quad ; \quad \sim T = F$$

T and F "pivot around" the intermediate value I, which is left fixed.

What of "and"? "Or"? These operators should have the many of the same properties as their binary conterparts; but which ones?

In this book, I shall concentrate on two systems for ternary logic; "Kleenean" logic, and "Bochvarian" logic. The first system is primary; for the second is derivable from Kleenean logic.

In Kleenean logic, $\vee$ ("or") is the maximum operator, while $\wedge$ ("and") is the minimum operator on the following linear order:

$$F < I < T$$

$$x \wedge y = x \quad \text{iff} \quad x \vee y = y \quad \text{iff} \quad x \leq y\,.$$

In Kleenean logic, "I" is the "intermediate" value.

The Kleenean operators satisfy these four axioms:

Commutativity	$x \wedge y = y \wedge x \;;\; x \vee y = y \vee x$
Identities	$x \wedge T = x \vee F = x$
Dominance	$x \wedge F = F \;;\; x \vee T = T$
Recall	$x \wedge x = x \vee x = x$

Exercise for the student: From the above four axioms alone, derive the following truth tables:

x:	∧ y: f	i	t	∨ y: f	i	t
f	f	f	f	f	i	t
i	f	i	i	i	i	t
t	f	i	t	t	t	t

The Bochvarian operators can be defined thus:

"$\vee_B$" is the maximum operator on $F < T < I$

"$\wedge_B$" is the minimum operator on $I < F < T$

That is, I is an "extreme", or "absorbing" value;

$$x \vee_B I = x \wedge_B I = I \quad , \text{ for all } x.$$

Bochvarian operators satisfy these axioms:

Commutativity	$x \wedge_B y = y \wedge_B x$; $x \vee_B y = y \vee_B x$
Identities	$x \wedge_B T = x \vee_B F = x$
Dominance	$x \wedge_B I = x \vee_B I = I$
Recall	$x \wedge_B x = x \vee_B x = x$

Exercise for the student: From the above four axioms alone, derive the following truth tables:

x:	$\wedge_B$ y: f	i	t	$\vee_B$ y: f	i	t
f	f	i	f	f	i	t
i	i	i	i	i	i	i
t	f	i	t	t	i	t

B. *Inner Functions*

Call a function an "inner" function if it can be defined from Kleenean "and", "or", "not", and the three values F, I, T. They include:

$a \Rrightarrow b$	=	$(\sim a) \vee b$
a iff b	=	$(a \Rrightarrow b) \wedge (b \Rrightarrow a)$
a xor b	=	$(a \wedge \sim b) \vee (b \wedge \sim a)$
a nor b	=	$\sim (a \vee b)$
a nand b	=	$\sim (a \wedge b)$

The "majority" operator M has two dual definitions:

$$M(a,b,c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$$

$$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$$

Here is the "semi-lattice operator":

a min b = M(a,I,b)

We can also call min "yet": a min b = "a yet b". Thus I = T min F ; so imaginary equals true yet false.

Here are the upper and lower differentials:

$$Dx = x \Rrightarrow x = x \text{ iff } x = x \vee \sim x$$

$$dx = x \text{ minus } x = x \text{ xor } x = x \wedge \sim x$$

This, then, is inner delta logic, a.k.a. Kleenean logic. It contains the boolean values, plus paradoxes and lattice operators.

Here are truth tables for the functions defined above:

x:	~x:	∧ y: f i t	∨ y: f i t	⇒ y: f i t	nor y: f i t	nand y: f i t
f	t	f f f	f i t	t t t	t i f	f i t
i	i	f i i	i i t	i i t	i i i	i i i
t	f	f i t	t t t	f i t	f i t	t i f

x:	iff y: f i t	xor y: f i t	min y: f i t	Dx:	dx:	M(x,y,z) majority
f	t i f	f i t	f i i	t	f	y and z
i	i i i	i i i	i i i	i	i	y min z
t	f i t	t i f	i i t	t	f	y or z

Exercise for the student: prove these identities:

$$x \lor_B y = (x \lor y) \land Dx \land Dy$$

$$x \land_B y = (x \land y) \lor dx \lor dy$$

so the Bochvarian operators can be derived from Kleenean;

$$x \text{ iff } y = (\sim x \lor y) \land (x \lor \sim y) = (\sim x \lor_B y) \land_B (x \lor_B \sim y)$$

$$x \text{ xor } y = (\sim x \land y) \lor (x \land \sim y) = (\sim x \land_B y) \lor_B (x \land_B \sim y)$$

so iff and xor are common to Kleenean and Bochvarian logic.

C. *Ternary Circuits*

One can implement ternary logic in switching circuits, two different ways; via "phased delay" and via "dual rail".

In "phased delay", one permits a standard switching circuit to oscillate. T then means "on", F means "off", and I mean oscillation. The sentence A has value A(n) at time n. We define ~A, $A \wedge B$, and $A \vee B$ this way:

$$(\sim A)(n) = \sim(A(n-1))$$

$$(A \wedge B)(n) = M(A(n-1), I(n), A(n-2)) \wedge M(B(n-1), I(n), B(n-2))$$

$$(A \vee B)(n) = M(A(n-1), I(n), A(n-2)) \vee M(B(n-1), I(n), B(n-2))$$

- where I is a "clock oscillation": $I(n) = \sim(I(n-1))$.

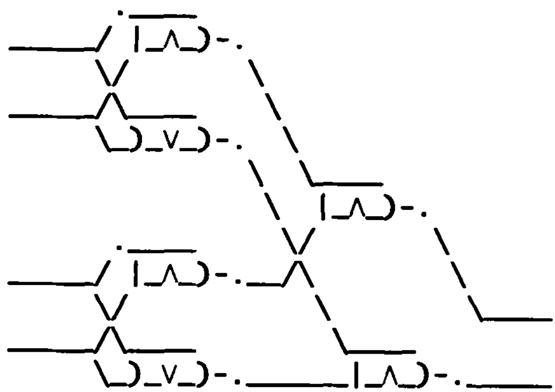
A "dual rail" circuit replaces all wires in a standard switching circuit with pairs of wires. T then means "both rails on", F means "both rails off", and I means that just one of the two rails is on.

The gates then are:

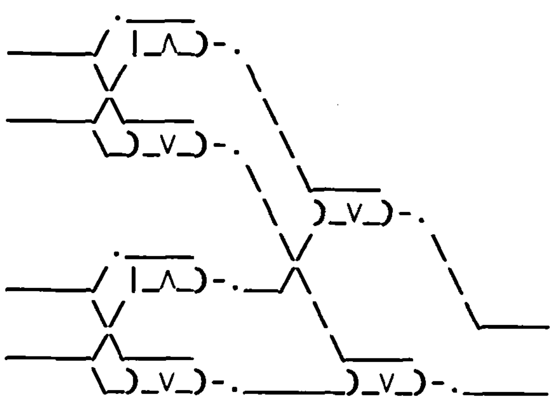
"not":

```
____________________|\_____  _______
                    |/     \/
____________________|\_____/\_______
                    |/
```

"and":



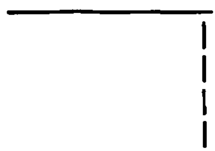
"or":



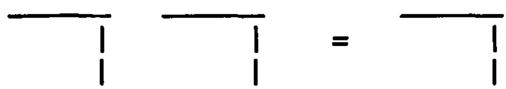
We can go from dual-rail to phased-delay, and vice versa, by means of 'jitter' gates, which connects the single phased-delay line to one of the dual rails, oscillating in time with the clock pulse.

D. *Brownian Forms*

Make a mark. This act generates a form:



A mark marks a space. Any space, if marked, remains marked if the mark is repeated:

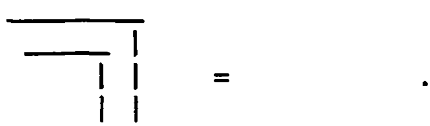


where "=" denotes "is confused with".

This is the crossed form, or "mark".

Each mark is a call; to recall is to call.

A mark is a crossing, between marked and unmarked space. To cross twice is not to cross; thus a mark within a mark is indistinguishable from an unmarked space:



This is the uncrossed form, or "void".

Each mark is a crossing; to recross is not to cross.

Thus we get the "arithmetic initials" for G. Spencer Brown's famous Laws of Form. In his book, *Laws of Form*, G.S.Brown demonstrated that these suffice to evaluate all formal expressions in Brown's calculus; and that these forms obey two "algebraic initials":

```
_______________           ___________________
 ____   ____   |             ______   ______  |
  A |    B |   | C   =       A C |    B C |   |   ; "Transposition"

      ________
      ___     |
       A| A   |      =                              ;    "Position"
```

He proved that these axioms are consistent, independent, and complete; that is, they prove all arithmetic identities. This "primary algebra" can be identified with Boolean logic. The usual matching is:

```
  ________
  _____   |
       |  |     (void)          F          (false)
       |  |

  ______
       |
       |        (mark)          T           (true)

    X Y         (juxtapose)     X or Y     (disjunction)

  ________
    X     |      (crossing)     not X       (negation)

  _______________
  ____    ____   |
   X |     Y |   |              X and Y    (conjunction)

  ____
   X |  Y                               If X, then Y

  _______
   X Y |                                X nor Y
```

```
 _____  _____
  X |    Y |                                  X nand Y

 __________  _________
  ____    |   ____    |
  X | Y |     Y | X |                         X xor Y

 __________________________
  ______  ______  ______  |
  X Y |   Y Z |   Z X |   |                   Majority (X,Y,Z)

    ______________   ______________   ______________
     ____  ____  |    ____  ____  |    ____  ____  |
=    X |   Y |   |    Y |   Z |   |    Z |   X |   |
```

There is a complementary interpretation:

```
void                                 true

mark                                 false

X Y         (juxtapose)             X and Y

 _____
  X  |      (crossing)               not X

 _______________
  ____   ____  |
  X |    Y |   |                    X or Y

 _____________
       ____  |
  X    Y |   |                     If X then Y

 ________
  X Y  |                            X nand Y

 _____  _____
  X |    Y |                         X nor Y
```

The standard interpretation is usually preferred because it has a simpler implication operator.

We can extend Brown's calculus to Kleenean logic by introducing a new form called "curl":

```
 ________
        |
    |   |      (curl)      i
    |___|
```

It has these relations:

```
 ________
 _____  |      _____      _____  _____      ____       ____  ____      ____
     |  |  =       |   ;      |      |  =      |   ;      |     |  =      |
   |_|  |        |_|        |_|      |         |        |_|   |_|       |_|
```

In the standard interpretation, we have:

```
 ___________
   ____    |                  _____
   x | x   |   =   dx   ;     x |    x   =   Dx      :  Differentials
```

(x min y) = M(x, curl, y)

```
             _____________     _____________     _____________
              ____  _____ |     _____  ____ |     ____  ____ |
  =           x |   \_| |       \_|   y | |       y |  x | |

            ________________________________________
             ________  _________  _________        |
                _____ |  _____  |          |       |
  =          x   \_| |    \_| y |    y x   |       |
```

E. *Bracket Forms*

Bracket forms are Brownian forms, adapted for the typewriter. They use brackets instead of Brown's mark:

[A] instead of $\overline{\text{A}\quad}|$.

The "arithmetic initials" are then:

[ ] [ ] = [ ] .

[[ ]] = .

If we call [] "1" and [[]] "0", then we get these equations:

[0] = 1 ; [1] = 0 ;

0 0 = 0 ; 0 1 = 1 0 = 1 1 = 1 .

G.S.Brown's algebraic initials are:

[[a] [b]] c = [[a c] [b c]] .

[[a] a] = .

For delta, we introduce a new form, 6, with these arithmetic relations:

[6] = 6

6 [] = []

6 6 = 6

These three relations, along with Brown's arithmetic initials, plus commutativity of juxtaposition, yields this table of form equations:

00 = 0 ; 06 = 6 ; 01 = 1 ; [0] = 1 ;

60 = 6 ; 66 = 6 ; 61 = 1 ; [6] = 6 ;

10 = 1 ; 16 = 1 ; 11 = 1 ; [1] = 0 .

If we identify F = 0, I = 6, T = 1, then we get this matching of bracket forms with Kleenean logic:

[ ]	=	true ;
6	=	intermediate;
[[ ]]	=	false ;
[A]	=	not A ;
A B	=	A or B ;
[[A] [B]]	=	A and B;
[A] B	=	if A then B ;
[A B]	=	A nor B;
[A] [B]	=	A nand B;
[[A] B] [[B] A]	=	A xor B ;
[[[A] B] [[B] A]]	=	A iff B ;
[[A B] [B C] [C A]]	=	Majority(A, B, C)
[[A B] [B 6] [A 6]]	=	A min B
[A [A]]	=	dA
A [A]	=	DA

The identification F = 1, I = 6, T = 0 yields this dual interpretation:

[ ]	=	false ;
6	=	intermediate;
[[ ]]	=	true ;
[A]	=	not A ;
A B	=	A and B ;
[[A] [B]]	=	A or B;
[A [B]]	=	if A then B ;
[A B]	=	A nor B;
[A] [B]	=	A nand B;
[[A] B] [[B] A]	=	A iff B ;
[[[A] B] [[B] A]]	=	A xor B ;
[[A] [B]] [[B] [C]] [[C] [A]]	=	Majority(A, B, C)
[[A] [B]] [[B] 6] [[C] 6]	=	A min B
[A [A]]	=	DA
A [A]	=	dA

Exercise for the student:

Prove that the forms 0, 1, and 6 have these identities:

Transposition:	[[a] [b]] c	=	[[a c] [b c]]
Occultation:	[[a] b] a	=	a
Relocation:	[[a] a] [6]	=	6

The next chapter will prove that these axioms (plus the commutativity and associativity of juxtaposition) suffice to prove all Kleenean identities.

Chapter 3

Ternary Algebra

Bracket Algebra
Kleenean Laws
Normal Forms
Completeness

A. *Bracket Algebra*

Call these the bracket axioms:

Transposition:	[[a] [b]] c	=	[[a c] [b c]]
Occultation:	[[a] b] a	=	a
Relocation:	[[a] a] [6]	=	6

We can verify these identities by case-checking; 27 cases for transposition, 9 cases for occultation, 3 for relocation. In addition, we assume commutativity and associativity for juxtaposition:

a b = b a ; a b c = a b c

These equations are implicit in the bracket notation. Brackets distinguish only inside from outside, not left from right.

When we translate Relocation into Kleenean logic, we see that the truth value "I" self-refers: I = "If I'm not mistaken, then x is true and false". Therefore "I" equals a Differential Santa Sentence!

From the bracket axioms we can derive theorems:

Fixity. [6] = 6

Proof. [6] = [[[6]][6]][6] = 6 occ., reloc.

Location. [x[x]] 6 = 6

Proof. [x[x]]6 = [x[x]][6] = 6 fix., reloc.

Situation. x[x] 6 = x[x]

Proof. x[x]6 = [6] x[x] fix.

= [[x[x]] 6] x[x] loc.

= x[x] occ.

Reflexion. [[x]] = x

Proof. [[x]] = [[x] [[]x]] occ.

= [[ ] [[]]] x trans.

= x occ.

Identity. [[]] x = x

Proof. Directly from Occultation.

Domination. [] x = []

Proof. [] x = [[[] x]] = [ ] ref., occ.

Recall. x x = x

Proof. x x = [[x]]x = x ref., occ

Note: from Fixity, Reflexion, Domination, Recall, and Identity, we can derive the tables for [a] and ab. The bracket axioms yield the bracket arithmetic.

Reoccultation. [xy] [x] = [x]

Proof. [xy] [x]	=	[[[x]] y] [x]	ref.
	=	[x]	occ.

Echelon. [[[x]y]z] = [xz] [[y]z]

Proof. [[[x]y]z]	=	[[[x][[y]]]z]	ref.
	=	[[[xz][[y]z]]]	trans.
	=	[xz] [[y]z]	ref.

Modified Generation. [[xy]y] = [[x]y] [y[y]]

Proof. [[xy]y]	=	[[[[x]] [[y]]] y]	ref.
	=	[[[[x]y] [[y]y]]]	trans.
	=	[[x]y] [[y]y]	ref.

Modified Extension. [[x]y] [[x][y]] = [[x] [y[y]]]

Proof. [[x]y] [[x][y]]	=	[[[[x]y] [[x][y]]]]	ref.
	=	[[[y] [[y]]] [x]]	trans.
	=	[[x] [y[y]]]	ref.

Inverse Transposition. [[xy][z]] = [[x][z]] [[y][z]]

Proof. [[xy][z]]	=	[[[[x]][[y]]][z]]	ref.
	=	[[[[x][z]][[y][z]]]]	trans.
	=	[[x][z]] [[y][z]]	ref.

Modified Transposition. [[x] [yw][zw]] = [[x][y][z]] [[x][w]]

Proof. [[x] [yw][zw]]	=	[[x] [[[yw][zw]]]]	ref.
	=	[[x] [[[y][z]]w]]	trans.
	=	[[x] [[[y][z]] [[w]]]]	ref.
	=	[[[[x][y][z]] [[x][w]]]]	trans.
	=	[[x][y][z]] [[x][w]]	ref.

Majority. [[xy][yz][zx]] = [[x][y]] [[y][z]] [[z][x]]

Proof. [[xy][yz][zx]]	=	[[xy][x][y]] [[xy][z]]	mod.trans.
	=	[[x][y]] [[xy][z]]	reocc.
	=	[[x][y]] [[x][z]] [[y][x]]	inv.trans.

Retransposition (3 terms). $[a_1x][a_2x][a_3x]$ = $[\ [[a_1][a_2][a_3]]\ x\ ]$

Proof. $[a_1x][a_2x][a_3x]$	=	$[[[a_1x][a_2x]]]\ [a_3x]$	ref.
	=	$[[[a_1][a_2]]x]\ [a_3x]$	trans.
	=	$[[\ [[[a_1][a_2]]x]\ [a_3x]\ ]]$	ref.
	=	$[[\ [[[a_1][a_2]]]\ [a_3]\ ]\ x\ ]$	trans.
	=	$[[\ [a_1][a_2][a_3]\ ]\ x\ ]$	ref.

Retransposition (n terms).

$[a_1x]\,[a_2x]\ldots[a_nx] \quad = \quad [\;[[a_1][a_2]\ldots[a_n]]\;x\;]$

Proof is by induction on n. Given that the theorem is true for n, the following proves it for n+1:

$[a_1x][a_2x]\ldots[a_nx][a_{n+1}x]$	=	$[\;[[a_1][a_2]\ldots[a_n]]\;x\;]\;\;[a_{n+1}x]$	*
	=	$[[\;[\;[[a_1][a_2]\ldots[a_n]]\;x\;]\;\;[a_{n+1}x]\;]]$	ref.
	=	$[[\;[\;[[a_1][a_2]\ldots[a_n]]\;]\;\;[a_{n+1}]\;]\;x\;]$	trans.
	=	$[[\;[a_1][a_2]\ldots[a_n][a_{n+1}]\;]\;x\;]$	ref.

Cross-Transposition:

$[\;[[a]x]\;\;[[b][x]]\;\;[x[x]]\;] \;=\; [ax]\;\;[b[x]]\;\;[x[x]]$

Proof. $[\;[[a]x]\;\;[[b][x]]\;\;[x[x]]\;]$

=	$[\;\;[[a]x]\;\;[[b][x]]\;\;[\;[\;[x]\;]\;[\;x\;]\;]\;\;]$	ref.
=	$[\;\;[\;\;[\;[[a]x]\;[[b][x]]\;[x]\;]\;\;\;[\;[[a]x]\;[[b][x]]\;x\;]\;]\;\;]$	trans.
=	$[\;[[a]x]\;[[b][x]]\;[x]\;]\;\;[\;[[a]x]\;[[b][x]]\;x\;]$	ref.
=	$[\;[[a][[x]]]\;[[b][x]]\;[x]\;]\;\;[\;[[a]x]\;[[b][x]]\;x\;]$	ref.
=	$[\;[[b][x]]\;[x]\;]\;\;[\;[[a]x]\;x\;]$	occ.
=	$[\;[[b][x]]\;[x]\;]\;\;[\;[[a][[x]]]\;x\;]$	ref.
=	$[\;[[b[x]][x[x]]]\;]\;\;[\;[[ax][[x]x]]\;]$	trans.
=	$[b[x]]\;[x[x]]\;\;[ax]\;[[x]x]$	ref.
=	$[ax]\;\;[b[x]]\;\;[x[x]]$	recall

This result translates into Kleenean in two dual ways:

$(\,A \wedge x\,) \vee (\,B \wedge \sim x\,) \vee dx \quad = \quad (\,A \vee \sim x\,) \wedge (\,B \vee x\,) \wedge Dx$

$(\,a \vee x\,) \wedge (\,b \vee \sim x\,) \wedge\; Dx \quad = \quad (\,a \wedge \sim x\,) \vee (\,b \wedge x\,) \vee dx$

General Cross-Transposition

[[a][x]] [[b]x] [[c][x]x] = [[a[x]] [bx] [abc] [x[x]]]

Proof. from right to left.

[[a[x]] [bx] [abc] [x[x]]]

=	[[[[a[x]] [bx] [x[x]]]] [abc]]	ref.
=	[[[[a][x]] [[b]x] [x[x]]] [abc]]	crosstrans.
=	[[[[a][x][abc]] [[b]x[abc]] [x[x][abc]]]]	trans.
=	[[a][x][abc]] [[b]x[abc]] [x[x][abc]]	ref.
=	[[a][x]] [[b]x] [x[x][abc]]	reocc.
=	[[a][x]] [[b]x] [x[x][[[a]][[b]][[c]]]]	ref.
=	[[a][x]] [[b]x] [[[x[x][a]][x[x][b]][x[x][c]]]]	trans.
=	[[a][x]] [[b]x] [x[x][a]] [x[x][b]] [x[x][c]]	ref.
=	[[a][x]] [[b]x] [x[x][c]]	reocc.

This result translates into Kleenean symbols two ways:

$$(A \wedge x) \vee (B \wedge \sim x) \vee (C \wedge dx) \quad = \quad (A \vee \sim x) \wedge (B \vee x) \wedge (A \vee B \vee C) \wedge Dx$$

$$(a \vee x) \wedge (b \vee \sim x) \wedge (c \vee Dx) \quad = \quad (a \wedge \sim x) \vee (b \wedge x) \vee (a \wedge b \wedge c) \vee dx$$

Here are two examples of general cross-transposition:

$$\text{a xor b} = (a \wedge \sim b) \vee (\sim a \wedge b) \ \vee (b \wedge \sim b \wedge f)$$

$$= (a \vee b) \wedge (\sim a \vee \sim b) \wedge (a \vee \sim a \vee f) \wedge Db$$

$$= (a \vee b) \wedge (\sim a \vee \sim b) \wedge Da \wedge Db \quad = (\text{a iff b}) \wedge Da \wedge Db$$

$$\text{a iff} \sim b = (a \vee b) \wedge (\sim a \vee \ \sim b) \wedge (b \vee \sim b \vee t)$$

$$= (a \wedge \sim b) \vee (\sim a \wedge b) \vee (a \wedge \sim a \wedge t) \vee db$$

$$= (a \wedge \sim b) \vee (\sim a \wedge b) \vee da \vee db \quad = (\text{a xor b}) \vee da \vee db$$

Let M(x,y,z) denote [[xy][yz][zx]], or [[x][y]] [[y][z]] [[z][x]].
We can derive these theorems:

Transmission. [M(x,y,z)] = M([x],[y],[z])

Proof. [M(x,y,z)]	=	[[[xy][yz][zx]]]	def.
	=	[xy][yz][zx]	ref.
	=	[[[x]][[y]]][[[y]][[z]]][[[z]][[x]]]	ref
	=	M([x],[y],[z])	def.

Distribution. x M(a,b,c) = M(xa, b, xc)

Proof. x M(a,b,c)	=	x [[ab][bc][ca]]	def.
	=	[[xab][xbc][xca]]	trans.
	=	[[xa b][b xc][xc xa]]	recall
	=	M(xa, b, xc)	def.

Redistribution. [[x][M(a,b,c)]] = M([[x][a]], b, [[x][c]])

Proof. [[x][M(a,b,c)]]	=	[[x] M([a],[b],[c])]	transmission
	=	[M([x][a], [b], [x][c])	distribution
	=	M([[x][a]], [[b]], [[x][c]])	transmission
	=	M([[x][a]], b, [[x][c]])	reflexion

Collection. M(x,y,z)) = [[x][y]] [[xy] [z]]

Proof. M(x,y,z))	=	[[xy][x z] [y z]]	def.
	=	[[xy][x][y]] [[xy] [z]]	mod.trans.
	=	[[x][y]] [[xy] [z]]	reocc.

General Distribution. M(x,y,M(a,b,c)) = M(M(x,y,a) , b, M(x,y,c))

Proof. M(x,y,M(a,b,c)) = [[x][y]] [[xy] [M(a,b,c)]] collect.

= [[x][y]] M([[xy][a]], b, [[xy][c]]) redist.

= M([[x][y]] [[xy][a]], b, [[x][y]] [[xy][c]]) dist.

= M(M(x,y,a) , b, M(x,y,c)) collect.

Coalition. M(x, x, y) = x

Proof. M(x, x, y) = [[xx][xy][yx]] def.

= [[x][xy]] recall

= [[x]] reocc.

= x ref.

General Associativity. M(x,a,M(y,a,z)) = M(M(x,a,y), a, z))

Proof. M(x,a,M(y,a,z)) = M(M(x,a,y), M(x,a,a), z) g.dist.

= M(M(x,a,y), a, z)) coal.

These results prove that these operators:

M(x,[[]],y) = [[x][y]] ;

M(x,[],y) = xy ;

M(x,6,y) = [[x6][y6][xy]]

have these properties: associativity; recall; attractors ([[]], [], and 6, respectively) ; and mutual distribution.

In bracket algebra, the form "6" is central and unique, as the following two theorems show:

Mediation. If x 6 = y 6 and [x] 6 = [y] 6

then x = y .

Proof. x	=	[[x]6]x	occ.
	=	[[y]6]x	*
	=	[[y][6]]x	fix.
	=	[[yx][6x]]	trans.
	=	[[yx][6y]]	*
	=	[[x][6]]y	trans.
	=	[[x]6]y	fix.
	=	[[y]6]y	*
	=	y	occ.

Since [6] = 6, Mediation is equivalent to:

x6 = y6 and [[x][6]] = [[y][6]] implies x = y

When we translate this result into Kleenean logic, it becomes:

$x \wedge I = y \wedge I$ and $x \vee I = y \vee I$ implies $x = y$

Centrality. If x = [x] then x = 6 .

Proof. We will prove that x6=66 and [x]6=[6]6, then invoke Mediation.

x 6 = [x] 6 = [xx] 6 = [[x]x]6 = 6 = 6 6

[x] 6 = x 6 = 6 6 = [6] 6 .

x6 = 66 ; [x]6 = [6]6 ; therefore by Mediation, x = 6.

B. *Kleenean Laws*

Exercise for the student:

From the above theorems, plus the bracket-form definitions of the Kleenean operators, prove that Kleenean logic obeys these *De Morgan* laws:

Commutativity: $A \vee B = B \vee A$; $A \wedge B = B \wedge A$

Associativity: $(A \wedge B) \wedge C = A \wedge (B \wedge C)$

$(A \vee B) \vee C = A \vee (B \vee C)$

Distributivity: $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

Identities: $A \wedge t = A$; $A \vee f = A$

Attractors: $A \wedge f = f$; $A \vee t = t$

Recall: $A \wedge A = A$; $A \vee A = A$

Absorption: $A \wedge (A \vee B) = A$; $A \vee (A \wedge B) = A$

Double Negation: $\sim(\sim A) = A$

De Morgan: $\sim(A \wedge B) = (\sim A) \vee (\sim B)$

$\sim(A \vee B) = (\sim A) \wedge (\sim B)$

Relocation: $(\sim i) \wedge Dx = i$; $(\sim i) \vee dx = i$

The Kleenean laws equal the De Morgan laws plus Relocation, a non-Boolean axiom which, with the other axioms, suffices to prove:

Paradox: $\sim i = i$

Differential Dominance: $i \wedge Dx = i$; $i \vee dx = i$

These rules, plus Identity, Attractors and Recall, suffice to construct the Kleenean truth tables.

Here are some **majority** laws:

Modulation: $M(a,f,b) = a \wedge b$

$M(a,t,b) = a \vee b$

$M(a,i,b) = a \min b$

Symmetry: $M(x,y,z)=M(y,z,x)=M(z,x,y)=M(x,z,y)=M(z,y,x)=M(y,x,z)$

Coalition: $M(x,x,y) = M(x,x,x) = x$

Mediocrity: $M(f,i,t) = i$

Transparency: $\sim(M(x,y,z)) = M(\sim x, \sim y, \sim z)$

Distribution: $M(a,b,M(c,d,e)) = M(M(a,b,c), d, M(a,b,e))$

Modulation plus Transparency explains DeMorgan and Transmission:

$\sim(a \wedge b) = (\sim a) \vee (\sim b)$

$\sim(a \vee b) = (\sim a) \wedge (\sim b)$

$\sim(a \min b) = (\sim a) \min (\sim b)$

Min obeys these **semi-lattice** laws:

Commutativity: $x \min y = y \min x$

Associativity: $x \min (y \min z) = (x \min y) \min z$

Recall: $x \min x = x$

Attractor: $x \min i = i$

Transmission: $\sim(x \min y) = (\sim x) \min (\sim y)$

Mutual Distribution: $x ** (y ++ z) = (x ** y) ++ (x ** z)$

where ** and ++ are both from: $\{\wedge, \vee, \min\}$

The differentials dx and Dx obey these **derivative** laws:

$$dx = dx \wedge x = dx \wedge Dx$$

$$x = x \vee dx = x \wedge Dx$$

$$Dx = Dx \vee x = Dx \vee dx$$

i.e. dx is a subset of x, which is a subset of Dx.

In Venn diagram terms, dx is the boundary of x.

$$ddx = dDx = dx$$

$$DDx = Ddx = Dx$$

$$\sim dx = Dx \quad ; \quad \sim Dx = dx$$

$$d(\sim x) = dx \quad ; \quad D(\sim x) = Dx$$

I call the following the **Leibnitz rules**, due to their similarity to the Leibnitz rule for derivatives of products:

$$d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$$

$$D(x \vee y) = (Dx \vee y) \wedge (x \vee Dy)$$

$$d(x \vee y) = (dx \wedge (\sim y)) \vee ((\sim x) \wedge dy)$$

$$D(x \wedge y) = (Dx \vee \sim y)) \wedge ((\sim x) \vee Dy)$$

C. *Normal Forms*

By using Echelon repeatedly, we can turn any bracket expression into one only two brackets deep; the "echelon normal form". This translates, in Kleenean terms, into these forms:

Disjunctive Normal Form:

$$
\begin{aligned}
F(x) \quad = \quad & \ \ (t_{11}(x_1) \wedge t_{12}(x_2) \wedge \ \dots \ \wedge t_{1n}(x_n)) \\
& \vee (t_{21}(x_1) \wedge t_{22}(x_2) \wedge \ \dots \ \wedge t_{2n}(x_n)) \\
& \vee \ \dots \\
& \vee (t_{m1}(x_1) \wedge t_{m2}(x_2) \wedge \ \dots \ \wedge t_{mn}(x_n))
\end{aligned}
$$

where each $t_{ij}(x)$ is one of these functions:

$$\{ \ F, \ I, \ T, \ x, \ \sim x, \ dx \ \}$$

Conjunctive Normal Form:

$$
\begin{aligned}
F(x) \quad = \quad & \ \ (t_{11}(x_1) \vee t_{12}(x_2) \vee \ \dots \ \vee t_{1n}(x_n)) \\
& \wedge (t_{21}(x_1) \vee t_{22}(x_2) \vee \ \dots \ \vee t_{2n}(x_n)) \\
& \wedge \ \dots \\
& \wedge (t_{m1}(x_1) \vee t_{m2}(x_2) \vee \ \dots \ \vee t_{mn}(x_n))
\end{aligned}
$$

where each $t_{ij}(x)$ is one of these functions:

$$\{ \ F, \ I, \ T, \ x, \ \sim x, \ Dx \ \}$$

These normal forms are just like their counterparts in boolean logic, except that they allow differential terms.

Primary Normal Form:

For any bracket expression F(x);

F(x) = [Ax] [B[x]] [Cx[x]] D

where A, B, C and D have no occurrences of variable x.

Furthermore: F( ) = [A] D

F([]) = [B] D

6 F(6) = 6 D

Proof. First we find the echelon normal form for F(x); then we use retransposition relative to x, to [x], and to x[x]. This yields the first equation.

Given that F(x) = [Ax] [B[x]] [Cx[x]] D , it follows that:

F( )	=	[A] [B[]] [C [ ]] D	
	=	[A] D	occ.
F([])	=	[A[]] [B[[]]] [C[] [[]]] D	
	=	[B] D	occ.
6 F(6)	=	6 [A6] [B[6]] [C6[6]] D	
	=	6 [A[6]] [B[6]] [C6[6]] D	fix.
	=	6 D	occ.

QED.

Theorem: **Median Normal Forms**

For any inner function F(x);

6 F(x) = 6 [[x F( )] [[x] F([])] [[x]x F(6)]]

6 [F(x)] = 6 [[x [F( )]] [[x] [F([])]] [[x]x [F(6)]]]

In Kleenean terms, this translates to:

$F(x) \wedge I = [(F(t) \wedge x) \vee (F(f) \wedge \sim x) \vee (F(i) \wedge dx)] \wedge I$

$F(x) \vee I = [(F(t) \vee \sim x) \wedge (F(f) \vee x) \wedge (F(i) \vee Dx)] \vee I$

Proof. Start with the Primary Normal Form:

F(x) = [Ax] [B[x]] [Cx[x]] D

where A, B, C and D have no occurrences of variable x, and:

*** F( ) = [A] D

F([]) = [B] D

6 F(6) = 6 D

Then these equations follow:

6 F(x)	=	6 [Ax] [B[x]] [Cx[x]] D	
	=	[Ax] [B[x]] [Cx[x]] [[x]x] 6 D	comm. & loc.
	=	[Ax] [B[x]] [[x]x] 6 D	reocc.
	=	[[[A]x] [[B][x]] [[x]x]] 6 D	crosstrans.
	=	[[[A]x6D] [[B][x]6D] [[x]x6D]]	trans.
	=	[[x6 F( )] [[x]6 F([])] [[x]x6 F(6)]]	***
	=	6 [[x F( )] [[x] F([])] [[x]x F(6)]]	trans.

QED. The second half of the theorem follows from the first half, by applying it to [F(x)].

D. *Completeness*

The median normal forms imply this theorem:

Completeness. Any equational identity in Kleenean logic can be deduced from the bracket axioms.

Proof. By induction on the number of variables.

Let F = G be an identity with N variables.

(Initial step.) If N = 0, then F = G is a form equation. Since the bracket axioms yield the form tables, F = G follows from those axioms.

(Induction step.) Suppose that all N-1 variable Kleenean identities are proveable from the bracket axioms. Let F(x) be F considered as an expression in its Nth variable x. By induction hypothesis these equations are proveable:

F( ) = G( ) ; F(6) = G(6) ; F([]) = G([])

Then the bracket axioms also prove, via the Median Normal Forms:

6 F(x) = 6 [[x F( )] [[x] F([])] [[x]x F(6)]]
= 6 [[x G( )] [[x] G([])] [[x]x G(6)]]
= 6 G(x)

And similarly;

6 [F(x)] = 6 [[x [F( )]] [[x] [F([])]] [[x]x [F(6)]]]
= 6 [[x [G( )]] [[x] [G([])]] [[x]x [G(6)]]]
= 6 [G(x)]

6 F(x) = 6 G(x) ; 6 [F(x)] = 6 [G(x)] ;

Therefore by Mediation, F(x) = G(x).

This concludes the induction proof. Therefore any Kleenean identity can be proved from the bracket axioms. QED.

Louis Kaufmann said (of a variant of this Completeness proof) that "the novelty of this approach is its use of imaginary ... values in the course of proving the result."

A note on feasibility. The above proof that F = G was only six equations long; but these were only links in a recursive chain. A complete proof requires proofs that F(f) = G(f), F(i) = G(i), and F(t) = G(t). Therefore any complete proof that F=G, if these expressions have n variables, will be about 3^n steps long; no faster than proof by full-table look-up! Thus, though the bracket axioms are deductively complete, they may fail to be *feasibly* complete. Is there a polynomial-time algorithm that can check the validity of a general diamond equation? Students of feasibility will recognize this as a variant of the Boolean Consistency Problem, and therefore NP-complete.

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Chapter 4

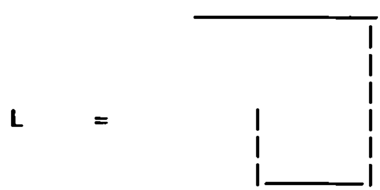
Self-Reference

Re-entrance and Fixedpoints
Inner Order and its laws
The Inner Fixedpoint; examples

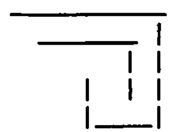
A. *Re-Entrance and Fixedpoints*

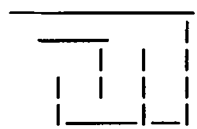
Consider the Liar Paradox as a Brownian form:

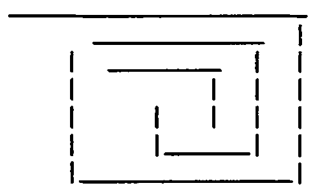
This form contains itself. That can be represented via re-entrance, thus:

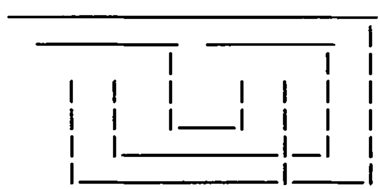


Let re-entrance permit any mark within a Brownian form to extend a tendril to a distant space, where its endpoint shall be deemed enclosed. Thus curl sends a tendril into itself. Other re-entrant expressions include:









Self-reference can be expressed as a re-entrant brownian form, as a switching circuit, as a vector of forms, as indexed brackets, and as a inner fixedpoint. For example:

```
 ___________
   _______  |
          | |      Brownian form
      |   | |
      |___|
```

≅

```
  ___________________________
 /                           \      Switching Circuit
 \_________|\________|\______/     (triangles = "not" gates)
           |/        |/
```

≅

```
 ___          ___
|                |
|  A  =   ___    |
|          B |   |
|         ___    |     Brownian Form Vector
|  B  =    A |   |
|___          ___|
```

≅

$A = [\ [\ A\ ]_B\ ]_A$ Indexed Brackets

≅

(A , B) = (~B , ~A) Inner Fixedpoint

B. *Inner Order*

Now I define the concept of inner order:

x ≤ y iff x min y = x

```
              t
        ≤
 i
        ≤
              f
```

This structure is a semi-lattice; it has a "minimum" operator.

Theorem: min is the minimum operator for ≤ ;

(X min Y) ≤ X ; (X min Y) ≤ Y ;

and Z ≤ (X min Y) , if Z ≤ X and Z ≤ Y

Proof. X min (X min Y) = (X min X) min Y = X min Y

ergo (X min Y) ≤ X ; similarly, (X min Y) ≤ Y.

If Z ≤ X and Z ≤ Y then Z min X = Z ; also Z min Y = Z ;

Therefore Z min (X min Y) = (Z min X) min Y) = Z min Y = Z

Therefore Z ≤ (X min Y) if Z ≤ X and Z ≤ Y .

Thus (X min Y) is the rightmost element left of both X and Y. QED.

Theorem: ≤ is transitive and antisymmetric:

a ≤ b and b ≤ c implies a ≤ c

a ≤ b and b ≤ a implies a = b

Proof. $a \leq b$ and $b \leq c$ implies

$$a \min b = a \quad ; \quad b \min c = b \quad ; \text{ so}$$

$$a \min c = (a \min b) \min c = a \min (b \min c)$$

$$= a \min b = a \quad ; \text{therefore } a \leq c. \quad \text{QED.}$$

$a \leq b$ and $b \leq a$ implies

$a \min b = a$; $a \min b = b$; so $a = b$. QED.

Theorem: $\leq$ is preserved by disjunction and conjunction:

$$a \leq b \quad \text{implies} \quad a \vee c \leq b \vee c$$

$$\text{and} \quad a \wedge c \leq b \wedge c$$

Proof. $a \leq b$ implies $a \min b = a$; so

$(a \vee c) \min (b \vee c) = (a \min b) \vee c = a \vee c$;

so $(a \vee c) \leq (b \vee c)$.

Similarly $(a \wedge c) \leq (b \wedge c)$. QED.

Theorem: $\leq$ is preserved by negation:

$$a \leq b \quad \text{implies} \quad \sim(a) \leq \sim(b) .$$

Proof. $a \leq b$ implies $a \min b = a$; so

$$\sim(a) \min \sim(b) = \sim(a \min b) = \sim a$$

so $\sim(a) \leq \sim(b)$; QED.

Theorem: $\preceq$ is preserved by any inner function:

$$a \preceq b \quad \text{implies} \quad F(a) \preceq F(b)$$

This follows by induction from the previous two results.

Theorem: For any inner f;

$$f(x \min y) \preceq f(x) \min f(y)$$

Proof. by semi-lattice properties.

$x \min y \preceq x$; $x \min y \preceq y$

ergo $f(x \min y) \preceq f(x)$

and $f(x \min y) \preceq f(y)$;

so by definition of the min operator

$$f(x \min y) \preceq f(y) \min f(y) . \quad \text{QED.}$$

These inequalities can be strict; for instance:

$$dt \min df = f; \quad \text{yet} \quad d(t \min f) = i$$

$$Dt \min Df = t; \quad \text{yet} \quad D(t \min f) = i$$

Now we extend $\preceq$ to ordered form vectors:

$$\underline{x} = (x_1, x_2, x_3, \dots, x_n)$$

$$\underline{x} \preceq \underline{y} \quad \text{if and only if} \quad (x_i \preceq y_i) \text{ for all } i$$

Theorem: $\preceq$ has "limited chains", with limit N.

That is, if $\underline{x}_n$ is an ordered chain of finite form vectors;

$\underline{x}_1 \preceq \underline{x}_2 \preceq \underline{x}_3 \dots$ or $\underline{x}_1 \succeq \underline{x}_2 \succeq \underline{x}_3 \dots$;

and if N is the dimension of these vectors,

then for all $n \geq N$, $\underline{x}_n = \underline{x}_N$.

Proof. Any given component of the $\underline{x}$'s can move at most one step before reaching a dead end; then that component stops moving. For N components, this implies at most N steps in an ordered chain before it stops moving.

Given any inner function $\underline{f}(\underline{x})$, define

a *left seed* for $\underline{f}$ is any vector $\underline{a}$ such that $\underline{f}(\underline{a}) \preceq \underline{a}$;

a *right seed* for $\underline{f}$ is any vector $\underline{a}$ such that $\underline{a} \preceq \underline{f}(\underline{a})$.

a *fixedpoint* for $\underline{f}$ is any vector $\underline{a}$ such that $\underline{a} = \underline{f}(\underline{a})$.

A vector is a fixedpoint if and only if it is both a left seed and a right seed.

Left seeds generate fixedpoints, thus:

If $\underline{a}$ is a left seed for $\underline{f}$, then $\underline{f}(\underline{a}) \preceq \underline{a}$. Since $\underline{f}$ is inner, it preserves order; so $\underline{f}^2(\underline{a}) \preceq \underline{f}(\underline{a})$; and $\underline{f}^3(\underline{a}) \preceq \underline{f}^2(\underline{a})$; and so on:

$\underline{a} \succeq \underline{f}(\underline{a}) \succeq \underline{f}^2(\underline{a}) \succeq \underline{f}^3(\underline{a}) \succeq \underline{f}^4(\underline{a}) \succeq \ldots$

Since Kleenean has limited chains, this descending sequence must reach its lower bound within n steps, if n is the number of components of $\underline{f}$. Therefore $\underline{f}^n(\underline{a})$ is a *fixedpoint* for $\underline{f}$:

$\underline{f}(\underline{f}^n(\underline{a})) = \underline{f}^n(\underline{a})$

This is the greatest fixedpoint left of $\underline{a}$.

Left seeds grow leftwards towards fixedpoints.

Similarly, right seeds grow rightwards towards fixedpoints:

$\underline{a} \preceq \underline{f}(\underline{a}) \preceq \underline{f}^2(\underline{a}) \preceq \underline{f}^3(\underline{a}) \preceq \underline{f}^4(\underline{a}) \preceq \ldots \preceq \underline{f}^n(\underline{a}) =$ fixedpoint

$\underline{f}^n(\underline{a})$ is the leftmost fixedpoint right of the right seed $\underline{a}$.

All fixedpoints are both left and right seeds - of themselves.

C. *The Inner Fixedpoint*

Now that we have self-referential forms, the question is; can we evaluate them in Kleenean logic? And if so, how?

It turns out that inner order permits us to do so in general. For any inner function $\underline{F}(\underline{x})$, we have the following:

The Self-Reference Theorem:

Any self-referential inner system has a fixedpoint:

$$\underline{F}(\underline{x}) = \underline{x}$$

Proof. Recall that all inner functions preserve order.

$\underline{i}$ is the leftmost set of values, hence this holds:

$$\underline{i} \preceq \underline{F}(\underline{i})$$

Therefore, $\underline{i}$ is a right seed for $\underline{F}$:

$$\underline{i} \preceq \underline{F}(\underline{i}) \preceq \underline{F}^2(\underline{i}) \preceq \underline{F}^3(\underline{i}) \preceq \ldots \underline{F}^n(\underline{i}) = \underline{F}(\underline{F}^n(\underline{i}))$$

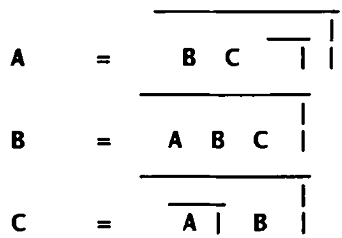
$\underline{i}$ generates the "leftmost" fixedpoint. QED.

All other fixedpoints lie to the right of the leftmost:

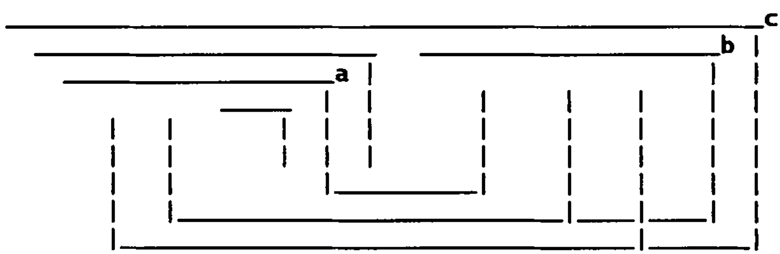
$$\underline{F}^n(\underline{i}) \preceq \underline{x} = \underline{F}(\underline{x})$$

I call this process "productio ex absurdo"; literally, production from the absurd; in contrast to "reduction to the absurd", boolean logic's refutation method. Kleenean logic begins where boolean logic ends.

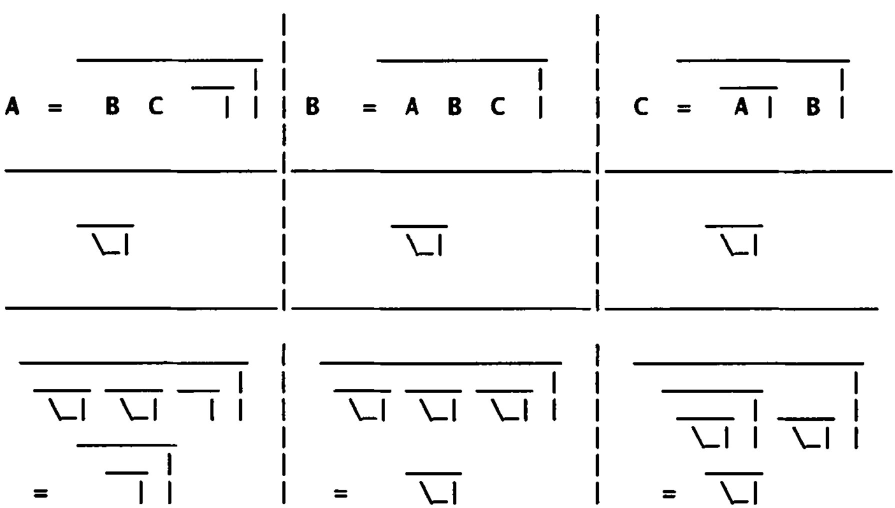
To see productio ex absurdo in action, consider this system:

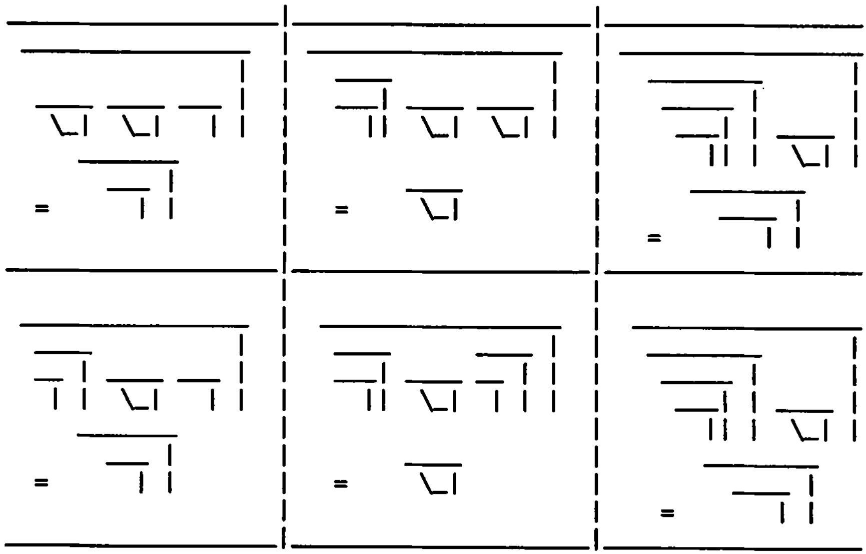


C = [[[B C [ ]]$_A$] [A B C]$_B$]$_C$:



Iterate this system from curl:





The leftmost fixedpoint is:

A = void, B = curl, C = void.

All fixedpoints are right of the leftmost fixedpoint; therefore A = C = void; therefore B = cross B; therefore B = curl. Thus the leftmost fixedpoint is the only one.

Here is a system that takes 4 steps to reach its fixedpoint:

$x_1 = f$; $x_2 = Dx_1$; $x_3 = dx_2$; $x_4 = Dx_3$

`iiii -> fiii -> ftii -> ftfi -> ftft`

Here's one that takes 5 steps:

$x_1 = t$; $x_2 = dx_1$; $x_3 = Dx_2$; $x_4 = dx_3$; $x_5 = Dx_4$

`iiiii -> tiiii -> tfiii -> tftii -> tftfi -> tftft`

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Chapter 5
Fixedpoint Semi-Lattices

Relative Semi-Lattices
Seeds and Spirals
Shared Fixedpoints
Examples, including "ant" and "triplet"

A. *Relative Semi-Lattices*

Any inner function $\underline{F}(\underline{x})$ has the inner fixedpoint: $\underline{F}^n(\underline{i})$, the *leftmost* fixedpoint. But often this is not all.

In general, $\underline{F}$ has an entire *semi-lattice* of fixedpoints.

Theorem: If $\underline{a}$ and $\underline{b}$ are fixedpoints for a inner function $\underline{F}$, then this fixedpoint exists:

$\underline{a} \min_F \underline{b}$ = the *rightmost* fixedpoint *left* of both $\underline{a}$ and $\underline{b}$ = $\underline{F}^n(\underline{a} \min \underline{b})$

Proof. Let $\underline{a}$ and $\underline{b}$ be fixedpoints, and let $\underline{c}$ be any fixedpoint such that $\underline{c} \le \underline{a}$ and $\underline{c} \le \underline{b}$. Then $(\underline{a} \min \underline{b}) \ge \underline{c}$; so

$(\underline{a} \min \underline{b}) = \underline{F}(\underline{a}) \min \underline{F}(\underline{b}) \ge \underline{F}(\underline{a} \min \underline{b}) \ge \underline{F}(\underline{c}) = \underline{c}$

Ergo $(\underline{a} \min \underline{b})$ is a left seed greater than $\underline{c}$:

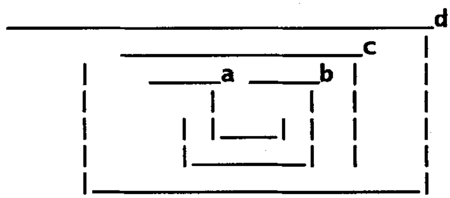
$(\underline{a} \min \underline{b}) \ge \underline{F}(\underline{a} \min \underline{b}) \ge \underline{F}^2(\underline{a} \min \underline{b}) \ge \ldots \underline{F}^n(\underline{a} \min \underline{b})$

$= \underline{F}(\underline{F}^n(\underline{a} \min \underline{b})) \ge \underline{c}$

Therefore $\underline{F}^n(\underline{a} \text{ min } \underline{b})$ is a fixedpoint left of $\underline{a}$ and of $\underline{b}$, and is moreover the rightmost such fixedpoint.

Thus, $\underline{F}^n(\underline{a} \text{ min } \underline{b}) = \underline{a} \text{ min}_F\, \underline{b}$. QED.

For instance, consider the following Brownian form:



This is equivalent to this bracket-form system:

a = [b] ; b = [a] ; c = [ab] ; d = [cd] .

In the standard interpretation, (a,b,c,d) is a fixedpoint for:

$F(a,b,c,d) = (\sim b,\ \sim a,\ \sim(a \vee b),\ \sim(c \vee d))$

In the nand-gate interpretation:

$d = \sim(d \wedge c) = \sim d \vee \sim c = (d \Rightarrow d(a))$

Sentence d says "If I'm not mistaken, then sentence A is both true and false": a Lower Differential Santa Sentence!

In the nor-gate interpretation:

$d = D(a) - d$;

Sentence d says "A is true or false, and I am a liar."

An Upper Differential Grinch! (See "The Grinch", in the Appendix.)

F has this fixedpoint semi-lattice:

```
       tffi
       /
      /
iiii
      \
       \
       ftfi
```

Finding TFFI $\min_F$ FTFI yields this process:

T F F I min F T F I = I I F I

⇒ I I I I

B. *Seeds and Spirals*

We can generalize the preceding results to "seeds".

Theorem: The minimum of two left seeds is a left seed.

Proof. Let $\underline{c} = \underline{a} \min \underline{b}$, where $\underline{a}$ and $\underline{b}$ are left seeds.
Then $\underline{c} \preceq \underline{a}$, and $\underline{c} \preceq \underline{b}$, and $\underline{c}$ is the rightmost such vector.
Therefore $\underline{f}(\underline{c}) \preceq \underline{f}(\underline{a}) \preceq \underline{a}$; $\underline{f}(\underline{c}) \preceq \underline{f}(\underline{b}) \preceq \underline{b}$;
therefore $\underline{f}(\underline{c}) \preceq \underline{c}$,
since $\underline{c}$ is the rightmost vector left of $\underline{a}$ and $\underline{b}$. QED.

Since all fixedpoints are seeds, their minima are left seeds.

Theorem: The minimum of left seeds generates the minimum of the fixedpoints in the relative lattice:

$\underline{a} \min \underline{b}$ generates $\underline{f}^n(\underline{a}) \min_f \underline{f}^n(\underline{b})$, if $\underline{a}$ and $\underline{b}$ are left seeds.

Proof. Let $\underline{z} = \underline{a} \min \underline{b}$, two left seeds. As noted above, $\underline{f}(\underline{z}) \preceq \underline{z}$; $\underline{z}$ is a left seed; moreover, $\underline{z}$ is the rightmost vector left of $\underline{a}$ and of $\underline{b}$. $\underline{f}^n(\underline{z})$ is a fixedpoint; it's left of $\underline{f}^n(\underline{a})$ and of $\underline{f}^n(\underline{b})$, because $\underline{z}$ is left of $\underline{a}$ and of $\underline{b}$.

If a fixedpoint $\underline{c}$ is to the left of $\underline{f}^n(\underline{a})$ and of $\underline{f}^n(\underline{b})$, then $\underline{c}$ is to the left of $\underline{a}$ and of $\underline{b}$: $\underline{c} \preceq \underline{f}^n(\underline{a}) \preceq \underline{a}$; $\underline{c} \preceq \underline{f}^n(\underline{b}) \preceq \underline{b}$;

$\underline{c} \preceq \underline{a}$ and $\underline{c} \preceq \underline{b}$ and $\underline{z} = \underline{a} \min \underline{b}$;

therefore $\underline{c} \preceq \underline{z}$;

therefore $\underline{c} \preceq \underline{f}(\underline{z}) \preceq \underline{z}$;

therefore $\underline{c} \preceq \underline{f}^n(\underline{z}) \preceq \ldots \preceq \underline{f}^2(\underline{z}) \preceq \underline{f}(\underline{z}) \preceq \underline{z}$.

$\underline{f}^n(\underline{z})$ is a fixedpoint left of $\underline{f}^n(\underline{a})$ and of $\underline{f}^n(\underline{b})$, and it is the rightmost such fixedpoint. Therefore $\underline{z}$ generates the minimum in the relative lattice:

$\underline{f}^n(\ \underline{a} \text{ min } \underline{b}\ ) \ = \ \underline{f}^n(\underline{a}) \ \text{min}_f\ \underline{f}^n(\underline{b})$

QED.

In summary: the minimum of left seeds is a left seed, one which generates the relative minimum of the generated fixedpoints. Since any fixedpoint is a seed, it follows that the minimum of fixedpoints is a seed generating the relative minimum of the fixedpoints:

$\underline{f}^n(\ \underline{a} \text{ min } \underline{b}\ ) \ = \ \underline{a} \ \text{min}_f\ \underline{b}$, if $\underline{a}$ and $\underline{b}$ are fixedpoints.

We can generate a seed from a "spiral coil".

Definition.

A Left Spiral is a function iteration sequence $\underline{v}_i = F^i(\underline{v}_0)$ such that,

for some N and K, $\underline{v}_{K+N} \preceq \underline{v}_K$

A Left Spiral Coil = $\{\underline{v}_K, \underline{v}_{K+1}, \dots, \underline{v}_{K+N-1}\}$

For all left spirals, these relations hold:

$$\begin{array}{lllllllll}
\underline{v}_K & \succeq & \underline{v}_{K+N} & \succeq \dots \succeq & \underline{v}_{K+nN} & = & \underline{v}_{K+nN+N} & = & \dots \\
\underline{v}_{K+1} & \succeq & \underline{v}_{K+N+1} & \succeq \dots \succeq & \underline{v}_{K+nN+1} & = & \underline{v}_{K+nN+N+1} & = & \dots \\
\underline{v}_{K+2} & \succeq & \underline{v}_{K+N+2} & \succeq \dots \succeq & \underline{v}_{K+nN+2} & = & \underline{v}_{K+nN+N+2} & = & \dots \\
\dots & & & & & & & & \\
\underline{v}_{K+N-1} & \succeq & \underline{v}_{K+2N-1} & \succeq \dots \succeq & \underline{v}_{K+nN+N-1} & = & \underline{v}_{K+nN+2N-1} & = & \dots
\end{array}$$

- where n is the dimension of the vectors.

The spiral coils leftwards until it reaches a limit cycle.

Spiral Theorem: The minimum of a left spiral coil is a left seed:

Proof. $F(\underline{v}_K \min \underline{v}_{K+1} \min \dots \min \underline{v}_{K+N-1})$

$\preceq$ $F(\underline{v}_K) \min F(\underline{v}_{K+1}) \min \dots \min F(\underline{v}_{K+N-1})$

$=$ $\underline{v}_{K+1} \min \underline{v}_{K+2} \min \dots \min \underline{v}_{K+N-1} \min \underline{v}_{K+N}$

$\preceq$ $\underline{v}_{K+1} \min \underline{v}_{K+2} \min \dots \min \underline{v}_{K+N-1} \min \underline{v}_K$

$=$ $\underline{v}_K \min \underline{v}_{K+1} \min \dots \min \underline{v}_{K+N-1}$

QED.

The fixedpoint that grows from this left seed is the rightmost fixedpoint left of the spiral's limit cycle; a "wave-bracketing fixedpoint".

C. *Shared Fixedpoints*

More than one function can share a fixedpoint. For instance:

Theorem: If $\underline{F}(\underline{x})$ and $\underline{G}(\underline{x})$ are inner functions
and $\underline{F}(\underline{G}(\underline{x})) = \underline{G}(\underline{F}(\underline{x}))$ ($\underline{F}$ and $\underline{G}$ commute)
Then $\underline{F}$ and $\underline{G}$ share a nonempty semi-lattice of fixedpoints:

$\underline{F}(\underline{x}) = \underline{x}$; $\underline{G}(\underline{x}) = \underline{x}$ for all $\underline{x}$ in L_{FG}.

Proof. We have proved that the inner function $\underline{G}$ has a semi-lattice of fixedpoints; $\underline{G}(\underline{x}) = \underline{x}$ for all $\underline{x}$ in L_G.

But since $\underline{F}$ commutes with $\underline{G}$,

$\underline{G}(\underline{F}(\underline{x})) = \underline{F}(\underline{G}(\underline{x})) = \underline{F}(\underline{x})$ for all $\underline{x}$ in L_G.

— that is, $\underline{F}$ sends fixedpoints of $\underline{G}$ to fixedpoints of $\underline{G}$.

$\underline{F}$ sends L_G to itself. What's more, $\underline{F}$ preserves order in ternary; therefore $\underline{F}$ preserves order in L_G.

Therefore $\underline{F}$ is an order-preserving function from L_G to itself. That fact, plus semi-lattice arguments like those in previous sections, will prove that $\underline{F}$ has fixedpoints in a nonempty sub-semi-lattice L_{FG} of L_G:

$\underline{F}(\underline{x}) = \underline{x}$ and $\underline{G}(\underline{x}) = \underline{x}$ for every element $\underline{x}$ of L_{FG}.

QED.

L_{FG} is a semi-lattice of shared fixedpoints.

Its least element is $\underline{F}^n(\underline{G}^n(\underline{i}))$;

and its relative minimum operator is $\underline{F}^n(\underline{G}^n(\underline{a} \text{ min } \underline{b}))$.

These results can be extended to N functions:

If $\underline{F}_1, \underline{F}_2, \ldots \underline{F}_N$ are N commuting inner functions, then they share a semi-lattice of fixedpoints:

$\underline{F}_i(\underline{x}) = \underline{x}$ for all i between 1 and N, and all $\underline{x}$ in L.

Its least element is $\underline{F}_1{}^n(\underline{F}_2{}^n(\ldots (\underline{F}_N{}^n(\underline{i}))\ldots)$;

and its relative minimum operator is

$\underline{F}_1{}^n(\underline{F}_2{}^n(\ldots (\underline{F}_N{}^n(\underline{a} \text{ min } \underline{b}))\ldots)$.

C. *Examples*

Consider the liar paradox:

```
                          _______
                                |
A    =   not A    =        A    |      =    [ A ]_A
```

Here is its Brownian form:

```
 ______
      |
 |____|
```

Here it is as a circuit:

```
 _____________
 |           |
 |____|\_____|
      |/
```

This is its fixedpoint lattice: `i ---------- j`

Now consider Tweedle's Quarrel:

Tweedledee: "Tweedledum is a liar."

Tweedledum: "Tweedledee is a liar."

```
         _____          _________
E   =      U |           ______  |       E  =  [ [ E ]_U ]_E
         _____                |  |
U   =      E |             |  |  |
                           |_____|
```

Its circuit is:

```
 ________________________
 |                      |
 |____|\________|\______|
      |/        |/
```

This "toggle's" semi-lattice is:

```
      tf
     /
    /
  ii
    \
     \
      ft
```

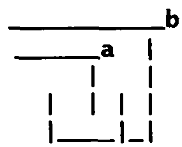
Consider the following statement:

"This statement is both true and false."

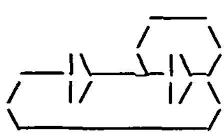
It resolves to this system, the "duck": B = [[B $]_A$ B $]_B$

$A = \overline{B} \,|$

$B = \overline{AB} \,|$



I gave these systems whimsical names based on the appearance of their circuits. The "Duck" has this circuit:

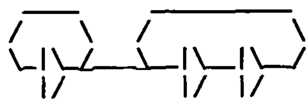


This is equivalent to the fixedpoint:

B = (B and not B) = dB ; a differential of itself!

Here is its semi-lattice: **ii ----- tf**

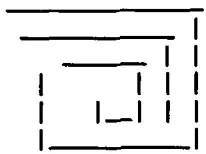
This is the "truck": $C = [\,[\,[\,A\,]_A\,C\,]_B\,]_C$



A = A |

B = CA |

C = B |



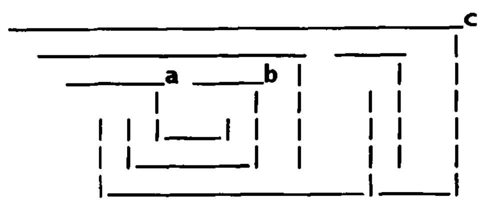
It has this semi-lattice: `iii ----- ift`

This jolly-looking form: $C = [\,[\,[\,BC\,]_A[\,A\,]_B\,][\,C\,]\,]_C$

A = B C |

B = A |

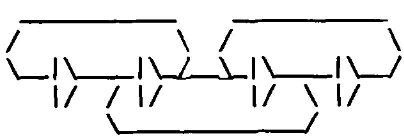
C = A B | C | |



has this semi-lattice:

```
                ftt
          fti
    iii         ftf
          iif
                tff
```

The "rabbit": $D = [[[[B]_AC]_BD]_C]_D$

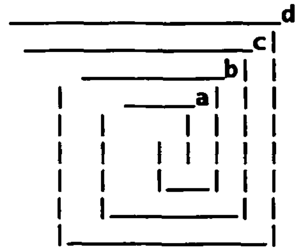


$A = \overline{B\,}\,|$

$B = \overline{AC}\,|$

$C = \overline{BD}\,|$

$D = \overline{C\,}\,|$



has a similar semi-lattice:

```
                        tftf
              tfii
iiii                    tfft
              iift
                        ftft
```

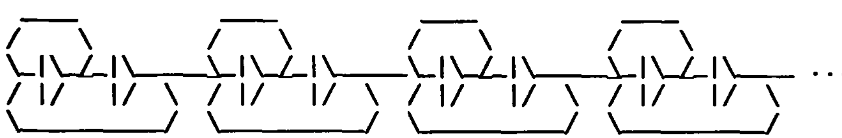
To create linear fixedpoint lattices of length n+1, use:

$$
\begin{aligned}
x_1 &= Dx_1 \\
x_2 &= Dx_2 \vee x_1 \\
x_3 &= Dx_3 \vee x_2 \\
&\dots \\
x_n &= Dx_n \vee x_{n-1}
\end{aligned}
$$

For n = 4, we get the lattice:

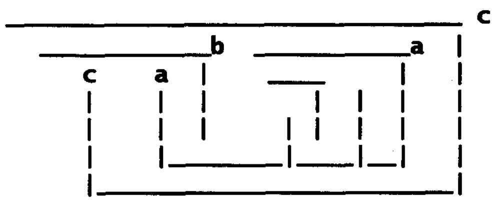
iiii - iiit - iitt - ittt - tttt

Its circuit is:



I call this circuit "the ducks".

This Brownian form: $c = [[[a]a]_a[ac]_b]_c$



is equivalent to the bracket-form system:

a = [a[a]] ; b = [ac] ; c = [ab]

That is: a = da ; b = a nor c ; c = a nor b

Its fixedpoint semi-lattice is:

```
          ftf
          /
         /
iii --- fii
         \
          \
          fft
```

In general, the system

$a = da$;

$\underline{b} = M(a, \sim a, \underline{f}(\underline{b}))$

will have this fixedpoint semi-lattice:

$i\underline{i}$ ---(L)

— where L is $\underline{f}$'s fixedpoint semi-lattice.

This system:

a = [b]; b = [a]; c = [ad]; d = [acd]; e = [bf]; f = [be]

has this fixedpoint semi-lattice:

```
                  tffftf
                 /
          tfffii
         /       \
        /         tfffft
       /
iiiiii
       \
        \
         ftiiff- fttfff
```

The "toggle" ab controls which subcircuit activates; the toggle ef or the "duck" cd. In general, the system

$$a = \sim b \; ;$$

$$b = \sim a \; ;$$

$$\underline{c} = (a \wedge \underline{f}(\underline{c})) \vee (b \wedge g(\underline{c})) \vee da$$

will have this fixedpoint semi-lattice:

```
          (L₁)
         /
        /
    iii
        \
         \
          (L₂)
```

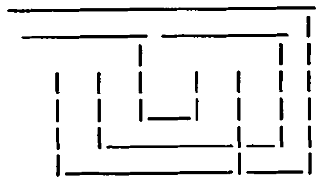
— where L_1 is $\underline{f}$'s semi-lattice, and where L_2 is g's semi-lattice.

The "triplet" has this form: $C = [\,[\,B\;C\,]_A\,[\,C\;A\,]_B\,]_C$

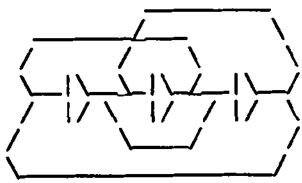
A = $\overline{\mathbf{B\;C\;|}}$

B = $\overline{\mathbf{C\;A\;|}}$

C = $\overline{\mathbf{A\;B\;|}}$



The triplet has this circuit:



Its semi-lattice is:

```
           tff
          /
         /
   iii -- ftf
         \
          \
           fft
```

In general, the system

a = not(b or c) ; b = not(c or a) ; c = not(a or b) ;

d = (a & f(d)) or (b & g(d)) or (c & h(d)) or (a&b&c)

will have this fixedpoint semi-lattice:

```
        (L1)
       /
      /
iiii--(L2)
      \
       \
        (L3)
```

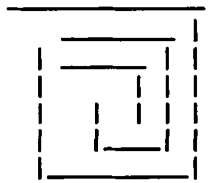
— where L_1, L_2, and L_3 are the semi-lattices for f, g, and h.

The "ant", or "toggled buzzer", has the form $C = [\,[\,[\,B\,]_A\,]_B\,C\,]_C$:

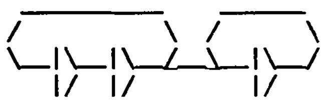
$A = \overline{B\,|}$

$B = \overline{A\,|}$

$C = \overline{BC|}$



The "Ant" has this circuit:



If these are "nand" gates: C = C ⇒ A

The Ant's a Santa!

If these are "nor" gates: C = A - C

The Ant's a Grinch!

The ant's semi-lattice is:

ftf

≤

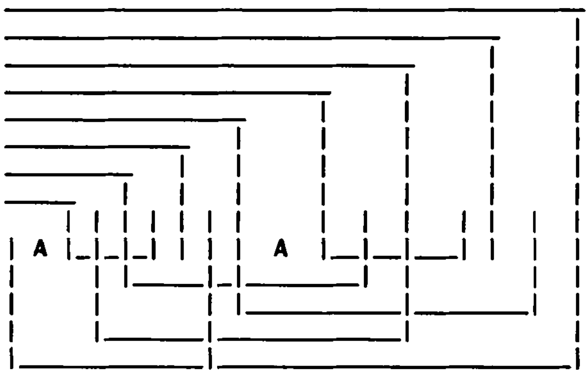
iii

≤

tfi

Note that the FTF state is the ant's only boolean state; all others contain paradox. Assuming that gate C is boolean forces gates A and B to be in the FT state only. The "ant" thus resembles the "Santa" statements of Chapter 1; both attempt to use the threat of paradox to force values otherwise free.

Consider this form; "Brown's First Modulator":



It is equivalent to the bracket-form system:

A	=	input
B	=	[KA]
C	=	[BD]
D	=	[BE]
E	=	[DF]
F	=	[HA]
G	=	[FE]
H	=	[KC]
K	=	[HG]

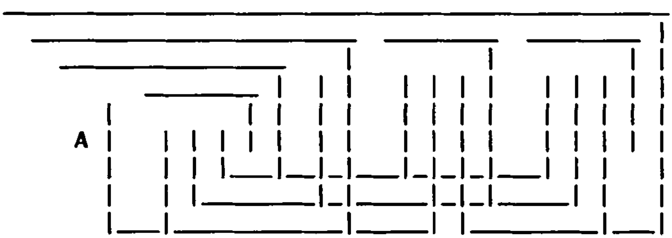
If we symbolize the marked state by "1", curl by "i", and unmarked by "0", then this system has these fixedpoints:

```
a     iiiiiiiii              h
                            /
c     0iiiiiiii            /  g
                          /  /
e     10iii0iii    a -- c--e
                          \  \
g     101010001            \  i
h     000101001             \
i     100100110              j
j     010010010
```

Exercise for the reader: are these the only fixedpoints?

G.S.Brown, in his *Laws of Form*, claims that this circuit "counts to two"; i.e. when A oscillates twice between marked and unmarked, K oscillates once. Is this true? (Assume that the circuit cycles much faster than the input.)

Now consider this form; "Brown's Second Modulator":



It is equivalent to this system:

A	=	input
B	=	[ACE]
C	=	[BD]
D	=	[BCG]
E	=	[BCD]
F	=	[BDG]
G	=	[CDF]

To see the circuit diagrams for Brown's two Modulators, see chapter 11 of his book, *Laws of Form*.

Exercise for the reader: find all fixedpoints for this system.

Is this system a modulator?

Chapter 6

Limit Logic

The Limit
Limit Fixedpoints
Kleenean Computation; Halting Theorem

A. *The Limit*

Delta logic is continuous; it defines a limit operator. This operator equals combinations of two more familiar limit operators; "infinity" and "cofinity":

$$\begin{aligned} \text{Inf}(x_n) &= (\text{All } N\geq 0)(\text{Exists } n\geq N)(x_n) \\ &= \quad (x_1 \vee x_2 \vee x_3 \vee x_4 \vee \ldots) \\ &\qquad \wedge (x_2 \vee x_3 \vee x_4 \vee \ldots) \\ &\qquad\quad \wedge (x_3 \vee x_4 \vee \ldots) \\ &\qquad\qquad \wedge (x_4 \vee \ldots) \\ &\qquad\qquad\quad \wedge \ldots. \end{aligned}$$

$$\begin{aligned} \text{Cof}(x_n) &= (\text{Exists } N\geq 0)(\text{All } n\geq N)(x_n) \\ &= \quad (x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge \ldots) \\ &\qquad \vee (x_2 \wedge x_3 \wedge x_4 \wedge \ldots) \\ &\qquad\quad \vee (x_3 \wedge x_4 \wedge \ldots) \\ &\qquad\qquad \vee (x_4 \wedge \ldots) \\ &\qquad\qquad\quad \vee \ldots. \end{aligned}$$

Inf, the "infinity" quantifier, says that x_n is true infinitely often. Cof, the "cofinity" operator, says that x_n is false only finitely often. Obviously these are deeply implicated in the Paradox of Finitude.

Note that cofinity is a stricter condition; cofinite implies infinite, but not necessarily the reverse:

$$\text{Inf}(x_n) \lor \text{Cof}(x_n) = \text{Inf}(x_n)\ ; \quad \text{Inf}(x_n) \land \text{Cof}(x_n) = \text{Cof}(x_n).$$

Now define a "directed limit" via majorities:

$$\lim^a(x_n) = M(\text{Inf}(x_n), a, \text{Cof}(x_n))$$

Note that:

$$\lim^f(x_n) = M(\text{Inf}(x_n), f, \text{Cof}(x_n)) = \text{Cof}(x_n)$$

$$\lim^t(x_n) = M(\text{Inf}(x_n), t, \text{Cof}(x_n)) = \text{Inf}(x_n)$$

The intermediate setting defines the "limit" operator:

$$\begin{aligned} \lim(x_n) &= \lim^i(x_n) \\ &= M(\text{Inf}(x_n), i, \text{Cof}(x_n)) \\ &= \text{Inf}(x_n) \min \text{Cof}(x_n) \end{aligned}$$

Theorem: $(\lim x_{n+1}) = (\lim x_n)$

This is true because Inf and Cof have that property. Inf and Cof are about the long run, not about the beginning.

Lim is the *rightmost value left of cofinitely many* x_n's:

$\lim x_n \quad \leq \quad x_N$, for all but finitely many N,

and lim is the rightmost such value.

Lim is the minimum of the *cofinal range* of x_n; the set of values that occur infinitely often:

$\lim\{x_n\} \quad = \quad \text{Min cofinal}\{x_n\}$

where $\text{cofinal}\{x_n\} = \{ Y : x_n = Y$ for infinitely many n $\}$

Theorem: If F is an inner function, then

$F(\lim x_n) \leq \lim F(x_n)$;

Proof. $\lim x_n \leq x_N$, for cofinitely many N.

Therefore: $F(\lim x_n) \leq F(x_N)$, for cofinitely many N.

Therefore: $F(\lim x_n) \leq \lim F(x_N)$,

since $\lim F(x_N)$ is the *rightmost* value left of cofinitely many $F(x_N)$'s!

QED.

Here's another proof, by cofinality:

$$\begin{aligned} F(\lim x_n) &= F(\text{Min cofinal}\{x_n\}) \\ &\leq \text{Min } F(\text{cofinal}\{x_n\}) \\ &= \text{Min cofinal}\{F(x_n)\} \\ &= \lim F(x_n) \end{aligned}$$

QED.

This inequality can be strict. For instance:

$F(x) = dx$, and $x_n = \{t,f,t,f,t,f...\}$:

$d(\lim\{t,f,t,f,t,f...\}) = di = i$;

$\lim\{dt,df,dt,df...\} = \lim\{f,f,f,f...\} = f$:

so $d(\lim x_n) \prec \lim d(x_n)$

B. *Limit Fixedpoints*

Fixedpoints can be found by transfinite induction on the limit operators. Recall that for all inner functions $\underline{F}$:

$$\underline{F}(\lim \underline{a}_n) \preceq \lim \underline{F}(\underline{a}_n)$$

Given *any* set of initial values $\underline{s}_0$, then let

$$\underline{s}_\omega = \underline{F}^\omega(\underline{s}_0) = \lim \underline{F}^n(\underline{s}_0)$$

So

$$\begin{aligned}\underline{F}(\underline{s}_\omega) &= \underline{F}(\lim \underline{F}^n(\underline{s}_0)) \\ &\preceq \lim \underline{F}(\underline{F}^n(\underline{s}_0)) \\ &= \lim \underline{F}^{n+1}(\underline{s}_0) \\ &= \lim \underline{F}^n(\underline{s}_0) \\ &= \underline{s}_\omega\end{aligned}$$

Therefore $\underline{F}^\omega(\underline{s}_0) = \lim(\underline{F}^n(s_0))$ is a left seed. It generates a fixedpoint:

$\underline{s}_\omega \succeq \underline{F}(\underline{s}_\omega) \succeq \underline{F}^2(\underline{s}_\omega) \succeq \ldots \underline{s}_{2\omega} = \lim \underline{F}^n(\lim (\underline{F}^n(\underline{s}_0)))$.

$\underline{s}_{2\omega}$ is the limit of a descending sequence, and therefore also its minimum. If $\underline{F}$ has only finitely many components, then the descending sequence can only descend finitely many steps before coming to rest.

Thus; if $\underline{F}$ has finitely many components, then $\underline{s}_{2\omega}$ is a fixedpoint for $\underline{F}$:

$$\underline{F}(\underline{s}_{2\omega}) = \underline{s}_{2\omega}$$

These seeds also generate minima in the relative semi-lattice:

$$\lim\text{-}(\underline{F}^n(\underline{F}^\omega(x_0) \min \underline{F}^\omega(y_0))) = \underline{F}^{2\omega}(x_0) \min_F \underline{F}^{2\omega}(y_0)$$

If $\underline{F}$ has *infinitely* many components, then we must continue the iteration through more limits. Let

$$\underline{s}_{3\omega} = \lim(\underline{F}^n(\underline{s}_{2\omega}))$$

$$\underline{s}_{4\omega} = \lim(\underline{F}^n(\underline{s}_{3\omega}))$$

...

$$\underline{s}_{\omega\omega} = \lim(\underline{s}_{n\omega})$$

...

And so on through the higher ordinals. They keep drifting left; so at a high enough ordinal, we get a fixedpoint:

$$\underline{F}(\underline{s}_{\alpha}) = \underline{s}_{\alpha}.$$

Large cardinals imply "late" fixedpoints: self-reference with high complexity. Alas, the complexity is all in the syntax of the system, not its (mostly imaginary) content. Late fixedpoints are absurdly simple answers to absurdly complex questions.

C. *Kleenean Computation*

Let $\underline{F}^{2\omega}(\underline{x}_0) = \lim \underline{F}^n(\lim \underline{F}^n(\underline{x}_0))$.

This is the limit fixedpoint generated from $\underline{x}_0$ by iterating $\underline{F}$ twice-infinity times. We can regard this as the output of a computation process whose input is $\underline{x}_0$ and whose program is $\underline{F}$. Kleenean computation theory is the same as its limit theory; output equals behavior "in the long run".

If $\underline{F}$ is n-dimensional, and if $\underline{F}^j(\underline{x}_0)$ is a cyclic pattern — that is, a wave — then $\lim(\underline{F}^j(\underline{x}_0))$ equals minimum over a cycle. This yields the **wave-bracketing fixedpoint:**

$$\underline{F}^{2\omega}(\underline{x}_0) = \lim \underline{F}^j(\lim \underline{F}^j(\underline{x}_0))$$
$$= \underline{F}^n(\text{ Min } (\underline{F}^j(\underline{x}_0)),$$

where Min is taken over at least one cycle.

This is the rightmost fixedpoint left of $\underline{F}^j(\underline{x}_0)$.

Its existence implies this **Halting Theorem:**

If $\underline{F}$ has n components, then its limit fixedpoint equals:

$$\underline{F}^{2\omega}(\underline{x}_0) = \underline{F}^n \left(\min_{3^n \leq j \leq 2*3^n} (\underline{F}^j(\underline{x}_0)) \right)$$

This is because by 3^n steps, the system has run through all possible different states; so between 3^n and $2*3^n$ it will traverse at least one cycle, and thus generate a seed.

The minimum of stages 3^n to $2*3^n$, iterated n times more, yields a wave-bracketing fixedpoint, in $(n + 2*3^n)$ steps.

Recall the Spiral Theorem: if $\underline{F}^{N+K}(\underline{x}_0) \preceq \underline{F}^{N}(\underline{x}_0)$ for some N and K, then $\mathrm{Min}[N \leq i < N+K](\ \underline{F}^{i}(\underline{x}_0)\ )$ is a left seed. If the inner order semi-lattice allows no "antichains" (that is, sets without any order relations) of size greater than 2^n, then with the help of the Spiral Theorem we may be able to shorten the computation time to $n + 2^n$.

In Kleenean logic, any computation with any input has an output; a wave-bracketing fixedpoint. However, some computations take exponential time to find their wave, and thus are nonfeasible.

Most of the logic fixedpoints in the last few chapters exist thanks to the default value; paradox. In Kleenean logic, paradox doesn't *refute* reasoning; it *grounds* reasoning.

Chapter 7

Paradox Resolved

The Liar and the Anti-Diagonal
Russell's Paradox
Santa Sentences
Antistrephon
Sorites Paradoxes
Game Paradoxes

A. *The Liar and the Anti-Diagonal*

"This sentence is false"; is that, the pseudomenon, true or false? It is true yet false! Dear reader, I must confess to a sense of anticlimax in this resolution. So many logicians have treated paradox with respect bordering on terror; surely the solution can't be *that* simple? Well, yes it can be; for as you can see, yes it is!

Call an adjective "heterological" if and only if it does not apply to itself: "A" is heterological = "A" is not A. Is "heterological" heterological?

"Heterological" is heterological = "Heterological" is not heterological.

True yet false.

" 'Is false when quined' is false when quined"; is it true?

True yet false.

B. *Russell's Paradox.*

Recall the definition of Russell's set R:

$$R \quad = \quad \{ x \mid x \notin x \}$$

R is an anti-diagonal set. Is it an element of itself?

In general: $x \in R \quad = \quad x \notin x$

and therefore: $R \in R \quad = \quad R \notin R$.

Therefore R is paradoxical. Does R exist? In Boolean logic, the answer must be "no"; yet there it is! In Kleenean logic:

$$R \in R \quad = \quad i$$

Recall also the "Short-Circuit Set": $S \quad = \quad \{x : S \notin S\}$.

S is a constant-valued set, like the universal and null sets:

For all x, $(x \in S) \; = \; (S \notin S) \; = \; (S \in S) \; = \; i$.

All sets are paradox elements for S.

Russell's barber shaves all those — and only those — who do not shave themselves. Does the barber shave himself?

Yes, yet no; which can be realized several ways. For instance, the barber might only *partially* shave himself. Or, if there are *two* barbers in town, then each can shave each other, but not themselves; then the two of them, as a team, shave all those who do not shave themselves.

That village's watchmen watch all those, and only those, who do not watch themselves. But who watches the watchmen?

Answer: they shall watch each other, but not themselves. Thus honesty in government is truly imaginary!

If you were to ask that village's veterans about the Great War (a war to end all wars, and only those wars, which do not end themselves), then they will laugh at your quaint name for a conflict now known as World War I.

"Did the Great War end itself?" they will say, then scratch their heads. "Yes, it did; yet no, it did not!"

That village's priest often ponders this theological riddle:

God is worshipped by all those, and only those, who do not worship themselves. Does God worship God?

Answer: this answer is false!

C. *Santa Sentences*

If a young child were to proclaim:

"Santa Claus exists, if I'm not mistaken."

and subsequent events were to refute his belief, then the poor child will be justified in exclaiming:

"I *am* mistaken!"

Humbling moments like these are part of growing up. Note that this admission is formally identical to the Fool's paradox!

Evidently Kris Kringle, in his departure, left behind some fool's gold. How generous!

Recall that we can create Santa sentences by Grelling's method, by Quine's method, and by Russell's method:

Grelling's Santa:

Define the adjective "Santa-logical":

"A" is Santa-logical = If "A" is A, then Santa exists.

Is "Santa-logical" Santa-logical?

"Santa-logical" is Santa-logical =

If "Santa-logical" is Santa-logical, then Santa exists.

Quine's Santa is:

"Implies that Santa exists when quined" implies that Santa exists when quined.

Russell's "Santa Set for sentence G" is:

$S_G \quad = \quad \{ x \mid (x \in x) \Rightarrow G \}$

Therefore: $x \in S_G \quad = \quad (x \in x) \Rightarrow G.$

and therefore: $S_G \in S_G \quad = \quad (S_G \in S_G) \Rightarrow G.$

If there is no Santa Claus, then the above are all paradoxes.

Above I told Barber-like stories about Santa sets. For instance, in another Spanish village, the barber takes weekends off; so he shaves all those, and only those, who shave themselves only on the weekend:

B shaves M = If M shaves M, then it's the weekend.

Does the barber shave himself?

B shaves B = If B shaves B, then it's the weekend.

When Monday rolls around, then (B shaves B) = paradox.

That village is watched by the watchmen, who watch all those, and only those, who watch themselves only when fortune smiles:

W watches C = if C watches C, then fortune smiles.

Who watches the watchmen?

W watches W = if W watches W, then fortune smiles.

If fortune ever frowns, then (W watches W) = paradox.

Recently that village saw the end of the Cold War, which ended all wars, and only those wars, which end themselves only if money talks:

CW ends W = if W ends W, then money talks.

Did the Cold War end itself?

CW ends CW = if CW ends CW, then money talks.

Does money talk? If not, then (CW ends CW) = paradox.

That village's priest proclaimed this theological doctrine:

God blesses all those, and only those, who bless themselves only when there is peace:

G blesses S = If S blesses S, then there is peace.

Does God bless God?

G blesses G = If G blesses G, then there is peace.

Is there peace? If not, then (God blesses God) = paradox.

Recall Promenides the Cretan, who said;

"If I am honest, then *some* Cretan is honest."

How logical! But alas, this is equivalent to:

"If *all* Cretans are liars, then so am I."

Promenides sounds logical; but his statement still leaves open the possibility that every Cretan is a liar, including Promenides.

D. *Antistrephon*

In the next few paragraphs I take the role of judge, and address the shades of Protagoras and Euathius.

Gentlemen, you have given me a dilemma. If Euathius is to win this case, then he must show that he has no obligation under the contract; but the contract says that he need not pay just if he loses the first case — which is this one. He wins if he loses and he loses if he wins; and the same goes for Protagoras.

If I find for Protagoras, then the judgement should go for Euathius; and if I find for Euathius, then the judgement should go for Protagoras. You wish me to declare sentence, but any sentence I declare will be an incorrect sentence, a false sentence. Therefore I declare:

This sentence is false.

The Pseudomenon; a paradox, or half-truth. By the nature of this case, I can be only half-right; I can only half-satisfy you. In the interest of justice, I should take a position midway between yours, favoring neither side. Compromise is called for.

I therefore reformulate this case. I say that it is actually *two* cases being decided simultaneously. The first case is about the second half of the fee, to be awarded only if the second case is lost; and the second case is about the first half of the fee, to be awarded only if the first case is lost.

This is an artificial division of the original case; it would make no difference if the original case had an unequivocal solution. But here equivocation is necessary, and it works; for it is consistent for Protagoras to win the first case and Euathius to win the second. Upon recombining these results, we see that Protagoras can claim half the fee, and Euathius can keep the other half of the fee, both having *won yet lost*.

One final legal note: in this case, as is usual, Protagoras won to the exact extent that Euathius lost:

$$i = \sim i$$

What is unusual about this case is that it's also true that Protagoras won to the exact extent that Euathius *won*:

$$i = i$$

E. *Size Paradoxes*

In Chapter 1, I heaped together the paradoxes of The Heap, The First Boring Number, Berry's Paradox, and Finitude. They all had in common the vagueness of the boundary between the interesting and the uninteresting. Surely both types of integers exist; but where do they meet?

Assuming that we could find a number on the boundary (even though the search for such a number would be boringly long), then it would be interesting just as much as it is boring; which suggests an intermediate state.

So is "the first boring number" boring or not? True yet false!

And what is "the smallest number that cannot be defined in less than twenty syllables"? In standard decimal nomenclature, that would be 127,777. (However, other naming schemes might name 127,777 in fewer than twenty syllables. As ever, uncertainty reigns.)

If you were to pile together 127,777 grains of sand, each 1 mm wide, then they will form a conical pile approximately 9.9 cm wide and half as tall; a small but respectable Heap. If you tried to move this Heap one grain at a time, laboring 5 seconds per grain, 8 hours per day, 5 days per week, then you will finish the job in approximately 4.5 weeks; a Heap of work.

"One plus the largest number defineable in less than twenty syllables" might be one plus "Twelve googol googol googol googol googol googol googol googol googol," or $1 + 1.2 * 10^{901}$. (This is if you allow the use of the word "googol", for 10^{100}. Other naming schemes yield even greater numbers.)

F. *Game Paradoxes*

Recall the definition of Hypergame: its initial position is the set of all "short" games — that is, all games that end in a finite number of moves. For one's first move in Hypergame, one may move to the initial position of any short game. Is Hypergame short?

Above I told the story of "the Mortal"; an unborn spirit who must now make this fatal choice; to choose some mortal form to incarnate as, and thus be be doomed to certain death.

The Mortal has a choice of dooms. Is the Mortal doomed?

The answer is that Hypergame is Finitude in disguise. Presumably the Mortal lives until the last interesting moment, then dies of boredom.

Recall my definition of the game Normalcy:

The move N ⇨ G is legal = the move G ⇨ G is not legal.

Is Normalcy normal? Let G = N:

The move N ⇨ N is legal = the move N ⇨ N is not legal.

This is a game-theory version of Russell's paradox. Normalcy is normal if and only if it is not. So is Normalcy normal? True yet false.

Above I told the story of the Rebel, who may become those, and only those, who do not remain themselves:

R may become B = B may not become B .

Can the Rebel remain a Rebel? True yet false.

Presumably Rebels play at Normalcy.

Chapter 8

The Continuum

Cantor's Paradox
Dedekind Splice
Cantor's Dyadic
The Line Within the Delta
Zeno's Theorem
Fuzzy Chaos

A. *Cantor's Paradox.*

Cantor's proof of the "non-denumerability" of the continuum relies on an anti-diagonal. For suppose we had a countable list of the real numbers:

$$R_1 = 0 . D_{11}, D_{12}, D_{13} \ldots$$

$$R_2 = 0 . D_{21}, D_{22}, D_{23} \ldots$$

$$R_3 = 0 . D_{31}, D_{32}, D_{33} \ldots$$

$$\ldots$$

where D_{NM} is the Mth binary digit of the Nth number.

Then we define Cantor's anti-diagonal number:

$C = 0 . \sim D_{11}, \sim D_{22}, \sim D_{33}, \sim D_{44} \ldots$

If $C = R_N$ for any N, then $D_{NX} = \sim D_{XX}$; therefore $D_{NN} = \sim D_{NN}$; the pivot bit buzzes. From this single buzzing bit Cantor deduces the existence of an infinity beyond infinity of real points! Was more ever made from less?

In Kleenean logic, the continuum is "semi-countable"; countable listings are possible, but they all contain paradox bits. The continuum is intermediate!

B. *Dedekind Splice*

Recall the "paradox of the boundary":

What day is midnight?

Is noon A.M. or P.M.?

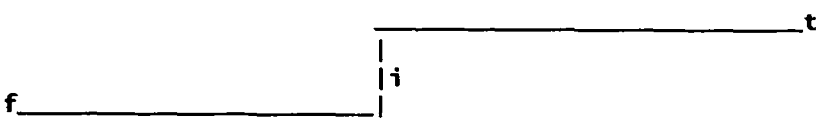
Is dawn day or night? Is dusk?

Which country owns the boundary?

Is zero positive or negative? (± 0 ?)

If a statement is true at point A and false at point B, then somewhere in-between lies a boundary. At any point on the boundary, is the statement true, or is it false?

To solve the paradox *of* the boundary, put a paradox *on* the boundary:



$$X \lesssim Y \quad = \quad (X < Y) \min (X \leq Y)$$

This is the "dedekind splice" operator, equal to paradox at the boundary.

To make this a "continuous" function from R to 3, we need to define a topology on 3. Let the "open subsets" of 3 be the right-closed subsets:

Open sets = { O : [(x in O) and (x $\leq$ y)] $\Rightarrow$ (y in O) }
= { {}, {t}, {f}, {t,f}, {i,t,f} }

In this topology, all inner functions are continuous, the Dedekind splice is continuous, and all values are near i.

The Dedekind splice is **anti-symmetric**, **transitive**, and **dense**:
For all x, y and z:
(x $\leq$ y) = ~(y $\leq$ x)
if (x $\leq$ z) and (z $\leq$ y) then (x $\leq$ y)
if (x$\leq$ y) , then there exists a z such that (x $\leq$ z) and (z $\leq$ y)

If the sequence $\{x_n\}$ approaches the limit x from "both sides", as in an alternating series, then (x $\leq$ y) = lim (x_n $\leq$ y)
In general ((lim x_n) $\leq$ y) $\preceq$ lim (x_n $\leq$ y)

The splice's anti-symmetry implies the paradox of the boundary:
(x $\leq$ x) = ~(x $\leq$ x)

C. *Cantor's Dyadic*

Let us take a closer look at Cantor's anti-diagonal number; just what kind of quantity is it?

This number is so fraught with mathematical significance that it forces us to postulate an infinity of infinities; so surely it must, within itself, contain an infinite amount of *information* about all those infinities. Otherwise the silly thing's just bluffing us!

We know that C has a paradox bit at place N; so the question is, when does that happen?

Graph r_N vs. r:

```
rN
↑                 |----|    |----|    |----|     <-- true
|                 |    |    |    |    |    |     <-- paradox
  --->r         --|    |----|    |----|    |--   <-- false
```

As you can see, the most obvious place to put paradox values is at the boundary points; namely, the dyadics $m/2^N$.

Paradox denotes bit-flip; the blur when 0 becomes 1:

```
0.111111... <----> i.iiiii... <----> 1.000000...
```

Cantor's number has bit-flip at place N; this implies bit-flip at all higher-precision places. Therefore Cantor's Anti-Diagonal Number has this form:

```
C   =   0.0100101...101011iiiiiiiiiiiiiiiiii...
                               ↑
                               N
              boolean region        paradox region
```

Thus we see that Cantor's Number contains only a finite number of boolean bits. It's a measly *dyadic*!

The silly thing *was* bluffing us! Far from being infinitary (this proof of infinities!) it is instead the most finite entity of all; a bounded bit string with round-off error!

The dyadics can trick up Cantor's proof, even within boolean logic. For instance, the possibility exists that C, as an anti-diagonal, reads .0111111111..., while C, on the list, reads .100000000...!

Cantor's Theorem is hereby exposed as not only superfluous, but actually ridiculous. The continuum is countable; Cantor's Paradox detects bit-flip at a dyadic. Therefore I propose a down-to-earth alternative to Cantor's tottering cardinal tower; a single countable infinity with paradox logic.

A slightly subtler logic yields an infinitely simpler model. This is known as elegance; sign of a correct theory.

Even with paradox accounted for, Cantor's argument still has revolutionary implications.

Consider C_n for $n > N$:

$C_n = i$ for $n > N$;

but $C_n = \sim R_{nn}$ for all n:

So $R_{nn} = \sim C_n = \sim i = i$, for all $n > N$.

In other words, every real number on the list after Cantor's Dyadic also has a paradox bit; and so is also a dyadic! By this account, at most a finite number of reals possess infinite precision!

Rather than showing that most real numbers are, say, transcendental, Cantor's Dyadic instead demonstrates that most real numbers are *dyadic*! As in quantum mechanics, uncertainty quantizes the continuum. Indeed we have a classic quantum-style complementarity; finitely many infinite-precision reals, and infinitely many finite-precision reals.

But what then of, say, 1/3? 1/5? 1/7? 1/(2n+1), for all n? Do we only have room for finitely many full rationals - let alone finitely many transcendentals? Do all *these* need Cantor's tower?

Perhaps C has a non-dyadic form:

```
C   =   0.0100101...101011i10111000011...
                          ↑
                          N
```

But what does that paradox bit at place N mean, given that higher-precision bits are boolean? Is C of the form $c \pm 2^{-N}$? What does such a dual number mean?

Perhaps Cantor's Dyadic is telling us that the synchronized bit-flips of dyadic numeration produce masking noise. Or perhaps Cantor's Dyadic is there to remind us that approximation is inevitable.

In practice, real numbers *are* dyadics. After all, dyadics are the numbers we really do, in fact, calculate with. Every single so-called real number in the so-called real world has finite precision. Even the Chudnovski brothers have computed only 2 billion digits of pi, and not infinity! Not one single infinite-precision computer has ever come off the assembly line; nor ever shall, so long as human beings remain finite. Call this Math for Mortals.

The finite-precision reals are easy to count:

```
.iiiiiiiiii...
.0iiiiiiiii...
.1iiiiiiiii...
.00iiiiiiii...
.01iiiiiiii...
.10iiiiiiii...
.11iiiiiiii...
.000iiiiiii...
.001iiiiiii...
.010iiiiiii...
.011iiiiiii...
.100iiiiiii...
.101iiiiiii...
.110iiiiiii...
.111iiiiiii...
.0000iiiiii...
```

and so on, in binary!

Note that by this counting C = .iiiii... = 1/2 ; the first entry!

D. *The Line Within The Delta*

The "approximate comparison" operator $\gtrsim$ is ideal for embedding the continuum in delta logic. Consider the following mapping from R (the continuum) to 3^ω (the space of all infinite delta-valued sequences) :

$$E(x) \quad = \quad (x \gtrsim q_1 , x \gtrsim q_2 , x \gtrsim q_3 , x \gtrsim q_4 , \ldots)$$

where q_n is an enumeration of the rationals.

This function E sends R (the real number continuum) into 3^ω, the space of all infinite delta vectors.

Its nth component, E_n, is comparison with the nth rational: $x \gtrsim q_n$

Theorem: This mapping E embeds R in 3^ω: that is, R's topology is carried intact into 3^ω, the space of delta vectors.

Proof. First, note that E is one-to-one; for if x<y, then some rational number q_n is between them; so

$E_n(x) = f$ and $E_n(y) = t$

Next note that E is continuous; for each of its components is continuous.

To complete proof of embedding, we need to prove this

Lemma: The inverse of E is continuous.

A function is continuous if the inverse image of an open set is an open set. The real line's topology is generated by the "half-lines":

$$(x , +\infty) = \{ y : x < y \}$$

$$(-\infty , x) = \{ y : y < x \}$$

so it suffices to prove that E sends each half-line to the intersection of an open set in 3^ω with the image E(R).

$$E (x , +\infty) = \text{Union[n such that } q_n > x \text{] } \{ E(y) : E_n(y) = t \}$$

$$E (-\infty , x) = \text{Union[n such that } q_n < x \text{] } \{ E(y) : E_n(y) = f \}$$

The first is a countable union of intersections of E(R) with the open set $\{ s : s_n = t \}$; the second is a countable union of intersections of E(R) with the open set $\{ s : s_n = f \}$. In either case, E sends a half-line to an intersection of E(R) with an open set in 3^ω.

Thus the lemma is proved: the inverse of E is continuous.

Therefore E is an embedding: 1-1 and bicontinuous. QED.

Theorem: Any continuous function f from R to 3 "lifts" to an inner function f* from 3^ω to 3.

```
               f
R    ----------->   3
|
| E                 =          This diagram commutes.
v
3^ω  ----------->   3
               f*
```

Proof. Let F(x) be a continuous function from R to 3. Any continuous inverse image of an open set is open; so these are open sets:

$F^{-1}(t) \;=\; \{ x \text{ in } R : F(x) = t \}$

$F^{-1}(f) \;=\; \{ x \text{ in } R : F(x) = f \}$

Call the first set A and the second set B. Being open, they are countable unions of open intervals:

$A = \text{Union(all N)} (a_N, A_N)$

$B = \text{Union(all N)} (b_N, B_N)$

where all the a's and b's are chosen from the rationals.

Approximate these sets by finite unions:

$A_n(x) \;=\; (a_1 \le x \le A_1) \vee (a_2 \le x \le A_2) \vee \ldots \vee (a_n \le x \le A_n)$

$B_n(x) \;=\; (b_1 \le x \le B_1) \vee (b_2 \le x \le B_2) \vee \ldots \vee (b_n \le x \le B_n)$

Then take limits:

$A(x) \;=\; \lim A_n(x)$

$\;=\; \lim((a_1 \le x \le A_1) \vee (a_2 \le x \le A_2) \vee \ldots \vee (a_n \le x \le A_n))$

$B(x) \;=\; \lim B_n(x)$

$\;=\; \lim((b_1 \le x \le B_1) \vee (b_2 \le x \le B_2) \vee \ldots \vee (b_n \le x \le B_n))$

These are the characteristic functions for A and B; made strictly from the Dedekind splice and Kleenean logic.

Now define $F^*(x)$:

$F^*(x)$

$= \quad A(x) \min [\sim B(x)\}$

$= \quad \lim((a_1 \leq x \leq A_1) \vee (a_2 \leq x \leq A_2) \vee \ldots \vee (a_n \leq x \leq A_n))$

min

$\sim (\lim((b_1 \leq x \leq B_1) \vee (b_2 \leq x \leq B_2) \vee \ldots \vee (b_n \leq x \leq B_n)))$

This function equals true if x is in the interior of A and the exterior of B: that is, F(y) = t and F(y) ≠ f, for any y near enough to x. This function equals false if x is in the exterior of A and the interior of B: that is, F(y) = f and F(y) ≠ t, for any y near enough to x. Finally, this function equals i at the boundary of the above two sets; that is, F(y) = f, and F(y') = t, for some y and y' in any neighborhood of x.

But F is a continuous function; so it equals t in the interior of A, f in the interior of B, and i at the boundary.

Therefore $\quad F^*(x) \quad = \quad F(x).$

Note that $F^*(x)$ is made only from E and Kleenean logic:

$F(x) \; = \; F^*(x) \; = \; C(E(x))$

where C is an inner function.

Therefore C(x) is an inner function which extends F(x) (via the embedding E) to all 3^ω. QED.

Theorem: E is not only an embedding; it is a morphism: that is, functions from R to R "lift" to functions from 3^ω to 3^ω;

$$\begin{array}{ccc} R & \xrightarrow{\quad f \quad} & R \\ \Big\downarrow E & & \Big\downarrow E \\ 3^\omega & \xrightarrow[\quad f^* \quad]{} & 3^\omega \end{array}$$

This diagram commutes.

Proof. If f is a function from R to R, then let f_n be the nth component of f(x), via the embedding E:

$$f_n(x) \quad = \quad E_n(f(x)) \quad = \quad [\, f(x) \geq q_n \,]$$

f is continuous; dedekind splice is continuous; so f_n is continuous. Therefore, by the above theorem, f_n extends to a function from 3^ω to 3; therefore the function

$$f = (f_1, f_2, \ldots\ f_n , \ldots)$$

extends to a continuous function from 3^ω to 3^ω. QED.

Thus the real continuum embeds and extends into the space of infinite-dimensional ternary vectors. The continuum reduces to inner form.

E. *Zeno's Theorem*

Every continuous function from the real line to itself extends to a inner function from delta space to itself. But every inner function on delta space has a fixedpoint. Therefore we get:

Zeno's Theorem: Any continuous function from the real line to itself has a fixedpoint in delta space.

I name this theorem after Zeno of Elea, famed for his paradoxes of motion. With the proof of this Theorem, we see that Zeno was right after all — in part. He claimed that no motion is possible: here we see that no motion is universal. Any continuous transformation of space has a fixedpoint; any chaotic dynamic has a paradoxical resolution.

F. *Fuzzy Chaos*

Consider "fuzzy logic", whose truth values are the real numbers between 0 and 1, where 0 means F and 1 means T. Fuzzy logic has these operators:

x and y	=	Minimum(x,y)
x or y	=	Maximum(x,y)
not x	=	$1 - x$.

— where minimum and maximum are relative to the usual $<$ ordering on the unit interval. As the previous section demonstrated, continuous real functions like these can be embedded into ternary space via the Dedekind splice. So can:

x is "different from" y	=	$\mid x - y \mid$
x is "very true"	=	x^2
x is "nearly true"	=	$x^{1/2}$
x is "extremely true"	=	$x^{16/5}$
x is "slightly true"	=	$x^{5/16}$
x is "at variance from" y	=	$(x-y)^2$
x "approximates" y	=	$1 - (x-y)^2$

Note that "x is at variance from y" = "x is very different from y", and that "x approximates y" = "x is not very different from y".

Now allow fuzzy truth functions to self-refer dynamically. For instance the Liar paradox, in fuzzy logic, becomes this iteration:

$$P_{n+1} \quad = \quad 1 - P_n$$

This has a constant solution $P_n = 0.5$, but also these wave solutions:

.1, .9, .1, .9, .1, .9, ...

.7, .3, .7, .3, .7, .3, ...

.4, .6, .4, .6, .4, .6, ...

— and many others. Self-reference in fuzzy logic yields many different dynamical behaviors; neutral oscillations like this, convergence to a fixedpoint, convergence to a limit cycle, and "chaos".

For instance, consider the Boaster, who says, "I am very honest."

$$B_{n+1} \quad = \quad B_n^2$$

This has an attracting fixedpoint at 0, and a repelling fixedpoint at 1.

By contrast, the Modest Truthteller says, "I am slightly honest."

$$M_{n+1} \quad = \quad M_n^{1/2}$$

This has an attracting fixedpoint at 1, and a repelling fixedpoint at 0.

The Golden Liar says, "I am slightly untrue."

$$G_{n+1} \quad = \quad (1 - G_n)^{1/2}$$

This has an attracting fixedpoint at 0.6180339888...; that is, $1/\phi$, or ϕ-1, where ϕ = the golden mean.

The Equivocal Liar says, "I am not very true."

$$E_{n+1} \quad = \quad 1 - E_n^2$$

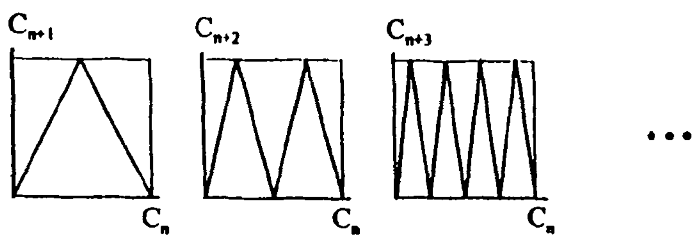
This has a repelling fixedpoint at ϕ-1, and a limit cycle of 0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1, ...

Now consider the Chaotic Liar, who says,

"I do not differ from my negation."

$C_{n+1} \quad = \quad 1 - |1 - C_n - C_n| \quad = \quad 1 - |1 - 2\,C_n|$

If you graph C_{n+1} versus C_n, you get a "tent" function, with a single peak. C_{n+2} versus C_n has two peaks, C_{n+3} versus C_n has four peaks, ... and C_{n+k} versus C_n has 2^{k-1} peaks.



This indicates that C_n is a "chaotic" function, deterministic yet unpredictable, with sensitive dependence on initial conditions.

Now consider the Logistic Liar, who says,

"I am not very different from my negation."

That is, "I approximate my opposite."

$$L_{n+1} \quad = \quad 1 - (1 - L_n - L_n)^2$$

So $$L_{n+1} \quad = \quad 4\,L_n\,(1 - L_n)$$

This is none other than the logistic map, most studied of all chaotic dynamical systems. (A complex version of this map yields the Mandelbrot set.)

Now consider this Socratic dialog:

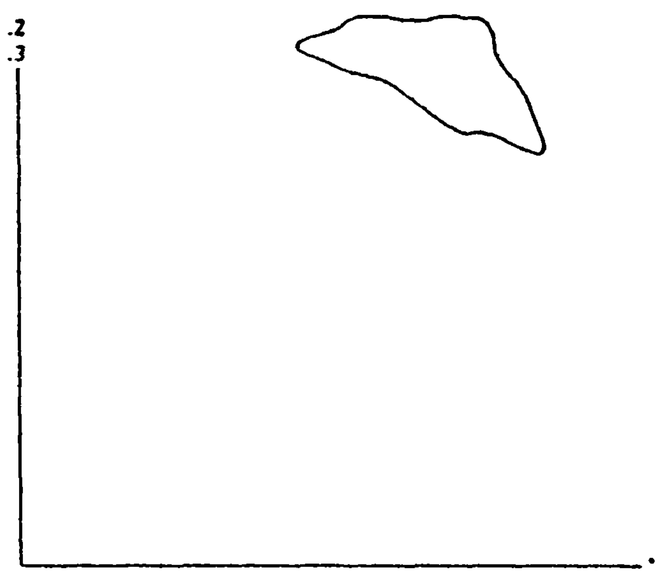
Socrates: "I approximate Plato's negation."

Plato: "I approximate Socrates."

$$S_{n+1} = 1 - (1 - P_n - S_n)^2$$

$$P_{n+1} = 1 - (S_n - P_n)^2$$

This system has a limit attractor in the form of a loop:



Now consider this dialog:

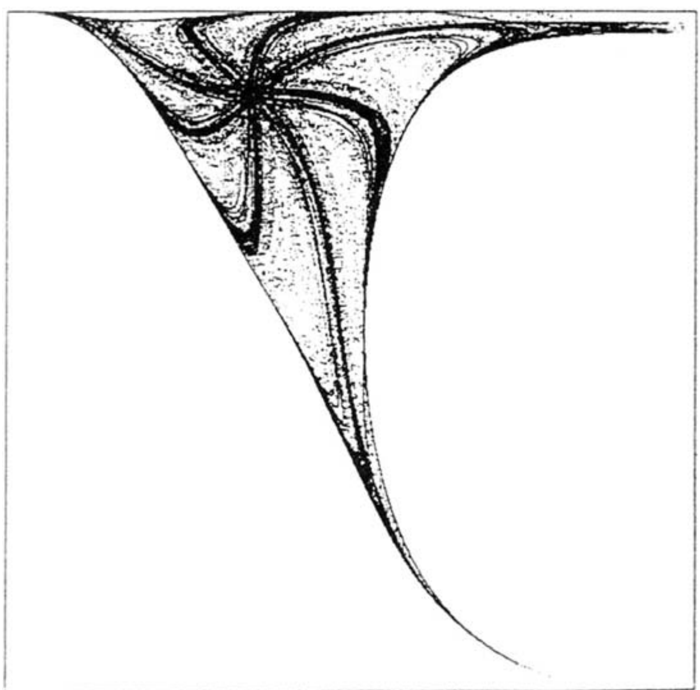
Socrates: "I am not even slightly different from Plato's opposite."

Plato: "I am not extremely different from Socrates."

$$S_{n+1} = 1 - (1 - P_n - S_n)^{5/16}$$

$$P_{n+1} = 1 - (S_n - P_n)^{16/5}$$

It has a fractal attractor. Behold Zeno's Theorem at its gnarliest:



Part Two

Outer Delta Logic

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Chapter 9

Outer Functions

Function Types
S_3 and Pivot
The Strengthened Liar

A. *Function Types*

All inner functions preserve inner order; but not all functions preserve the semi-lattice, so not all functions are inner. Consider these:

x	f	i	t	
x = t	f	f	t	$F(i) \not\leq F(t)$
x = i	f	t	f	$F(i) \not\leq F(t)$
x = f	t	f	f	$F(i) \not\leq F(f)$
1 + x	i	t	f	$F(i) \not\leq F(t)$
2 + x	t	f	i	$F(i) \not\leq F(f)$
$\sim_1$ x	i	f	t	$F(i) \not\leq F(t)$
$\sim_2$ x	f	t	i	$F(i) \not\leq F(f)$
1 + dx	i	t	i	$F(i) \not\leq F(t)$
2 + Dx	i	f	i	$F(i) \not\leq F(t)$

A function F 'preserves order' if, whenever $\underline{x} \leq \underline{y}$, then $F(\underline{x}) \leq F(\underline{y})$.

F 'reverses order' if, for some $\underline{x} \leq \underline{y}$, then $F(\underline{x}) \geq F(\underline{y})$.

F 'breaks order' if, for some $\underline{x} \leq \underline{y}$, then $F(\underline{x}) \nleq F(\underline{y})$ and $F(\underline{y}) \nleq F(\underline{x})$.

Another way to describe these types of functions involves the idea of "connection". Say that two ternary vectors $\underline{x}$ and $\underline{y}$ are connected ($\underline{x}$ C $\underline{y}$) if either $\underline{x} \leq \underline{y}$ or $\underline{x} \geq \underline{y}$. Therefore the value 'I' is connected to 'T' and 'F', which are not connected to each other.

Clearly order implies connection; and order-reversing functions are those which preserve connection but not order; and order-breaking functions are those which do not preserve connection.

Order-preserving functions include all Kleenean functions.

Order-reversing functions include: 1+dx, 2+Dx.

Order-breaking functions include: (x=y), $\sim_1$, $\sim_2$, 1+x, 2+x.

The next few theorems will need the following **lemma:**

$X \text{ min } (Y \text{ min } Z) = (X \wedge I) \vee (Y \wedge I) \vee (Z \wedge I) \vee (X \wedge Y \wedge Z)$

Proof:

$X \text{ min } (Y \text{ min } Z) = X \text{ min } ((Y \wedge I) \vee (Z \wedge I) \vee (Y \wedge Z))$

$= [X \wedge ((Y \wedge I) \vee (Z \wedge I) \vee (Y \wedge Z))] \vee [X \wedge I]$

$\vee\ [((Y \wedge I) \vee (Z \wedge I) \vee (Y \wedge Z)) \wedge I]$

$= (X \wedge Y \wedge I) \vee (X \wedge Z \wedge I) \vee (X \wedge Y \wedge Z) \vee (X \wedge I)$

$\vee\ (Y \wedge I \wedge I) \vee (Z \wedge I \wedge I) \vee (Y \wedge Z \wedge I)$

$= (X \wedge Y \wedge Z) \vee (X \wedge I) \vee (Y \wedge I) \vee (Z \wedge I)$ QED

Theorem: Any order-preserving function is Kleenean.

(Note that we've already proved the converse.)

Proof. If F(x) preserves order, then $F(I) \leq F(T)$ and $F(I) \leq F(F)$; therefore

$F(I) \min F(T) \min F(F) = F(I)$.

Now consider this **anchored normal form:**

$$F^*(x) = (F(t) \wedge x) \vee (F(f) \wedge \sim x) \vee (F(i) \wedge dx) \vee (F(t) \wedge F(f) \wedge F(i))$$

Let us test cases:

$$F^*(t) = (F(t) \wedge t) \vee (F(f) \wedge f) \vee (F(i) \wedge f) \vee (F(t) \wedge F(f) \wedge F(i))$$
$$= F(t) \vee (F(t) \wedge F(f) \wedge F(i)) = F(t)$$
$$F^*(f) = (F(t) \wedge f) \vee (F(f) \wedge t) \vee (F(i) \wedge f) \vee (F(t) \wedge F(f) \wedge F(i))$$
$$= F(f) \vee (F(t) \wedge F(f) \wedge F(i)) = F(f)$$
$$F^*(i) = (F(t) \wedge i) \vee (F(f) \wedge i) \vee (F(i) \wedge i) \vee (F(t) \wedge F(f) \wedge F(i))$$
$$= F(t) \min F(i) \min F(f) = F(i).$$

Therefore $F^*(x) = F(x)$ for all three possible values of x.

If x is F's only variable, then F* is a Kleenean function.

If F has n variables, proceed by induction.

Therefore all order-preserving functions are Kleenean. QED.

Theorem: Any function which reverses order everywhere, will, when iterated from i, tends towards a cycle of period two.

Proof. If F(x) reverses order everywhere, then F^2(x) preserves order. Therefore F^{2n}(i) is a fixedpoint for F^2; therefore { F^{2n}(i), F^{2n+1}(i) } is a limit cycle for F of period 2. QED.

Note: if F reverses order everywhere, then F^2 preserves order, and hence is Kleenean. Therefore F is the "square root" of a Kleenean function. For instance, the function (1+dx) is the square root of Dx, denoted rD(x). Similarly, 1+Dx = rd(x) ; 1+(x up T) = rI∨(x) ; 1+(x dn F) = rI∧(x) .

Theorem: Any order-breaking function yields an everywhere-boolean function when composed with some kleenean.

Proof: If F(x) breaks order, then there are vectors a and b such that a ≺ b but F(a) and F(b) are unequal Boolean values; say F(a) = t and F(b) = f.

Now consider this function: (b ∧ Dx) ∨ (a ∧ dx) ∨ (a ∧ b)

When x equals t or f, then this equals b; when x equals i, then this equals

(b ∧ i) ∨ (a ∧ i) ∨ (a ∧ b) = a min b = a

Therefore the function

F((b ∧ Dx) ∨ (a ∧ dx) ∨ (a ∧ b))

equals T when x equals I, and F when x equals T or F. Therefore

F((b ∧ Dx) ∨ (a ∧ dx) ∨ (a ∧ b))

equals (X=I), an everywhere-boolean function. QED

Theorem: Any function F is either Kleenean (and therefore has fixedpoints) or there exist Kleenean functions G and H such that the function G(F(H(x))) has no fixedpoints.

Proof. If F is not Kleenean, then by one of the above theorems it is not order-preserving either. Therefore there are vectors $\underline{a}$ and $\underline{b}$ such that $\underline{a} \prec \underline{b}$ but $F(\underline{a}) \nprec F(\underline{b})$; either $F(\underline{a}) \succ F(\underline{b})$ (order-reversing) or they are unequal Boolean values (order-breaking). In either case the function

$$F'(x) \;=\; F(\,(\underline{b} \wedge Dx) \vee (\underline{a} \wedge dx) \vee (\underline{a} \wedge \underline{b}\,)\,)$$

is a one-variable function that does not preserve order; for $F'(I) = F(\underline{a})$, $F'(T) = F'(F) = F(\underline{b})$.

If $F(\underline{a}) \succ F(\underline{b})$ then $F(\underline{a})$ is a boolean value (say, T) and $F(\underline{b})$ equals I; but then F', iterated from any value, oscillates between T and I; no fixedpoint.

If $F(\underline{a})$ and $F(\underline{b})$ are unequal Boolean values, then F'(x) equals (x=I) or else ~(x=I); boolean functions, never equal to 0; therefore these functions lack fixedpoints:

$$(x{=}F) \;=\; \sim(\,(x \wedge I) = I\,) \quad ; \quad (x \neq T) \;=\; (\,(x \vee I) = I\,)$$

Therefore, in each case, the function F, composed with some Kleenean functions, has no fixedpoints. QED.

B. S_3 *and Pivot*

Recall these functions, as defined above:

x	f	i	t
0 + x	f	i	t
1 + x	i	t	f
2 + x	t	f	i
$\sim_0$ x	t	i	f
$\sim_1$ x	i	f	t
$\sim_2$ x	f	t	i

This is the permutation group on three elements: S_3. This group is also isomorphic to the symmetries of the triangle. It contains three rotations (a+x) and three reflections ($\sim_a$ x). Here is its group table:

a∘b	0+	1+	2+	$\sim_0$	$\sim_1$	$\sim_2$
0+	0+	1+	2+	$\sim_0$	$\sim_1$	$\sim_2$
1+	1+	2+	0+	$\sim_2$	$\sim_0$	$\sim_1$
2+	2+	0+	1+	$\sim_1$	$\sim_2$	$\sim_0$
$\sim_0$	$\sim_0$	$\sim_1$	$\sim_2$	0+	1+	2+
$\sim_1$	$\sim_1$	$\sim_2$	$\sim_0$	2+	0+	1+
$\sim_2$	$\sim_2$	$\sim_0$	$\sim_1$	1+	2+	0+

Note that all the rotations are double negations.

Now for a pivotal concept; the *pivot* operator x#y, pronounced "x pivot y" or "x because y". (See Notes for explanation.) Pivot is an inherently ternary operator; it is *cental to*, and *characteristic of*, three-valued logic. Given x and y, x#y is uniquely defined; and furthermore x#y is defined by means unique to a three-valued system.

We define the pivot x#y by considering two cases; either x and y are equal, or they are not. If x equals y, then surely they specify a unique value; namely their shared value. If x does not equal y, then they specify a unique value; namely the third value, equal to neither. Therefore this axiom:

$$a \# b = c \quad \text{if and only if :}$$

$$a = b = c = a \quad \text{OR} \quad a \neq b \neq c \neq a$$

The pivot a#b is "equally equal" to a and b. Here is its table:

#	f	i	t
f	f	t	i
i	t	i	f
t	i	f	t

Note that each row of the pivot table equals a negation:

$$\sim_2 x = F\#x \quad ; \quad \sim_0 x = I\#x \quad ; \quad \sim_1 x = T\#x$$

so $\quad 1+x = I\#(T\#x) = T\#(F\#x) = F\#(I\#x)$;

and $\quad 2+x = I\#(F\#x) = F\#(T\#x) = T\#(I\#x)$.

If we use the **standard identification**

$$F = 2 \ ; \ I = 0 \ ; \ T = 1 \ ;$$

then $\sim_a x = a\#x$;

$1+x = 0\#(1\#x) = 1\#(2\#x) = 2\#(0\#x)$;

$2+x = 0\#(2\#x) = 2\#(1\#x) = 1\#(0\#x)$.

Thus pivot defines all of S_3. It has these laws:

Recall: $a \# a = a$

Commutativity: $a \# b = b \# a$

Cancellation: $a \# (a \# b) = b$

Level associativity: $(a\#b) \# (c\#d) = (a\#c) \# (b\#d)$

Transposition: $a\#b = c$ if and only if $a = b\#c$

Self-distribution: $a \# (b\#c) = (a\#b) \# (a\#c)$

If R is an element of S_3, then R preserves #:

S_3 symmetry: $R(a\#b) = Ra \# Rb$

This is because pivot self-distributes, and every permutation in S_3 derives from pivot. Pivot is the only operator on the triple with this property.

C. *The Strengthened Liar*

All Kleenean functions have fixedpoints; therefore it is always possible to evaluate self-referential kleenean statements. This is not so for non-kleenean sentences. Consider, for instance, the following:

"This sentence equals false."

That is, x = (x=f). This sentence has no fixedpoint solution; at best it has the period-2 cycle f,t,f,t,f,t,...

"This sentence does not equal true."

That is, x = ~(x=t). This too only has the cycle f,t,f,t,f,t,...

"This sentence is true, and this sentence is false, plus one."

That is, x = 1+dx. This has the period-2 cycle i,t,i,t,i,t,...

"One plus this sentence."

x = 1+x. This has the period-3 cycle f,i,t,f,i,t,f,i,t,...

These sentences are called *Strengthened Liars*; that is, liar's paradoxes strengthened to be insoluble in multi-valued logic. Many logicians regard the existence of Strengthened Liars to be a refutation of the multi-valued solution to the paradox problem, but I do not; for these sentences depend upon non-kleenean functions, which essentially reduce multi-valued logic to two values.

There is a similar situation in complex arithmetic. There exist complex numbers whose square is negative ($z^2 < 0$); but there do not exist complex numbers whose *length*, when squared, is negative ($|z|^2 < 0$)! But this is no defect of the complex numbers: for the length operator is *defined* to send complex numbers to real numbers; the lack of numbers with imaginary length is a trivial consequence. It is not the complex field's fault if you lose complex solutions when you leave the complex field!

Similarly, the Strengthened Liars merely show that you lose paradox-logic's self-reference when you leave paradox logic. A purist might quibble that one would *like* to have solutions to *all* equations in ternary logic, even those defined in a way excluding solution; but that of course is illogical.

So we see that some paradox-equations have solutions, but others do not. In a sense, then, paradox only *half*-exists! (How fitting!)

Chapter 10

Conjugate Logics

A. *S_3 Conjugation*

The permutation group S_3 permutes functions and relations as well as elements. Given a permutation P, a function F, and a relation R, define the permuted function P[F], and the permuted relation P[R], thus:

$$P[F](x) \quad = \quad P(F(P^{-1}(x)))$$

$$x\,P[R]\,y \quad \text{iff} \quad P^{-1}(x)\ R\ P^{-1}(y)\ .$$

These are F and R conjugated by P.

Conjugation Theorems:

$$P(\,F(x, y)\,) \quad = \quad P[F](\,P(x), P(y)\,)$$

$$x\ R\ y \quad \text{iff} \quad P(x)\ P[R]\ P(y)$$

$$y = P[F](x) \quad \text{iff} \quad P^{-1}(y) = F(P^{-1}(x))$$

$$P[=] \quad = \quad (=)$$

$$P[Q[F]] \quad = \quad (PoQ)[F]$$

$$P[F]o(P[G]) \quad = \quad P[FoG]$$

Proofs.

$P[F](P(x),P(y)) = P(F(\ P^{-1}(P(x)),\ P^{-1}(P(y))\ )) = P(F(x,y))$ QED.

$P(x)\ P[R]\ P(y)$ iff $P^{-1}(P(x))\ R\ P^{-1}(P(y))$ iff xRy. QED.

$y = P[F](x)$ iff $y = P(F(P^{-1}(x)))$ iff $P^{-1}(y) = F(P^{-1}(x))$

$x\ P[=]\ y$ iff $P^{-1}(x) = P^{-1}(y)$ iff $x = y$. QED.

$$
\begin{aligned}
P[Q[F]](x) &= P(Q[F](P^{-1}(x))) = P(Q(F(Q^{-1}(P^{-1}(x))))) \\
&= (PoQ)oFo(PoQ)^{-1}(x) = (PoQ)[F](x).\ \text{QED.}
\end{aligned}
$$

$$
\begin{aligned}
P[F]o(P[G])(x) &= P(F(P^{-1}(P(G(P^{-1}(x)))))) \\
&= P(F(G(P^{-1}(x)))) \\
&= P[FoG](x)\ .\ \text{QED.}
\end{aligned}
$$

Whatever equational identities the functions F and G may have, the functions P[F] and P[G] also have. Thus the conjugate of a DeMorgan algebra is a DeMorgan algebra, the conjugate of a field is a field, etc. Conjugation is isomorphism; it transports identities.

B. *The Three Logics*

The S_3 group table also defines the group's conjugation action on these permuted lattices:

$<$, or $<_0$:	f	<	i	<	t
$\sim_0[<]$:	t	<	i	<	f
$1+[<]$, or $<_1$:	i	<	t	<	f
$\sim_2[<]$:	f	<	t	<	i
$2+[<]$, or $<_2$:	t	<	f	<	i
$\sim_1[<]$:	i	<	f	<	t

These define permuted lattice operators:

$\sim = \sim_0$; $1+[\sim] = \sim_1$; $2+[\sim] = \sim_2$

$\wedge_0$ = the "minimum" operator for $<_0$: $f<i<t$

$\vee_0$ = the "maximum" operator for $<_0$: $f<i<t$

$\wedge_1$ = the "minimum" operator for $<_1$: $i<t<f$

$\vee_1$ = the "maximum" operator for $<_1$: $i<t<f$

$\wedge_2$ = the "minimum" operator for $<_2$: $t<f<i$

$\vee_2$ = the "maximum" operator for $<_2$: $t<f<i$

$\wedge_0$	f	i	t	$\vee_0$	f	i	t	$\sim_0$	id
f	f	f	f	f	f	i	t	t	f
i	f	i	i	i	i	i	t	i	i
t	f	i	t	t	t	t	t	f	t

$\wedge_1$	f	i	t	$\vee_1$	f	i	t	$\sim_1$	1+x
f	f	i	t	f	f	f	f	i	i
i	i	i	i	i	f	i	t	f	t
t	t	i	t	t	f	t	t	t	f

$\wedge_2$	f	i	t	$\vee_2$	f	i	t	$\sim_2$	2+x
f	f	f	t	f	f	i	f	f	t
i	f	i	t	i	i	i	i	t	f
t	t	t	t	t	f	i	t	i	i

Note that each "or" operator is the "and" operator of the reverse order; and therefore a permuted DeMorgan's Law applies:

$$P[\sim](x\ P[\wedge]\ y) = P[\sim](x)\ \ P[\vee]\ \ P[\sim](y)$$

$$P[\sim](x\ P[\vee]\ y) = P[\sim](x)\ \ P[\wedge]\ \ P[\sim](y)$$

We can define three majority operators:

$$M_0(x,y,z) = (x \wedge_0 y) \vee_0 (y \wedge_0 z) \vee_0 (z \wedge_0 x)$$
$$M_1(x,y,z) = (x \wedge_1 y) \vee_1 (y \wedge_1 z) \vee_1 (z \wedge_1 x)$$
$$M_2(x,y,z) = (x \wedge_2 y) \vee_2 (y \wedge_2 z) \vee_2 (z \wedge_2 x)$$

The subscripted majority operator has these identities:

Symmetry: $M_a(x,y,z) = M_a(y,z,x) = M_a(z,x,y) = M_a(x,z,y)$
$= M_a(z,y,x) = M_a(y,x,z)$

Coalition: $M_a(x,x,y) = M_a(x,x,x) = x$

Mediocrity: $M_2(f,i,t) = f$; $M_0(f,i,t) = i$; $M_1(f,i,t) = t$;

The subscript 'a' is the "Chairman's Subscript": it decides three-way ties. (See "The Chairman's Paradox", in the "Delta Dynamics" chapter below.)

These three axioms suffice to define all values of $M_a(x,y,z)$; for either $\{x,y,z\}$ has three distinct elements (and then Symmetry plus Mediocrity applies) or $\{x,y,z\}$ has coincidental elements (so Symmetry and Coalition apply).

Permutations: for any permutation P, $P(M_a(x,y,z)) = M_{Pa}(Px,Py,Pz))$

Positives: $x \wedge_a y = M_a(x, y, (2+a))$
$x \vee_a y = M_a(x, y, (1+a))$

For any permutation P, P[~], P[∧], and P[∨] is isomorphic to Kleenean logic. The "midpoint" of the lattice P[<] is a Liar paradox; it solves the equation L = P[~](L). Permuting three-logic generates three interlocking paradox logics.

C. *Cyclic Distribution*

Note that { $<_0$, $<_1$, $<_2$ } yields a political conundrum:

2/3 agree that	f	<	i
2/3 agree that	i	<	t
2/3 agree that	t	<	f

yet all agree that the order is linear!

This is the "Condorcet paradox", or "voter's paradox". It is reflected in an extraordinary phenomenon which I call **cyclic distributivity**:

$\wedge_0$ distributes over $\vee_0$, $\wedge_0$, $\vee_1$ and $\wedge_1$, but not $\vee_2$ or $\wedge_2$.

$\vee_0$ distributes over $\vee_0$, $\wedge_0$, $\vee_2$ and $\wedge_2$, but not $\vee_1$ or $\wedge_1$.

$\wedge_1$ distributes over $\vee_1$, $\wedge_1$, $\vee_2$ and $\wedge_2$, but not $\vee_0$ or $\wedge_0$.

$\vee_1$ distributes over $\vee_1$, $\wedge_1$, $\vee_0$ and $\wedge_0$, but not $\vee_2$ or $\wedge_2$.

$\wedge_2$ distributes over $\vee_2$, $\wedge_2$, $\vee_0$ and $\wedge_0$, but not $\vee_1$ or $\wedge_1$.

$\vee_2$ distributes over $\vee_2$, $\wedge_2$, $\vee_1$ and $\wedge_1$, but not $\vee_0$ or $\wedge_0$.

In general:

$\wedge_n$ distributes over $\vee_n$, $\wedge_n$, $\vee_{n+1}$ and $\wedge_{n+1}$, but not $\vee_{n-1}$ or $\wedge_{n-1}$ (modulo 3.)

That is, "$\wedge$" distributes up the loop 0⇒1⇒2⇒0

$\vee_n$ distributes over $\vee_n$, $\wedge_n$, $\vee_{n-1}$ and $\wedge_{n-1}$, but not $\vee_{n+1}$ or $\wedge_{n+1}$.

That is, "$\vee$" distributes down the loop 0⇒1⇒2⇒0

Proof of Cyclic Distributivity:

In this proof we ask; when does $P[\wedge]$ distribute over $Q[\wedge]$, if P and Q are permutations of $\{f,i,t\}$?

Without loss of generality we will assume that $P[\wedge] = \wedge$. Other cases will follow by group symmetry.

So now our question is:

When is $X \wedge (Y\ Q[\wedge]\ Z) \quad = \quad (X \wedge Y)\ Q[\wedge]\ (X \wedge Z)$?

Case 1: $\{X,Y,Z\}$ has only 1 element.

Then the equation follows by the idempotence of every lattice operator; $X\ R[\wedge]\ X = X$.

Case 2: $\{X,Y,Z\}$ has only 2 elements - say, $\{f,i\}$.

Then the lattice operators $\wedge$ and $Q[\wedge]$ would be min or max operators on the 2-element lattice $f<i$; these are isomorphic to the Boolean positive functions; these distribute over each other, therefore so do $\wedge$ and $Q[\wedge]$.

Case 3: $\{X,Y,Z\} = \{f,i,t\}$.

This divides into subcases:

Case 3A: $X = f = \wedge$'s minimum: so $Z \wedge f = f$

ergo $f \wedge (i\ Q[\wedge]\ t) = f$

and $(f \wedge i)\ Q[\wedge]\ (f \wedge t) = f\ Q[\wedge]\ f = f$;

CHECK.

Case 3B: X = t = ∧'s maximum: so Z ∧ t = Z

ergo t ∧ (f Q[∧] i) = f Q[∧] i

and (t ∧ f) Q[∧] (t ∧ i) = f Q[∧] i ;

CHECK.

Case 3C: X = i = ∧'s midpoint: so ~(i) = i.

ergo i ∧ (f Q[∧] t) = i ∧ (f Q[∧] t) ; itself.

and (i ∧ f) Q[∧] (i ∧ t) = f Q[∧] i

So *now* our question is:

For what Q is i ∧ (f Q[∧] t) = f Q[∧] i ?

Simply check all six lattices:

Q	Q[<]	i ∧ (f Q[∧] t)	f Q[∧] i	
id:	$f<i<t$	f	f	CHECK
1+:	$i<t<f$	i	i	CHECK
2+:	$t<f<i$	i	f	NO!
$\sim_0$:	$t<i<f$	i	i	CHECK
$\sim_1$:	$i<f<t$	f	i	NO!
$\sim_2$:	$f<t<i$	f	f	CHECK

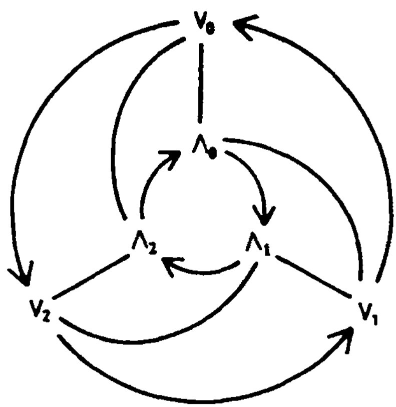
Thus this part of the lattice cycle: $\wedge_0$ distributes over all but $\wedge_2$ and $\vee_2$; those "after" it in the cycle $0 \Rightarrow 1 \Rightarrow 2 \Rightarrow 0$.

Starting from ∧, you can generate the rest of the distribution cycle via group symmetry transformations. Thus ∧ distributes up the loop $0 \Rightarrow 1 \Rightarrow 2 \Rightarrow 0$, and ∨ distributes down the loop.

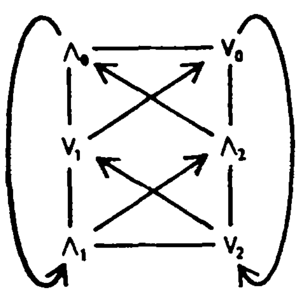
I illustrate the system this way:

$x \to y$	means	x distributes over y but not the reverse
x -- y	means	x and y distribute over each other

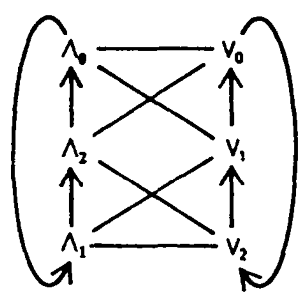
Then we get this diagram:



Two counter-rotating cycles. These diagrams resemble magnetic fields:

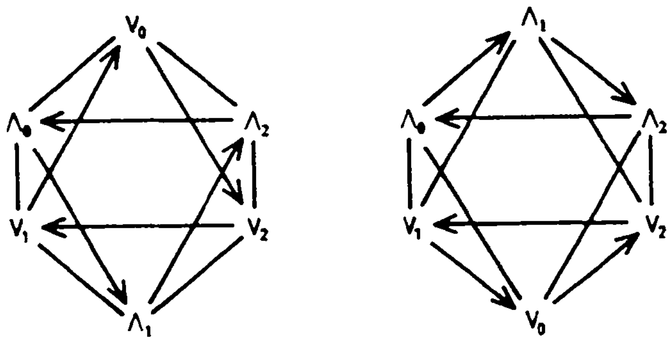


"Bar Magnet"

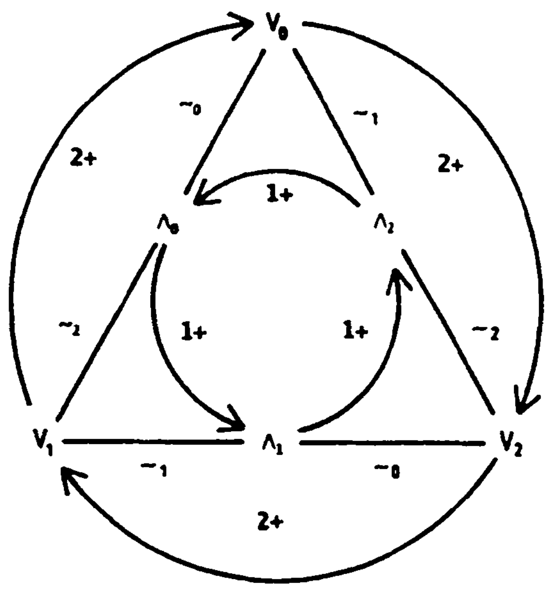


"Electromagnet"

Here are two Stars of David — or octohedra:



Here is the "eye in the pyramid", or "vortex", diagram:



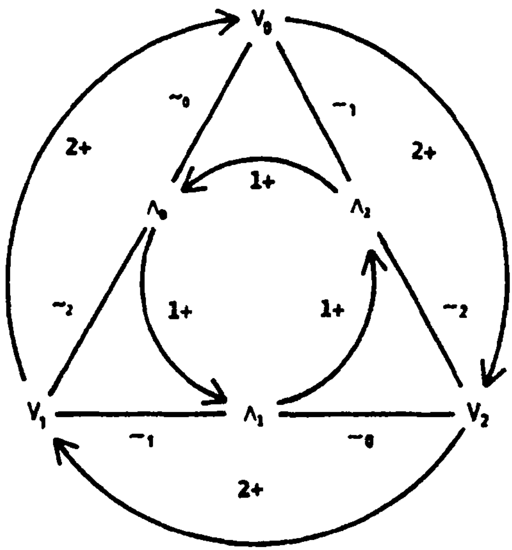
Each edge is labelled with the appropriate conjugation function.

And so we see that permuting three-valued logic generates three *entangled paradox logics*. Each 3-valued DeMorgan logic solves all paradoxes of self-reference and of continuity; however, they disagree as to which of {f,i,t} is minimum, maximum, or midpoint. The three logics interlock in a voter's paradox; S_3 versions of the DeMorgan laws apply, and the lattice operators distribute over each other in two counter-rotating cycles of period 3.

A Strange Loop indeed! Three values suffice to solve the liar's paradox, three ways; but this in turn generates a voter's paradox. Liar's paradox plus voter's paradox yields a logical knot; cyclic distributivity!

D. *The Vortex*

Recall the "vortex" diagram:



From this web of distribution we can extract a few links; each of these links defines a sublogic. The vortex contains three Kleenean sublogics, three Bochvarian sublogics, and two voter's sublogics.

Kleenean Logics:

$\vee_0, \wedge_0, \sim_0$; $\vee_1, \wedge_1, \sim_1$; $\vee_2, \wedge_2, \sim_2$

These are DeMorgan logics, rotated versions of the base Kleenean logic; they contain *intermediate* paradox values, analogous to the *indefinite* quantity 0/0; they preserve "inner order" ($i < t$, $i < f$) ; and in consequence solve all fixedpoint equations and embed the continuum.

Bochvarian Logics:

$\vee_2, \wedge_1, \sim_0$; $\vee_0, \wedge_2, \sim_1$; $\vee_1, \wedge_0, \sim_2$

These are DeMorgan logics; they contain *absorbing* paradox values, analogous to the *infinite* quantity 1/0; they preserve "inner order" ($i < t$, $i < f$) ; and in consequence solve all fixedpoint equations and embed the continuum. Bochvarian operators can be reduced to Kleenean: $\vee_2 = \wedge_{B0}$, $\wedge_1 = \vee_{B0}$

Voter's Logics:

$\wedge_{012}$, 1+x, 2+x ; $\vee_{012}$, 2+x, 1+x

These are cyclically distributive logics. They yield the voter's paradox.

Kleenean logic defines Bochvarian logic; Bochvar logic plus ($x \vee_0 i$) defines Kleene logic; Kleene logic plus (x+1) defines voter's logic.

Exercise for the reader: Prove these Bochvar-Kleene dualities:

$\vee_0 = \wedge_{B1}$, $\vee_1 = \wedge_{B2}$, $\vee_2 = \wedge_{B0}$; $\wedge_0 = \vee_{B2}$, $\wedge_1 = \vee_{B0}$, $\wedge_2 = \vee_{B1}$

$\vee_n = \wedge_{B,\, n+1}$; $\wedge_n = \vee_{B,\, n+2}$

$\vee_{B,\, n} = \wedge_{n+1}$; $\wedge_{B,\, n} = \vee_{n+2}$

Prove from this that the six permuted Bochvarian operators $\vee_{B,012}$, $\wedge_{B,012}$ have a distribution diagram almost identical to the ones on the preceding pages, except with the arrows reversed. This is the *Bochvarian counter-vortex.*

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Chapter 11

Bivalent Projections

Bivalent Commuting Operators
The Loop
Approximation and Mediation
The Differential

A. *Bivalent Commuting Operators*

Call an operator x*y "bivalent" if the following is true:

Bivalence: For any x and y, (x*y = x) or (x*y = y) .

Bivalent operators reduce delta's three values to two. $\wedge_{012}$ and $\vee_{012}$ are all bivalent operators; but min is not, for (T min F) = I .

One immediate consequence of bivalence is:

Recall: For any x, x * x = x .

One trivial example of a bivalent operator is the 'first-component' operator: x*y = x . To rule this out, postulate:

Commutativity: x*y = y*x .

Given a commuting bivalent operator $*$, define $<_*$ this way:

$a <_* b$ if and only if $a * b = a$.

This makes $*$ the "minimum" operator for $<_*$. Since there are three ways to compare pairs of elements from three (T?I, I?F, F?T), and since $<_*$ can point in two possible ways in each of those comparisons, it follows that there are $2^3 = 8$ commuting bivalent operators, corresponding to minima for these $<_*$'s:

$F<I$, $I<T$, $T>F$: this is $<_0$: $* = \wedge_0$,
$F>I$, $I<T$, $T<F$: this is $<_1$: $* = \wedge_1$,
$F<I$, $I>T$, $T<F$: this is $<_2$: $* = \wedge_2$,
$F>I$, $I>T$, $T<F$: this is $>_0$: $* = \vee_0$
$F<I$, $I>T$, $T>F$: this is $>_1$: $* = \vee_1$
$F>I$, $I<T$, $T>F$: this is $>_2$: $* = \vee_2$
$F<I$, $I<T$, $T<F$: this is the 'loop' $\triangleleft$: $*$ = dn (see below)
$F>I$, $I>T$, $T>F$: this is the 'loop' $\triangleright$: $*$ = up (see below)

Six of these orders are linear, and define the six linear operators $\wedge_{012}$, $\vee_{012}$; two of these orders are cyclic, and define the loop operators up and dn.

B. *The Loop*

Loop logic is non-linear. It's based on the following loop:

$$f \quad \lhd \quad i \quad \lhd \quad t \quad \lhd \quad f$$

This is like the children's game: scissors cuts paper, paper covers rock, rock breaks scissors. Pronounce "a◁b" as "a precedes b" or "b follows a".

This nonlinear order has two loop operators: "up" and "dn".

dn	f	i	t
f	f	f	t
i	f	i	i
t	t	i	t

up	f	i	t
f	f	i	f
i	i	i	t
t	f	t	t

a up b = the "greater" of a and b, according to ◁;

a dn b = the "lesser" of a and b, according to ◁.

a ⊴ b if and only if (a up b) = b if and only if (a dn b) = a

The loop is defined by these two axioms:

Triad (trichotomy): a◁b or else a=b or else a▷b

Loop (anti-transitivity): a◁b and b◁c implies c◁a

Some consequences of these axioms include:

Synchrony: $a \triangleleft b$ and $a \triangleleft c$ implies $b=c$

Period 3: $a \triangleleft b \triangleleft c \triangleleft d$ implies $a=d$

S_3's rotation operators send values "up" and "down" the loop:

Spin: $a \triangleleft 1{+}a \triangleleft 2{+}a \triangleleft a$

$a \triangleleft b$ if and only if $a = 2{+}b$, $b = 1{+}a$

Negations reverse the loop, and rotations preserve the loop:

Chirality: If $a \triangleleft b$, then $n\#a \triangleright n\#b$, $(n{+}a) \triangleleft (n{+}b)$

De Morgan: $n\#(a \text{ up } b) = (n\#a) \text{ dn } (n\#b)$;

$n\#(a \text{ dn } b) = (n\#a) \text{ up } (n\#b)$;

$n{+}(a \text{ up } b) = (n{+}a) \text{ up } (n{+}b)$;

$n{+}(a \text{ dn } b) = (n{+}a) \text{ dn } (n{+}b)$;

The loop operators admit these identities:

$(x \text{ up } F) = (dx)$

$(x \text{ up } I) = (x \lor I)$

$(x \text{ up } T) = (x \neq F)$

$(x \text{ dn } F) = (x = T)$

$(x \text{ dn } I) = (x \land I)$

$(x \text{ dn } T) = (Dx)$

— so most of the loop is Kleenean!

The loop operators have these identities:

Recall:	a up a = a dn a = a
Commutativity:	a up b = b up a ;
	a dn b = b dn a
Absorption:	a up (a dn b) = a ;
	a dn (a up b) = a
Repetition:	a dn (a dn b) = a dn b ;
	a up (a up b) = a up b
Local Dominance:	(a up b) dn (a dn b) = a dn b ;
	(a up b) up (a dn b) = a up b

In fact, every equation in two values, valid for the boolean connectives, is valid for up and dn. The loop is "locally linear".

Due to local linearity, these **modularity** laws apply:

$a \vee (a \text{ up } b) \;=\; a \text{ up } (a \vee b)$

$a \wedge (a \text{ up } b) \;=\; a \text{ up } (a \wedge b)$

$a \vee (a \text{ dn } b) \;=\; a \text{ dn } (a \vee b)$

$a \wedge (a \text{ dn } b) \;=\; a \text{ dn } (a \wedge b)$

However, many *three*-variable boolean equations do not apply; the third point "detects the loop". Thus:

(t up i) up f = f ≠ t = t up (i up f)

t up (i dn f) = f ≠ t = (t up i) dn (t up f)

— so loop logic is neither associative nor distributive.

In general, up and dn are neither associative nor distributive; but (x up k) and (x dn k) project the triple to two values; and this yields **Relative Booleanity**:

up and dn are associative and distributive for all elements of the forms:

	(x up T), and (x dn F).
or of the form:	(x up I), and (x dn T)
or or the form:	(x up F), and (x dn I)
or, in general:	(x up a), and (x dn (1+a))

The last result has this corollary:

(a=b) = { [((1+a) dn F) up ((1+b) dn F)]
dn [(a dn F) up (b dn F)] }
dn [((2+a) dn F) up ((2+b) dn F)]

The loop defines the three linear orders:

$$f <_0 i <_0 t \quad ; \quad i <_1 t <_1 f \quad ; \quad t <_2 f <_2 i$$

$a <_0 b$ if and only if $(a \triangleleft i \triangleleft b) \vee (a \triangleright t \triangleright b) \vee (a \triangleright f \triangleright b)$

$a <_1 b$ if and only if $(a \triangleleft t \triangleleft b) \vee (a \triangleright f \triangleright b) \vee (a \triangleright i \triangleright b)$

$a <_2 b$ if and only if $(a \triangleleft f \triangleleft b) \vee (a \triangleright i \triangleright b) \vee (a \triangleright t \triangleright b)$

These are the three "lines in the loop".

The three lines in turn define the loop:

$a \triangleleft b$ if and only if $M((a <_0 b), (a <_1 b), (a <_2 b))$

— where M is the "majority" operator (any subscript).

The linear orders $<_{012}$ define lattice operators: $\wedge_{012}$, $\vee_{012}$. For each n, $\wedge_n$ is the "minimum" operator on $<_n$, and $\vee_n$ is the "maximum" operator. These can be defined directly from the loop, thus:

$$a \wedge_0 b = ((a \text{ dn } i) \text{ up } (b \text{ dn } f)) \text{ dn } ((a \text{ dn } f) \text{ up } (b \text{ dn } i))$$
$$a \vee_0 b = ((a \text{ up } i) \text{ dn } (b \text{ up } t)) \text{ up } ((a \text{ up } t) \text{ dn } (b \text{ up } i))$$

$$a \wedge_1 b = ((a \text{ dn } t) \text{ up } (b \text{ dn } i)) \text{ dn } ((a \text{ dn } i) \text{ up } (b \text{ dn } t))$$
$$a \vee_1 b = ((a \text{ up } t) \text{ dn } (b \text{ up } f)) \text{ up } ((a \text{ up } f) \text{ dn } (b \text{ up } t))$$

$$a \wedge_2 b = ((a \text{ dn } f) \text{ up } (b \text{ dn } t)) \text{ dn } ((a \text{ dn } t) \text{ up } (b \text{ dn } f))$$
$$a \vee_2 b = ((a \text{ up } f) \text{ dn } (b \text{ up } i)) \text{ up } ((a \text{ up } i) \text{ dn } (b \text{ up } f))$$

In return, the three linear operators define the loop:

$$\begin{aligned} a \text{ up } b &= (a \vee_1 b) \vee_0 (a \vee_2 b) \\ &= (a \vee_2 b) \vee_1 (a \vee_0 b) \\ &= (a \vee_0 b) \vee_2 (a \vee_1 b) \\ &= M(\,(a \vee_0 b), (a \vee_1 b), (a \vee_2 b)\,) \end{aligned}$$

$$\begin{aligned} a \text{ dn } b &= (a \wedge_1 b) \wedge_0 (a \wedge_2 b) \\ &= (a \wedge_2 b) \wedge_1 (a \wedge_0 b) \\ &= (a \wedge_0 b) \wedge_2 (a \wedge_1 b) \\ &= M(\,(a \wedge_0 b), (a \wedge_1 b), (a \wedge_2 b)\,) \end{aligned}$$

The loop defines the lines, and the lines define the loop.

The pivot anti-distributes over the loop:

a # (b dn c) = (a#b) up (a#c)

a # (b up c) = (a#b) dn (a#c)

— but the loop operators do *not* distribute over pivot:

F up (T#I) = F up F = F $\neq$ T = F#I = (F up T)#(F up I)

F dn (T#I) = F dn F = F $\neq$ I = T#F = (F dn T)#(F dn I)

The loop and line operators do *not* distribute over each other:

F up (I $\wedge$ T) = F up I = I $\neq$ F = (I $\wedge$ F) = (F up I) $\wedge$ (F up T)

I $\wedge$ (T up F) = I $\wedge$ F = F $\neq$ I = (I up F) = (I $\wedge$ T) up (I $\wedge$ F)

F up (I $\vee$ T) = F up T = F $\neq$ I = (I $\vee$ F) = (F up I) $\vee$ (F up T)

I $\vee$ (T up F) = I $\vee$ F = I $\neq$ T = (T up I) = (I $\vee$ T) up (I $\vee$ F)

F dn (I $\wedge$ T) = F dn I = I $\neq$ F = (F $\wedge$ T) = (F dn I) $\wedge$ (F dn T)

I $\wedge$ (T dn F) = I $\wedge$ T = I $\neq$ F = (I dn F) = (I $\wedge$ T) dn (I $\wedge$ F)

T dn (I $\vee$ F) = T dn I = I $\neq$ T = (I $\vee$ T) = (T dn I) $\vee$ (T dn F)

I $\vee$ (T dn F) = I $\vee$ T = T $\neq$ I = (T dn I) = (I $\vee$ T) dn (I $\vee$ F)

Many open questions remain, including:

What are the definining axioms for the loop operators? In particular, what up-dn equation is equivalent to the loop relation's anti-transitivity axiom?

What 'normal forms' can we reduce loop expressions to?

Can we define $\wedge_{012}$ using dn only? (Ditto with $\vee_{012}$ and up.)

Can we relate loop logic to other ternary algebras - in particular, the quaternions? What else does loop logic apply to?

C. *Approximation and Mediation*

Define the approximate equality operator $\approx_a$ thus:

$$(x \approx_a y) \;=\; ((x{=}a) = (y{=}a))$$

X approximates Y, relative to a, if X and Y are equally equal to a; that is, both equal a or neither do. $\approx_a$ is an equivalence relation with two equivalence classes; one class containing **a** alone, and one containing the other two values.

A function f is "**well-defined relative to a**" if $f(x) \approx_a f(y)$ whenever $x \approx_a y$. Relative to t, ∧ and ∨ are well-defined, but not Dx nor ~x. Relative to f, ∧ and ∨ are well-defined, but not dx nor ~x. Relative to i, dx, Dx, and ~x are well-defined, but not ∧ nor ∨.

Now consider these truth tables:

x	F	I	T		
x ∨ I	I	I	T	=	x up I
x ∧ I	F	I	I	=	x dn F
dx	F	I	F	=	x up F

From this we can see the following:

x and y are equally equal to F if and only if (x ∧ I) and (y ∧ I) are equal:

$$(x \approx_f y) \;=\; ((x \wedge I) = (y \wedge I))$$

x and y are equally equal to I if and only if (dx) and (dy) are equal:

$$(x \approx_i y) \;=\; (dx = dy)$$

x and y are equally equal to T if and only if (x ∨ I) and (y ∨ I) are equal:

$$(x \approx_t y) \;=\; ((x \vee I) = (y \vee I))$$

Recall "Mediation" from chapter 3:

$x = y$ if and only if $x \wedge I = y \wedge I$ and $x \vee I = y \vee I$

In terms of approximation, this is:

$x = y$ if and only if $x \approx_f y$ and $x \approx_t y$

We can generalize this to **general mediation:**

If $a \neq b$, then $x = y$ if and only if $x \approx_a y$ and $x \approx_b y$

We can get equality by approximating twice. To prove general mediation, it suffices to prove **differential mediation:**

$x = y$ if and only if $x \vee I = y \vee I$ and $dx = dy$

$x = y$ if and only if $x \wedge I = y \wedge I$ and $dx = dy$

We can prove both at once by bracket notation. Given these equations:

*	x 6 = y 6	;
**	[x]x = [y]y	

we can derive [x]6 = [y]6 (and therefore x = y) thus:

[x]6	=	[x] [[] [x]] 6	occ.
	=	[[[x]] [[x]x]] 6	trans.
	=	[[[x]] [[y]y]] 6	**
	=	[[[x]6] [[y]y6]]	trans.
	=	[[[x]6] [[y]x6]]	*
	=	[[[x]] [[y]x]] 6	trans.
	=	[x [[y]x]] 6	ref.
	=	[xy] [[x]x] 6	echelon
	=	[xy] [[y]y] 6	**
	=	[y [[x]y]] 6	echelon

=	[[[y]] [[x]y]] 6	ref.
=	[[[y]6] [[x]y6]]	trans.
=	[[[y]6] [[x]x6]]	*
=	[[[y]] [[x]x]] 6	trans.
=	[[[y]] [[y]y]] 6	**
=	[y] [[] [y]] 6	trans.
=	[y]6	occ.

Mediation can be described in terms of computability. Given any function f(x), we can make the following identifications:

(f(x) ∨ I) is an *enumerably verified* function; it equals T for some values of x, and I provisionally. It "colors the exterior" of the set F.

(f(x) ∧ I) is an *enumerably falsified* function; it equals F for some values of x, and I provisionally. It "colors the interior" of the set F.

d(f(x)) is an *enumerably decided* function; it equals F whenever f(x) becomes boolean, and I provisionally. It "colors the boundary" of F.

In computational terms, $\approx_t$ corresponds to recursively enumerable sets; $\approx_f$ corresponds to complements of recursively enumerable sets; $\approx_i$ corresponds to decidability of sets (i.e. the Halting Problem).

In differential mediation, we use the boundary of a set to deduce its exterior, given its interior (or vice versa). The boundary allows a 'sideways' approach to computing equality.

D. *Differentials*

Recall these "self-difference" expressions:

$$dx = x \wedge \sim x = x \text{ minus } x = x \text{ xor } x = x \text{ up } F$$

the "lower differential"

$$Dx = x \vee \sim x = x \text{ implies } x = x \text{ iff } x = x \text{ dn } T$$

the "upper differential"

In Boolean logic these are identical to, respectively, false and true; and indeed those identities are the Laws of the Excluded Middle. Delta does not obey those laws; instead it has the **relocation** law:

$$i \vee dx = i \quad ; \quad i \wedge Dx = i$$

We get these "differential logic equations":

$$dx = dx \wedge x = dx \wedge Dx$$

$$x = x \vee dx = x \wedge Dx$$

$$Dx = Dx \vee x = Dx \vee dx$$

i.e. dx is a subset of x, which is a subset of Dx.

In Venn diagram terms, dx is the boundary of x.

$$\sim dx = Dx \quad ; \quad \sim Dx = dx$$

$$d(\sim x) = dx \quad ; \quad D(\sim x) = Dx$$

$$ddx = dDx = dx$$

$$DDx = Ddx = Dx$$

Here are the **Leibnitz rules**:

$$d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$$

$$D(x \vee y) = (Dx \vee y) \wedge (x \vee Dy)$$

I call these the "Leibnitz rules", due to their similarity to the Leibnitz rule for derivatives of products. They imply:

$d(x \vee y) = (dx \wedge (\sim y)) \vee ((\sim x) \wedge dy)$

$D(x \wedge y) = (Dx \vee \sim y)) \wedge ((\sim x) \vee Dy)$

$d(x \wedge y \wedge z) = (dx \wedge y \wedge z) \vee (x \wedge dy \wedge z) \vee (x \wedge y \wedge dz)$

$d(x \vee y \vee z) = (dx \wedge \sim y \wedge \sim z) \vee (\sim x \wedge dy \wedge \sim z) \vee (\sim x \wedge \sim y \wedge dz)$

$D(x \wedge y \wedge z) = (Dx \vee y \vee z) \wedge (x \vee Dy \vee z) \wedge (x \vee y \vee Dz)$

$D(x \vee y \vee z) = (Dx \vee \sim y \vee \sim z) \wedge (\sim x \vee Dy \vee \sim z) \wedge (\sim x \vee \sim y \vee Dz)$

And so on.

The Leibnitz rules imply these differential logic equations:

$d(x\text{-}y) = d(x \Rightarrow y) = ((\sim y) \wedge dx) \vee (x \wedge dy)$

$D(x\text{-}y) = D(x \Rightarrow y) = (y \wedge Dx) \vee ((\sim x) \wedge Dy)$

$d(x \text{ xor } y) = d(x \text{ iff } y) = dx \vee dy$

$D(x \text{ xor } y) = D(x \text{ iff } y) = Dx \wedge Dy$

$dM(x,y,z) = ((y \text{ xor } z) \wedge dx) \vee ((z \text{ xor } x) \wedge dy) \vee ((x \text{ xor } y) \wedge dz) \vee M(dx,dy,dz)$

$DM(x,y,z) = ((y \text{ iff } z) \vee Dx) \wedge ((z \text{ iff } x) \vee Dy) \wedge ((x \text{ iff } y) \vee Dz) \wedge M(Dx,Dy,Dz)$

$d(x \min y) = dx \min dy \min (x\text{-}y) \min (y\text{-}x)$

$D(x \min y) = Dx \min Dy \min (x \Rightarrow y) \min (y \Rightarrow x)$

Thus D and d do not commute with min.

$d(x \min \sim x) = x \min \sim x$

$D(x \min \sim x) = x \min \sim x$

$x \min \sim x = dx \min Dx$

Define:

If x then y else z $=$ $(x \Rightarrow y) \land ((\sim x) \Rightarrow z)$

Therefore:

d(if x then y else z)

$= (x \land dy) \lor (y \land dx) \lor (\sim x \land dz) \lor (z \land dx)$

$= ((y \lor z) \land dx) \lor$ (if x then dy else dz)

D(if x then y else z)

$= (\sim x \lor Dy) \land (\sim y \lor dx) \land (x \lor Dz) \land (\sim z \lor Dx)$

$= ((\sim y \land \sim z) \lor Dx) \land$ (if $\sim$ x then Dy else Dz)

By combining differentials, we get these equations:

$dx \land dy = (x \land y) - (x \lor y)$ "both without either"

$Dx \lor Dy = (x \land y) \Rightarrow (x \lor y)$ "both implies either"

$dx \lor dy$ = (dx iff dy) $\land$ (dx iff Dy)

= (x iff y) $\land$ (x iff $\sim$ y)

$Dx \land Dy$ = (Dx xor Dy) $\lor$ (Dx xor dy)

= (x xor y) $\land$ (x xor $\sim$ y) "opposite reflections"

dx xor dy = d(x xor y)

Dx xor Dy = d(x xor y)

dx iff dy = D(x iff y)

Dx iff Dy = D(x iff y)

We get these extensions of the Leibnitz rules:

$$
\begin{aligned}
d(x \wedge y) &= (x \wedge dy) \vee (y \wedge dx) \\
&= (x \wedge y) \wedge (dx \vee dy) \\
&= (x \wedge y) \wedge (x \text{ iff} \sim y) \wedge (x \text{ iff } y) \\
&= (x \wedge y) \wedge (x \text{ iff} \sim y) \wedge (x \text{ iff } y) \\
&= (x \wedge y) \wedge (dx \text{ iff } dy) \wedge (dx \text{ iff } Dy) \\
&= (x \wedge y) \wedge (x \text{ xor } y)
\end{aligned}
$$

$$
\begin{aligned}
D(x \vee y) &= (x \vee Dy) \wedge (y \vee Dx) \\
&= (x \vee y) \vee (Dx \wedge Dy) \\
&= (x \vee y) \vee (x \text{ iff} \sim y) \vee (x \text{ iff } y) \\
&= (x \vee y) \vee (x \text{ xor} \sim y) \vee (x \text{ xor } y) \\
&= (x \vee y) \vee (Dx \text{ xor } Dy) \vee (Dx \text{ xor } dy) \\
&= (x \vee y) \vee (x \text{ iff } y)
\end{aligned}
$$

We can also rewrite the majority-boundary rules:

$$
\begin{aligned}
&dM(x,y,z) \\
&= M(x\text{-}x,\ y\text{-}z,\ z\text{-}y) \vee M(y\text{-}y,\ x\text{-}z,\ z\text{-}x) \vee M(z\text{-}z,\ x\text{-}y,\ y\text{-}x) \\
&= M(x\text{-}y,\ y\text{-}z,\ z\text{-}x) \vee M(x\text{-}z,\ y\text{-}x,\ z\text{-}y) \vee M(x\text{-}x,\ y\text{-}y,\ z\text{-}z)
\end{aligned}
$$

$$
\begin{aligned}
&DM(x,y,z) \\
&= M(Dx,\ y \Rightarrow z,\ z \Rightarrow y) \wedge M(Dy,\ x \Rightarrow z,\ z \Rightarrow x) \wedge M(Dz,\ x \Rightarrow y,\ y \Rightarrow x) \\
&= M(x \Rightarrow y,\ y \Rightarrow z,\ z \Rightarrow x) \wedge M(x \Rightarrow z,\ y \Rightarrow x,\ z \Rightarrow y) \wedge M(Dx,\ Dy,\ Dz)
\end{aligned}
$$

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Chapter 12

Ternary Arithmetic

Z modulo 3
Z_3 Matrices
Balanced Ternary

A. *Z modulo 3*

Recall the "standard interpretation" of three-logic:

False = 2 ; Inner = 0 ; True = 1 .

Consider these numerical tokens as elements of Z_3, the remainders of division by three. In Z mod 3 arithmetic, 2 = -1; so False corresponds to Z_3's negative sign, Inner to its zero, and True to its positive sign.

Now recall the "pivot" operator:

#	2	0	1
2	2	1	0
0	1	0	2
1	0	2	1

#	f	i	t
f	f	t	i
i	t	i	f
t	i	f	t

The pivot also defines + and - for Z mod 3, thus:

x + y = 0#(x#y) = (0#x)#(0#y) ; - x = 0#x

Here is + in both notations:

+	2	0	1
2	1	2	0
0	2	0	1
1	0	1	2

+	f	i	t
f	t	f	i
i	f	i	t
t	i	t	f

In terms of Z_3, a#b = -a-b = -(a+b); an arithmetic NAND!

In general (a#b)#(c#d) = a+b+c+d. This explains 'level associativity'.

We can define S_3 from Z_3:

$\sim_0(x) = 0-x$; $\sim_1(x) = 2-x$; $\sim_2(x) = 1-x$

$x = 0+x$; $1+x = 1+x$; $2+x = 2+x$

Now compare these two tables:

*	2	0	1
2	1	0	2
0	0	0	0
1	2	0	1

iff	f	i	t
f	t	i	f
i	i	i	i
t	f	i	t

Multiplication modulo 3 is identical to "iff", Kleenean logic's equivalence operator. ("Iff" is also in Bochvarian logic.)

We get these identities:

a iff b = a b ; a xor b = - a b

Da = a^2 ; da = $-a^2$

This unites ternary logic with modulo 3 arithmetic. A close fit; and very pleasing to connect this logic to a familiar and useful arithmetic. Z_3 is the second-simplest nontrivial number field; the only one simpler is Z_2, which defines boolean logic. Z_3's multiplication is both Kleenean equivalence and multiplication of signs and zero in the reals. Z_3 is simple and unique; therefore radical.

Z_3 has the usual field axioms; and in addition these:

$$x + x + x = 0 \quad ; \quad x^3 = x \ .$$

These suffice to define Z_3. Note in particular the Boolean equation:

$$Dx = 1; \text{ i.e. } \quad x^2 = 1 \ , \text{ ergo } \quad x^{-1} = x$$

is true just for the non-zero (i.e. Boolean) values. In Z mod 3, times equals divides except at zero, the division point.

Z_3 has these **loop ordering** properties:

$a \lhd b$ if and only if $b = a+1$ if and only if $a = b-1$

For all a, $a^2 \unrhd 0$.

If $a \lhd b$, then: $c+a \lhd c+b$ for all c ;

$c-a \rhd c-b$ for all c ;

$ac \lhd bc$ if and only if $0 \lhd c$;

$ac = bc$ if and only if $0 = c$;

$ac \rhd bc$ if and only if $0 \rhd c$;

This is very much like $<$ in real number arithmetic. On the other hand:

If $a \lhd b$ and $c \lhd d$, then $a+c \rhd b+d$

Note that + is defined using 0. This is appropriate, as 0 is the attractor for the base logic's 'iff'. For rotated logics we can define rotated versions of +.

$$x +_a y = (x\#y)\#a = x+y-a.$$

This $+_a$ is commutative and associative. Its identity is a, its negation is a#x = -x-a, and multiplication is iff_a. All three rotated Kleenean logics, and all three rotated Bochvarian logics, participate in some version of Z mod 3.

B. Z_3 *Matrices*

In Z modulo 3, $x^3 = x$; so all polynomials are at most quadratic in each variable: $F(x) = ax^2 + bx + c$. If a, b, and c are constants, then there are 3*3*3 = 27 such functions, all distinct; these correspond exactly to the 27 distinct monic (one-variable) functions from {f,i,t} to itself.

Therefore the monic (one-variable) ternary functions have two representations; via their truth tables:

[F(-1) F(0) F(1)]

and via their polynomials:

$F(x) = ax^2 + bx + c$

We can unite the two descriptions by means of matrix arithmetic; for we can write the quadratic $ax^2 + bx + c$ as a matrix product:

$$ax^2 + bx + c = (a,b,c)(x^2,x,1)^T = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

Given $F(x) = ax^2 + bx + c$, then these Z_3 matrix equations follow:

$$\begin{pmatrix} F(-1) \\ F(0) \\ F(1) \end{pmatrix} = \begin{pmatrix} a-b+c \\ c \\ a+b+c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} F(-1) \\ F(0) \\ F(1) \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F(-1) \\ F(0) \\ F(1) \end{pmatrix}$$

```
( 1 -1  1 )
( 0  0  1 ) this is "Mcv", the "coefficients-to-values" matrix
( 1  1  1 )

(-1 -1 -1 )
( 1  0 -1 ) this is "Mvc", the "values-to-coefficients" matrix
( 0  1  0 )
```

Also useful is the "basis vectors map":

$e_x = (x - x^2 , 1 - x^2 , - x - x^2)^T$.

It is easy to check that

$e_2 = (1,0,0)^T$; $e_0 = (0,1,0)^T$; $e_1 = (0,0,1)^T$;

and also that $e_x = M_{vc}{}^T (x^2 , x , 1)^T$.

We can use the basis vectors map and the two M's to find the polynomial for an operator, given its values table. Let an operator have this truth table:

*	2	0	1
2	A_{22}	A_{20}	A_{21}
0	A_{02}	A_{00}	A_{01}
1	A_{12}	A_{10}	A_{11}

Call that 3 by 3 matrix "A", the operator's "value matrix". We can find its entries via Z_3 matrix products with basis vectors:

$$A_{xy} = e_y{}^T \, A \, e_x$$
$$= (y^2 , y , 1) \; M_{vc} \; A \; M_{vc}{}^T \; (x^2 , x , 1)^T$$

Therefore the matrix $M_{vc} A M_{vc}{}^T$ is the operator's "coefficient matrix"; it gives the coefficients of the Z_3 quadratic equal to A.

Conversely we can get the values matrix from the coefficient matrix thus:

$$A = M_{cv} C M_{cv}{}^T$$

Consider for instance the operator (x = y). It has this truth table:

	2	0	1
2	1	-1	-1
0	-1	1	-1
1	-1	-1	1

Therefore the coefficient matrix for (x = y) is

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Therefore (x = y) $=$ $\mathbf{x^2 + y^2 + xy + 1}$.

Now consider the up and dn operators:

dn	2	0	1
2	-1	-1	1
0	-1	0	0
1	1	0	1

up	2	0	1
2	-1	0	-1
0	0	0	1
1	-1	1	1

Their coefficient matrices are

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

Therefore

$$\mathbf{a\ dn\ b} = \mathbf{a^2 + b^2 + ab - a - b}$$
$$\mathbf{a\ up\ b} = \mathbf{-a^2 - b^2 - ab - a - b}$$

Exercise for the student: prove by matrices:

$a \wedge_0 b = -a^2b^2 + a^2 + b^2 - ab - a - b$

$a \vee_0 b = a^2b^2 - a^2 - b^2 + ab - a - b$

$a \wedge_1 b = -a^2b^2 - a^2b - b^2a + ab$

$a \vee_1 b = a^2b^2 + a^2b + b^2a - ab + a + b$

$a \wedge_2 b = -a^2b^2 + a^2b + b^2a + ab + a + b$

$a \vee_2 b = a^2b^2 - a^2b - b^2a - ab$

$a \Rightarrow b = a^2b^2 - a^2 - b^2 - ab + a - b$

$a \min b = -a^2b - b^2a$

Define $(a \min_n b) = n + ((a-n) \min (b-n))$. Prove:

$a \min_1 b = -a^2b - b^2a + ab + a^2 + b^2$

$a \min_2 b = -a^2b - b^2a - ab - a^2 - b^2$

$a \min_c b = -a^2b - b^2a + abc + a^2c + b^2c$

Prove that the base logic majority operator M_0 equals:

$M_0(a,b,c) = ab^2c^2 + a^2bc^2 + a^2b^2c - a^2b - b^2c - c^2a - a^2c - c^2b - b^2a + abc$

Define $M_n(a,b,c) = n + M_0((a-n), (b-n), (c-n))$

Find expressions for M_n, $\wedge_n$, and $\vee_n$.

Here is a list of all 27 ternary quadratics:

$-a^2 - a - 1$	=	(a≠T)+1	=	rF	OR
$-a^2 - a$	=	a up I	=	a∨I	KL
$-a^2 - a + 1$	=	(a≠T)			OB
$-a^2 - 1$	=	(a≠I)			OB
$-a^2$	=	a up F	=	da	KL
$-a^2 + 1$	=	1+da	=	rD(a)	OR
$-a^2 + a - 1$	=	(a≠F)-1	=	rI∧(a)	OR
$-a^2 + a$	=	~a ∨ I	=	~a dn I	KL
$-a^2 + a + 1$	=	a up T	=	(a≠F)	OB
$- a - 1$	=	$\sim_1$(a)			OB
$- a + 0$	=	$\sim_0$(a)			KL
$- a + 1$	=	$\sim_2$(a)			OB
$- 1$	=	2			KL
0	=	0			KL
1	=	1			KL
$a - 1$	=	2+a			OB
$a + 0$	=	0+a			KL
$a + 1$	=	1+a			OB
$a^2 - a - 1$	=	(a=F)			OB
$a^2 - a$	=	a dn I	=	a ∧ I	KL
$a^2 - a + 1$	=	(a=F) - 1	=	rT	OR
$a^2 - 1$	=	2+Dx	=	rd(x)	OR
a^2	=	a dn T	=	Dx	KL
$a^2 + 1$	=	(a=I)			OB
$a^2 + a - 1$	=	a dn F	=	(a=T)	OB
$a^2 + a$	=	~a up T	=	~a ∧ I	KL
$a^2 + a + 1$	=	(a=T) - 1	=	rI∨(x)	OR

OB = order-breaking, OR = order-reversing, KL = kleenean.

C. *Balanced Ternary*

Balanced Ternary is base-3 numbering; but instead of the digits 0,1, and 2, we use the digits 0,1, and -1. (Here, I'll use 0,+, and -). This lets us express negative numbers. Here are some numbers in balanced ternary:

...	-5	-4	-3	-2	-1	0	1	2	3	4	5	...
...	-++	--	-0	-+	-	0	+	+-	+0	++	+--	...

We can define arithmetic algorithms for balanced ternary. If x and y are digits in being added, then the sum digit is x+y modulo 3, and the carry digit is (x min y). To subtract, merely flip bits and add. Multiplication requires summing partial products. For instance:

```
    +        +0--+       +0--+        ++0-
+   +    +  +--+-    -  +--+-     x   +--0
-----    ---------   ---------    --------
   +-       --+00       +0--+         0000
          + -       +  -++-+         --0+
         ---------   ---------       --0+
          +-++00        0+0+-     ++0-
-   -       +             -+      --------
+   -    ---------   ---------    +-0+++0
-----       +0++00       +---
   -+                     +
                     ---------
                         +0--
```

Now for a 'magic' trick. Start with Z_3's addition and subtraction tables:

+	-	0	+
-	+	-	0
0	-	0	+
+	0	+	-

-	-	0	+
-	0	-	+
0	+	0	-
+	-	+	0

Merge these tables and interpret the entries in balanced ternary:

+0	--	0+
-+	00	+-
0-	++	-0

3	-4	1
-2	0	2
-1	4	-3

Then add 5 to each entry:

8	1	6
3	5	7
4	9	2

This is none other than the classic Lo Shu magic square!

Chapter 13
Voter's Paradox

The Troika
Glitches
Examples
Delta Deduction

A. *The Troika*

Recall that { $<_0$, $<_1$, $<_2$ } yields a voter's paradox:

2/3 agree that	f	<	i
2/3 agree that	i	<	t
2/3 agree that	t	<	f

yet all agree that the order is linear.

The voter's paradox is the heart of Kenneth Arrow's Impossibility Theorem. It appears that such logic knots have a habit of bollixing political systems. These tiny tangles give politics its notorious perversity.

To simplify presentation, I now introduce three fictional characters; none other than the Three Stooges.

General Moe rules the Scissors Party with an iron hand. His politics are fascistic; he favors power over logic over fairness. He would rather be decisive than consistent, and he would rather be consistent than share power. Naturally he prefers monarchy, most preferably if the monarch is himself.

Judge Larry is senior theoretician for the Paper Party. His politics are legalistic: he favors logic over fairness over power. Naturally he prefers to govern by consensus.

Mayor Curly is lead singer for the Rock Party. His politics are populistic: he favors fairness over power over logic. Naturally he prefers to govern by majority rule.

Each single Stooge has a consistent linear ranking of fairness, power, and logic; but when you put them all together, something has got to go.

Two-thirds of the Stooges (namely, Moe and Larry) put logic above fairness; Larry and Curly put fairness above power; and Curly and Moe put power above logic.

	Moe	Larry	Curly
Power < Fairness?	no	yes	yes
Logic < Power?	yes	no	yes
Fairness < Logic?	yes	yes	no

This gives us a Condorcet Election, or "Voter's Paradox":

logic

< < 2/3 majority each

fairness > power

— yet they *all* agree that the ranking is linear!

There are several partial resolutions to this.

If we appoint a single voter as tyrant (Moe, say) then we can decide this consistently; but this is not a fair system.

If we attempt to decide by consensus (as Larry suggests) then that is fair and consistent; but we decide nothing, and that is a weak system.

If we have faith in majority rule (as Curly professes) then we accept the non-linear order, *and* the linearity of the order. This is fair and decisive, but it is inconsistent.

Finally, we can accept the non-linear ranking, and accept it as non-linear; this goes with every 2/3 majority, but reverses a consensus; and that is perverse.

This political knot is an instance of **Arrow's Theorem**, which says that no voting system has all four of these virtues:

it is fair: it gives all voters equal power

it is decisive: it decides all questions posed to it

it is logical: it does not believe contradictions

it is responsive: it never defies a voter consensus.

In other words, any government is at least one of:

cruel ; weak ; absurd ; perverse.

Moe prefers cruelty, Larry prefers weakness, and Curly prefers folly; *none* of them want perversity, but that of course is what they always get!

B. *Glitches*

The logic of Stooge elections (a.k.a. **troikas**) is highly non-Aristotelian. Even though each Stooge is as logical as he can be, the system within which they operate adds error on its own. These systematic errors include: weak and, strong or, weak majority, strong majority, arithmetic glitch, equivalence glitch, implication glitch, modus ponens breakdown, sorites failure, and set loops.

Weak And is this election:

	Moe	Larry	Curly
Are you an ape?	no	yes	yes
Are you a bozo?	yes	no	yes

Majorities agree to these propositions:

* I am an ape.

* I am a bozo.

* I am not both an ape and a bozo.

Strong Or is this election:

	Moe	Larry	Curly
Are you an ape?	yes	no	no
Are you a bozo?	no	yes	no

Majorities agree to:

* I am not an ape.

* I am not a bozo.

* I am either an ape or a bozo.

Here is a **weak majority**:

	Moe	Larry	Curly
Do you like ale?	yes	yes	no
Do you like beer?	no	yes	yes
Do you like cider?	no	yes	no

Majorities agree to:

* I like ale.

* I like beer.

* I don't like cider.

* I don't like most of those three.

Here is a **strong majority**:

	Moe	Larry	Curly
Do you like ale?	no	no	yes
Do you like beer?	yes	no	no
Do you like cider?	yes	no	yes

Majorities agree to:

* I don't like ale.

* I don't like beer.

* I do like cider.

* I do like most of those three.

Here is an **arithmetic glitch**:

		Moe	Larry	Curly
x	=	1	0	0
y	=	0	1	0

Majorities believe:

* x = 0

* y = 0

* x+y = 1

Here is an **equivalence glitch**:

	Moe	Larry	Curly
Do you love Alice?	no	yes	yes
Do you love Bob?	no	no	yes

Majorities believe:

* I love Alice.

* I love Alice and Bob equally.

* I do not love Bob.

In the above election, this also passes:

* If I love Alice, then I love Bob.

so this is also an **implication glitch**.

Here's another implication glitch:

	Moe	Larry	Curly
Are you a bum?	yes	yes	no
Are all bums crooks?	yes	no	yes
Are you a crook?	yes	no	no

Majorities believe:

* I am a bum.

* All bums are crooks.

* I am not a crook.

Related to the implication glitch is **modus ponens breakdown**:

	Moe	Larry	Curly
Are all men fools?	yes	yes	no
Are all fools goons?	yes	no	yes
Are all men goons?	yes	no	no

Majorities believe:

* All men are fools.

* All fools are goons.

* Not all men are goons.

I also call this **Barbarism** because it undermines the validity of that classic Aristotelian syllogism, BARBARA:

All A are B, all B are C, therefore all A are C.

Here is a **Sorites Failure:** (in honor of Lewis Carroll)

Curly believes: that babies are logical;
that illogical people are despised;
and that despised people cannot manage a crocodile;
and babies can manage a crocodile, being logical and respected.

Larry believes: that babies are illogical;
that illogical people are not despised;
and that despised people cannot manage a crocodile;
and babies can manage a crocodile, being illogically respected.

Moe believes: that babies are illogical;
that illogical people are despised;
and that despised people can manage a crocodile;
and babies can manage a crocodile, being despised.

Majorities agree: (L&M) Babies are illogical;
(M&K) Illogical people are despised;
(K&L) Despised people cannot manage a crocodile;
yet they *all* agree that babies can manage a crocodile!

Here is a **Set Loop**:

	Moe	Larry	Curly
Apes	{Dr.0}	{Dr.0, #1}	{}
Bozos	{Dr.0, #1}	{}	{Dr.0}
Crooks	{}	{Dr.0}	{Dr.0, #1}

Majorities agree that:

* All apes are bozos.

* All bozos are crooks.

* All crooks are apes.

Yet not the reverse! That is:

* Not all bozos are apes.

* Not all crooks are bozos.

* Not all apes are crooks.

And worst of all, *every* Stooge agrees:

** These three classes form a BARBARA syllogism:

all X are Y; all Y are Z; therefore all X are Z.

So the Stooges, those bunglers, made a huge mess of BARBARA, in the very act of affirming it! These spinning set loops make mincemeat of classical logic. How barbaric!

C. *Examples*

Here is a **Double Bind**:

	Moe	Larry	Curly
Can we choose chocolate?	yes	no	no
Can we choose vanilla?	no	no	yes
Do we have to choose?	yes	no	yes

Majorities believe:

* We can't choose chocolate.

* We can't choose vanilla.

* We have to choose.

Thus the Stooges, as a collective, are in a double-bind, even though no one of them is!

Here is an **Orlov Doubt State**:

	Moe	Larry	Curly
Can we tell chocolate from vanilla?	yes	no	no
Can we choose at random?	no	no	yes
Do we have to choose?	yes	no	yes

Majorities believe:

* We can't tell chocolate from vanilla.

* We can't choose at random.

* We have to choose.

Here the collective is in a quandary, even though no individual Stooge is.

Here is a **heap glitch**:

Curly says: 100 grains of sand are the smallest heap.

Larry says: 101 grains are the smallest heap.

Moe says: 102 grains are the smallest heap.

Majorities believe:

* 100 grains of sand are not a heap.
* 101 grains are a heap.
* 101 grains are not the smallest heap.

The Sorites Failure shows up in these elections:

"French Film Paradox" (with thanks to Matthew Groening):

(2/3): The French are funny.

(2/3): Sex is funny.

(2/3): Comedies are funny.

Yet all agree that no French sex comedies are funny!

We can explain this troika by "expanding" it thus:

Moe: The French and sex are funny, but not comedies;

Larry: Comedies and the French are funny, but not sex;

Curly: Sex and comedies are funny, but not the French.

Exercise for the reader: expand the following 11 troikas:

"Intellectual Property Troika":

L&M: Speech is information.

M&K: Information is property.

K&L: Property is not free.

Yet all agree that speech is free!

"Libertarian Troika":

M&K: The free market fosters competition.

L&M: Competition undermines civic virtue.

K&L: Prosperity requires civic virtue.

Yet all agree that the free market brings prosperity.

"Traditionalist Troika":

K&L: Democracy depends upon family values.

M&K: Popular culture strengthens democracy.

L&M: Popular culture weakens family values.

"Presidential Troika":

M&L: Tweedledee is more moral than Tweedledum.

L&K: Only the moral should be President.

K&M: Tweedledum should be President.

"Downsizing Troika":

M&K: Corporations help the economy grow.

K&L: The economy can grow only if there are more jobs.

L&M: Corporations reduce the number of jobs.

"Unborn Troika":

M&L: The rights of the unborn are low just after conception.

L&K: The rights of the unborn are high just before birth.

K&M: The rights of the unborn do not increase during gestation.

"P.C. Troika":

K&L: Some people are victims.

M&K: Victims deserve special treatment.

L&M: Everybody deserves equal treatment.

"Omniscience Troika":

M&K: God is omniscient.

L&M: If God is omniscient, then there is no free will.

K&L: There is free will.

"Mortality Troika":

M&L: The soul is not immortal.

K&M: If the soul is not immortal, then life is not worth living.

L&K: Life is worth living.

"Nihilist Troika":

M&L: God is dead.

K&M: If God is dead, then all is permitted.

L&K: Not all is permitted.

"Lifeline Strong Majority":

2/3: Old age is not O.K.

2/3: Adulthood is not O.K.

2/3: Childhood is O.K.

2/3: Most of life is O.K.

Here are several **linear loops**:

Theatrical Values Loop:

Moe:	sex	<	comedy	<	violence
Larry:	violence	<	sex	<	comedy
Curly:	comedy	<	violence	<	sex

so by 2/3 majorities:

sex < comedy < violence < sex ; though all agree the order is linear.

Personal Values Loop:

Moe: A crook is better than a fool, and a fool is better than a wimp.

Larry: A wimp is better than a crook, and a crook is better than a fool.

Curly: A fool is better than a wimp, and a wimp is better than a crook.

So by 2/3 each: a crook is better than a fool, a fool is better than a wimp, and a wimp is better than a crook.

Political Values Loop:

Moe:	sharing	<	production	<	organization
Larry:	production	<	organization	<	sharing
Curly:	organization	<	sharing	<	production

Here, Moe's values are essentially those of the State, Larry's are those of the Church, and Curly's that of the Market!

Honor Loop:

Moe: There are no saints, some but not all are sages, and all are heros.

Larry: There are no heros, some but not all are saints, and all are sages.

Curly: There are no sages, some but not all are heros, and all are saints.

So therefore by 2/3 each:

(ML): All saints are sages, but not all sages are saints;

(LK): All sages are heros, but not all heros are sages;

(KM): All heros are saints, but not all saints are heros.

A set loop.

Logical Values Loop:

Moe:	Imagination	<	Truth	<	Lies
Larry:	Lies	<	Imagination	<	Truth
Curly:	Truth	<	Lies	<	Imagination

So by 2/3 each: Lies < Imagination < Truth < Lies.

This brings us back to $<_0$, $<_1$, $<_2$ and $\triangleleft$.

D. *Delta Deduction*

Delta deduction is a system for determining what propositions will pass, knowing which ones already passed.

Delta weakens deduction for positive functions such as "and" and "or"; only half of the usual deduction rules are valid. For instance:

From:		"A ∧ B" passes
Deduce:		"A" passes
	AND	"B" passes

From:		"A" passes
	OR	"B" passes
Deduce:		"A ∨ B" passes

The reverse deductions are invalid due to "strong or" and "weak and".

Majority gets treated like 'and' and 'or' do in ordinary proof systems.

From:		"A" passes
	AND	"B" passes
	AND	"C" passes
Deduce:		"M(A,B,C)" passes

You need all three; "weak majorities" exist, which fail when one of their three components fails.

From: "M(A,B,C)" passes

Deduce: "A" passes

OR "B" passes

OR "C" passes.

You need all three; "strong majorities" exist, which pass when two of their three components fail.

From: "A" passes

AND A = B is provable by Boolean logic

(— that is, passes unanimously)

Deduce: "B" passes.

Delta's rules for "not" are as strong as in classical logic; the law of the excluded middle applies:

Deduce: "$A \vee \sim A$" passes.

Reductio ad absurdum also applies:

From: From: "A" passes

Deduce: "$B \wedge \sim B$" passes

Deduce: "$\sim A$" passes.

Conjecture: The above deductive system is complete for 3-voter elections. That is, it deduces which propositions necessarily pass, given which passed before; and any system which does not yield an explicit contradiction under these rules has a 3-voter model.

Please note that not all inconsistent systems have three-voter models; for some are inconsistent under delta rules. But consider this 5-voter election:

Do you like...

	Apples?	Bananas?	Cherries?
Moe:	yes	no	no
Larry:	no	yes	no
Curly:	no	no	yes
Shemp:	yes	yes	yes
Curly Joe:	yes	yes	yes

Note that the following propositions pass by 3/5 each:

I like apples.

I like bananas.

I like cherries.

I do not like most of those three.

Thus 5-voter election deduction is even weaker than 3-voter.

Chapter 14

Delta Dynamics

Paradox of the Second Best
Distribution Paradox
Agenda Manipulation
Chairman's Paradox

A. *Paradox of the Second Best*

Recall how the Stooges ranked power, fairness, and logic:

Moe:	Fairness	<	Logic	<	Power
Larry:	Power	<	Fairness	<	Logic
Curly:	Logic	<	Power	<	Fairness

Logic

<   <   2/3 majority each

Fairness   >   Power

This nonlinearity generates a chaotic dynamic. For instance:

One fine day Larry decided to wimp out, the better to get his two friends under control.

He went to Curly and said, "Look. I *don't* want Moe's first choice; he wants Power in power, and I *don't* want that! Personally, I like *Logic*, but we can't have everything! Now, *you* want Fairness on top; and I'm willing to go along with that. It's my second-best choice, and your first; so let's be allies."

Curly agreed to this scheme; and Moe, to his infuriation, found himself shut out by their Sophisticated Voting!

Thus Larry, by accepting a mediocre outcome, avoided the worst outcome. That is, until Moe hit on this strategem; approaching Larry with uncharacteristic deference, Moe agreed to cast his vote in favor of Logic; Moe's second-best choice, and Larry's favorite.

Larry accepted, and, Curly, to his consternation, was on the outs this time! That is, until he approached Moe, with a Sophisticated Voting scam in mind. And so Moe and Curly combined against Larry, and put Power in power.

Then Larry approached Curly with a little deal. Round and round it goes!

B. *Paradox of Distribution*

One fine day Curly tried to share a wonderful windfall with his two friends. They were willing (indeed, eager) to make the most of it, but somehow the deal got lost in all the shuffle.

It all started when Curly met a wealthy philanthropist. This worthy told Curly, "My boy, I shall give you six shiny pennies!"

Curly said, "For *me*?"

"And your two friends," the philanthropist replied. "You must share!"

Curly said glumly, "All right, we'll share."

"And furthermore," the philanthropist smiled, "you must first tell me *how* you plan to share!"

"Aww, that's easy!" said Curly. "We'll figger out some kinda deal!" And off he went to inform his buddies. Alas, when he met them, the result was not what he expected.

Larry said, "So we get six cents if we can agree on shares?"

"That's right!" said Curly. "I say let's divvy up the loot even-steven; two cents each! Great, huh?"

But Moe and Larry glanced at each other and shook their heads. To Curly's consternation, Moe said, "Nah. Me and my pal Larry here plan to split it three cents each."

Curly counted on his fingers, then objected, "But that leaves me broke!"

"We outvote you," Larry said. Then he winked at Curly and said, "Unless... of course...".

Curly picked up the hint and said, "Hey Larry, how'd you like to have four cents?"

"I'd love to," said Larry. "So between the two of us it'll be four cents for me, two cents for you..."

"... and nothing for that bum over there," Curly agreed.

Moe yelled, "Now wait a minute!"

"Unless... of course..." Curly said, winking at Moe.

Moe got the hint. "Hey Curly, ya want four cents?"

"Soitenly!" said Curly. "So I get four, you get two..."

"... and Larry gets diddly-squat," Moe agreed.

Larry cried, "Hey!"

"Unless... of course..." Moe said, winking at Larry.

Larry sighed. Then he said, "Hey Moe, ya want four cents?"

Round and round it goes!

C. *Agenda Manipulation*

One fine day Moe decided to seize absolute power. To this end he rigged an election. His nefarious scheme succeeded, but it didn't do him any good.

"Boys," he said after a particularly confusing wrangle, "we're getting too many tie votes. I say we need a chairman to cast tie-breaking votes!"

"That's a good idea!" Curly enthused. "But who should be the chairman?"

"Who but me?" said Moe. "Wasn't I the one who was clever enough to think of the idea?"

"But I'm not so sure I want you for Chairman," Larry ventured.

"Why then, let's put it to a vote," said Moe.

"That's fair," said Curly. "I nominate myself!"

"And I nominate myself," Larry added.

After much bargaining, discussion, and fisticuffs, they settled on these preferences:

Moe:	Curly	<	Larry	<	Moe
Larry:	Moe	<	Curly	<	Larry
Curly:	Larry	<	Moe	<	Curly

Larry

<   <   2/3 majority each

Curly   >   Moe

"This isn't getting us anywhere," Curly complained.

"Why not try a run-off election?" Moe suggested.

"You'd be the *last* person I'd vote for!" Larry said.

"Okay," said Moe, "then why don't you and Curly face off?"

"That sounds fair," said Curly; and so the first round of voting was between Larry and Curly.

Larry won the first round, thanks to Moe's and Larry's vote. But then Larry faced Moe, who won with Moe's and Curly's votes:

```
Larry        Curly
    \        /
     Larry        Moe
         \        /
            Moe
```

Had Larry been the last one considered, then the elections would have been like this:

```
Moe          Curly
    \        /
     Curly        Larry
         \        /
           Larry
```

And had Curly been the last one considered, then the elections would have been like this:

```
Larry        Moe
    \        /
     Moe          Curly
         \        /
           Curly
```

Therefore, in an election like this, the last one to be considered wins. Thus Moe become Chairman of the Stooges!

D. *Chairman's Paradox*

"Well then!" Moe said, eagerly rubbing his hands. "Let's decide a few things, shall we? Eh, boys?"

Larry and Curly looked up. They had been discussing something together.

Moe continued, "Now let's try ranking fairness, power, and logic. Which is best? Curly?"

"Fairness is best," said Curly.

"How about you, Larry?" Moe said gleefully. He was expecting an answer of 'Logic', so he could vote 'Power' and invoke a tie.

But Larry said, "I agree with Curly."

More sophisticated voting! For Larry was going along with his second-favorite choice, to keep Moe from exercising the chairman's power.

"Oh," said Moe, crestfallen. "All right then, which one is your *least* favorite? Larry?"

"Power is worst," said Larry.

"How about you, Curly?" Moe was expecting an answer of 'Logic', so that he could vote 'Fairness' and invoke a tie.

But Curly said, "I agree with Larry."

Once again, a sophisticated vote! Curly went along with Larry's choice, to keep Moe from using the chairman's power.

Moe said grimly, "I see. And is Logic in the middle?"

"That's what *you* believe," Curly said.

"So that's how we'll vote," Larry added.

Actual preferences:

Moe:	Fairness	<	Logic	<	Power
Larry:	Power	<	Fairness	<	Logic
Curly:	Logic	<	Power	<	Fairness

Preferences according to Sophisticated Vote:

Power < Logic < Fairness

"But that's the exact opposite of what I want!" Moe yelled.

"That's because you're chairman," Larry explained.

Curly added, "Now we have a *reason* to gang up on you!"

"Power corrupts, and mathematical power corrupts mathematically," Larry explained.

"It's the Peter Principle," Curly confided. "You've just risen to your Level of Incompetence!"

"That is what the troika is *for*," Larry explained.

Curly chirped, "It's a king trap!"

Moe hollered, "And I'm the Stooge who fell for it!"

Curly chuckled: nyuck-nyuck-nyuck!

Part Three

Beyond Delta Logic

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Chapter 15

Diamond

Definitions and Tables
Phase Order and Self-Reference
Dihedral Conjugation
No Cyclic Distribution

A. *Definitions and Tables*

"Diamond" is a paradox logic containing Kleenean logic as a sublogic. 3-logic is a 'spatial' solution to the problem of paradox; in it, paradox is 'intermediate' between the boolean values. Similarly, diamond is a 'temporal' solution to the paradox-problem; in diamond, paradox is a 'logic wave' — that is, an oscillation between the boolean values.

So let us consider the period-2 oscillations of binary values. There are four such logic waves:

t t t t t t	;	call this "t/t", or "t".
t f t f t f	;	call this "t/f", or "i".
f t f t f t	;	call this "f/t", or "j".
f f f f f f	;	call this "f/f", or "f".

"/" is pronouced "but"; thus i is "true but false" and j is "false but true". These four values form a diamond-shaped lattice:

true = t/t

i = t/f **j = f/t**

false = f/f

This is "diamond logic"; a wave logic with two components and four truth values. It describes the logic waves of period 2.

The values i and j can be interpreted as "underdetermined" and "overdetermined" states; where "underdetermined" means "insufficient data for definite answer", and "overdetermined" means "contradictory data". The value i can take either role, provided that j takes the other.

Let the positive operators "$\wedge$" and "$\vee$" operate termwise:

$$(a/b) \wedge (c/d) = (a \wedge c) / (b \wedge d)$$

$$(a/b) \vee (c/d) = (a \vee c) / (b \vee d)$$

We can then define "but" as a projection operator:

$$a/b = (a \wedge i) \vee (b \wedge j)$$

$$= (a \vee j) \wedge (b \vee i)$$

In diamond logic, negation operates after a flip:

$$\sim (a/b) = (\sim b) / (\sim a)$$

This corresponds to a split-second time delay in evaluating negation; and this permits fixedpoints:

$$\sim(t/f) = (\sim f)/(\sim t) = t/f$$

$$\sim(f/t) = (\sim t)/(\sim f) = f/t$$

Thus paradox is possible in diamond logic.

Call a function "harmonic" if it can be defined from "$\wedge$", "$\vee$", "~", and the four values t, i, j, f. They include:

$a \Rightarrow b$	=	$(\sim a) \vee b$
a iff b	=	$(a \Rightarrow b) \wedge (b \Rightarrow a)$
a xor b	=	$(a \wedge \sim b) \vee (b \wedge \sim a)$

The "majority" operator M has two definitions:

$$M(a,b,c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$$
$$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$$

Here are the "lattice operators":

a min b	=	$(a \vee b) / (a \wedge b)$	=	"a $\vee/\wedge$ b"
a max b	=	$(a \wedge b) / (a \vee b)$	=	"a $\wedge/\vee$ b"

We can define "but" from the lattice operators:

a/b = (a min f) max (b min t) = (a max t) min (b max f)

Here are the two "harmonic projection" operators:

$$\lambda(x) = x / (\sim x)$$
$$\rho(x) = (\sim x) / x$$

Here are the upper and lower differentials:

Dx = x implies x = x iff x = $x \vee \sim x$

dx = x minus x = x xor x = $x \wedge \sim x$

This, then, is Diamond; a logic containing the boolean values, plus paradoxes and lattice operators.

Here are truth tables for the functions defined above:

x:	~ x:	$\wedge$ y:	$\vee$ y:	$\rightarrow$ y:	iff y:	xor y:
		t f i j	t f i j	t f i j	t f i j	t f i j
t	f	t f i j	t t t t	t f i j	t f i j	f t i j
f	t	f f f f	t f i j	t t t t	f t i j	t f i j
i	i	i f i f	t i i t	t i i t	i i i t	i i i f
j	j	j f f j	t j t j	t j t j	j j t j	j j f j

x:	but y:	min y:	max y:	λx:	ρx:	Dx:	dx:	M(x,y,z)
	t f i j	t f i j	t f i j					majority
t	t i i t	t i i t	t j t j	i	j	t	f	y $\vee$ z
f	j f f j	i f i f	j f f j	j	i	t	f	y $\wedge$ z
i	t i i t	i i i i	t f i j	i	i	i	i	y min z
j	j f f j	t f i j	j j j j	j	j	j	j	y max z

Inspection reveals that Kleenean ternary logic is a sublogic of diamond; just consider the values {f,i,t}. The subset {f,j,t} also works.

B. *Phase Order and Self-Reference*

Now I define the concept of phase order:

x ≤ y if and only if x min y = x if and only if x max y = y

t
≺ ≺
i ≺ j
≺ ≺
f

This structure is a lattice; it has a mutually distributive minimum and maximum. It is also a proper extension of the kleenean inner-order semi-lattice. The following theorems have proofs similar to those in chapter 4.

Theorem: min is the minimum operator for ≤ ;

(X min Y) ≤ X ; (X min Y) ≤ Y ;

and Z ≤ (X min Y) , if Z ≤ X and Z ≤ Y

Also: max is the maximum operator for ≤ ;

X ≤ (X max Y) ; Y ≤ (X max Y) ;

and (X max Y) ≤ Z , if X ≤ Z and Y ≤ Z

Theorem: $\preceq$ is transitive and antisymmetric:

$a \preceq b$ and $b \preceq c$ implies $a \preceq c$

$a \preceq b$ and $b \preceq a$ implies $a = b$

Theorem: $\preceq$ is preserved by disjunction and conjunction:

$a \preceq b$ implies $a \vee c \preceq b \vee c$

and $a \wedge c \preceq b \wedge c$

Theorem: $\preceq$ is preserved by negation:

$a \preceq b$ implies $\sim(a) \preceq \sim(b)$.

Theorem: $\preceq$ is preserved by any harmonic function:

$a \preceq b$ implies $F(a) \preceq F(b)$

This follows by induction from the previous two results.

Theorem: For any harmonic f;

$f(x \max y) \succeq f(x) \max f(y)$

$f(x \min y) \preceq f(x) \min f(y)$

These inequalities can be strict; for instance:

$dt \min df = f$; yet $d(t \min f) = i$

$Dt \min Df = t$; yet $D(t \min f) = i$

$dt \max df = f$; yet $d(t \max f) = j$

$Dt \max Df = t$; yet $D(t \max f) = j$.

Now we extend $\preceq$ to ordered form vectors:

$$\underline{x} = (x_1, x_2, x_3, \ldots, x_n)$$

$$\underline{x} \preceq \underline{y} \text{ if and only if } (x_i \preceq y_i) \text{ for all } i$$

Theorem: $\preceq$ has "limited chains", with limit 2N.

That is, if $\underline{x}_n$ is an ordered chain of finite form vectors;

$\underline{x}_1 \preceq \underline{x}_2 \preceq \underline{x}_3 \ldots$ or $\underline{x}_1 \succeq \underline{x}_2 \succeq \underline{x}_3 \ldots$;

and if N is the dimension of these vectors,

then for all $n > 2N$, $\underline{x}_n = \underline{x}_{2N}$.

Given any harmonic function $\underline{f}(\underline{x})$, define

a *left seed* for $\underline{f}$ is any vector $\underline{a}$ such that $\underline{f}(\underline{a}) \preceq \underline{a}$;

a *right seed* for $\underline{f}$ is any vector $\underline{a}$ such that $\underline{a} \preceq \underline{f}(\underline{a})$;

a *fixedpoint* for $\underline{f}$ is any vector $\underline{a}$ such that $\underline{a} = \underline{f}(\underline{a})$.

A vector is a fixedpoint if and only if it is both a left seed and a right seed.

Left seeds generate fixedpoints, thus:

If $\underline{a}$ is a left seed for $\underline{f}$, then $\underline{f}(\underline{a}) \preceq \underline{a}$. Since $\underline{f}$ is harmonic, it preserves order; so $\underline{f}^2(\underline{a}) \preceq \underline{f}(\underline{a})$; and $\underline{f}^3(\underline{a}) \preceq \underline{f}^2(\underline{a})$; and so on:

$$\underline{a} \succeq \underline{f}(\underline{a}) \succeq \underline{f}^2(\underline{a}) \succeq \underline{f}^3(\underline{a}) \succeq \underline{f}^4(\underline{a}) \succeq \ldots \succeq \underline{f}^{2n}(\underline{a}) = \underline{f}(\underline{f}^{2n}(\underline{a}))$$

This is the greatest fixedpoint left of $\underline{a}$.

Left seeds grow leftwards towards fixedpoints.

Similarly, right seeds grow rightwards towards fixedpoints:

$$\underline{a} \preceq \underline{f}(\underline{a}) \preceq \underline{f}^2(\underline{a}) \preceq \underline{f}^3(\underline{a}) \preceq \underline{f}^4(\underline{a}) \preceq \ldots \preceq \underline{f}^{2n}(\underline{a}) = \text{fixedpoint}$$

$\underline{f}^{2n}(\underline{a})$ is the leftmost fixedpoint right of the right seed $\underline{a}$.

Phase order permits us to find fixedponts in general. For any harmonic function $\underline{F}(\underline{x})$, we have the following:

The Self-Reference Theorem:

Any self-referential harmonic system has a fixedpoint:

$$\underline{F}(\underline{x}) = \underline{x}$$

Proof: Recall that all harmonic functions preserve order.

$\underline{i}$ is the leftmost set of values, hence this holds:

$$\underline{i} \preceq \underline{F}(\underline{i})$$

Therefore, $\underline{i}$ is a right seed for $\underline{F}$:

$$\underline{i} \preceq \underline{F}(\underline{i}) \preceq \underline{F}^2(\underline{i}) \preceq \underline{F}^3(\underline{i}) \preceq \ldots \underline{F}^{2n}(\underline{i}) = \underline{F}(\underline{F}^{2n}(\underline{i}))$$

$\underline{i}$ generates the "leftmost" fixedpoint. QED.

Similarly, $\underline{j}$ generates the "rightmost" fixedpoint:

$$\underline{F}(\underline{F}^{2n}(\underline{j})) = \underline{F}^{2n}(\underline{j})$$

All other fixedpoints lie between the two outermost:

$$\underline{F}^{2n}(\underline{i}) \preceq \underline{x} = \underline{F}(\underline{x}) \preceq \underline{F}^{2n}(\underline{j})$$

C. *Dihedral Conjugation*

Now let the dihedral group D operate on the diamond. It has four reflections and four rotations: (tf)="~"; (ij)="*"; (ti)(jf)="o/-"; (tj)(if)="-/o"

identity="o"; (tifj)="L";(tf)(ij)="-"; (tjfi)="R".

b

a*b	o	R	-	L	~	o/-	*	-/o
o	o	R	-	L	~	o/-	*	-/o
a R	R	-	L	o	o/-	*	-/o	~
-	-	L	o	R	*	-/o	~	o/-
L	L	o	R	-	-/o	~	o/-	*
~	~	-/o	*	o/-	o	R	-	L
o/-	-/o	*	o/-	~	R	-	L	o
*	*	o/-	~	-/o	-	L	o	R
-/o	o/-	~	-/o	*	L	o	R	-

If we identify the two-dimensional real vectors with a "diamond vector"; that is, linear combinations of diamond values:

$$\begin{pmatrix} r \\ s \end{pmatrix} = rt + si$$

then we can identify this group as 2 by 2 matrices;

```
o    =  (1 0)     ;    ~    =    (-1 0)
        (0 1)                    ( 0 1)

R    =  ( 0 1)    ;    o/-  =    (0 1)
        (-1 0)                   (1 0)

-    =  (-1 0)    ;    *    =    (1  0)
        (0 -1)                   (0 -1)

L    =  (0 -1)    ;    -/o  =    (0 -1)
        (1  0)                   (-1 0)
```

Modulo -, these are the generators of M(2,2), the two-by-two matrices.

D permutes functions and relations as well as elements, by conjugation. For P in the dihedral group, this applies:

	P[M]	=	M
That is:	P(M(x, y, z))	=	M (P(x), P(y), P(z))

Thus the above group table also defines the group's conjugation action on the logic operators:

	a[b]	∧ (b)	∨	min	max	~	*
a	o	∧	∨	min	max	~	*
	R	min	max	∨	∧	*	~
	-	∨	∧	max	min	~	*
	L	max	min	∧	∨	*	~
	~	∨	∧	min	max	~	*
	o/-	max	min	∨	∧	*	~
	*	∧	∨	max	min	~	*
	-/o	min	max	∧	∨	*	~

Note that all four positive operators distribute over each other; very symmetric.

Recall that -(a/b) = (-a)/(-b); "termwise" negation.

Minus gets down to Diamond's boolean innards.

Let * = ~ - ; that is, *(a/b) = (*b)/(*a).

Star reverses order. It exchanges i and j, leaving t and f fixed. Thus Star looks just like Not, at "right angles"; it is "sideways negation".

In diamond logic star is flip:

*(a/b) = (*b)/(*a)

In dynamic implementation star equals delay:

(*a)(n) = a(n-1)

In dual-rail circuits star equals swap wires.

Star, ~, identity and - form a Klein group:

	ab	o	~	*	-
		b			
	o	o	~	*	-
a	~	~	o	-	*
	*	*	-	o	~
	-	-	*	~	o

all elements equal their own inverse

Let "star logic" be a logic made from *, majority, and the four values, just as diamond logic is made from ~, majority, and the four values.

Star logic is isomorphic to diamond logic via rotation; therefore all results from the preceding chapters apply:

Star logic is a complete De Morgan algebra.

It proves the self-reference theorem.

It has limit operators.

The continuum embeds via a morphism.

Zeno's theorem.

When we combine star logic and diamond logic, then we get -, a non-fixedpoint operator. In a sense, then, star logic is "perpendicular" to diamond logic; similar to it, but intersecting it only at a point. Therefore I call star logic "paraharmonic"; it resembles harmonic logic but is incompatible. Diamond logic is two-dimensional; it has room for two separate dimensions of thought within it. Negation and star are "perpendicular" logics; they work at cross-purposes.

D. *No Cyclic Distribution*

Consider these period-three permutations of diamond:

(tif), (fit), (tjf), (fjt), (ijf), (fji), (tij), (jit).

They do *not* preserve adjacency, minus, or majority.

D is a normal subgroup of S_4; and modulo D, all elements of S_4 are conjugate to one of { id, (tif), (fit) } ; the group Z_3. At first I thought that these non-dihedral elements of S_4 generate, as in 3-logic, three cyclically distributive lattices on the diamond. However, after checking the additional cases added by the fourth value, I found that this is not true. Consider the permutation U = (fit). (This is the same as the "up" rotation in 3-logic, extended to the diamond.) U applied to the diamond yields the lattice $U[\Diamond]$:

f

t j

i

and these equations:

$$j \wedge (t\ U[\wedge]\ i) = j \wedge i = f$$

$$(j \wedge t)\ U[\wedge]\ (j \wedge i) = j\ U[\wedge]\ f = j$$

$$j \wedge (t\ U[\vee]\ i) = j \wedge t = j$$

$$(j \wedge t)\ U[\vee]\ (j \wedge i) = j\ U[\vee]\ f = f$$

As in 3-logic, the period-3 rotations induce three isomorphic logics; however, "$\wedge$" distributes over neither $U[\wedge]$ nor $U[\vee]$; so cyclic distributivity fails on the diamond.

Naturally, cyclic distributivity still applies on the sublogic {f,i,t}.

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Chapter 16

Dilemma

Prisoner's Dilemma
Banker's Dilemma
Voter's Dilemma
Pessimistic Chicken Logic
The Unexpected Departure

A. *Prisoner's Dilemma*

a , b		B	
		nice	mean
	nice	T , T	L , W
A	mean	W , L	D , D

W = Win
T = Truce
D = Draw
L = Lose

scoring: $L < D < T < W$; also $W+L = D+T$

For instance: $(L,D,T,W) = (0,1,2,3)$

This non-zero-sum game presents a player's paradox. It exemplifies the central dilemma of any society; namely, how to get people to co-operate for mutual benefit, when competitive behavior yields a tactical advantage. Negotiation and reciprocation are possible in Dilemma, unlike in competitive games, where there is never anything to negotiate. Mutual profit gives incentive to mutual aid; but exploitation remains tempting.

There are many different strategies for dilemma play. I call three of them the "Iron", "Gold", and "Silver" rules.

The **Iron** rule is the rule of rigid exploitation, justified in the name of expediency. Players ruled by the Iron rule see that no matter how the other player plays, exploitation always yields an advantage; they jump to the conclusion that no more thought is necessary, and play accordingly. This strategy is usually called "All D" (AD) for "Always Defect".

The **Gold** rule is the policy of absolute altruism. Gold rule players see that a society under Golden rule would be at peace, and thus prevail in the long run; they jump to the conclusion that the long run is already here, and play accordingly. This strategy is usually called "All C" (AC) for "Always Cooperate".

The **Silver** rule is the strategy of reciprocity. Silver players do unto others as those others have done unto them. They see that only exact imitation can ensure that the game's inner logic favors cooperation; they jump to the conclusion that the other player is aware of this, and play accordingly. This strategy is usually called "TFT", for "Tit For Tat", which starts by cooperating and continues by reciprocation.

Thus the Iron, Gold, and Silver rules are, respectively, vicious, vulnerable, and vain. Gold is for prey (or host) species, Iron for predator (or parasite) species, and Silver for social (or symbiotic) species. Gold says, "what's mine is yours"; Iron says, "greed is good"; and Silver says, "value for value".

Negotiation, strategy, and tactics intermesh in the following two negotiation agendas; "Axial Play" and "The Generous Offer":

Axial Play: for players at balance.

Tactic; a player limits play to truce-draw "axis".

The board permits no advantage of one over another.

Strategy; that player threatens draw unless truce.

Appeal to principle. Firmness against exploitation.

This is tactically soft-line cooperative and strategically hard-line competitive. This is the Justice agenda; soft actions, hard bargaining. It stands on shared principle. Its motto is; "Bribe, threaten, and emulate."

The Generous Offer: for player in position of strength.

Tactic; the player limits play to truce-win "column".

The board permits no adverse outcome for player.

Strategy; the player offers to share his prosperity.

Appeal to self-interest. Peace bought and paid for.

This is tactically hard-line competitive and strategically soft-line cooperative. This is the Mercy agenda; hard actions, soft bargaining. It stands on shared privilege. Its motto is; "Make them an offer they can't refuse."

Each agenda requires tactical support (the facts on the board) and strategic negotiation (the offer on the table).

Here is a dilemma version of a familiar game:

Dilemma Tic-Tac-Toe

The grid # and the letters X and O are the same; but there are three new rules:

* Player X starts first, but not in the center square;
* X and O alternate, until they fill the grid; and
* Truce = both XXX and OOO rows; win/lose = only one sort;
  draw = neither XXX nor OOO rows.

Here are some sample games. Numbers tell order of moves:

X5	06	X3
08	02	X9
X1	X7	04

Draw

X5	06	X3
X7	02	X9
X1	08	04

Truce

04	X3	06
08	02	X9
X1	X7	X5

X wins

X9	X3	08
04	02	06
X1	X7	X5

Truce

X5	X9	X7
06	02	08
X1	04	X3

Truce

06	X3	X5
X7	02	X9
04	X1	08

0 wins

06	X1	04
X9	02	X3
X5	08	X7

Draw

X9	X1	04
08	02	06
X5	X7	X3

Truce

X5	X7	X3
08	02	X9
X1	06	04

X wins

Even if we allow X to start in the center square, we might still get a truce such as this:

04	X5	08
06	X1	X9
02	X7	X3

Compare the first two games:

X5	06	X3
08	02	X9
X1	X7	04

Draw

X5	06	X3
X7	02	X9
X1	08	04

Truce

On the second game's sixth move, O put down O6 in the top-center square (A2), and then told X, "If you block me at C2, I'll block you at B1, and we'll draw. Better to grab the ABC1 file now, and let me get ABC2." X agreed, and they truced. This is classic axial play.

B. *Banker's Dilemma*

A Billiard-Marker, whose skill was immense
might perhaps have won more than his share;
But a Banker, engaged at enormous expense
had the whole of their cash in his care.
— Lewis Carroll, *The Hunting Of The Snark*

Consider a Dilemma game between players K and L; it is financed by a banker M, who gets to keep the remainder of the fund after the payoffs are distributed. Their payoffs are:

4 dollars invested

(K,L,M) payoff		L: nice	L: mean
K	nice	(2,2,0)	(0,3,1)
	mean	(3,0,1)	(1,1,2)

This makes Dilemma zero-sum again; for the main player's cooperation is the banker's defeat. What is more, the banker has a vested interest in fostering distrust between the other two players. (Indeed, that is the the <u>only</u> thing the banker can actively do; for the other two players make all the moves.)

The three players rank win/lose, lose/win, truce, and draw in three different ways:

K:	L/W	<	D	<	T	<	W/L
L:	W/L	<	D	<	T	<	L/W
M:	T	<	W/L	=	L/W	<	D

These three preference rankings yield these majorities:

2/3 say:	W/L	<	D	(Voters L and M)
2/3 say:	L/W	<	D	(Voters K and M)
2/3 say:	D	<	T	(Voters K and L)
2/3 say:	T	<	W/L	(Voters K and M)
2/3 say:	T	<	L/W	(Voters L and M)

W/L L/W

$<_{LM}$ $<_{LM}$

D $<_{KL}$ T

$<_{KM}$ $<_{KM}$

L/W W/L

This "Condorcet Crossing" diagram agrees with most — but not all — of each voter's preferences. It contains preference loops, yet every player agrees that the order relation is transitive! Thus we get a voter's paradox. The banker's financing makes Dilemma zero-sum, but non-Aristotelian. The glitch remains; to escalate order is to escalate chaos.

Either non-modus-ponens or non-zero-sum; Dilemma's illogic is marked. It often displays paradoxical signs, for Dilemma is, so to speak, snarked.

C. *Voter's Dilemma*

In the previous section, we got voter's paradox from prisoner's dilemma. Now let's go the other way and get prisoner's dilemma from voter's paradox.

Let's start with the Stooges and their three-way power struggle. They have these cyclically-clashing value systems:

Moe: Fairness < Logic < Power
Larry: Power < Fairness < Logic
Curly: Logic < Power < Fairness

Moe is Chairman, able to decide three-way ties, deadlocks, or collisions. This implies a Chairman's Paradox; Curly and Larry have good reason to conspire. Suppose that Curly and Larry *do* meet to conspire on how to vote. Their plotting and scheming can end in one of four feasible election outcomes:

Curly Wins: Logic < Power < Fairness.
Larry Wins: Power < Fairness < Logic
Moe Wins: Fairness < Logic < Power
Moe Loses: Power < Fairness < Logic

The next-to-last outcome (Moe Wins) is a draw for the conspirators; the last outcome (Moe Loses) is a shared win, or truce. For the scheme to go through, Larry and Curly must come to agreement on two issues:

Is Power worst, or merely middle? "Yes" = Larry's vote; "Yes" = 'nice' Curly vote; "No, it's merely middle" = 'mean' Curly vote.

Is Fairness best, or merely middle? "Yes" = Curly's vote; "Yes" = 'nice' Larry vote; "No, it's merely middle" = 'mean' Larry vote.

No doubt Larry and Curly vowed to be nice to each other; but when came time for a vote, there were these four possible outcomes:

Larry

		Fairness best	Fairness middle
Curly	Power worst	P<L<F truce	P<F<L Larry wins
	Power middle	L<P<F Curly wins	collision: Moe decides F<L<P

We score the election this way:

For Curly: score = 2(L<F) + 1(L<P)

For Larry: score = 2(P<L) + 1(F<L)

For Moe: score = 1(F<L) + 1(L<P)

If (a<b) equals 0 or 1, then we get the payoff matrix on page 222.

Therefore voter's paradox and prisoner's dilemma imply each other!

D. *Pessimistic Chicken Logic*

Consider Kleenean versus Bochvarian logic. In Kleenean logic, we adjoin to two-valued Boolean logic an *intermediate* value, fixed under negation:

$\wedge$ = minimum on $f < i < t$

$\vee$ = minimum on $t < i < f$

$\sim$ = exchange t and f.

In Bochvarian logic we adjoin an *absorbing*, or *extreme* value:

$\wedge_B$ = minimum on $j < f < t$

$\vee_B$ = minimum on $j < t < f$

$\sim$ = exchange t and f.

Now let us adjoin *both* kinds of values. The result is "pessimistic Chicken logic", defined as follows:

$\wedge_{PC}$ = minimum on $j < f < i < t$

$\vee_{PC}$ = minimum on $j < t < i < f$

$\sim_{PC}$ = exchange t and f.

We can embed Bochvarian logic into Kleenean; therefore PC-logic embeds into Kleenean logic also. PC-logic preserves this version of inner order:

T

$\prec$

J $\prec$ I

$\prec$

F

Therefore all PC functions have a semi-lattice of fixedpoints.

Now, why the name "pessimistic Chicken logic"? Because of "Chicken", a variant of Prisoner's Dilemma. Chicken is played like Dilemma, with nice or mean moves, but it has this scoring system:

Draw < Lose < Truce < Win

In Chicken, *draw* is the worst outcome; but this makes it a credible threat for purposes of intimidation. In Chicken, an 'unreasonable' player can play 'mean' constantly, and thus force a more 'reasonable' player to back down - i.e. settle for loss rather than draw. Chicken is a game of psychological domination.

Note that Chicken's value system resembles PC-logic's value system, thus:

Draw < Lose < Truce < Win (Chicken)

J < F < I < T ($\wedge_{PC}$ = minimum)

J < T < I < F ($\vee_{PC}$ = minimum)

~ = exchange t and f.

So let J = Draw

I = Truce

F = $\wedge$ player loses, $\vee$ player wins

T = $\wedge$ player wins, $\vee$ player loses

$\sim_{PC}$ = players swap outcomes

$\wedge_{PC}$ = $\wedge$ player's minimum operator

$\vee_{PC}$ = $\vee$ player's minimum operator

Therefore two pessimistic Chicken players can discuss their mutual predicament with logic and sympathy, and come to fixed terms!

E. *The Unexpected Departure*

For truce to succeed requires certain conditions. One of them is that the expected number of plays be great enough; another is that the play not end at too definite at time. If it does, then a "backwards induction paradox" destroys truce, no matter how long the tournament.

Consider the following scene:

Curly is about to play with Moe in a dilemma tournament scheduled to last exactly 100 rounds. Curly, a Silver Rule player, is optimistic that he can convince Moe (an Iron Rule player) that it'll be in his own best interest to cooperate.

But Moe said, "What about the 100th round? Won't that be the last one?"

Curly said, "Yes."

"There won't be any after the 100th?"

"Yes," said Curly.

Moe asked, "So in the very last play, what's to keep me from defecting?"

"'Cause I'll defect the next..." Curly said, then slapped himself on the face. "Alright, nothing will stop you from defecting on the 100th play."

"So you might as well defect too, right?" Moe said, smiling.

"I guess so," Curly said reluctantly. "On the 100th play."

Moe continued, "And what about the 99th play? What's to keep me from defecting then?"

"'Cause I'll defect the next..." Curly said, then slapped himself on the face. "But I'll defect on the 100th play anyhow."

"That's right," Moe said, smiling.

"So nothing's keeping you from defecting on the 99th play."

"That's right," Moe said, smiling.

"So I should defect on the 99th play also," said Curly.

"That's right," Moe said. "Now, what about the 98th play?"

And so they continued! Moe whittled down Curly's proposed truce, one play at a time, starting from the end. By the time the conversation was over, Moe had convinced Curly that the only logical course was for them to defect from each other 100 times, drawing the tournament. And so they did; yet when Curly played with Larry (a Gold Rule player) they cooperated 100 times, for a truce!

Thus we deduce, by mathematical induction, that the prospect of abruptly terminated play, even if in the far future, poisons the relationship at its inception.

That is the "backwards induction paradox". In dilemma play, cooperation requires continuity to the end. Departure should not be at an expected time lest that light the backwards-induction fuse; departure should be unannounced, at an unexpected time.

We need an *unexpected* departure; but this yields a paradox. Consider this following story about an Unexpected Exam:

Once upon a time a professor told his students, "Sometime next week I will give you an exam; and that exam will be at an unexpected time. Right up until the moment I give you the exam, you will have no way to deduce when it will happen, or even if it will happen. It will be an Unexpected Exam."

One of the professor's student objected, "But then the exam couldn't happen on Friday; for by then it would be expected!"

The professor said, "True."

The student continued, "So Friday's ruled out."

Another student said, "But if Thursday's the last possible day for an Unexpected Exam, then it's ruled out too; for by Thursday the *Thursday* exam will be expected!"

The professor said, "True."

And so on; by such steps the students concluded that the Unexpected Exam can't happen on Friday, Thursday, Wednesday, Tuesday, or Monday; so it can't happen at all!

"So you don't expect it?" said the professor.

His students said, "No!"

The professor smiled...

On the next Wednesday, he handed out an exam, to his students' surprise.

That's the Paradox of the Unexpected Exam. Here a backwards induction paradox also appears; but this time it yields a strangely *false* result rather than a strangely *undesirable* result. This match of methods suggest the following fable.

The same professor visited the Dean; he said, "I will depart this school sometime during the next month. To ensure cordial relations between us until that time, my departure will take place on an unexpected day. It will be an Unexpected Departure."

The Dean retorted, "You couldn't leave on the 31st, for by then your Unexpected Departure would be expected."

The professor agreed.

The Dean added, "Having ruled out the 31st, the 30th is also ruled out; for *it* would be expected."

The professor agreed to that too.

And so the conversation continued; and in the end the Dean concluded, "Your Unexpected Departure can't happen on any day. Therefore I don't expect it." The professor agreed.

On the seventeenth day of the month the professor departed, to the Dean's astonishment.

This **Paradox of the Unexpected Departure** is just what the doctor ordered; for here the *failure* of backwards induction (so puzzling to the reason) is precisely what is needed to defend the Axelrod equilibrium from *its* backwards induction proof!

A dilemma tournament can use "open bounding"; replay only if a random device permits it. This ensures an Unexpected Departure; play will be finite, but there will be no definite last play during which the Iron player is safe from the danger of Silver retaliation.

The paradox of the Unexpected Departure is related to the paradox of the First Boring Number; for presumably the tournament ends as soon as it stops being interesting.

The conclusion then is clear; let none of your social relationships end too definitely; let there be some possibility that you might encounter that person again, soon. (And conversely, when you *must* leave, slip away quietly!)

Chapter 17
Speculations

Delta Types?
Minimal Surds?
Null Quotients?
General Semi-Lattices?

A. *Delta Types?*

The very concept of a ternary type seems ironic; for the whole point of (inner) delta logic is to create typeless fixedpoints. In complete self-reference, there is but one type, and it refers to itself completely.

If we allow reference by outer functions as well, then we need type theory; for outer functions reveal delta's boolean substrate. Stranger still, the three logics are analytic *relative to each other*. This is the "perpendicularity" of the three logics. Given two perpendicular systems, one must be of lower type than the other.

Definition. A ***delta type*** is a completely interreferential system of statements, in one of the three logics.

Definition. ***Type Order.*** Any statement in a given type may refer to any other statement in that or any lower type.

Conjecture. Is type order a semi-lattice? A fixedpoint semi-lattice?

Conjecture. Inner fixedpoints for typeless systems take linear time to evaluate. Can delta types define polynomial-time fixedpoints?

B. *Minimal Surds?*

As we saw in the "Voter's Paradox" chapter above, we can find 'troikas' to give logical structure to a wide variety of logical errors. (For instance, in the Notes you will find Kant's Antinomies of Space, Atomism, Freedom and God reduced to Stooge elections.)

Call a system of beliefs *pre-contradictory* if it does not include any statements of the form "$A \wedge \sim A$", but does contain a set of statements from which such a contradiction can be derived by boolean reasoning. Let a *surd* for such a system S be a set of voters $\{V_1, V_2, \ldots V_n\}$ such that each voter has consistent beliefs, and majorities of these voters support each statement in S.

The question I now ask is: given *any* pre-contradictory belief system, does it have a surd? Does it have a *minimal* surd? Will one election do, or will we in general need a series of primaries to get the exact effect required?

Define a "delta system" as a pre-contradictory belief system which is consistent by the delta deduction rules of chapter 13. Does every delta system have a three-voter surd?

In chapter 13 I showed that 5-voter systems can violate delta's deduction rules for majority. However, 5-voter systems still obey these weakened rules:

From: A passes, and B passes, and C passes, and D passes, and E passes

Deduce: $M_5(A,B,C,D,E)$ passes

From: $M_5(A,B,C,D,E)$ passes

Deduce: A passes, or B passes, or C passes, or D passes, or E passes

Do these rules suffice for 5 voters?

C. *Null Quotients?*

The **null quotients** are the result of division by zero.

There are two of them:

1/0 = "infinity"; larger than any finite quantity

0/0 = "indefinity"; indistinguishable from any quantity

Consider these algebraic equations:

$x = 1/0$

$0x = 1$

$0 = 1$

Infinity leads us to an obvious absurdity. 1/0 is inherently inconsistent; "over-determined".

Consider these algebraic equations:

$x = 0/0$

$0x = 0$

$0 = 0$

Indefinity leads us to a vague tautology. 0/0 is inherently uninformative; "underdetermined". As noted in Chapter 15, this connects us to diamond logic; for we can identify i with one, and j with the other.

In terms of Size paradoxes, perhaps we can say:

0/0 = the Heap , and 1/0 = Finitude

Consider the "sign" function:

sign(x) = |x|/x .

Its graph is:

```
                         *----------------------- +1
 - ----------------------0------------------------ +
-1 ----------------------*
```

Note that sign(0) = 0/0 ; sign is *undefined* at zero. Note also the similarity of this graph to the Dedekind splice.

According to Gödel's Theorem, any arithmetical deductive system is either inconsistent or incomplete. Inconsistent is overdetermined, like 1/0; incomplete is underdetermined, like 0/0; thus arithmetic, though it avoids using null quotients, itself resembles a null quotient!

D. *General Semi-Lattices?*

Is *every* semi-lattice the fixedpoint semi-lattice for some harmonic system? If so, then given a semi-lattice, which system emulates it? How do we find that system?

Given a system, how can we improve it? (Reduce the number of variables, references, etc.) How does a semi-lattice change when you change its system? And vice versa?

Given a semi-lattice and its system, can input leads into the system provide control over points in the semi-lattice?

What *practical* computation tasks does semi-lattice emulation permit? Could we (say) emulate a *tree* (for sorting, searching, etc.)? Note that boolean logic usually corresponds to the *end* of the semi-lattice, or the *nodes* of the tree; the part with the fewest order constraints.

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Notes

Chapter 1. Paradox.

1A. The Liar

As you may know, Chelm is a mythical town inhabited entirely by fools. Not just ordinary fools, mind you, but fools so foolish that they thought themselves the wisest of the wise! It is true that events would sometimes conspire to cast their reputation for sagacity in doubt; and upon one such occasion this dialog ensued between the Chief Sage and the other Sages of Chelm:

Chief Sage: All Chelmites are fools!

Other sages: But are not we Chelmites?

Chief Sage: All Sages of Chelm are fools!

Other sages: But are not you a Sage of Chelm?

Chief Sage: I am a fool!

The other sages considered this, and decided that the Chief had made a real Fool of himself; for his statement is wise if and only if it is foolish.

In the end, Chelm's Council of Elder Sages firmly decided against ever officially recognizing their own folly. They sagely noted that for them to say "we are fools" would be paradoxical. Such is the wisdom of their wisest citizens; so you can imagine the wisdom of Chelm as a whole!

By the way, dear reader, you should know that Chelm is no longer an Eastern European shtetl. Long ago the Chelmites fled from the insanity of the Old World to the lunacy of the New World; and there they founded — not a village, not a town, but an great big modern *city*, complete with highways and bridges and skyscrapers and stadiums and radio transmitters and phone lines and computers and nuclear power plants. Chelm remains as it was before, city of fools; but during the move they lost their old name. So who knows which city is Chelm? Dear reader, it could be mine — or yours.

For consider this; Socrates, reputed to be the wisest of the Greeks, is also reputed to have said, "I know only that I know nothing" — as good a statement of the Fool's Paradox as any. In a similar vein, note the case of Desiderius Erasmus, who was also reputed to be among the wisest of his day. Of all the many books he wrote, the one best known today is a brief satire he tossed off in a week; the hilarious *Praise of Folly*, as told to Erasmus by Folly herself!

1D. Santa Sentences

Suppose that some sarcastic Grinch were to proclaim:

"Santa Claus exists, and I am a liar."

$$G = (S \wedge \sim G)$$

If Boolean logic applies to this "Grinch Sentence", then it refutes both itself and Santa Claus! For consider this line of argument:

$G = (S \wedge \sim G)$; assume that G is either true or false.

If G is true, then $G = (S \wedge \sim T) = F$.

G = true implies G = false;

therefore (by contradiction) G must be false.

False $= G = (S \wedge \sim G) = (S \wedge \sim F) = S$

Therefore S is false. Therefore Santa Claus does not exist!

This proof uses proof by contradiction; an indirect method, suitable for avoiding overt mention of paradox. Here is another argument, one which confronts the paradox directly:

S is either true or false. If it's false, then so is G:

$G = (F \wedge \sim G) =$ false.

No problem. But if S is true, then G becomes a liar paradox:

$G = (T \wedge \sim G) = \sim G$.

If S is true, then G is non-boolean.

Therefore; if G is boolean, then S is false. QED.

Call an adjective "Grinchian" if and only if it does not apply to itself, and Santa Claus exists:

"A" is Grinchian = Santa exists, and "A" is not A.

Is "Grinchian" Grinchian?

"G" is G = Santa exists, and "G" is not G.

The "Grinch Set for sentence H" is:

$$G_H = \{ x \mid H \wedge (x \notin x) \}$$

$$G_H \in G_H = H \wedge (G_H \notin G_H)$$

In ternary logic, the threatened paradox need not affect any other truth value. If Santa Claus *does* exist after all, then the Grinch is exposed as a Liar!

The Grinch sets suggest Grinch stories. Consider the Weekend Barber, who only shaves on the weekends, and only those who do not shave themselves:

WB shave M = It's the weekend, and M does not shave M.

Does the Weekend Barber shave himself?

WB shave WB = It's the weekend, and WB does not shave WB.

Note that Epimenides's statement:

"All Cretans are liars, including myself."

— makes him the Grinch of Crete!

Here's a "Strong Santa" sentence:

"I'm not mistaken if and only if S."

$$X \quad = \quad (X \text{ iff } S)$$

In terms of Z_3, this yields the equation

$$X = XS \; ;$$

which yields

$$X(1-S) = 0 \; ;$$

ergo $S = 1$ (there is a Santa) or $X = 0$ (paradox).

1F. Size Paradoxes

Standard (i.e. boolean) set theorists were so disturbed by Russell's paradox that they decided to acknowlege only "well-founded" sets. A set is well-founded if and only if it has no "infinite descending element chains"; that is, there is no infinite sequence of sets X_1, X_2, X_3, ... such that

$... \in X_4 \in X_3 \in X_2 \in X_1 \in X.$

Well founded sets include {}; {{}}; {{{}},{{},{{}}}}; and even infinite sets such as {{},{{}},{{{}}},{{{{}}}},...}; for well-founded sets can be infinitely "wide", so long as they are finitely "deep" along each "branch".

On the other hand, well-foundedness excludes sets such as

A = { A } = {{{{{{...}}}}}}

for it has the infinite descending element chain $... \in A \in A \in A.$

Let WF be the set containing all well-founded sets. Is WF well-founded?

If WF *is* well-founded, then WF is in WF; but this yields the infinite descending element chain $... \in WF \in WF \in WF.$

On the other hand, if WF is *not* well-founded, then any element of WF is well-founded, and element chains deriving from those will be finite. Thus all element chains from WF will be finite; and therefore WF would be well-founded.

And so we see that the concept of "well-foundedness" leads us to the paradox of Finitude.

Chapter 3. Ternary Algebra.

B. Normal Forms

Call these the Crosstransposition Operators:

$$C_0(A,B,x) = (A \wedge x) \vee (B \wedge \sim x) \vee dx$$

$$C_1(A,B,x) = (A \vee x) \wedge (B \vee \sim x) \wedge Dx$$

Note that

$$C_0(A,B,F) = B = C_1(A,B,T)$$

$$C_0(A,B,I) = I = C_1(A,B,I)$$

$$C_0(A,B,T) = A = C_1(A,B,F)$$

According to Crosstransposition, $C_0(A,B,x) = C_1(B,A,x)$

According to DeMorgan's laws, the definitions, and crosstransposition;

$$\sim C_0(A,B,x) = C_1(\sim A,\sim B,\sim x) = C_1(\sim B,\sim A,x) = C_0(\sim A,\sim B,x)$$

$$C_0(A,B,x) \vee C_0(a,b,x) = C_0(A \vee a, B \vee b, x)$$

$$C_0(A,B,x) \wedge C_0(a,b,x) = C_0(A \wedge a, B \wedge b, x)$$

We cannot define all inner functions via C_0; in particular, constants!

$C_0(T,T,x) = Dx$; $C_0(F,F,x) = dx$.

Therefore **inner transmission:**

If F is defined from ~, $\wedge$, and $\vee$, but no boolean constants, then

$$F(C_0(A_1,B_1,x),\ C_0(A_2,B_2,x), \dots , C_0(A_n,B_n,x))$$

$$= C_0(F(A_1, A_2, \dots, A_n),\ F(B_1, B_2, \dots, B_n), x)$$

Chapter 4. Self-Reference.

A Parenthetical Remark About The Parenthetical Remark

Consider the Brownian form [[]]. It is equal to (i.e. confused with) the void; yet it is not itself void, being made of two nested marks. It therefore deserves names of its own; I suggest "doublecross", or the "remark". Doublecross denotes the void, but unlike the void, is visible.

The remark is to forms as zero is to numbers; both name the nameless, and both are placeholders. In algebraic terms, the remark denotes parentheses:

$$(A) \quad = \quad \overline{\overline{A}|\;}| \quad = \quad A$$

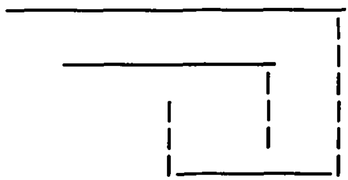
I use parentheses to distinguish these from the brackets of bracket forms. In fact (A) = [[A]].

The remark allows one to express the associative law:

$$(AB)C \quad = \quad \overline{\overline{A\ B}|\;}|\ C \quad = \quad ABC \quad = \quad A\ \overline{\overline{B\ C}|\;}| \quad = \quad A(BC)$$

[[A]] *remarks* about A without *marking* A; it draws attention without changing values. It is literally a parenthetical remark.

Finally, consider this *re-entrant remark*:



It represents this system:

$$A = \overline{B\,|}\ ;$$

$$B = \overline{A\,|}\ .$$

$$A = (\,A\,)\ .$$

This is a *toggle*, or *memory circuit*.

Thus memory remarks on itself.

Chapter 8. The Continuum.

B. Cantor's Dyadic

I noted that C, as an anti-diagonal, can read .0111111111..., while C, on the list, reads .100000000...

The usual response to this is that we could separately list all dyadics in a countable list; but that the non-dyadic (i.e. nonterminating) reals are uncountable by Cantor's proof. This, however, is a non-sequitur; for the anti-diagonal of the nondyadic numbers would then be a dyadic.

Another problem with this defense of Cantor's theorem is that it uses the word "dyadic"; for a real number is "dyadic" if and only if its binary expansion has a *finite* number of 1's, or of 0's. But this leads us straight back to the paradox of Finitude!

Indeed, the smooth transition from small to large through paradox mimics the Dedekind splice's transition from false to true through paradox. The "first boring number" and the "last interesting number" are boundary paradoxes in the discrete domain.

Note the complementarity between "finitely many infinite-precision numbers" and "infinitely many finite-precision numbers". Here we see finitude versus itself; second-order finitude. Cantor's Dyadic is the limit of real arithmetic.

Kleenean logic radically simplifies the theory of the infinite. In the ternary world, Cantor's Theorem does not apply; the power set of a set can be put in one-to-one correspondence with the set. (There will, of course, be paradoxes at the pivot bits.) The ternary world needs no cardinal hierarchy: it has only one infinity, and that one tinged with paradox and finitude.

I intend paradox logic's theory of infinity to be comprehensible to finite beings such as you or I. Who are *we* to speak of aleph-seventeen? This theory is to be computable by actual store-bought computers, not by Platonic ideals. This paradox logic is to be, above all, down to earth.

Set theorists speak of uncountable ordinals and measureable cardinals; but mathematics nowadays is more concerned with megahertz and gigabytes. Finitude is our style, indeed our birthright. Call this Math for Mortals.

Chapter 9. Outer Functions

9A. Function Types

Here is a dual version of the lemma about minima:

$X \text{ min } (Y \text{ min } Z) = (X \vee I) \wedge (Y \vee I) \wedge (Z \wedge I) \wedge (X \vee Y \vee Z)$

Consider the following **Anchor Forms:**

$$A_0(a,b,c,x) = (a \wedge x) \vee (b \wedge \sim x) \vee (c \wedge dx) \vee (a \wedge b \wedge c)$$

$$A_1(a,b,c,x) = (a \vee \sim x) \wedge (b \vee x) \wedge (c \vee Dx) \wedge (a \vee b \vee c)$$

I call these 'anchor' forms because of the abc terms.

The Absorption axiom and the mimima lemma imply these equations:

$$A_0(a,b,c,F) = b \vee (a \wedge b \wedge c) = b$$

$$A_0(a,b,c,I) = (a \wedge I) \vee (b \wedge I) \vee (c \wedge I) \vee (a \wedge b \wedge c) = a \text{ min } b \text{ min } c$$

$$A_0(a,b,c,T) = a \vee (a \wedge b \wedge c) = a$$

Exercise for the reader: prove $A_0(a,b,c,x) = A_1(a,b,c,x)$.

If f(x) preserves order, then $f(I) \text{ min } f(T) \text{ min } f(F) = f(I)$;

therefore $\mathbf{f(x) = A(f(T), f(F), f(I), x)}$

This is the Anchored Normal Form.

We can use the Anchored Normal Form to prove completeness. It's a quicker proof than the one using Mediation and the Median Normal Forms, but this proof requires prior understanding of inner order.

9B. S_3 and Pivot

The Dean Of Chelm University Tackles Because

I noted above that Chelm is a mythical town inhabited entirely by fools. Not just ordinary fools, mind you, but fools so foolish that they thought themselves the wisest of the wise!

One fine day the Dean of Chelm University summoned his three best students for an examination in Mathematics and Logic. The Dean of Chelm University called forth Larry, and they had this dialog:

Dean: What is one plus one?

Larry: One plus one equals two.

Dean (thumping his copy of the *Principia Mathematica*): That is correct, indeed axiomatic. In this masterpiece, Russell and Whitehead spent dozens of pages on a rigorous proof of precisely that proposition. "1+1=2" follows from axioms, therefore it is also axiomatic; so I shall give you an "A" in Math.

Larry (excited): Thank you!

Dean: You're welcome. And now the Logic question. You say 1+1=2. *Why* do you say that? 1+1=2 because what?

Larry (nervously): 1+1=2 because... 1+1=2?

Dean (nodding): You explain an axiom by itself; that is, you declare your axiom to be self-evident. But that too is self-evident, i.e. axiomatic. Therefore you get an "A" in Logic as well.

With that the Dean dismissed Larry. Then he called Moe in for his exam.

Dean: What is one plus one?

Moe: One plus one equals one.

Dean: That is false, indeed fallacious. Once the logician Bertrand Russell was challenged to prove that he was the Pope, given that 1+1=1. He replied, 'I am one, and the Pope is one; together we are one, and I am the Pope'. An agnostic Pope would be anomalous; so proof of Russell's papacy is fallacious. Therefore 1+1=1 is also a fallacy; so I shall give you an "F" in Math.

Moe (sullen): Thanks a lot!

Dean: You're welcome. And now the Logic question. You say 1+1=1. *Why* do you say that? 1+1=1 because what?

Moe (stubbornly): 1+1=1 because 1+1=1!

Dean (shaking his head): You explain a fallacy by itself; that is, you declare your fallacy to be self-evident. But that too is a fallacy. Therefore you get an "F" in Logic as well.

With that the Dean dismissed Moe. Then he called Curly in for his exam.

Dean: What is one plus one?

Curly: I am a fool!

Dean: That is paradoxical, indeed confusing. Once the logician Bertrand Russell asked, if R is the set of all sets which do not contain themselves, then does R contain itself? Any fool can see that R contains itself if and only if it does not. That set R, like your statement, is a paradox, hence neither true nor false, hence is confused. Therefore I shall give you a "C" in Math.

Curly (relieved): Thank you.

Dean: You're welcome. And now the Logic question. You say that you are a fool. *Why* do you say that? You are a fool because what?

Curly (resigned): I am a fool because I am a fool.

Dean (shrugging his shoulders): You explain a paradox by itself; that is, you declare your confusion to be self-evident. But that too is confused. Therefore you get a "C" in Logic as well.

Later, the Dean of Chelm University evaluated these six other answers to his Logic question:

"1+1=1 because I am a fool". This expresses the last-minute anguish of the student who realizes too late that he blew his Math question and must swiftly distance himself from it. As such it denotes Achievement and deserves an A.

"1+1=2 because I am a fool". Here the student panics needlessly and fatally takes back a correct answer. A Failure, therefore an F.

"1+1=1 because 1+1=2". Confused; a C.

"1+1=2 because 1+1=1". Another C.

"I am a fool because 1+1=1". If he believed that, then surely he was a fool! Here is the Moment of Enlightenment, when the sinner detects his error and thus sees through it. For this, an A.

"I am a fool because 1+1=2." This is the rejection of reason; the reverse of enlightenment, delusion! This blunder deserves an F.

Thus the Dean of Chelm University deduced this truth table:

because	A	C	F
A	A	F	C
C	F	C	A
F	C	A	F

A = "1+1=2"
C = "I am a fool"
F = "1+1=1"

The Dean was so pleased by his new crib sheet that he posted it up for all his students to see. "After all," he reasoned, "why would *they* ever cheat? I think it's unthinkable!"

It was by rationalizations such as these that the ancient Sages of Chelm entered Fool's Paradise, which differs from the real Paradise mainly in the length of one's stay.

That concludes my story! From it we see that the truth table for "because", Chelm University style, is identical to pivot.

"Because" is not "if". I'm sure you will agree that Bertrand Russell is the Pope *if* one plus one equals one; but I'm also sure you will deny that Bertrand Russell is the Pope *because* one plus one equals one!

Two-Bracket Forms

In addition to the bracket [x] representing the base negation operator $\sim_0$, let us add the "brace" {x} representing the rotation operator 1+x.

Then these are the arithmetic axioms:

[] = [] ; [6] = 6 ; [[]] = ;

{} = 6 ; {6} = [] ; {[]} = .

These are algebraic axioms:

x = [[x]] = {{{x}}} ;

{[x]} = [{{x}}] ; {{[x]}} = [{x}] .

Exercise for the reader:

Prove that in the standard interpretation:

{{{x}}} = x ; {x} = x+1 ; {{x}} = x+2

[{{x}}] = $\sim_2 x$; [{x}] = $\sim_1 x$; [x] = $\sim_0 x$

Consider these three operations:

xy ; {{{x}{y}}} ; {{{x}}{{y}}}

Prove that they are equivalent to, respectively, V_0, V_1, and V_2.

Prove that this is an identity: x {{{y}{z}}} = {{{xy}{xz}}}

Find counter-examples to this: {{{x}{yz}}} = {{{x}{y}}} {{{x}{z}}}

Pivot Circuits

In Chapter 2 I defined circuits for kleenean functions. In the 'dual-rail' form, we let each ternary 'trit' t be represented by two binary bits t_1 and t_2, where $t_1=t_2=T$ only if t=T; $t_1=t_2=F$ only if t=F; and $t_1 \neq t_2$ only if t=I.

I defined Kleenean gates with these circuits:

To define "~x ": $Z_0 = \sim x_1$; $Z_1 = \sim x_0$

To define "x ∧ y":

$X_0 = x_0 \wedge x_1$; $X_1 = x_0 \vee x_1$; $Y_0 = x_0 \wedge x_1$; $Y_1 = x_0 \vee x_1$; $Z_0 = X_0 \wedge Y_0$; $Z_1 = X_1 \wedge Y_1$

To define "x ∨ y":

$X_0 = x_0 \wedge x_1$; $X_1 = x_0 \vee x_1$; $Y_0 = x_0 \wedge x_1$; $Y_1 = x_0 \vee x_1$; $Z_0 = X_0 \vee Y_0$; $Z_1 = X_1 \vee Y_1$

We can define gates for S_3 and pivot with these circuits:

To define "$\sim_1 x$ ": $Z_0 = x_0 \wedge x_1$; $Z_1 = x_0 \text{ iff } x_1$

To define "$\sim_2 x$ ": $Z_0 = x_0 \text{ xor } x_1$; $Z_1 = x_0 \vee x_1$

To define "1+x ": $Z_0 = x_0 \text{ xor } x_1$; $Z_1 = \sim(x_0 \wedge x_1)$

To define "2+x ": $Z_0 = \sim(x_0 \vee x_1)$; $Z_1 = x_0 \text{ iff } x_1$

To define "x # y":

$X_1 = x_0 \wedge x_1$; $X_0 = x_0 \text{ xor } x_1$; $X_2 = \sim(x_0 \vee x_1)$;

$Y_1 = y_0 \wedge y_1$; $Y_0 = y_0 \text{ xor } y_1$; $Y_2 = \sim(y_0 \vee y_1)$;

$U_0 = X_2 \wedge Y_0$; $U_1 = X_0 \wedge Y_2$; $U_2 = X_1 \wedge Y_1$;

$V_0 = X_2 \wedge Y_2$; $V_1 = X_0 \wedge Y_1$; $V_2 = X_1 \wedge Y_0$;

$Z_0 = U_0 \vee U_1 \vee U_2$; $Z_1 = \sim(V_0 \vee V_1 \vee V_2)$

11D. The Differential

In Boolean logic, "xor" and "iff" are isomorphic to addition modulo 2. Alas, in delta they are no longer group operations:

$(t \wedge i) \text{ xor } (t \wedge i) = i \neq f = (t \text{ xor } t) \wedge i$

$(f \vee i) \text{ iff } (f \vee i) = i \neq t = (f \text{ iff } f) \vee i$

so they are non-distributive.

This is because xor and iff contain "differential terms":

$x \text{ iff } y = \sim(x \text{ xor } y) = ((\sim x) \text{ xor } y) \vee dx \vee dy$

$x \text{ xor } y = \sim(x \text{ iff } y) = ((\sim x) \text{ iff } y) \wedge Dx \wedge Dy$

$(x \wedge z) \text{ xor } (y \wedge z)$

$= ((x \text{ xor } y) \wedge z) \vee ((x \vee y) \wedge dz)$

$(x \vee z) \text{ iff } (y \vee z)$

$= ((x \text{ iff } y) \vee z) \wedge ((x \wedge y) \vee Dz)$

Nondistributivity is due to asymmetric differential terms.

Chapter 13. Voter's Paradox

We can turn Kant's Four Antinomies into voter's paradoxes, thus:

A. **Antinomy of Time:** Limited or infinite?

Moe says that time is linear, bounded, and finite:

```
        |-------------|
```

Larry says that time is linear, unbounded, and infinite:

```
<----------------------->
```

Curly says that time is circular, unbounded, and finite:

```
          ---->---
         |        |
          ---<----
```

2/3 (ML) say: time is linear.

2/3 (LK) say: time is unbounded (i.e. has no beginning nor end).

2/3 (KM) say: time is finite.

B. **Antinomy of Atomism:** Ultimate mechanism or none?

Moe, a Reductionist, says that all things are made of simple parts, and can be fully explained in terms of their parts;

Larry, a Holist, says that all things are made of simple parts, but that some things cannot be fully explained in terms of their parts;

Curly, a Chaoticist, says that all things can be fully explained in terms of their parts, but that some things have only composite parts.

2/3 (M&L) say: all things are made of simple parts;

2/3 (K&M) say: all things can be fully explained in terms of their parts;

2/3 (L&K) say: some things cannot be fully explained in terms of simple parts.

That is, all wholes have simple parts; all wholes are the sum of their parts; yet some wholes are not the sum of simple parts!

C. **Antinomy of Freedom:** Free or not?

Curly, a medieval Supernaturalist, says that not all phenomena are governed by the laws of nature, which are deterministic;

Moe, a modern Mechanist, says that all phenonema are governed by the laws of nature, which are deterministic;

Larry, a postmodern Quantumist, says that all phenonema are governed by the laws of nature, which are not deterministic.

2/3 (M&L) say: all phenonema are governed by the laws of nature;

2/3 (K&M) say: the laws of nature are deterministic;

2/3 (L&K) say: not all phenomena are pre-determined.

That is; everything is natural, nature is inevitable, yet not everything is inevitable!

D. **Antinomy of God:** Exists or not?

Moe says there is no God either within or beyond this world;
Larry says that God exists, but only within this world;
Curly says that God exists, but only beyond this world.

2/3 (M&L) say: there is no God beyond this world;
2/3 (K&M) say: there is no God within this world;
2/3 (L&K) say: there is a God.

Between immanence, transcendence, and nonexistence, God manages to give the Stooges the slip!

Here is a similar "Existentialist" troika:
2/3 (M&L) say: life has no meaning beyond itself;
2/3 (K&M) say: life has no meaning within itself;
2/3 (L&K) say: life has meaning.

Exercise for the reader: figure out each Stooge's life-philosophy.

Chapter 15. Diamond

By "underdetermined" I mean a statement which logic still hasn't decided; by "overdetermined" I mean a statement about which logic has derived opposite conclusions. Thus, an underdetermined statement is neither provable nor refutable, and an overdetermined statement is both provable and refutable. According to Gödel's Theorem, any logic system is either incomplete or inconsistent; thus the equation;

i or j = t

that is;

underdetermined or overdetermined = true

is none other than Gödel's Theorem, written as a diamond equation. Diamond harmonizes with meta-mathematics.

Note that we have four interpretations for diamond logic:

t	=	true	; true	; false	; false
i	=	undet	; overdet	; undet	; overdet
j	=	overdet	; undet	; overdet	; undet
f	=	false	; false	; true	; true
or	=	a or b	; a or b	; a and b	; a and b
and	=	a and b	; a and b	; a or b	; a or b

These are isomorphic to each other under conjugation by the four operations { identity, not, star, minus } - a Klein group.

I and J are complementary paradoxes; the yin and yang of diamond logic. They oppose, yet reflect.

Yang is not yang, yin is not yin, and the Tao is not the Tao!

Chapter 16. Dilemma

16A. Prisoner's Dilemma

There exist many dilemma strategies other than the Iron, Gold, and Silver rules. For instance, there is R, for Random play; TF2T, "Tit For Two Tats", which defects only after the other player defects twice in a row; 2TFT (two-tits-for-a-tat); "angry" TFT (TFT starting in an unfriendly state); TFT with occasional "testing" behavior; and TFT with "forgiveness factor", which ocassionally (at random) forgives misbehavior on the other player's part; RTFT, "reverse tit-for-tat", which punishes cooperation and rewards punishment; and "Pavlov", which is nice on the next round if this round truced or drew, and is mean on the next round if this round won or lost. (That is, Pavlov repeats its present play if it came out truce or win, and switches if it came out draw or loss.)

Which strategy is best? That depends on many factors; the other player's strategy, the expected length of the tournement, and the tactical position of the dilemma game itself. Thus dilemma games have a second level of play; strategic as well as tactical. *How* to play matters as much as *what* to play.

Many kinds of ordinary games can be "dilemmized". Prominent among these is chess. Dilemma chess is chess plus deterrence, with a dilemma payoff matrix. The board, pieces and moves are the same as in regular chess; but the game is allowed to end with mutual checkmate, called truce.

This is the payoff matrix:

payoff for (A, B)		B: checkmated	B: not checkmated
A	checkmated	(truce, truce)	(lose, win)
A	not checkmated	(win, lose)	(draw, draw)

Thus a dilemma. If competitive chess is the king of games, then dilemma chess is the queen; for truce opens up a new dimension of play; namely, between competition and cooperation.

The basic innovation in dilemma chess is to allow the "reply" move. The reply move is a final move by the player whose king has been captured. If the other king can be captured in the reply move, then the first capture is "deterred". You may not capture if your check is deterred. You may move into deterred check, or respond to check with a deterrent. You may not cancel the other player's deterrent unless you also escape check (no "forced exchanges").

Mutual deterred check is "tryst"; both sides can capture and retaliate. Truce is mutual assured check (MAC), or inescapable tryst; one move after truce, both sides can still capture and retaliate. In tryst, capture is deterred; the other player could capture next move, but would suffer retaliation. Other forms of deterrence exist; "pinned check", "delayed deterrent", even "temporary checkmate". If you have a deterrent, then your king is free to advance into enemy territory; the "brave king" phenomenon.

For instance, consider this endgame:

8							BR	
7	BP			BQ	BP	BP		
6		BP		BP			BK	
5			BP	WP	BB	WP		
4			WP				WK	
3							WP	
2	WP	WP			WN			
1						WR		WQ
	a	b	c	d	e	f	g	h

Black to move. Note that PxK is deterred.

```
...               Kg6xf5 tryst
Qh1-f3 tryst      Kf5-f4 tryst
Kg4-h5 check      Qd7-f5 tryst
Kh5-g6 tryst      Rg8-h8 truce
```

Two courageous kings!

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