GödelandPhysics

JohnD.Barrow
DAMTP
CentreforMathematicalSciences
CambridgeUniversity
WilberforceRd.,
CambridgeCB30WA
UK

Summary

Weintroducesomeearlyconsiderationsofphysical preludestoGödel'sincompletenesstheorems. Wecon thesetheoremsandtheirunderlyingassumptionsand theoremsbythoseseekingtodrawconclusionsfrom theoriesofphysics. Wearguethatthereisnoreas ohandicapthesearchforadescriptionofthelawso whatwecanpredictabouttheoutcomesofthoselaw discussthe'Gödeluniverse', asolutionofEinstei n's universewheretimetravelispossible, whichwasd roleitplayedinexposingthefullspectrumofpos space-timewouldreveal. Finally, weshowhowrecen howglobalspace-timestructuredeterminestheulti devices withinthem.

andmathematicalimpossibilityas
on sidersomeinformalaspectsof
discusssometheresponsestothese
themaboutthecompletabilityof
ontoexpectGödelincompletenessto
fNature,butwedoexpectittolimit
s,andexamplesaregiven.We
n'sequationsdescribingarotating
iscoveredbyGödelin1949,andthe
sibilitiesthataglobalunderstandingof
tstudiesofsupertaskshaveshown
matecapabilityofcomputational

1: SomeHistorical Background

PhysicalImpossibilities

Thereisalonghistoryofscientificandphil osophicalconsiderationofphysical impossibilities1. The Aristotelian world view out lawed the possibili infinitiesorlocalphysicalvacuacouldbecreated physicistsdevisedingeniousthoughtexperimentsto 'tricked'intoallowinganinstantaneousvacuumto possibilitywasstoppedfromoccurringbynaturalp invocationofaCosmicCensor,topreventitsappea ongoingalchemicaldebateaboutthepossibilityor basermetals, and engineering maintained an endurin perpetualmotionmachinesthatonlyfullyabatedwh thermodynamicsweresystematicallyunderstooddurin examples,likeMaxwell'ssortingdemon,stillremai exorcised by the application of the modern thermody 1961⁴.

tythatphysical orobserved ².DuringtheMiddleAges, trytoimaginehowNaturecouldbe form, and the narguing about how this rocessesor, if that failed, by the rance³. Chemistryhaditsown impossibilityofmakinggoldfrom gattachmenttothequestfora entheconsequencesoflawsof gthenineteenthcentury.Subtle neduntiltheywereeventuallyfully namictheoryofcomputationin

MathematicalImpossibilities

Mathematiciansalsooccasionallyconsidere contextofaseveralfundamentalproblemsofarithm Supposedly,inabout550BC,thePythagoreansfirst numberslike $\sqrt{2}$ which cannot be expressed as the ratio of two int originallymeaningsimply'notaratio',rathertha today). Legendhasitthatthisdiscoverywassuchascan Hippasos, was drowned by the members of the Pythago Thisgivesusthefirstglimpseofoperationsandq particularsetofrules. In the first quarter of th explicitformforthesolutionofageneralquintic coefficientswasprovedtohavenosolutioninvolvi radicals by the young Norwegian mathematician, Henr quadratic, cubic, or quartic equations, the general formula. Justafewyears later, in 1837, rigorous degreescouldnotbetrisectedjustbyuseofastr examples revealed for the first time, to those who hintsastothelimitationsofparticularaxiomatic

InthelightoftheongoingimpactofGödel'swork limitations of the human mind, it is interesting to psychologicaleffectsofsomeoftheseearlyresult wasofthedeepestconcerntothePythagoreans;how werenodeepphilosophicalquestionsaboutthelimi raisedbythedemonstrationthatthequinticcould Previously, the reweremanythings thought impossib despitemanyeffortstodoso.Butnowtherewerep done.

dthequestionofimpossibilityinthe etic, geometry, and algebra. encounteredthe 'irrationality' of egers('irrational' nbeyondreason, as might be suggested dal.thatthediscoverer. reanbrotherhoodforhistrouble. uestionswhichhavenoanswersgivena e19 th century, the problem of finding an algebraicequationintermsofits ngordinaryarithmeticoperationsand ikAbel ⁶.Unlikethecaseof quinticcannotbesolvedbyanyexact proofsweregiventhatanangleof60 aightedgeandpairofcompasses. These lookedatthemintherightway,some systems.

onspeculationsaboutthe reflectbrieflyonthesociologicaland s. The existence of irrational numbersever, as far as we can judge, there tationsofmathematicalreasoning notbesolved. Yet, there was a change. lethatcouldnotbesoproven roofsthatsomethingcouldnotbe

Axiomatics

Thedevelopmentofunderstandingofwhatconstructi outbylimitedmeans, such as ruler and compass con operations and radicals, showed that axioms mattere ofaxiomsdeterminedwhatitsallowedrulesofreas th century, the archetypalaxiomatic system was that Untilthe19 geometry. Butitimportant to appreciate that this today, as just one among many axiomatic possibiliti worldreallywas. It was part of the absolute truth specialstatusanditsconstructionsandelucidatio years, provided a style that was a ped by many works widespreadbeliefinitsabsolutetruthprovidedan theologiansandphilosophersthathumanreasoncoul nature of things. If challenged that this was beyon theycouldalwayspointtoEuclideangeometryasa thistypeofinsightintotheultimatenatureofth result, the discoveries, by Bolyai, Lobachevskii, G geometriesexisted, but in which Euclid's parallel impactoutsideofmathematics ⁷. The existence of other logically consistent geome meantthatEuclid'sgeometrywasnot thetruth. As a result, newforms of relativisms pr thatevenEuclid's ancient foundational system was geometries-andindeedoneofthesealternativesw describingthegeometryoftheEarth'ssurfacethan aboutnon-Euclideanmodelsofgovernmentandeconom ics. 'Non-Euclidean' becamea intellectualfashion ⁸.Later,newlogics bywordfornewandrelativetruth, the very latest wouldbecreatedaswell, by changing the axioms of theclassicallogicalsystemthat Aristotlehaddefined.

Outofthesestudiesemergedadeeperapprec consistentlydefinedandclearlystated. The tradit descriptionofhowtheworld'was'hadtobesupers thatrecognisedmathematicstobeanunlimitedsyst infinitenumberofpossibleaxiomaticsystemsthat patternsappeartobemadeuseofinNature,butmo Euclideangeometryhadbeenassumedtobepartoft anduniquelyrelatedtoreality.Butthedevelopmen non-standardlogicsmeantthatmathematicalexisten logicalself-consistency(ieitmustnotbepossibl anynecessaryrequirementofphysicalexistence.

onsandproofscouldbecarried struction, or using only arithmetic d.Thepowerandscopeofasystem oningcouldencompass. ofEuclidean systemwasnotthenviewed, asitis es.Euclideangeometrywashowthe abouttheUniverse.Thisgaveita n,largelyunchangedformorethan2000 ofphilosophyandtheology. The importantcornerstoneforthebeliefsof dgraspsomethingoftheultimate dthepowerofourmindstopenetrate, concreteexampleofhowandwhere ingshadalreadybeenpossible.Asa auss, and Riemann, that other postulatewasnotincluded.hadamajor thetruth:itwassimplyamodelforsomepartsof angup, nourished by the demonstration merelyoneofmanypossible asafarmoreappropriatemodelfor Euclid's.Curiousbooksappeared

> iationoftheneedforaxiomstobe ionalrealistviewofmathematicsasa ededbyamoresophisticatedview emofpatternswhicharisefromthe canbedefined.Someofthose starenot.Mathematicalsystemslike heabsolutetruthabouttheworld tofnon-Euclideangeometriesand cenowmeantnothingmorethan etoprovethat0=1).Itnolongerhad

Hilbert'sprogramme

Thecarefulstudyofaxiomaticsystemsrevealedtha developmentofplanegeometrymadeuseofunstated gaveaverysimpleexampleofanintuitively 'obvio

tevenEuclid'sbeautiful axioms.In1882,MoritzPasch us'propertyofpointsandlinesthat couldnotbeprovedfromEuclid'sclassicalaxioms straight line such that Blies between A and C andpossibletoprovethatBliesbetweenAandD.The inevitable but that is not a substitute for a proof

⁹.IfthepointsA,B,C,andDlieona CliesbetweenBandDthenitisnot pictureofthesetupmadeitappear

Α **BCD**

Paschwantedtodistinguishbetweenthelogicalcon andthosepropertiesthatwejustassumedwereintu argumentationshouldnotdependonanyphysicalint quantities involved. He was concerned that axiomati hasbeendescribedas'thefatherofrigoringeome DavidHilbert,thegreatestmathematician of writingsbothdirectlyandthroughtheireffectson beganasystematicprogrammein1899toplacemathe footing 12. This was an ewemphasis, conveyed by Hilbert's re 'Onemustbeabletosay..--insteadofpoints,s andbeermugs' ¹³.Hebelievedthatitwouldbepossibletodetermin underlyingeachpartofmathematics(andhenceoft axiomsareself-consistent, and then show that the deductionsformedfromtheseaxiomsisbothcomplet systemis consistentifwecannotprovethatastatementSanditsnega truetheorems. Itis complete if for every statement Swecan forminits langua oritsnegation, ~Sisatruetheorem.Itis formedinitslanguage, we can prove whether Sist

Hilbert's formalistic vision of mathematics wa outwithimpeccablelogicalconnectionsfromthede was defined to be the collection of all those deductions. Hilb formalisationofmathematicswiththehelpofother ultimatelybepossibletoextenditsscopetoinclu builtuponappliedmathematics. Hebegan with Eucli placingitonarigorousaxiomaticbasis. Hisprogr systembyaddingadditionalaxioms, showing at each decidabilityremained,untileventuallythesystem encompassthewholeofarithmetic.

decidableitmustbecomplete.

Hilbert'sprogrammebeganconfidentlyandhe oftimebeforeallofmathematicswascorralledwit worldwassoonturneduponitsheadbytheyoungKu oftheearlystepsinHilbert'sprogrammeasparto consistencyandcompletenessof1 wouldshowthatitwasnotdecidable).Butthenext fameasthegreatestlogicianofmoderntimes.Far toachieveitskeyobjective-aproofofthecompl thatanysystemrichenoughtocontainarithmeticm

sequencesoftheaxiomsofgeometry itivelytrue.Forhim,mathematical erpretationorvisualisationofthe csystemsshouldbecompleteand try'byFreudenthal ¹⁰. theday,felttheinfluence ofPasch's Peano'swork ¹¹from1882to1899,and maticsuponaformalaxiomatic markthatinmathematics traightlinesandplanes--tables, chairs, etheaxioms hewhole).demonstratethatthese resultingsystemofstatementsand eanddecidable.Moreprecisely,a tion, ~S, are both ge,eitherS decidable if for every statement Sthat can be rueorfalse. Thus, if asystemis

> sofatightwebofdeductionsspreading finingaxioms.Indeed,mathematics ertsetouttocompletethis s,andbelievedthatitwould descienceslikephysics ¹⁴whichwere deangeometryandsucceededin ammethenimaginedstrengtheningthe stepthatconsistencyand hadbecomelargeenoughto

believedthatitwouldjustbeamatter hinitsformalisticweb.Alas.the rtGödel.Gödelhadcompletedone fhisdoctoralthesis, by proving the st-orderlogic(laterAlonzoChurchandAlanTuring stepsthathetookhaveensuredhis fromextendingHilbert'sprogramme etenessofarithmetic-Gödelproved ustbeincompleteandundecidable.

Thistookalmosteveryonebysurprise,includingJo theconferenceinKönigsberg(Hilbert'shometown) brieflycommunicatedhisresults,andquicklyappre onlytofindthatGödelhadalreadymadetheextens Finsler(whotriedunsuccessfullytoconvinceGödel beforehim),andeffectivelykilledHilbert'sprogr

hnvonNeumann,whowaspresentat on7 thSeptember1930whenGödel ciatedthem--evenextendingthem, ioninaseparatepaper--andPaul thathehaddiscoveredtheseresults ammewithonestroke.

Theory	IsitConsistent?	IsitComplete?	IsitDecidable?
Propositional	Yes	Yes	Yes
calculus			
Euclidean	Yes	Yes	Yes
geometry			
1 st orderlogic	Yes	Yes	No
Arithmetic	Yes	Yes	Yes
(+,-)only			
Arithmetic	??	No	No
Infull(+,-, \times ,÷)			

Table 1: Summary of the results established about the decidability of simple logical systems.

heconsistency, completeness, and

2:SomeMathematicalJujitsu

TheOptimistsandthePessimists

Gödel'smonumentaldemonstration,thatsystem infiltratedthewayinwhichphilosophersandscien understandit.Somecommentatorsclaimedthatitsh Universemustbelimited.Scienceisbasedonmathe truths;thereforesciencecannotdiscoveralltruth sequence with the sequence of th

'Clearlythennoscientificcosmology,whichofnec itsproofofconsistencywithinitselfasfarasma consistency,allmathematicalmodels,alltheories quarksandgluons...fallinherentlyshortofbeing

sofmathematicshavelimits,gradually tistsviewedtheworldandourquestto owsthatallhumaninvestigationsofthe matics;mathematicscannotdiscoverall s.OneofGödel'scontemporaries,Hermann sconstantdrainontheenthusiasmand '.Hebelievedthatthisunderlying Hilberthadissuedtomathematiciansin 1900, ifferenttowhattheirscientificendeavours ng,sufferingandcreativeexistenceinthe yandsciencewithatraditional Catholic preventsusfromgaininganunderstanding of

essitymustbehighlymathematical,canhave thematicsgoes.Intheabsenceofsuch ofelementaryparticles,includingthetheoryof thattheorywhichshowsinvirtueofitsapriori

in

truththattheworldcanonlybewhatitisandnot happenedtoaccountwithperfectaccuracyforallp particular time. 16

hingelse. This is true even if the theory henomena of the physical world known at a

Itconstitutesafundamentalbarriertounderstandi

ngoftheUniverse,for:

ItseemsthatonthestrengthofGödel'stheoremt symbolicconstructionsofmathematicalphysicswill ofthinkingcharacterizedbothbythewisdomandby thespeculativephysicistthisimpliesthattherea thepurethinkingoftheoreticalphysicsthereisa bethescientisthimself,asathinker..' 17

hattheultimatefoundationsofthebold

Il remainembeddedforeverinthatdeeperlevel thehazinessofanalogiesandintuitions.For relimitstotheprecisionofcertainty,thateven boundary...Anintegralpartofthisboundaryis

Intriguingly, and just to show the important ro the significance of limits, other scientists, like places limits on our ability to discover the truths interpret this as ensuring that science will go on theorem as an insurance policy against the scient if much, coming to a self-satisfied end; for

lehumanpsychologyplaysinassessing FreemanDyson,acknowledgethatGödel ofmathematicsandscience,but forever.Dysonseestheincompleteness icenterprise,whichheadmiresso

Gödelprovedthattheworldofpuremathematicsis ofinferencecaneverencompassthewholeofmathem meaningfulmathematicalquestionswhichtheaxioms situationexistsinthephysicalworld. If myview ophysics and astronomy is also in exhaustible; no mat always benew things happening, new information comexpanding domain of life, consciousness, and memory

inexhaustible;nofinitesetofaxiomsandrules
atics;givenanysetofaxioms,wecanfind
leaveunanswered.Ihopethatananalogous
ofthefutureiscorrect,itmeansthattheworldo
terhowfarwegointothefuture,therewill
ingin,newworldstoexplore,aconstantly

Thus, we see epitomised the optimistic and 'optimists', like Dyson, see his result as a guaran human investigation. They sees cientific research a spirit which, if it were completed, would have a dijust a sit did upon Weyl. The 'pessimists', like Ja interpret Gödelasest ablishing that the human mind most) of the secrets of Nature.

Gödel'sownviewwasasunexpectedasever. can'see'truthsofmathematicsandscience,wasa asformallyandreverentlyaslogicitself, hepessimisticresponsestoGödel.The torofthenever-endingcharacterof spartofanessentialpartofthehuman sastrousde-motivatingeffectuponus– ki,Lucas ¹⁹,andPenrose ²⁰,bycontrast, cannotknowall(maybenoteven

He thought that intuition, by which we tool that would one day be valued just

Idon'tseeanyreasonwhyweshouldhavelesscon mathematicalintuition,thaninsenseperception,w andtoexpectthatfuturesenseperceptionswillag questionnotdecidablenowhasmeaningandmaybed

fidenceinthiskindofperception,i.e.,in hichinducesustobuildupphysicaltheories reewiththemand,moreover,tobelievethata ecidedinthefuture.'

However, it is easy to use Gödel's theorem in ways underlying assumptions of his theorem. Many specula spanning the fields of philosophy, theology, and coinal ucideritical fashion by the late Torkel Franzé

thatplayfastandloosewiththe la tiveapplicationscanbefound mputingandtheyhavebeenexamined zén²².

Gödelwasnotmindedtodrawanystrongconclu sionsforphy incompletenesstheorems. Hemadenoconnectionswit quantummechanics, which was advertised as another ability to know, and which was discovered by Heisen made his discovery. In fact, Gödelwas rather hosti mechanics at all. Those who worked at the same Inst really worked with Gödel) believed that this was are sult of his frequent to disbelieving quantum mechanics and the Uncerta records 23 this account of Wheeler's attempt to draw Gödelou to the reisa connection between Gödelin completeness and Heisen Principle,

sionsforphysicsfromhis
t htheUncertaintyPrincipleof
greatdeductionwhichlimitedour
bergjustafewyearsbeforeGödel
letoanyconsiderationofquantum
ituteforAdvancedStudy(noone
altofhisfrequ entdiscussionswith
wthemboth)'brainwashedGödel'
intyPrinciple.GregChaitin
delou tonthequestionofwhether
andHeisenberg'sUncertainty

'Well,onedayIwasattheInstituteforAdvanced wasGödel.ItwaswinterandGödelhadanelectric Isaid'ProfessorGödel,whatconnectiondoyousee Heisenberg'suncertaintyprinciple?'AndGödelgot

Study,andIwenttoGödel'soffice,andthere heaterandhadhislegswrappedinablanket. betweenyourincompletenesstheoremand angryandthrewmeoutofhisoffice!'

Theclaimthatmathematicscontainsunprovable mathematics--thereforephysicswillnotbeablet beenaroundforalongtime.Moresophisticatedver whichexploitthepossibilityofuncomputablemathe makepredictionsaboutobservablequantities.From hasconjecturedthat ²⁵

statements--physicsisbasedon odiscovereverythingthatistrue,has sionsofithavebeenconstructed maticaloperationsbeingrequiredto thisvantagepoint,StephenWolfram,

'Onemayspeculatethatundecidabilityiscommonin Evensimplyformulatedproblemsintheoreticalphys

allbutthemosttrivialphysicaltheories. icsmaybefoundtobeprovablyinsoluble.'

Indeed,itisknownthatundecidabilityistherule truthsofarithmetic ²⁶.

rather than the exception amongst the

DrawingtheLineBetweenCompletenessandIncomplet eness

Withtheseworriesinmind,letuslookalit tlemorecloselyatwhatGödel'sresult mighthavetosayaboutphysics.Thesituationisn wouldoftenhaveusbelieve.Itisusefultolayout thepreciseassumptionsthatunderlie Gödel'sdeductionofincompleteness.Gödel'stheore msaysthatifaformalsystemis

- 1. finitelyspecified
- 2. largeenoughtoincludearithmetic
- 3. consistent

thenitis incomplete.

Condition1 means that there are not an uncomputa for instance, choose our system to consist of all the cause this collection cannot be finitely listed in Condition2 means that the formal system includes a rithmetic. The symbols are 0, 'zero', S, 'success'

bleinfinityofaxioms. We could not, he true statements about arithmetic nthe required sense.

esallthesymbolsandaxiomsusedin orof',+, ×,and=.Hence,thenumber

twoisthesuccessorofthesuccessorofzero, writ twoequalsfourisexpressedasSS0+SS0=SSSS0. Thestructureofarithmeticplaysacentralro Specialproperties of numbers, like their factorisa befactoredinonlyonewayastheproductofprim usedbyGödeltoestablishacrucialcorrespondence and statements about mathematics. Thereby, linguist couldbeembedded,likeTrojanhorses,withinthes logicalsystemswhicharerichenoughtoincludear encoding of statements about themselves to be madeAgain, it is instructive to see how the sereq pickedatheorythatconsistedofreferencesto(a numbers(0,1,2,3,4,5,6,7,8,9)witharithmetic modul amini-arithmeticiscomplete.Arithmeticmakesst terms(likeSS0,above). If a system does not have Euclideangeometry, only makes statements about ac lines,ingeneral,thenitcannotsatisfyCondition showed, Euclidean geometry is complete. There is no Euclideannatureofthegeometryeither:thenon-Eu surfaces are also complete. Completeness can belon geometryinvolvingnsymbolscantakeuptoexp[exp truthorfalsitydetermined ²⁷.Forjustn=10,thisnumberamountstoastagger 10⁹⁵⁶⁵; for comparison, there have only been about 10 beginningoftheUniverse's expansion history. Similarly, if we had a logical theory dealing of'greaterthan', without referring to any specifi can determine the truth or falsity of any statement'greaterthan'relationship.

Anotherexampleofasystemthatissmallerth multiplication, x, operation. This is called Presburger calledPeanoarithmeticafterthemathematicianwho 1889). Atfirst this sounds strange, in our every da nothingmorethanashorthandwayofdoingaddition thefulllogical system of arithmetic, in the prese exists'or'forany', multiplication permits constr toasuccessionofadditions.

Presburgerarithmeticiscomplete:allstat numberscanbeprovedordisproved; all truthscan Similarly, if we create another truncated version o addition, butretains multiplication, this is also multiplicationaresimultaneouslypresentthatinco systemfurtherbyaddingextraoperationslikeexpo operationsmakesnodifference.Incompletenessrema itisfound. Arithmeticisthewatershedincomplex TheuseofGödeltoplacelimitsonwhata anythingelse)canultimatelytellusseemsafairl

tenasthe termSS0,andtwoandplus

leintheproofofGödel'stheorem. tionsandthefactthatanynumbercan edivisors(eg.130=2 $\times 5 \times 13$), were betweenstatementsofmathematics icparadoxeslikethatofthe'liar' tructureofmathematicsitself.Only ithmeticallowthisincestuous withintheirownlanguage. uirementsmightfailtobemet.Ifwe ndrelationsbetween)onlythefirstten o10.thenCondition2failsandsuch atements about individual numbers, or individualtermslikethisbut,like ontinuumofpoints, circles, and 2. Accordingly, as Alfred Tarski first thingmagical about the flat, clideangeometriesoncurved g-windedthough.Astatementof

[n]]computationalstepstohaveits ing $9.44 \times$ ²⁷nanosecondssincetheapparent

withnumbersthatonlyusedtheconcept cnumbers, then it would be complete: we aboutrealnumbersinvolvingjustthe

anarithmeticisarithmeticwithoutthe ²⁸arithmetic(thefullarithmeticis firstexpresseditaxiomatically, in yencounterswithmultiplicationitis $(eg2+2+2+2+2+2=2 \times 6)$, but in nceoflogicalquantifierslike'there uctionswhicharenotmerelyequivalent

ementsabouttheadditionofnatural bereachedfromtheaxioms farithmetic, which does not have complete. It is only when addition and mpletenessemerges.Extendingthe nentiationtotherepertoireofbasic insbutnointrinsicallynewformof ity. mathematicaltheoryofphysics(or ystraightforwardconsequence.Butas

onelooksmorecarefullyintothequestion,things themoment, that all the conditions required for Gö Whatwouldincompletenesslooklikeinpractice?W havingaphysicaltheorywhichmakesaccuratepredi observedphenomena:wemightcallit'thestandard surprisedbyanobservationaboutwhichithasnoth accommodated within its framework. Examples are pro unified theories' in particle physics. Some early e propertythatallneutrinosmusthavezeromass.No non-zeromass(asexperimentshavenowconfirmed)t cannotbeaccommodated within our original theory. encounteredacertainsortofincompleteness, butw modifyingthetheorytoincludethenewpossibiliti looksverymuchlikeinadequacyinatheory.Itwou incompletenessifwecouldfindnoextensionofthe observedfact.

Aninterestingexampleofananalogousdilem mathematics. Duringthesixteenthcentury, mathemat happened when they added to gether infinite lists of getlarger then the sum will 'diverge', that is, as ther so does the sum. An example is the sum

arenotquitesosimple.Suppose,for del'stheoremtoholdareinplace.
earefamiliarwiththesituationof ctionsaboutawiderangeof model'.Oneday,wemaybe ingtosay.Itcannotbe oro videdbysomesocalled'grand ditionsofthesetheorieshadthe wifaneutrinoisobservedtohavea henweknowthatthenewsituation Whatdowedo?Wehave erespondtoitbyextendingor es.Thus,inpractice,incompleteness ldbecomemorelikeGödel theorythatcouldpredictthenew

maisprovidedbythehistoryof mat iciansstartedtoexplorewhat numbers.Ifthequantitiesinthelist thenumberoftermsapproachesinfinity

However, if the individual terms gets maller and sm sum of an infinite number of terms can get closer a which we shall call the sum of the series; for exam

1+1/9+1/25+1/36+1/49+....=

Thisleftmathematicianstoworryaboutamostpecu

allersufficientlyrapidly ³⁰thenthe ndclosertoafinitelimitingvalue ple

$$\pi^2/8=1.2337005...$$

liartypeofunendingsum,

Ifyoudivideuptheseriesintopairsoftermsit
Thisisjust0+0+0+...=0andthesumiszero
+1-...}anditlookslike1-{0}=1.Weseemto
Mathematicianshadavarietyofchoiceswhen
Theycouldrejectinfinitiesinmathematicsanddea
asCauchyshowedintheearlynineteenthcentury,t
mustbedefinedbyspecifyingmorecloselywhatis
ofthesummustbespecifiedtogetherwiththeproc
contradiction0=1arisesonlywhenoneomitstos
thesum.Inbothcasesitisdifferent,andsothe
theyariseindifferentaxiomaticsystems.Thus,he
limitisside-steppedbyenlargingtheconceptwhic

lookslike(1-1)+(1-1)+....andsoon. .Butthinkoftheseriesas1-{1-1+1-1 haveprovedthat0=1.

facedwithambiguoussumslikethis. lonlywithfinitesumsofnumbers,or, hesumofaserieslikethelastone meantbyitssum. Thelimitingvalue edureusedtocalculateit. The pecifytheprocedureusedtoworkout twoanswersarenotthesamebecause reweseeasimpleexampleofhowa hseemstocreatelimitations.

Divergentseriescanbedealtwithconsistentlyso seriesissuitablyextended ³¹. AnotherpossiblewayofevadingGödel'stheore

useofthedecidablepartofmathematics. Weknowt possiblestructures. Onlysomeofthosestructures and application in the physical world. It may be the decidable truths. Things may be even better protect patterns are instantiated in physical reality? It is also possible that the conditions requires the might be that the laws of physics are not listable aradical departure from the situation that we thin fundamental laws is believed to be not just listable always possible that we are just scratching the sur only the top of which has significant effects upon an unlistable in finity of physical laws then we wouthant hat of incompleteness.

Anequallyinterestingissueisthatoffinite physical possibilities is finite, although astronom largethenumberofprimitivequantitiestowhicht theresultingsystemofinter-relationshipswillbe althoughwehabituallyassumethatthereisaconti isjustanassumptionthatisveryconvenientfort nodeepreasontobelievethatspaceandtimearec mostfundamentalmicroscopiclevel;infact,there thatassumethattheyarenot.Quantumtheoryhasi inanumberofplaceswhereoncewebelievedinac ifwegiveupthiscontinuity, so that there is not twosufficientlyclosepointsyoucaretochoose.S morecomplicated because continuous functions can b rationals.Manymorethingscanhappen.Thisquesti upwiththequestionofwhethertheuniverseisfin of elementary particles (or whatever the most eleme finiteorinfiniteinnumber. Thus, theremighton whichtheultimatelogicaltheoryofthephysicalw complete.

Afurtherpossibilitywithregardtotheappli cathatCondition2oftheincompletenesstheoremmigh Althoughweseemtomakewideuseofarithmetic,an structures,whenwecarryoutscientificinvestigat meanthattheinnerlogicofthephysicalUniverse Itisundoubtedlyconvenientforustouselargema conceptslikeinfinitybutthismaybeananthropom Universemayberootedinamuchsimplerlogicthan complete.Allthiswouldrequirewouldbefortheu additionormultiplicationbutnotboth.Recalltha

longastheconceptofasumfora

misifthephysicalworldonlymakes hatmathematicsisaninfiniteseaof andpatternsappeartofindexistence attheyareallfromthesubsetof edthanthat:perhapsonlycomputable

iredtoproveGödelincompletenessdonot
eaxiomsofthetheorytobelistable.It
inthispredictablesense.Thiswouldbe
kexists,wherethenumberof
e,butfinite(andverysmall).Butitis
faceofabottomlesstoweroflaws,
ourexperience.However,iftherewere
Idfaceamoreformidableproblem

ness.Itmaybethattheuniverseof icallylarge ³². However, nomatter how helawsrefer, solong as they are finite, complete. We should stress that nuumofpointsofspaceandtimethis heuseofsimplemathematics. Thereis ontinuous,ratherthandiscrete,attheir aresometheoriesofquantumgravity ntroduceddiscretenessandfiniteness ontinuumofpossibilities. Curiously, necessarilyanotherpointinbetweenany pace-timestructurebecomes infinitely edefinedbytheirvaluesonthe onoffinitenessmightalsobebound iteinvolumeandwhetherthenumber ntaryentitiesmightbe)ofNatureare lyexistafinitenumberoftermsto orldapplies.Hence,itwouldbe

cationofGödeltothelawsofphysicsis
gh tnotbemet.Howcouldthisbe?
n dmuchlargermathematical
ionsofthelawsofNature,thisdoesnot
needstoemploysuchalargestructure.
thematicalstructurestogetherwith
orphism.Thedeepstructureofthe
thatoffullarithmetic,andhencebe
nderlyingstructuretocontaineither
tallthesumsthatyouhaveeverdone

haveusedmultiplicationsimplyasashorthandfor Presburgerarithmeticaswell. Alternatively, abas simplerelationshipsofageometrical variety, orw than'relationships, or subtle combinations of them factthatEinstein'stheoryofgeneralrelativityr andweightby geometrical distortions in the fabric of space-time may well h clueaboutwhatispossiblehere.

Thereisasurprisinglyrichrangeofpossib mathematicalphysicsintermsofsystemswhichmigh Tarskishowedthat,unlikePeano'sarithmeticofad numbers.thefirst-ordertheoryofrealnumbersund decidable. This is rather surprising and may gives basedontherealsorcomplexnumberswillevadeun wentontoshowthatmanymathematicalsystemsused projective geometry, and Abelian group theory ared Abeliangrouptheoryarenot ³⁴.Littleconsiderationseemstobehavegiventoth consequences of these results to the development of Thereisanotherimportantaspectofthesit logical system is complete, it always contain sunpr whicharechosentodefinethesystem. And after th candoisdeduceconclusionsfromthem.Insimplel theaxiomsseemreasonablyobviousbecausewearet somethingthatwehavebeendoingintuitivelyfort subjectlikephysics, there are parallels and diffe aretheprimetargetofphysicsresearch. They are becausetheygovernregimesthatcanliefaroutsid thoselawsareunpredictableincertaincircumstanc breakings. Trying to deduce the laws from the outco everdouniquelyandcompletelybymeansofacomp Thus, we detect a completely different emphasi physicalscience.Inmathematicsandlogic,westar lawsofdeduction. Then, we might try to show that anddeduceasmanytheoremsaswecanfromtheaxio topickanylogicalsystemoflawsthatwechoose. lawsandaxioms(assumingthereisone--ormoret theoutcomesthatwesee. As west ressed earlier, i lawswhichwillgiverisetoanysetofobservedou unprovablestatementsthatthelogiciansandthema lawsofdeduction--thatthescientistismostint assuming. The only hope of proceeding as the logici thereisonlyonepossiblesetofaxiomsorlawsof likely³⁵; evenifitwere we would not be able to prove.

addition. Theywould be possible in icstructureofrealitythatmadeuseof hichderivedfrom'greaterthan'or'less 33 .The allcouldalsoremaincomplete eplacesmanyphysicalnotionslikeforce oldsome

ilitiesforabasicrepresentationof tbedecidableorundecidable. ditionandmultiplicationofnatural eradditionandmultiplicationis omehopethattheoriesofphysics decidabilityingeneral. Tarskialso inphysics, likelattice theory, ecidable, while others, notably nonultimatetheoriesofphysics. uationtobekeepinview.Evenifa ovable'truths'. These are the axioms eyarechosen, all the logical system ogicalsystems, like Peanoarithmetic, hinkingbackwards--formalising housandsofyears. When we look at a rences. Theaxioms, or laws, of physics bynomeansintuitivelyobvious, eofourexperience. The outcomes of esbecausetheyinvolvesymmetry mesisnotsomethingthatwecan uterprogramme. sinthestudyofformalsystemsandin tbydefiningasystemofaxiomsand thesystemiscompleteorincomplete, ms.Inscience, wearenotatliberty Wearetryingtofindthesystemof hanoneperhaps)thatwillgiveriseto tisalwayspossibletofindasystemof tcomes.Butitistheverysetof thematiciansignore--theaxiomsand erestedindiscoveringratherthansimply ansdo, would be if for some reason

physics.Sofar,thisdoesnotseem

3:LawsversusOutcomes

Symmetry Breaking

Thestructureofmodernphysicspresentswithanim portantdichotomy.Itis importanttoappreciatethisdivisioninordertou nderstandthesignificanceofGödel incompletenessforphysics. The fundamental laws of Naturegoverningtheweak, strong, electromagnetic, and gravitational forces, are all localgaugetheoriesderivedfromthe maintenanceofparticularmathematicalsymmetries. Astheseforcesbecomeunified,the numberofsymmetriesinvolvedwillbereduceduntil ultimately(perhaps)thereisonly oneover-archingsymmetrydictatingtheformthela wsofNature-aso-called'Theoryof Everything',ofwhichMtheoryisthecurrentcandi date.ThusthelawsofNatureareina realsense'simple'andhighlysymmetrical. Theult imatesymmetrywhichunitesthem mustpossessanumberofpropertiesinordertoacc ommodateallthelow-energy hatlooklikeelementary'particles'with manifestations of these parate forces, the statest alltheirproperties, and it must be big enough for themalltofitin. ThereisnoreasonwhyGödelincompleteness shouldhamperthesearchforthisallencompassingsymmetrygoverningthe *laws* of Nature. This search is, at root, as earch forapattern, perhaps agroupsymmetry or some oth ermathematicalprescription.Itneed notbecomplicated and it probably has a particular mathematicalpropertythatmakesit specially(orevenuniquely)fittedforthispurpos Inreality, wenever'see'lawsofNature: theyinhabitaPlatonicrealm.Rather,we witnesstheir outcomes. This is an important distinction, because the ou tcomesarequite differenttothelawsthatgovernthem. They areas ymmetricalandcomplicatedandneed possessnoneofthesymmetriesdisplayedbythelaw s. This is fortunate because, we reit notso, we could not exist. If the outcomes of the lawsofNaturepossessedallthe symmetriesofthelawsthennothingcouldhappenwh ichdidnotrespectthem. There andplaces, nodirectional asymmetries, couldbenostructureslocatedatparticulartimes and nothing happening at anyone moment. All would beunchangingandempty. Thisdichotomybetweenlawsandoutcomesis whatIwouldcall'thesecretofthe universe'. Itiswhatenablesa Universeto begove rnedbyaverysmallnumber(perhaps justone)ofsimpleandsymmetricallaws, yetgive risetoanunlimitednumberofhighly complex, asymmetrical states—of which we are one variety³⁶. Thus, whereas there is no reason to worry aboutGödelincompletenessfrustrating thesearchforthemathematicaldescriptionsofthe lawsofNature, we might well expect Gödelincompletenesstoariseinourattemptstode scribesomeofthecomplicated sequencesofeventsthatariseasoutcomestothel awsofNature.

UndecidableOutcomes

Specificexampleshavebeengivenofphysical theirunderlyinglawsareundecidable. Asonemight theydonotinvolveaninabilitytodeterminesomet thelawsofphysics, or eventhemostelementary pa aninability to perform some specific mathematical to determine the course of events in a well-defined the problem may be mathematically well defined, this create the precise conditions required for the unde

problemsinwhichtheoutcomesof expectfromwhathasjustbeensaid, hingfundamentalaboutthenatureof rticlesofmatter.Rather,theyinvolve calculation,whichinhibitsourability physicalproblem.However,although sdoesnotmeanthatitispossibleto cidabilitytoexist. Aninterestingseriesofexamplesofthissor mathematicians, Francisco Doriaand N. da Costa posedbytheRussianmathematicianVladimirArnold, possibletohaveageneralmathematicalcriterionw equilibriumwasstable. Astable equilibrium is as ofabasin-displaceitslightlyanditreturnsto aneedlebalancedvertically-displaceitslightly Whentheequilibriumisofasimplenaturethispro sciencestudentslearnaboutit.But, when the equ complicatedcouplingsbetweenthedifferentcompeti becomes more complicated than the situation studied thereareonlyafewcompetinginfluencesthestabi decidedbyinspectingtheequationsthatgovernthe discoveranalgorithmwhichtellsusifthiscanal competing influences there are, and no matter how c 'discover'hemeantfindaformulaintowhichyouc theequilibriumalongwithyourdefinitionofstabi answer'stable'orunstable'.

DaCostaandDoriadiscoveredthattherecane equilibriacharacterisedbyspecialsolutionsofma undecidable. In order for this undecidability to ha interestinmathematicalphysicstheequilibriahav numbersofdifferentforces. Whilesuchequilibria arisenyetinrealphysicalproblems.DaCostaand problemswheretheanswertoasimplequestion, lik chaotic',isGödelundecidable.Theycanbeviewed thetheoremsofRice ⁴⁰andRichardson ⁴¹whichshow,inaprecisesense,thatonlytrivial properties of computer programs are algorithmically identifyformallyundecidableproblems.Gerochand quantumgravitythatpredictthevaluesofpotentia termswhoselistingisknowntobeaTuringuncompu Richards⁴³showedthatverysimpledifferentialequations,wh physics, like the wave equation, can have uncomputa notverysmooth. This lack of smoothness gives rise posed'problem. It is this feature that gives rise andWozniakowski ⁴⁴haveshownthateveryill-posedproblemiswell-po averageunderrathergeneralconditions. Wolfram undecidabilityarisingincondensedmatterphysics istypicalinphysicaltheories.

ThestudyofEinstein's'generaltheoryof problemifthemathematicalquantitiesinvolvedare exactsolutionofEinstein'sequationsitisalways another.knownsolutionthatiswritteninadiffe this by hand, but for complicated solutions compute requirecomputersprogrammedtocarryoutalgebraic variousquantitiestodiscoverifagivensolution

thavebeencreatedbytheBrazilian ³⁷.Respondingtoachallengeproblem theyinvestigatedwhetheritwas hichwoulddecidewhetherornotany ituationlikeaballsittinginthebottom thebottom;anunstableequilibriumislike anditmovesawayfromthevertical blemisveryelementary; first-year ilibriumexistsinthefaceofmore nginfluences,theproblemsoon bysciencestudents ³⁹.Solongas lityoftheequilibriumcanstillbe situation.Arnold'schallengewasto waysbedone,nomatterhowmany omplextheirinter-relationships.By anfeedtheequationswhichgovern lity, and out of which will popt he

xistnosuchalgorithm.Thereexist thematicalequationswhosestabilityis veanimpactonproblemsofreal etoinvolvetheinterplayofverylarge cannotberuledout, they have not Doriawentontoidentifysimilar e'willtheorbitofaparticlebecome asphysicallygroundedexamplesof decidable.Othershavealsotriedto Hartlehavediscussedproblemsin llyobservablequantitiesasasumof tableoperation ⁴².Pour-Eland icharewidelyusedin bleoutcomeswhentheinitialdatais towhatmathematicianscallan'illtotheuncomputability. However, Traub sedonthe givesexamplesofintractabilityan andevenbelievesthatundecidability

relativityalsoproducesanundecidable unrestricted ⁴⁶. Whenone finds an necessarytodiscoverwhetheritisjust rentform. Usually, one can investigate rscanhelp.Forthispurposewe manipulations. They can check isequivalenttoonealreadysittinginits

memorybankofknownsolutions.Inpracticalcases procedurecomesupwithadefiniteresultafteras thecomparisonisanundecidableprocesseguivalent problemofpuremathematics, 'thewordproblem' of Dehn⁴⁷in1911andshowntobeundecidablein1955 Thetentativeconclusionsweshoulddrawfr physicsmakesuseofmathematics.itisbvnomeans upontheoverallscopeofphysicstounderstandthe thatNaturemakesuseofmaybesmaller, and simple rearitshead.

encounteredsofar, this checking mallnumberofsteps.But,ingeneral, toanotherfamousundecidable grouptheory, first posed by Max

omthisdiscussionisthat, just because requiredthatGödelplacesanvlimit lawsofNature ⁴⁹.Themathematics rthanisneededforundecidabilityto

4:GödelandSpace-timeStructure

Space-timeinaSpin

AlthoughKurtGödelisfamousamongstlogiciansfor hisincompletenesstheorems, uitedifferentreason.In1949. heisalsofamousamongstcosmologists, butforaq inspiredbyhismanyconversationswithEinsteinab outthenatureoftimeandMach's principle, Gödelfoundanewandcompletelyunsusp ectedtypeofsolutiontoEinstein's equationsofgeneralrelativity ⁵⁰.Gödel'ssolutionwasauniversethatrotatesand permits timetraveltooccurintothepast.

Thiswasthefirsttimethatthepossibilityof thecontextofatheoryofphysics. Theideaofti famous 1895 story, The Time Machine, but it was widely suspected that backwards travelintothepastwouldinsomewaybeinconfli universeshowedthatwasnotnecessarilyso:itcou obeying all the conservation laws of physics. Time uncontroversialmatterandisjustanotherwayofd dilationinspecialrelativity.

Gödel'suniverseisnottheonethatwel notexpanding:foranother,thereisnoevidenceth thenitsrateofspinmustbeatleast10 theisotropyofthemicrowavebackgroundradiation akeydiscoveryinthestudyofspace-timeandgrav perhapsitcouldariseinotheruniverseswhichare ButtheinfluenceofGödel'ssolutionon indirect.Itrevealedforthefirsttimethesubtle particularlywhenrotationispresent. Previously, studiedtendedtobespatiallyhomogeneouswithsim globalsymmetrythatruledoutordisguisedglobal Penrosewouldapplypowerfulnewmethodsofdiffere provethefirstsingularitytheoremsincosmology. curvesinspace-timethatGödelhadrevealedmeant insomeofthesetheoremstoexcludetheirpresence geodesicscouldbeavoidedbyperiodicallyreappear universethatfirstshowedhowunusualspace-times physicallyandfactuallyconsistent.Priortoitsd

timetravel(intothepast)hademergedin metravelfirstappearedinH.G.Wells' ⁵¹time ctwiththelawsofNature.Gödel's ldariseasaconsequenceofatheory travelintothefutureisarelatively escribingtheobservedeffectsoftime

ivein.Foronething,Gödel'suniverseis atourUniverseisrotating-andifitis, ⁵timesslowerthanitsexpansionratebecauseof ⁵².Nonetheless, Gödel's universe was itation.Iftimetravelwaspossible, viabledescriptionsofourown? thedevelopmentofthesubjectwasmore tyofthe *global*structureofspace-time, thecosmologicalmodelsthatwere pletopologiesandhighdegreesof structure.Later,in1965,Roger ntialtopologytothisproblemand Thepossibility of closed time-like thatspecificvetoshadtobeincluded ,otherwise pastin completeness of inginthefuture.ItwasGödel's couldbewhilststillremaining iscovery, physicists and philosophers of

scienceregardedtimetraveltothepastasthenec essaryharbingeroffactual contradictions.ButGödel'ssolutionshowsthatthe reexistself-consistenthistorieswhich are periodicins paceand time ⁵³.It continues to be studied as a keyexample of an intrinsically general-relativistic effect and its ull stability properties have been elucidated only recently ⁵⁴. Some of its unusual properties are explained int heaccompanying article by Wolfgang Rindler.

Inrecentyears, Gödel's study of space -ti incompleteness of logical systems have been pulled been shown that there is a link between the global of computations that can be completed within them. strange old problem with a new name: is it possible finite amount of time? And then ewn ame for such a 'supertask'.

-timestructureandhisworkonthe
ed togetherinafascinatingway.Ithas
structureofaspace-timeandthesorts
n. Thisunexpectedlinkarisesfroma
todoaninfinitenumberofthingsina
remarkableoldactivityisa

Supertasks

Theancients, beginning with Zeno, were challenged bytheparadoxesofinfinities onmanyfronts ⁵⁵.Butwhataboutphilosopherstoday?Whatsortofp roblemsdothey worryabout? There are live is sue son the interface betweenscienceandphilosophythat areconcernedwithwhetheritispossibletobuild an"infinitymachine"thatcanperform aninfinitenumberoftasksinafinitetime.Ofco urse, this simple questionneeds some "tasks","number","infinite","finite" clarification: what exactly is meant by "possible", and, by nomeans least, by "time". Classical physic sappearstoimposefewphysical limitsonthefunctioningofinfinitymachinesbeca usethereisnolimittothespeedat which signals can travelors witches can move. New t on'slawsallowaninfinitymachine. ThiscanbeseenbyexploitingadiscoveryaboutNe wtoniandynamicsmadein1971by theUSmathematicianJeffXia ⁵⁶.Firsttakefourparticlesofequalmassandarran gethem intwobinarypairsorbitingwithequalbutopposit ely-directedspinsintwoseparate parallelplanes, so the overall angular momentum is zero.Now,introduceafifthmuch lighterparticlethatoscillatesbackandforthalo ngaperpendicularlinejoiningthemass centresofthetwobinarypairs.Xiashowedthatsu chasystemoffiveparticleswill expandtoinfinitesizeina finitetime!

Howdoesthishappen?Thelittleoscillatingpartic lerunsbackandforthbetween thebinarypairs, eachtimecreatinganunstableme particlethengetskickedbackandthebinarypair recoilsoutwardstoconserve momentum.Thelighterparticlethentravelsacross totheotherbinaryandthesame *ménageàtrois* isrepeatedthere.Thiscontinueswithoutend, acc eleratingthebinarypairs apartsostronglythattheybecomeinfinitelysepar atedwhilethelighterparticle finitetimebeforethesystemachieves infinitesize.

Unfortunately(orperhapsfortunately),thisbehavi ourisnotpossiblewhen relativityistakenintoaccount.Noinformationca lightandgravitationalforcescannotbecomearbitr motionandgravitation;norcanmassesgetarbitrar thereisalimittohowcloseseparationcanget,a enclosestheparticlestoformablackhole solution in the stransmitted fasterthanthespeed of arilystrongin Einstein's theory of ilyclosetoeachotherandrecoil-fterwhichan eventhorizon surface enclosestheparticlestoformablackhole solution in the stransmitted fasterthanthespeed of arilystrongin Einstein's theory of ilyclosetoeachotherandrecoil-fterwhichan eventhorizon surface enclosestheparticlestoformablackhole solution in the stransmitted fasterthanthespeed of arilystrongin Einstein's theory of ilyclosetoeachotherandrecoil-fterwhichan eventhorizon surface encloses the particles to formation to the outside world solution.

Butthisdoesnotmeanthatallrelativisticinfini Indeed, the Einsteinian relativity of time that is whattheirmotion, open supsome interesting new po tasksinfinitetime.Coulditbethatonemovingo computationsoccurringeventhoughonlyafinitenu someoneelse?Misner ⁵⁹andBarrowandTipler entireuniversesinwhichaninfinitenumberofosc singularitiesinspace-timebutitisnecessaryfor ineffect, the whole universe is the infinity machi infinitymachinecouldexistandsendussignalsas numberofoperationsinafiniteamountofourtime

Thefamousmotivatingexampleofthissortoftempo 'twinparadox'. Twoidentical twins are given diffe athomewhileTweedleawaygoesawayonaspaceflig light. When they are eventually reunited, relativit Tweedlehometobemucholder. The twinshave experi andtimebecauseoftheaccelerationanddecelerati hisroundtrip.

Socanweeversendacomputeronajourneysoextr aninfinitenumberofoperationsbythetimeitret urnstoitsstav-at-homeowner?Itamar Pitowskyfirstargued ⁶¹thatifTweedleawaycouldacceleratehisspaceship strongly, then he could record a finite amount of t whilehistwinrecordsaninfiniteamountoftimeo permittheexistenceofa"Platonistcomputer"-on numberofoperationsalongsometrajectorythrough thatwecouldseebackhome. Alas, there is a probl withthecomputer, healso has to accelerate dramat information. Eventually the gravitational forces be

Notwithstandingtheseproblemsacheck-listofpro universesthatcanallowaninfinitenumberoftask "supertasks" astheyhave become known. These arec universesafterDavidMalament,aUniversityofChi Hogarth⁶², a former Cambridge University research student, w theconditionsunderwhichtheyweretheoretically fascinatingprospectoffindingorcreatingconditi thingscanbeseentobeaccomplishedinafinitet forcomputerscienceandmathematicsbecauseitwou computableanduncomputableoperations. It is somet (seeFigure 1) are self-consistent mathematical pos properties that suggest they are not realistic phys somedisturbingnotions, such as the prospect of th travelbackwardsthroughtime.

Themostserious by-product of being allowed to bui rathermorealarmingthough. Observers who stray in findthatbeingabletoperformaninfinitenumber meansthatanyamountofradiation, nomatterhows

tymachinesareforbidden arequirementofallobservers, nomatter ssibilitiesforcompletinginfinite bservercouldseeaninfinitenumberof mberhadoccurredaccordingto ⁶⁰haveshownthatthereareexamplesof illationsoccuronapproachto theentireuniversetohitthesingularity; ne.Itstillremainstoaskwhetheralocal aresultofcompletinganinfinite

ralrelativityistheso-called rentfuturecareers. Tweedlehomestays htataspeedapproachingthatof ypredictsthatTweedleawaywillfind enceddifferentcareersinspace onthatTweedleawayunderwenton

emethatitcouldaccomplish

sufficiently

heuniverse'shistoryonhisownclock nhisclock.Doesthis, hewondered, ethatcouldcarryoutaninfinite spaceandtimeandprintoutanswers em-forthereceivertostayincontact icallytomaintaintheflowof comestupendousandheistornapart. pertieshasbeencompiledfor stobecompleted infinite time, or alledMalament-Hogarth(MH) cagophilosopher, and Mark ho,in1992,investigated possible. Supertasks ⁶³openthe onsunderwhichaninfinitenumberof ime. This has all sorts of consequences Idremovethedistinctionbetween hingofasurprisethatMHuniverses sibilitiesbut,unfortunately,have icalpossibilitiesunlessweembrace ingshappeningwithoutcauses, and

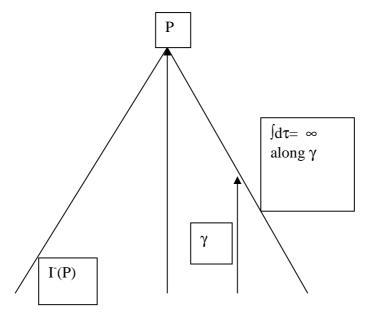
Idaninfinitymachineis tobadpartsoftheseuniverseswill ofcomputationsinafinitetimealso mall, gets compressed to zero

wavelengthandamplifiedtoinfinitefrequencyand computationaltrail. Thus any attempt to transmitt computations will zap the receiver and destroyhim. ruleoutthepracticalityofengineeringarelativi wecouldsafelyreceiveandstoretheinformation. tasksarepossibleinfinitetimeincludesatypeo structureoftheverysuperstringtheoriesthatloo Ifyoucouldseetheoutputfromaninfini youhavethepossibilityofdecidingundecidablepr infinitecatalogueofpossibilities:Turing'suncom completeableinafiniteamountofourwristwatcht Remarkably, Hogarthshowed ⁶⁴insomespace-timesitwaspossibletodecideGöde undecidablequestionsbydirectsearchbysendinga path, y, sothatitcould printout and sendy outheanswe hierarchyofnspace-timestructuresofascendingc sequenceallowsasupertasktobecompletedwhichc th but not the (n+1) arithmeticalassertionmadeinthen logicalhierarchy, by which logicians calibrate the expressions. There is an eatone-to-one corresponde andthecomplexityofthelogicalstatementsthatt Nemeti⁶⁵showedthatsomerelationsonnaturalnumberswhic co-universal, canbedecided in Kerrspace-times. We resultstoshowthatthecomputationalcapabilityo thatofarithmetictohyper-arithmetics, and showed computationalabilityinanyspace-timewhichisa space-time.

Thus,inconclusion,wefindthatGödel's programmesandunsuspectedpropertiesoftheworlds incompletenesstheoremsshouldnotbeadrainonou thelawsofNature:thereisnoreasonforthemto symmetriesofNatureinanysignificantway.But,b complexity, wedoexpectto find that Gödelin compl usethoselawstopredictthefuture,carryoutspe incompletenessbesetstheoutcomesofverysimplel universes, then Gödel's impact will always be felt geometryofspaceandtimewiththeextraordinaryp structureallows. Space-timestructure defines what

energyalongtheinfinite heoutputfromaninfinitenumberof Sofar, the sedire problems seem to sticinfinitymachineinsuchawaythat Buttheuniversesinwhichinfinite fspacethatplaysakeyroleinthe kedsoappealinglyfinite. tymachinethatcompletessupertasksthen oblemsbydirectsearchthroughthe putableoperationsseemtobecome ime.Isthisreallypossible? computeralongacertainspace-time rtothequestion.Now,createa omplexities such that then thin the ancheckthetruthofany st quantifier arithmetic in Kleene's complexityofpossiblelogical ncebetweenthelistofspace-times heycandecide.Subsequently,Etesiand hareneitheruniversalnor lch⁶⁶recentlygeneralisedthese fspace-timescouldberaisedbeyond thatthereisupperboundonthe universalconstantdefinedbythe

ideasarestillprovokingnewresearch oflogicalandphysicalreality.His renthusiasmtoseekoutandcodify limitthatsearchforthefundamental ycontrast, insituations of sufficient etenessplaceslimitsonourabilityto cificcomputations, or build algorithms: awsofNature.Finally,ifwestudy aswetrytoreconcilethesimplelocal ossibilitiesthatitsexoticglobal canbeprovedinauniverse.



 $\begin{tabular}{ll} Figure 1: The space-time of an MH space-time with time mapp & edvertically and space \\ (compressed to one dimension) horizontally. We are & located at Pandour causal past, \\ $\Gamma(P)$ consists of all the events that can influence u s. If there is a path in our past, γ, such that there is an infinite amount of its own time part of the point where it intersects our past line cone, thee deposit $\Gamma(P)$. }$

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                                                        tiallylong,though.Thatis,thecomputationaltim
 Thedecisionprocedureisingeneraldouble-exponen
                                                                                                           e
                                               ^{N)N}. Presburgerarithmetical low sustotal kabout pos
requiredtocarryoutNoperationsgrowsas(2
                                                                                                       itive
integers, and variables whose values are positive i
                                                   ntegers.Ifweenlargeitbypermittingtheconcept
                                                                                                      ofsets
ofintegerstobeused, then the situation becomes
                                                   almostunimaginablyintractable. It has been shown
                                                                                                        that
thissystemdoesnotadmitevenaK-foldexponentia
                                                      lalgorithm,foranyfiniteK.Thedecisionproblem
                                                                                                         is
saidtobenon-elementaryinsuchsituations. Thei
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                                                                                                       an
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